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Optimism, pessimism and financial bubbles*

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Abstract

This paper shows that it is possible to extend the scope of the existence of rational bubbles when uncertainty is introduced associated with rank-dependent expected utility. This RDU assumption can be viewed as a transformation of probabilities depending on the pessimism/optimism of the agent. The results show that pessimism favors the existence of deterministic bubbles, when optimism may promote the existence of stochastic bubbles. Moreover, under pessimism, the RDU assumption may generate multiple bubbly equilibria. The RDU assumption also leads to new conditions ensuring the (absence of) Pareto-optimality of the competitive equilibrium without bubbles. These conditions still govern the existence of bubbles. JEL classification: D81, D9, G1, O41. Keywords: rational bubbles, RDU preferences.

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1 Introduction

This paper shows that the scope for the existence of rational bubbles can be extended when uncertainty and rank-dependent expected utility are introduced. In the framework of an overlapping generations model *à la* Diamond (1965), the seminal article by Tirole (1985) proves that bubbles can arise in economies for which the return on capital at a steady state is below the growth rate of output. The bubbleless economy must be in a state of overaccumulation that corresponds to dynamic inefficiency. Weil (1987) proposes a model of stochastic bubbles using the same framework as Tirole, and finds existence conditions that are even stronger. Different authors have introduced rational bubbles in richer frameworks with endogenous growth (e. g. Grossman and Yanagawa, 1993 and Olivier, 2000). But the existence of bubbles remains linked to the same condition between the growth rate and the interest rate. As empirical observations suggest that this condition is not fulfilled in general (see Abel et al. 1989), rational bubbles seem unlikely to arise. They may perhaps not be the pertinent explanation to understand bubble phenomena that actually are observed.

In recent contributions, Caballero and Hammour (2002) and Caballero et al. (2006) obtain the existence of bubbles under less stringent conditions at the price of a transformation of the notion of bubble. They build an overlapping generations model with an adjustment cost to capital leading to two long-run equilibria. They interpret the equilibrium corresponding to a higher valuation of the capital stock as a bubbly[1] equilibrium.

This paper intends to show that it is possible to extend the scope for the existence of rational bubbles when uncertainty is introduced associated with a rank-dependent expected utility. A simple overlapping generations model is studied in which the production technology depends on a technological shock: capital return is random. In order to have a very tractable model, there are only two possible states of the nature and the capital return may oscillate between a high and a low value. In an economy in which capital is the only asset, financial markets are incomplete as two states of the nature

[1] This paper uses the terminology introduced by Tirole (1985) and employs the expressions of "bubbly equilibrium" and "bubbleless equilibrium".

exist. The existence of a bubbly asset can make financial markets complete. Two types of bubbles are considered in this context. The first type, called a deterministic bubble, is an asset that has the same price in both states of the nature. The second type, called a stochastic bubble, is an asset whose existence is conditional to the occurrence of a particular state of the nature. As in Weil (1987), agents form their expectations according to a self-fulfilling prophecy which assumes that the bubble will burst if the other state arises.

Moreover, it is assumed that agents are endowed with a rank-dependent expected utility (RDU) function. This model has been introduced by Quiggin (1993) and developed by Chateauneuf (1999). A general presentation can be found in Cohen and Tallon (2000). The RDU model can be viewed as a generalization of the standard EU (Expected Utility) model, based on the Von Neumann Morgenstern's axioms. In a famous paper, Allais (1953) has showed through experiments that a majority of people do not behave according to the expected utility model, as their actions violate the independence axiom. The RDU model is based on a weaker form of this independence axiom that can reconcile the theory with some actual behaviors.

According to the RDU model, the distribution of probabilities is transformed by a probability weighting function (pwf). The utility is no longer linear with respect to the probabilities of the states of nature. This assumption can be viewed as a transformation of probabilities depending on the pessimism/optimism of the agent. A pessimistic agent will give more weight to the bad state of the nature, whereas an optimistic agent will give more weight to the good state. This assumption has two implications. Firstly, the deformation of probabilities may lead to quantitative changes with respect to the ones obtained with the standard EU (Expected Utility) model. More precisely, our results show that pessimism favors the existence of deterministic bubbles and of small stochastic bubbles, while optimism may promote the existence of big stochastic bubbles. Secondly, the deformation of probabilities depends on the *rank* of the consumptions in the different states of the nature. It will be shown that this property may lead to a multiplicity of bubbly equilibria.

Considering pessimistic agents in the case of a deterministic bubble, the transformation of probabilities weakens the existence conditions of a bub-

ble. The interpretation is simple. By assumption, the gross capital return is greater than 1 in the good state of the nature, while it is smaller than 1 in the bad state. Investing in the bubble provides a gross return equal to 1. Agents invest in the bubble in order to be protected against the occurrence of the bad state. In the case of pessimism, they put more weight on this state and invest more in the bubble. Therefore, pessimism may support the bubble.

In the case of a stochastic bubble, it is assumed that the existence of the bubble is conditional to the state with a low capital return, the bubble bursting in the state with a high capital return.[2] This assumption may reverse the rank of the states of the nature. With a deterministic bubble, as this asset provides the same gain in the two states of the nature, the state with a high capital return is always the best state of the nature. With a stochastic bubble, the state with a low capital return also corresponds to the continuation of the bubble. Therefore, if the bubble has a high value, it is possible that the state with a low capital return becomes the best state of the nature. But if the bubble has a low value, the state with a high capital return remains the best state of the nature. As the transformation of probabilities in the RDU model depends on the rank, two types of bubbly equilibria may exist, associated with a low value or a high value of the bubbly asset.

Optimism promotes the existence of an equilibrium with a high value of the bubble. As the good state for the consumer corresponds to the existence of the bubble, optimistic agents assign more weight to the bubbly state and invest more in the bubble. In the end, optimism favors stochastic bubbles. In contrast, pessimism plays in favor of the existence of an equilibrium with a low value of the bubble. In this case, the bubbly state is the bad state of nature for the consumer. A pessimistic agent assigns more weight to this state, which favors the existence of a bubbly equilibrium.

The case of a stochastic bubble is particularly interesting under the RDU assumption, as it leads to two types of bubbly equilibria associated with either a low price or a high price for the bubble. Increasing the degree of pessimism can have both a positive or a negative effect on the existence of a bubbly equilibrium. Pessimism promotes the existence of a bubble with a low

[2] A stochastic bubble conditional to the state with a high capital return could not exist.

price, whereas optimism plays in favor of a bubble with a high price. These results come from the property that the weights put on the different states of the nature depend on the rank within the RDU framework. A stochastic bubble with a low value does not change the ranking of the two states of the nature whereas a bubble with a high value does. Therefore, the same parameter - the degree of optimism - can have opposite effects depending on the type of bubbly equilibrium.

Finally, in the case of pessimism and stochastic bubbles, an equilibrium may exist associated with a value of the bubble such that the two states of nature lead to equal levels of consumption. Moreover, this bubbly steady state may be stable and there is convergence with oscillations. There exists an infinity of initial conditions for the value of the bubble and the bubbly equilibrium is indeterminate. The existence of such an equilibrium is due to the "kink" in the indifference curves which appears for equal levels of consumption, in the two states of nature in the RDU framework.

This result can be related to previous ones obtained in finance literature with rank dependent utility or Choquet utility: Tallon (1997) and Epstein and Wang (1994) also obtain the existence of multiple equilibria. The originality of this work is to obtain the result in a production economy with capital and a bubbly asset. In an exchange economy, the dynamics of asset prices is completely governed by the history of exogenous shocks. In a production economy with capital, the dynamics of asset prices also depend on the dynamics of capital accumulation. As simple parametric forms are used in the model, the dynamics of the bubbly asset can be explicitly determined: the price converges with oscillations toward its long run value. Bosi and Seegmuller (2010) have also developed a framework in which there exists an indeterminate bubbly equilibrium. In their model, indeterminacy is due to frictions introduced via a cash-in-advance constraint with financial market imperfections. In this work, indeterminacy is obtained in a model in which the only financial market imperfection comes from incomplete markets associated with RDU preferences.

This paper successively considers different assumptions: EU preferences, RDU preferences, deterministic bubbles, and stochastic bubbles. For each case, the existence conditions of bubbles are related to the Pareto optimality

properties of the economy without bubbles. The study of Pareto optimality is based particularly on previous articles that have studied Pareto optimality of allocations in overlapping generations models with stochastic shocks, such as Peled (1984), Peled and Aiyagari (1991), Wang (1993), Gottardi (1996) and Demange and Laroque (1999).

As expected, bubbles can only appear in an economy for which the competitive equilibrium is not Pareto optimal. For RDU preferences, the condition ensuring Pareto optimality also depends on the transformation of probabilities implied by the pessimism/optimism of the agents. Therefore, it cannot be reduced to a comparison between the interest rate and the growth rate of the economy. It provides an additional degree of freedom that may reconcile the existence of bubbles with parameters that are empirically relevant. The link between dynamic efficiency and the existence of bubbles has been questioned recently in contributions that introduce *capital market imperfections*: in Farhi and Tirole (2012), firms are financially constrained and demand and supply liquidity; in Martin and Ventura (2012), investors can be "productive" or "unproductive" and productive investors cannot borrow from unproductive ones. In these two contributions, the assumption of imperfect capital markets tends to disentangle the existence of bubbles from dynamic inefficiency.

Our approach can be related to a recent literature in behavioral economics that explains bubbles and financial anomalies by deviations from rationality. Irrationality can be related both to behaviors and to expectations. A general presentation can be found in Hommes (2006). In these models, anomalies in financial markets may result from the bounded rationality of agents associated with their heterogeneity. Agents behave according to different heuristic rules and make various expectations. In such a framework, Boswijk et al. (2007) shows that two expectation regimes may exist: a "fundamentalists regime" in which agents believe in the reversion of stock prices toward the fundamental value, and a "chartist, trend following regime" in which agents expect persisting deviations from the fundamental value. When behavioral heterogeneity exists, more hedging instruments may destabilize markets, as it is shown in Brock et al. (2009). Some experimental results may support

the behavioral assumption, as Hommes et al. (2008). This article presents a controlled experiment in which subjects must expect the price of a risky asset, a computer calculating the optimal trading decisions resulting from rational behaviors and the equilibrium price. In the experiment, prices may deviate from their fundamental value and bubbles may emerge, that seem inconsistent with rational expectations. With respect to this literature, our approach may be viewed as developing the foundation of behaviors, using the new theories of decision under uncertainty. As discussed in Machina (1989), the RDU assumption introduces some deviation with respect to the "full" rationality implied by the EU assumption. It is a way to give a precise sense to the notions of optimism and pessimism. A generalization of our model could be to incorporate heterogeneous agents in their degree of optimism/pessimism, in order to have different speculative behaviors. The assumption of rational expectations could also be relaxed, either by an ad-hoc expectation rules, or by the assumption of a reduced information set for the agents.

Our work can also give foundations to a long tradition in economics which emphasizes the role of optimism in the development of financial bubbles. In the famous book "Manias, panics and crashes: a history of financial crises", Kindleberger and Aliber argue that investors become more optimistic in expansion phases, which contributes to the development of bubbles. This view also corresponds to Minsky's model developed in 1982 that stresses the financial instability hypothesis. These theories have enjoyed some popularity in the media and it is often referred to waves of optimism or exuberance to explain bubbles phenomenon. As discussed above, the RDU assumption allows to give a precise sense to the notions of optimism and pessimism. In a general equilibrium model, the effect of optimism can be counterintuitive and can challenge the standard view: pessimism may play in favor of bubbles when the investors use the bubbly asset as a coverage strategy against risk.

Section 2 presents the basic framework with EU preferences and a competitive equilibrium without bubbles. A condition ensuring the Pareto optimality of this equilibrium is derived. Section 3 studies the existence of deterministic and stochastic bubbles in this framework. Section 4 concludes. All proofs of propositions are presented in Section 5. A last Appendix (Ap-

pendix 7 in Section 6) is available as online material for a supplementary file published online alongside the electronic version. It is also available upon request, or in Wigniolle (2012).

2 The basic model

The basic setup is an overlapping generations model with capital accumulation *à la* Allais (1947)-Samuelson (1958)-Diamond (1965). Agents live during two periods. They supply one unit of labor in the first period (when young) and they are retired and consume the proceeds of their savings in the second period (when old). The number of agents in each generation is normalized to 1. In this part, capital is the only asset that can be held by agents and there is no bubble.

2.1 Production

There is a single good in the economy, produced in period t with capital K_{t-1} (the capital stock results from $t-1$ investment) and labor L_t. In each period t exists one competitive firm using a linear production technology

$$Y_t = R(\sigma_t)K_{t-1} + wL_t$$

Capital depreciation is completed in one period. Labor productivity is constant and equal to w. Capital productivity $R(\sigma_t)$ follows a random process that depends on the state of the nature σ_t. At each period t, $\sigma_t \in \{1, 2\}$. State 1 occurs with probability π and state 2 with probability $1-\pi$. For $\sigma_t = i$, $i = 1, 2$, $R(i)$ will be denoted R_i and it is assumed that

$$R_1 > 1 > R_2 \tag{1}$$

In each period t, σ_t is known by agents before they make their choices. Under perfect competition, it is straightforward that w will be the equilibrium value of the wage and $R(\sigma_t)$ the capital gross rate of return.

As in Demange and Laroque (1999), a linear production technology is used. This assumption can also be interpreted as an exchange economy framework, with w as the endowment of each young agent, and $R(\sigma_t)$ as the

exogenous stochastic return of a storage technology. It allows to have explicit results only depending on the structural parameters of the economy.

We can wonder on the robustness of the results to other specifications for the production technology. This question is studied in an Appendix 7 that is available online as supplementary content. This appendix considers the two types of equilibria that are studied in the paper - equilibrium with a non-exploding bubble, equilibrium with an exploding bubble - with more general assumptions on the production technology. For the first type of bubble, two results are shown. With a Cobb-Douglas production function, uncertainty does not affect the existence of a long run bubbly equilibrium. Therefore, pessimism or optimism do not influence the existence of bubbles. With a CES production function, for a high elasticity of substitution, the results that were obtained with a linear production technology are kept: bubbly equilibria may exist and are favored by pessimism.

For exploding bubbles, assuming a CES production function with an elasticity of substitution greater than 1, two results of the paper are kept: pessimism favors the existence of small stochastic bubbles whereas optimism favors the existence of big stochastic bubbles.

These different results obtained with Cobb-Douglas or CES production technologies reveal the crucial role played by the elasticity of substitution in the existence of bubbly equilibria. In the case of non-exploding bubbles, the numerical simulations show that an increase in the elasticity of substitution gives the same type of result than an increase in pessimism. In the case of exploding bubbles, a formal condition can be derived that show that the elasticity of substitution must be higher than some threshold value in order to have bubbly equilibria.

The elasticity of substitution plays a role as it governs the relative impact for labor and capital incomes of a technology shock on the capital stock. In the Cobb-Douglas case, a technological shock on the capital stock affects in the same way the productivity of labor. This is the well-known property that a Solow-neutral technical progress is also Hicks-neutral and Harrod-neutral. From Abel et al. (1989), it is known that dynamic inefficiency needs that gross investment exceeds capital income. For a Cobb-Douglas technology associated with homothetic preferences, capital income and gross investment

are affected in the same way by a technology shock and uncertainty has no impact on the existence of bubbles.[3] For a CES production technology with an elasticity of substitution greater that 1, capital income and gross investment are not affected in the same way by the technological shock. Therefore, the conditions ensuring dynamic inefficiency and the existence of bubbles depend on uncertainty. In the limit case of a linear technology (infinite elasticity of substitution), labor income is not hit by the shock and uncertainty has the higher impact.

2.2 Budget constraints and preferences

An agent born in t knows the state of the economy for t, but not for $t+1$. c_t is the first period consumption, s_t is the amount of investment in physical capital, and d_{t+1} is the second period consumption that is a random variable in t. The budget constraints are:

$$c_t + s_t = w \tag{2}$$

$$d_{t+1} = R(\sigma_{t+1})s_t \tag{3}$$

The agent is endowed with an intertemporal utility function. By assumption, this function corresponds to a rank-dependent expected utility (RDU), which is a generalization of the standard expected utility model. In the expected utility framework, the preferences of a generation t are based on the function:

$$u(c_t) + \beta E_t \left[u(d_{t+1})\right] \tag{4}$$

with E_t the expectation taken in period t and u the Von Neumann and Morgenstern utility function (VNM). To have simple calculations, the particular case $u(x) = \ln(x)$ will be considered.

A more general assumption will be used through this paper, the assumption of a rank-dependent expected utility (RDU) function. This model was initiated by Quiggin (1993) and developed by Chateauneuf (1999). It can be viewed as a generalization of the standard EU (Expected Utility) model, based on the Von Neumann Morgenstern's axiomatic. As it was shown by

[3] With homothetic preferences, gross investment is a fraction of labor income.

Allais (1953) with his so-called Allais' Parodox, a majority of people do not behave according to the independence axiom of the EU theory. The RDU model is based on a weaker form of this independence axiom that can reconcile the theory with some actual behaviors.

Following Quiggin (1993), the distribution of probabilities is transformed by a probability weighting function (pwf) ϕ. ϕ is a continuous increasing function from $[0,1]$ to $[0,1]$ such that $\phi(0)=0$ and $\phi(1)=1$.

With this assumption, the preceding utility function (4) must be transformed according to the following rules.

Denoting c, d^1 and d^2, the consumption levels when young, when old in the state 1 and old in the state 2, the utility function is:

- if $d^1 > d^2$,

$$\begin{aligned} & \ln c + \beta\phi(\pi)\ln(d^1) + \beta\left[1-\phi(\pi)\right]\ln(d^2) \\ = \; & \ln c + \beta\ln(d^2) + \beta\phi(\pi)\left[\ln(d^1)-\ln(d^2)\right] \end{aligned} \tag{5}$$

- if $d^1 < d^2$,

$$\begin{aligned} & \ln c + \beta\left[1-\phi(1-\pi)\right]\ln(d^1) + \beta\phi(1-\pi)\ln(d^2) \\ = \; & \ln c + \beta\ln(d^1) + \beta\phi(1-\pi)\left[\ln(d^2)-\ln(d^1)\right] \end{aligned} \tag{6}$$

In each case, the second formulation allows to have more intuition on the RDU assumption. If $d^1 > d^2$, the term $\beta\ln(d^2)$ represents the minimum utility level that the agent will have without risk in the second period. With a probability π, she/he will enjoy an additional gain of $\beta\left[\ln(d^1)-\ln(d^2)\right]$. In the calculation of the expected gain, this probability π is transformed in $\phi(\pi)$. If $d^1 < d^2$, the minimum utility level is now $\beta\ln(d^1)$. With a probability $1-\pi$ that is transformed in $\phi(1-\pi)$, he/she will enjoy the additional gain of $\beta\phi(1-\pi)\left[\ln(d^2)-\ln(d^1)\right]$. As a consequence of these assumptions, the formulation of the utility function depends on the rank of the variables d^1 and d^2.

In abbreviated form, the utility function will be denoted by:

$$\ln c + \beta E_\phi \ln(d) \tag{7}$$

where E_ϕ is the expected value calculated with the transformed probabilities according to (5) and (6), and d is the random variable $(d^1, d^2; \pi, 1-\pi)$.

The property $\phi(\pi) < \pi$ can be interpreted as pessimism and $\phi(\pi) > \pi$ as optimism. An optimistic agent puts more weight on the best state of the nature, whereas a pessimistic agent puts more weight on the worst state.

The following notations will be used from now: $\pi_1 = \phi(\pi)$ and $\pi_2 = 1 - \phi(1-\pi)$. Therefore, for a pessimistic agent, $\pi_1 < \pi < \pi_2$, and for an optimistic agent $\pi_1 > \pi > \pi_2$. For $\phi(\pi) = \pi$, the EU model is recovered and $\pi_1 = \pi = \pi_2$. The function ϕ gives to the RDU model an additional degree of freedom.

The RDU assumption has two main consequences. The first one corresponds to the deformation of probabilities, that may lead to quantitative changes: with respect to the EU model, the agents behave as if they did not take into account the true probabilities. The second consequence corresponds to the existence of a kink in the indifference curves for $d^1 = d^2$. Figures 1 and 2 represent an indifference curve in the plane (d^1, d^2), for a given value of c, in the case of a pessimistic or optimistic agent. When the curve crosses the line $d^1 = d^2$, the slope in absolute value is $[1 - \phi(1-\pi)]/\phi(1-\pi)$ at left and $\phi(\pi)/[1-\phi(\pi)]$ at right. This feature comes from the formulation of the utility function that depends on the rank of the variables. In the pessimistic case, for $d^1 = d^2$, different values of prices may be admissible. This property may imply qualitative changes in the results: in the literature on RDU preferences, it often generates multiple equilibria. In the optimistic case, preferences are no more convex, as for a given point (c, d^1, d^2), the set $\{(c', d^{1\prime}, d^{2\prime}) \text{ such that } (c', d^{1\prime}, d^{2\prime}) \succcurlyeq (c, d^1, d^2)\}$ is not convex (see Figure 2).

2.3 The competitive equilibrium without bubbles

From condition (1), it is clear that state 1 is the good state of the nature and state 2 the bad state in the economy without bubbles. Therefore, the utility function is always given by (5). Maximizing (5) under budget constraints (2)

and (3) gives the results:

$$\begin{aligned} c_t &= \frac{w}{1+\beta} \\ s_t &= \frac{\beta w}{1+\beta} \\ d_{t+1} &= \frac{R(\sigma_{t+1})\beta w}{1+\beta} \end{aligned}$$

Results do not depend on $\phi(\pi)$ as the utility function is log-linear. In this simple case, the RDU assumption gives the same results as the EU model.

The capital stock used in period $t+1$ results from the investment in physical capitals in t :

$$K_t = \frac{\beta w}{1+\beta}$$

Therefore, after one period, the capital stock reaches a constant value.

In $t = 0$, the initial value of the capital stock K_{-1} is given, as the consumption level of the first old agent: $d_0 = R(\sigma_0)K_{-1}$.

2.4 Pareto optimality of the competitive path

2.4.1 Definition

A standard result in OLG deterministic models (Tirole, 1985) is that the existence of financial bubbles is possible only in economies that are dynamically inefficient. If uncertainty is removed from the model ($R(1) = R(2) = R$), dynamic efficiency is obtained for $R > 1$.

By dynamic efficiency, Tirole's article refers to the criterion introduced by Cass (1972). This criterion is concerned with the efficiency of aggregate consumption in a deterministic model of growth. A more accurate concept for overlapping generations models has been developed by Homburg (1992) and De la Croix and Michel (2002). It takes into account the utility levels of the agents of the different generations and it corresponds to Pareto optimality.

In stochastic overlapping generations models, Zilcha (1991) derived a criterion of dynamic efficiency based on aggregate consumption that generalizes Cass's criterion. But the appropriate concept for Pareto optimality in such frameworks is *interim optimality,* as it refers to the expected utility of the

agents and not only on aggregate consumption. It has been developed and used by Peled (1984) and Demange and Laroque (1999). It is the natural extension of the standard notion of Pareto optimality to a dynamic framework with uncertainty. The formal definition of this concept is given in the proof of Proposition 1 (in Appendix 1). In order to keep the exposition simple, only a non formal definition is given in the text.

Interim optimality is defined on *feasible allocations*. In period t, the resource constraint of the economy can be expressed as:

$$c_t + d_t + K_t = R(\sigma_t)K_{t-1} + w \tag{8}$$

A *feasible allocation* is an allocation $(c_t, d_t, K_t)_{t\geq 0}$, starting from a given value for K_{-1}, that satisfies the resource constraint (8) for all t. Note that c_t, d_t and K_t are random variables that may take different values according to the all history of technological shocks.

A feasible allocation is *interim optimal* if there does not exist another feasible path that gives higher expected utility (calculated with transformed probabilities) for all period t and all histories, with a strict improvement for at least one period and one state of the nature.

2.4.2 The expected utility framework

The Pareto optimality of the equilibrium is first studied under the EU assumption, with the utility function (4).

Proposition 1 *If*

$$\frac{\pi}{R_1} + \frac{1-\pi}{R_2} < 1 \tag{9}$$

the competitive equilibrium is interim Pareto optimal.

Proof. See Appendix 1. ■

The proposition shows that interim Pareto optimality is preserved if the low value of R_2 in the bad state is compensated by a high enough value of R_1 in the good state. It is interesting to note that if (9) holds, then $\pi R_1 + (1-\pi)R_2 > 1$. Therefore, Pareto optimality needs a stronger condition than an average interest rate higher than the growth rate of capital. Earlier

contributions in different frameworks have also pointed out this property (see e. g. Blanchard and Weil (2001) for instance). (9) can be written under the form:

$$\pi > \frac{R_1 (1 - R_2)}{R_1 - R_2} \tag{10}$$

A high probability for the good state plays in favor of the Pareto optimality of the equilibrium.

2.4.3 The RDU framework

What can be said about the Pareto optimality of the equilibrium with RDU preferences? It is necessary to study separately the cases of pessimistic and optimistic agents. In both cases, the utility function is not differentiable at a point such that $d^1 = d^2$. But, if agents are pessimistic, their preferences remain convex, whereas this property is lost for optimistic agents.

In the first case, it is easy to adapt the result of Proposition 1.

Proposition 2 *If agents are pessimistic and*

$$\frac{\pi_1}{R_1} + \frac{1 - \pi_1}{R_2} < 1 \tag{11}$$

the competitive equilibrium without bubbles is interim Pareto optimal.

Proof. See Appendix 2. ■

This condition is the same as the one of Proposition 1, except that π is replaced by π_1. This is due to the transformation of probabilities in the utility function of the agent. Considering the inequality under the form (10), as $\pi_1 < \pi$ for a pessimistic agent, it is clear that pessimism is not favorable to the Pareto optimality of the competitive equilibrium.

When agents are optimistic, as agents' preferences are no longer convex, the preceding efficiency condition (11) is necessary but not sufficient. Another condition is needed that can be interpreted as a "moderate" optimism, or a weak transformation of the probabilities.

Proposition 3 *Assuming optimistic agents, if*

$$\frac{\pi_1}{R_1} + \frac{1 - \pi_1}{R_2} < 1 \tag{12}$$

and

$$\frac{R_2}{R_1} < \left[\frac{\pi_1^{\pi_2}\left(1-\pi_1\right)^{(1-\pi_2)}}{\pi_2^{\pi_2}\left(1-\pi_2\right)^{(1-\pi_2)}}\right]^{\frac{1}{\pi_1-\pi_2}} \tag{13}$$

the competitive equilibrium without bubbles is interim Pareto optimal.

Proof. See Appendix 3. ■

Condition (13) guarantees that no other feasible allocation dominating the competitive equilibrium exists in the zone in which $d^1 < d^2$. More precisely, it is possible to represent in the plane (d^1, d^2) the point of the indifference curve corresponding to the equilibrium. When (13) holds, this point is out of the zone delimited by the tangent line to the indifference curve (see Figure 3).

A better intuition can be achieved in particular cases. The following corollary studies the limit condition obtained from (13) when the transformation of probabilities vanishes. Then, it takes a particular assumption for ϕ

$$\phi(\pi) = \pi^{\rho}, \text{ with } 0 < \rho < 1$$

The lower ρ, the more optimistic the agent is.

Corollary 1 *1. In the limit case $\pi_1 \to \pi$ and $\pi_2 \to \pi$, Condition (13) becomes $R_2/R_1 < 1$, which is true by assumption. By continuity, (13) is fulfilled if the transformation of probabilities is moderate.*

2. Let us assume that $\phi(\pi) = \pi^{\rho}$ with $\rho \in (0,1)$. There exists a value $\bar{\rho} \in (0,1)$ such that (13) is satisfied if and only if $\rho > \bar{\rho}$.

Proof. See Appendix 4. ■

The first part of the corollary shows that Condition (13) is satisfied in the limit case $\pi_1 \to \pi$ and $\pi_2 \to \pi$. By continuity, it is satisfied if the transformation of probabilities is not too strong. The second part introduces a particular function ϕ that allows the transformation of probabilities to be measured by the parameter ρ. ρ can be interpreted as the degree of pessimism (or as the opposite of the degree of optimism). The corollary allows a lower bound $\bar{\rho}$ on ρ to be defined that represents a limit value for the degree of optimism.

3 The equilibrium with financial bubbles

This section assumes the existence of a bubble asset. Following Tirole (1985) and Weil (1987), this asset is a pure bubble, with a fundamental value equal to 0. It plays the role of store of value but yields no transactions services. It is called "pieces of paper" by Tirole or "money" by Weil, and can be any good that cannot be consumed or used.

Two types of bubbles are studied: one is called a bubble "*à la Tirole*" which is "deterministic", while the second is a bubble "*à la Weil*" which is "stochastic". The deterministic bubble is a deterministic asset that has the same price in the two states of the nature. The stochastic bubble only exists in state 2 (the bad state of the nature). Its existence is therefore conditional to the continuation of this state and at each period, the bubble has a probability π of exploding in the next period.

3.1 Equilibrium with a deterministic bubble

It is assumed that a bubble asset is available in the economy in a fixed quantity normalized to 1. Its price is p_t in period t. The budget constraints of a generation t agent become:

$$c_t + s_t + p_t x_t = w \tag{14}$$

$$d_{t+1} = R(\sigma_{t+1}) s_t + p_{t+1} x_t \tag{15}$$

x_t is the demand for the bubble asset and s_t is the investment in physical capital.

For a deterministic bubble, it is clear that second period consumption in state 1 will always be greater than second period consumption in state 2, as:

$$d^1_{t+1} = R_1 s_t + p_{t+1} x_t > d^2_{t+1} = R_2 s_t + p_{t+1} x_t$$

Therefore, the utility function is defined by (5).

The consumer problem is studied under the assumptions that $R_2 < p_{t+1}/p_t < R_1$, as one focuses on equilibria with $s_t > 0$ and $p_t > 0$. Indeed, if $p_{t+1}/p_t < R_2$, the bubble is a dominated asset and the solution $x_t = 0$ is immediate. If $p_{t+1}/p_t > R_1$, capital is a dominated asset and the solution $s_t = 0$ follows.

Maximizing (5) under budget constraints (14) and (15) gives the results:

$$c_t = \frac{w}{1+\beta} \tag{16}$$

$$s_t = \frac{\beta w}{1+\beta}\left(\frac{\pi_1}{1-\frac{p_t R_2}{p_{t+1}}} - \frac{1-\pi_1}{\frac{p_t R_1}{p_{t+1}}-1}\right) \tag{17}$$

$$p_{t+1}x_t = \frac{\beta w}{1+\beta}\left[\frac{(1-\pi_1)R_1}{\frac{p_t R_1}{p_{t+1}}-1} - \frac{\pi_1 R_2}{1-\frac{p_t R_2}{p_{t+1}}}\right] \tag{18}$$

Equilibrium conditions on the bubble and capital markets imply:

$$x_t = 1 \tag{19}$$

$$K_t = s_t \tag{20}$$

Condition (19) with (18) gives the dynamics of the price of the bubble:

$$p_{t+1} = \frac{\beta w}{1+\beta}\left[\frac{(1-\pi_1)R_1}{\frac{p_t R_1}{p_{t+1}}-1} - \frac{\pi_1 R_2}{1-\frac{p_t R_2}{p_{t+1}}}\right]$$

This equation is simplified in defining the variable $\delta_t = p_t(1+\beta)/(\beta w)$, and δ_t follows the dynamic equation:

$$\delta_{t+1} = \frac{(1-\pi_1)R_1}{\frac{\delta_t R_1}{\delta_{t+1}}-1} - \frac{\pi_1 R_2}{1-\frac{\delta_t R_2}{\delta_{t+1}}} \tag{21}$$

A positive stationary state of this equation corresponds to a steady-state equilibrium with bubble, with

$$\delta = \frac{(1-\pi_1)R_1}{R_1-1} - \frac{\pi_1 R_2}{1-R_2}$$

At this state, the value of the investment in physical capital is:

$$s = \frac{\beta w}{1+\beta}\left(\frac{\pi_1}{1-R_2} - \frac{1-\pi_1}{R_1-1}\right)$$

This stationary state exists only if p (or equivalently δ) and s are positive, which gives:

$$\frac{\pi_1}{R_1} + \frac{(1-\pi_1)}{R_2} > 1 \tag{22}$$

$$\pi_1 R_1 + (1-\pi_1)R_2 > 1 \tag{23}$$

The following lemma analyzes the dynamics of the bubbly equilibrium.

Lemma 1 *Under conditions (22) and (23), the dynamics is well defined and has two steady states,* 0 *which is stable and* δ *which is unstable. Starting from* $\delta_0 > 0$*, if* $\delta_0 < \delta$*,* δ_t *converges toward* 0*; if* $\delta_0 > \delta$*,* δ_t *converges toward* $+\infty$*.*

Proof. See Appendix 5. ■

Under the assumptions of the lemma, there exists a multiplicity of equilibria that can be classified in three types. In the first type, the economy stays at the bubbly steady state. In the second type, the economy stays at the bubbleless equilibrium. In the third type, the economy starts with a price of the bubble p_0 that is smaller than p, the sequence (p_t) is decreasing and tends to 0. The economy converges towards the bubbleless stationary equilibrium.[4]

When condition (22) is satisfied but (23) does not hold, it is straightforward enough to show that a bubbly equilibrium exists without investment in capital ($s = 0$). This equilibrium is associated with a constant value of the bubble:

$$p = \frac{\beta w}{1 + \beta}$$

The existence of this equilibrium is related to the assumption of a linear technology that allows production to occur without using capital.

Condition (22) defines an upper bound on π_1:

$$\pi_1 < \frac{R_1 (1 - R_2)}{R_1 - R_2}$$

The results can be summarized by a proposition:

Proposition 4 *If condition (22) holds, there exists an equilibrium of the economy associated with a deterministic bubble.*

- *If (23) is satisfied, agents hold both capital and the bubble asset at equilibrium.*
- *If (23) is not satisfied, a bubbly equilibrium exists with no investment in capital.*

[4] The case in which p_t (and δ_t) converges towards $+\infty$ cannot be an equilibrium as it implies a negative value of capital.

When (23) is not fulfilled, two different equilibria may exist: a first one with capital and no bubble, a second one without capital and with the bubbly asset. This second bubbly equilibrium Pareto-dominates the first one (in the sense of interim Pareto optimality). To prove this result, it is useful to have in mind that the RDU model can be interpreted in this part as an EU model in which the probability π is replaced by π_1. This interpretation is correct because the equilibrium with a deterministic bubble always satisfies $d_{t+1}^1 > d_{t+1}^2$. For the bubbly equilibrium without capital, there is no risk on the gross return of the asset that is equal to 1. For the equilibrium with capital and no bubble, the model can be interpreted as an EU model with a probability π_1. The gross return on capital is risky with an "average return" $\pi_1 R_1 (1 - \pi_1) R_2 < 1$. Therefore, the bubbly equilibrium Pareto dominates the equilibrium with capital and no bubble.

Remark 1 *This paper focuses only on two types of bubbles: a deterministic one that has the same value in all states of the nature; a stochastic one that cancels out if state 1 occurs. More generally, the case of a bubbly asset that takes two positive different values depending on the state of the nature could be studied. It is easy to check that this type of solution does not exist in our framework.*

The case of EU preferences

Condition (22) with $\pi_1 = \pi$ is the converse of the condition that ensures interim Pareto optimality of the competitive equilibrium (1). As expected, when the bubbleless equilibrium is not interim Pareto optimal, there exists a bubbly equilibrium. Figure 4 gives an illustration of the different cases depending on the values of R_1 and R_2. The curves are drawn for the value $\pi = 1/2$.

The case of pessimism

Agents are assumed to be pessimistic: $\pi_1 < \pi$. If

$$\pi_1 < \frac{R_1 (1 - R_2)}{R_1 - R_2} < \pi$$

a deterministic bubble exists in an economy in which there would be no bubble if agents did not "transform" the probabilities. The interpretation of this result is simple. Investing in the bubble provides a gross return equal to 1, which is greater than the capital return in the bad state of the nature R_2. Agents invest in the bubble in order to be protected against the occurrence of state 2. In the case of pessimism, they put more weight on this state and invest more in the bubble. Therefore, pessimism can play in favor of the existence of a deterministic bubble. From Proposition 2, a bubbly equilibrium can only exist in an inefficient economy.

Figure 5 gives an illustration of the effect of pessimism on the existence of the different regimes in the plane (R_1, R_2). The curves are obtained under the assumption that the pessimism generates a transformation of π from $\pi = 1/2$ to $\pi_1 = 0.4$.

The case of optimism

The condition ensuring the existence of a bubbly equilibrium remains Condition (22). Optimism is unfavorable to the existence of bubbles, as agents put more weight on the good state of the nature.

The relation between the existence of bubbles and interim Pareto optimality is more complex in the case of optimism. The usual way to analyze the impact of a bubble is to interpret it as an intergenerational transfer. In the basic economy with standard EU preferences, when (9) is not fulfilled, the existence of a bubble constitutes an intergenerational transfer from the young to the old agents and this transfer is Pareto improving. This analysis can also be used to understand the case of pessimistic agents. But, with optimistic agents, preferences are no longer convex. If (12) holds and (13) is not fulfilled, it may be possible that the economy is not interim Pareto optimal and that no bubbly equilibrium exists. Considering the proof of Proposition 3, the competitive equilibrium without bubbles can be inefficient in this case, because the technology does not allow agents to redistribute consumption from state 1 in favor of state 2. A deterministic bubble cannot solve this problem as the bubble carries out a transfer among generations, and not a transfer among the two states of nature.

3.2 Equilibrium with a stochastic bubble

In this part, a stochastic bubble *à la* Weil (1987) is introduced. Following Weil, the asset has some probability to explode at each period. More precisely, agents form their expectations according to a self-fulfilling prophecy which assumes that the bubble will continue if some state of the nature occurs, and will burst if the other state arises. In all the section, we assume that the continuation of the bubble is conditional to the realization of state 2, and that the bubble bursts if state 1 occurs. This assumption is taken because it is the most favorable to the existence of the bubble. Indeed, agents can arbitrate between the two assets, the productive capital and the bubbly asset. Investing in the stochastic bubble allows to be insured against the occurrence of the state with a low return of productive capital. It would be impossible to have a stochastic bubble which existence is conditional to state 1. This asset would be dominated by productive capital, or its value should grow at a factor greater than R_1. This last case would be unsustainable.

3.2.1 The bubble *à la* Weil (1987)

A bubble asset *à la* Weil (1987) is available in the economy in a fixed quantity normalized to 1. Its price is p_t in period t, conditional to the realization of state 2. Assuming that the economy is in period t in state 2, agents expect a price p_{t+1} in period $t+1$ conditional to the realization of state 2, and a price 0 if state 1 occurs. In the case of the bubble exploding, the dynamics of the economy after the explosion becomes the same as in the economy without the bubble.

Assuming that state 2 occurs in period t, the budget constraints of a generation t agent are:

$$c_t + s_t + p_t x_t = w \tag{24}$$

$$d^1_{t+1} = R_1 s_t \tag{25}$$

$$d^2_{t+1} = R_2 s_t + p_{t+1} x_t \tag{26}$$

From these three equations, by elimination of s_t and x_t, one gets the in-

tertemporal budget constraint:

$$c_t + \frac{d^1_{t+1}}{R_1}\left(1 - R_2\frac{p_t}{p_{t+1}}\right) + d^2_{t+1}\frac{p_t}{p_{t+1}} = w \tag{27}$$

3.2.2 Stochastic bubbles in the EU model

In the EU model, an agent maximizes (4) under budget constraints (24), (25) and (26). The results are:

$$\begin{aligned} c_t &= \frac{w}{1+\beta} \\ s_t &= \frac{\beta w}{1+\beta}\frac{\pi}{1-\frac{p_t R_2}{p_{t+1}}} \\ x_t &= \frac{\beta w}{1+\beta}\left[\frac{(1-\pi)}{p_t} - \frac{\pi R_2}{p_{t+1} - p_t R_2}\right] \end{aligned}$$

Equilibrium conditions on the bubble market ($x_t = 1$) and on the capital market $K_t = s_t$ imply:

$$1 = \frac{\beta w}{1+\beta}\left[\frac{(1-\pi)}{p_t} - \frac{\pi R_2}{p_{t+1} - p_t R_2}\right] \tag{28}$$

$$K_t = \frac{\beta w}{1+\beta}\frac{\pi}{1-\frac{p_t R_2}{p_{t+1}}} \tag{29}$$

With the change of variable $\delta_t = p_t(1+\beta)/(\beta w)$, equation (28) gives:

$$\delta_{t+1} = R_2\delta_t\frac{1-\delta_t}{1-\pi-\delta_t}$$

This equation has 2 stationary states: 0 which is stable, and

$$\delta = \frac{1-\pi-R_2}{1-R_2}$$

which is unstable. This last steady state exists only if $\delta > 0$,

$$R_2 < 1 - \pi \tag{30}$$

Proposition 5 *If condition (30) holds, there exists an equilibrium of the economy associated with a stochastic bubble conditional to the continuation of state 2.*

This condition is stronger than (22): the deterministic bubble is more likely to exist than the stochastic bubble. Indeed, the stochastic bubble has a positive return only if state 2 arises. Moreover, the stochastic bubble cannot exist in an efficient economy: if (30) holds, (9) does not hold. A graphical illustration of these results is shown in Figure 4, with a value $\pi = 1/2$.

As for the case of deterministic bubbles, when (30) holds, there exists a multiplicity of equilibria: either the economy stays at the bubbly steady state; either the economy stays at the bubbleless equilibrium; or, starting from a value $\delta_0 < \delta$, the economy converges towards the bubbleless stationary equilibrium.

3.2.3 Two types of bubbly equilibria in the RDU model

A stochastic bubble conditional to state 2 may induce some redistribution of consumption between the two states of nature. For the equilibrium with a deterministic bubble, state 1 always remains the best state of the nature. For an equilibrium with a stochastic bubble conditional to state 2, it is possible that state 2 becomes the good state of the nature, as the bubble bursts in state 1. More precisely, the size of the bubble may determine which is the best state. For a small bubble, state 1 will always lead to more consumption than state 2. For a large bubble, the inequality can be reversed. Therefore, it is *a priori* possible to obtain two types of bubbly equilibria, associated either with $d^2 < d^1$ or with $d^2 > d^1$.

Equilibrium such that $d^2 > d^1$

Assuming that the equilibrium is such that $d^2_{t+1} > d^1_{t+1}$, the program of the agent consists in maximizing (6) under the three budget constraints (24), (25) and (26). The results are:

$$\begin{aligned} c_t &= \frac{w}{1+\beta} \\ s_t &= \frac{\beta w}{1+\beta}\frac{\pi_2}{1-\frac{p_t R_2}{p_{t+1}}} \\ x_t &= \frac{\beta w}{1+\beta}\left[\frac{(1-\pi_2)}{p_t} - \frac{\pi_2 R_2}{p_{t+1}-p_t R_2}\right] \end{aligned}$$

The equilibrium condition on the bubble market ($x_t = 1$) leads to the dynamics of the price of the bubble. Using the variable $\delta_t = p_t(1+\beta)/(\beta w)$, δ_t follows the dynamic equation:

$$\delta_{t+1} = R_2 \delta_t \frac{1-\delta_t}{1-\pi_2-\delta_t} \tag{31}$$

The condition $d^2_{t+1} = R_2 s_t + p_{t+1} > d^1_{t+1} = R_1 s_t$ leads to:

$$\delta_{t+1} > R_2 \delta_t + (R_1 - R_2)\pi_2$$

With (31), this condition gives:

$$\delta_t > \frac{R_1 - R_2}{R_1}(1-\pi_2) \equiv \hat{\delta}_l \tag{32}$$

The bubble must be large enough to change the ranking of the states 1 and 2.

The bubbly steady state corresponds to

$$\hat{\delta} = \frac{1-\pi_2-R_2}{1-R_2}$$

It exists only if $\hat{\delta} > 0$ or

$$\pi_2 < 1 - R_2 \tag{33}$$

Moreover, condition (32) must be checked along this equilibrium. It leads to the constraint:

$$\pi_2 < \frac{1-R_2}{R_1+1-R_2} \tag{34}$$

This last condition is stronger than (33) as $R_1 + 1 - R_2 > 1$.

Equilibrium such that $d^2 < d^1$

Assuming that the equilibrium is such that $d^2_{t+1} < d^1_{t+1}$, the program of the agent gives:

$$\begin{aligned}
c_t &= \frac{w}{1+\beta} \\
s_t &= \frac{\beta w}{1+\beta} \frac{\pi_1}{1-\frac{p_t R_2}{p_{t+1}}} \\
x_t &= \frac{\beta w}{1+\beta}\left[\frac{(1-\pi_1)}{p_t} - \frac{\pi_1 R_2}{p_{t+1}-p_t R_2}\right]
\end{aligned}$$

The equilibrium condition on the bubble market ($x_t = 1$) leads to the dynamics of the price of the bubble. The variable $\delta_t = p_t(1+\beta)/(\beta w)$ follows the dynamic equation:

$$\delta_{t+1} = R_2 \delta_t \frac{1-\delta_t}{1-\pi_1-\delta_t} \tag{35}$$

The condition $d^2_{t+1} = R_2 s_t + p_{t+1} < d^1_{t+1} = R_1 s_t$ leads to:

$$\delta_{t+1} < R_2 \delta_t + (R_1 - R_2)\pi_1$$

With (35), this condition gives:

$$\delta_t < \frac{R_1 - R_2}{R_1}(1-\pi_1) \equiv \check{\delta}_l \tag{36}$$

In this case, the value of the bubble must be low enough to not change the ranking of the states.

The bubbly steady state corresponds to

$$\check{\delta} = \frac{1-\pi_1-R_2}{1-R_2}$$

It exists only if $\check{\delta} > 0$:

$$\pi_1 < 1 - R_2 \tag{37}$$

Moreover, $\check{\delta}$ must satisfy (36):

$$\pi_1 > \frac{1-R_2}{R_1+1-R_2} \tag{38}$$

The case of a stochastic bubble associated with RDU preferences leads to new results that could not occur in the standard EU framework. The RDU framework gives birth to two types of bubbly equilibria associated with either a low price or a high price for the bubble. These results come from the property that the weights put on the different states of the nature depend on the rank within the RDU framework. A stochastic bubble with a low value does not change the ranking of the two states of the nature whereas a bubble with a high value does.

3.2.4 Stochastic bubbles and optimism

For an optimistic agent, $\pi_1 > \pi > \pi_2$, and thus, $\hat{\delta} > \check{\delta}$. Depending on the value of the parameters, it is possible to obtain a bubbly equilibrium with $d^1 < d^2$ and a high price level of the bubble, or a bubbly equilibrium with $d^1 > d^2$ and a low price level of the bubble.[5] The following propositions (6, 7 and 8) show that, for given values of parameters, only one type of equilibrium is possible, either associated with $d^1 < d^2$, or with $d^1 > d^2$.

Using conditions (33), (34) , (37) and (38), the following results are obtained.

Proposition 6 *Assume that* $\pi_2 < \frac{1-R_2}{R_1+1-R_2}$ *and* $\pi_1 > 1-R_2$ *or* $\pi_1 < \frac{1-R_2}{R_1+1-R_2}$. *There exists a unique bubbly equilibrium with* $d^1 < d^2$ *and a price of the bubble equal to* $\hat{\delta}(\beta w)/(1+\beta)$.

Proposition 6 shows that optimism can play in favor of the existence of bubbles. Assume that $\pi > 1 - R_2$. Under this condition, stochastic bubbles cannot exist in the economy with EU preferences. If agents are optimistic, it is possible that they transform probabilities with $\pi_2 = 1-\phi(1-\pi) < \frac{1-R_2}{R_1+1-R_2}$. In this case a stochastic bubble may exist. The price of the bubble is high enough in such a way that consumption in state 2 (the bubble exists) is higher than consumption in state 1 (the bubble explodes). As agents are optimistic, they put more weight on the good state and invest more in the bubble.

Proposition 7 corresponds to the converse case of a low price of the bubble such that d^1 remains higher than d^2.

Proposition 7 *Assume that* $\pi_1 < 1 - R_2$ *and* $\pi_2 > \frac{1-R_2}{R_1+1-R_2}$. *There exists a unique bubbly equilibrium with* $d^1 > d^2$ *and a price of the bubble equal to* $\check{\delta}(\beta w)/(1+\beta)$.

As $\pi < \pi_1 < 1 - R_2$, a necessary condition for the existence of such an equilibrium is that there exists a bubbly equilibrium in the economy with

[5] It is not possible to obtain a bubbly equilibrium such that $d^1 = d^2$, because it is never optimal for the consumer to choose such an allocation in case of optimism.

EU preferences (see condition (30)). Moreover agents need to be not too optimistic, in such a way that π_1 and π_2 remain in the interval $\left(\frac{1-R_2}{R_1+1-R_2}, 1-R_2\right)$.

A last case remains to be studied, if

$$\pi_2 < \frac{1-R_2}{R_1+1-R_2} < \pi_1 < 1-R_2$$

In this case, all preceding conditions (33), (34) , (37) and (38) are fulfilled. But, it does not imply that the two types of bubbly equilibria can exist together. Indeed, in the case of optimistic agents, preferences are not convex. It is possible that two different solutions satisfy the marginal conditions of the consumer program, one with $d^1 < d^2$, the other one with $d^1 > d^2$. Therefore, it is necessary to compare the utility levels associated with the two solutions. This is done in the following proposition.

Proposition 8 *Assume that* $\pi_2 < \frac{1-R_2}{R_1+1-R_2} < \pi_1 < 1-R_2$. *The function* ζ *is defined according to:*

$$\zeta(\pi) = \pi \ln\left(\pi \frac{R_1}{1-R_2}\right) + (1-\pi)\ln(1-\pi)$$

- *If* $\zeta(\pi_2) > \zeta(\pi_1)$ *then a bubbly equilibrium with* $d^1 < d^2$ *exists and the price of the bubble is equal to* $\hat{\delta}(\beta w)/(1+\beta)$.
- *If* $\zeta(\pi_2) < \zeta(\pi_1)$ *then a bubbly equilibrium with* $d^1 > d^2$ *exists and the price of the bubble is equal to* $\check{\delta}(\beta w)/(1+\beta)$.

Proof. See Appendix 5. ■

To summarize the impact of optimism on the existence of stochastic bubbles, the most interesting result is obtained in Proposition 6. Optimism may favor stochastic bubbles with respect to EU preferences. The price of the bubble must be high enough in such a way that $d^1 < d^2$. In that case, state 2 (the bubble exists) can be better than state 1 (the bubble bursts). Optimistic agents put more weight on the good state and invest more in the bubble.

In overlapping generations models *à la* Tirole (1985), there exists a maximum size of the bubble that corresponds to the saddle path converging towards the long run bubbly steady state. There also exists a multiplicity of equilibria starting with a value of the bubble between 0 and the maximum

value. All these equilibria converge towards the long run bubbleless equilibrium. In our model with RDU preferences and a stochastic bubble, it is possible that these properties are no more satisfied. Under the assumptions of proposition (6), there exists a stationary bubbly equilibrium with a price of the bubble equal to $\hat{\delta}(\beta w)/(1+\beta)$. But any initial value below the steady state value is not admissible. For an optimistic agent, as $\pi_1 > \pi_2$, the two constraints (32) and (36) are such that $\hat{\delta}_l > \check{\delta}_l$. Therefore, any initial value δ_0 such that $\check{\delta}_l < \delta_0 < \hat{\delta}_l$ is not admissible as it cannot correspond to an equilibrium path.[6]

3.2.5 Stochastic bubbles and pessimism

For a pessimistic agent, $\pi_1 < \pi < \pi_2$. Using conditions (33), (34), (37) and (38), it is possible to obtain existence conditions for the bubbly equilibria that satisfy the marginal conditions of the consumer program, as in Propositions 6 and 7.[7] But, in the case of pessimism, another situation may exist that corresponds to $d^1_{t+1} = d^2_{t+1}$. This case results from the existence of a kink for $d^1_{t+1} = d^2_{t+1}$.

Considering the maximization of the RDU function under the intertemporal budget constraint (27), the optimal solution with $d^1_{t+1} = d^2_{t+1}$ is obtained if:

$$\frac{\pi_1}{1-\pi_1} < \frac{1 - R_2 \frac{p_t}{p_{t+1}}}{R_1 \frac{p_t}{p_{t+1}}} < \frac{\pi_2}{1-\pi_2} \tag{39}$$

It corresponds to the solution:

$$\begin{aligned} c_t &= \frac{w}{1+\beta} \\ d^1_{t+1} &= d^2_{t+1} = \frac{\frac{\beta w}{1+\beta}}{\frac{1}{R_1} - \frac{R_2}{R_1}\frac{p_t}{p_{t+1}} + \frac{p_t}{p_{t+1}}} \end{aligned}$$

The equilibrium price of the bubble satisfies:

$$p_t = \frac{\beta w}{1+\beta} \frac{\frac{p_t}{p_{t+1}}(R_1 - R_2)}{1 + \frac{p_t}{p_{t+1}}(R_1 - R_2)}$$

[6] I am grateful to an anonymous referee that has pointed out this property.

[7] The case studied in Proposition 8 does no more exist as it was related to the non convexity of preferences when agents are optimistic.

With the change of variable $\delta_t = p_t(1+\beta)/(\beta w)$, this equation gives:

$$\begin{aligned} \delta_t &= 0 \text{ or} \\ \delta_{t+1} &= -\delta_t(R_1 - R_2) + (R_1 - R_2) \end{aligned} \tag{40}$$

(40) has a stationary state

$$\tilde{\delta} = \frac{R_1 - R_2}{1 + R_1 - R_2}$$

If $R_1 - R_2 > 1$, this stationary state is unstable. If $R_1 - R_2 < 1$, it is stable. In this case, it is possible to observe convergence towards the stationary state, with oscillations. In the case of instability, there is only one stationary bubbly equilibrium. In the case of stability, a multiplicity of bubbly equilibria exists.

Finally, the following results have been obtained in the case of a pessimistic agent:

Proposition 9 *Assume that* $\pi_1 < \pi < \pi_2$.

- *If*
$$\pi_2 < \frac{1 - R_2}{R_1 + 1 - R_2}$$
a bubbly equilibrium with $d^1 < d^2$ *exists and the price of the bubble is equal to* $\hat{\delta}(\beta w)/(1+\beta)$.

- *If*
$$\frac{1 - R_2}{R_1 + 1 - R_2} < \pi_1 < 1 - R_2$$
a bubbly equilibrium with $d^1 > d^2$ *exists and the price of the bubble is equal to* $\check{\delta}(\beta w)/(1+\beta)$.

- *If*
$$\pi_1 < \frac{1 - R_2}{R_1 + 1 - R_2} < \pi_2$$
a stationary bubbly equilibrium exists associated with a price of the bubble $\tilde{\delta}(\beta w)/(1+\beta)$.

  - *If* $R_1 - R_2 > 1$, *this stationary state is unstable.*

- *If $R_1 - R_2 < 1$, it is stable. In this case, a multiplicity of bubbly equilibria exists that converge towards the steady state with oscillations.*

To summarize, pessimism may favor stochastic bubbles such that $d^1 > d^2$ with respect to EU preferences. To illustrate this point, assume that $\pi > 1 - R_2$. Under this condition, stochastic bubbles cannot exist in the economy with EU preferences. If agents are pessimistic, it is possible that they transform probabilities in such a way that $\pi_1 = \phi(\pi) < 1 - R_2$. In this case a stochastic bubble can exist. The price of the bubble is low enough in such a way that consumption in state 2 (the bubble exists) remains lower than consumption in state 1 (the bubble explodes). As agents are pessimistic, they put more weight on the bad state and invest more in the bubble.

The previous section has showed that optimism may favor stochastic bubbles with a high enough price in such a way that $d^1 < d^2$. This section proves that pessimism may favor stochastic bubbles with a low price. Therefore, the same parameter (the degree of optimism) can have opposite effects depending on the type of the bubble. This result comes from the property that the weights put on the different states of the nature depend on the rank within the RDU framework. A stochastic bubble with a low value does not change the ranking of the two states of the nature whereas a bubble with a high value does. In the first case, pessimism favors the existence of the bubble whereas optimism favors the existence of the bubble in the second case.

Moreover it is possible to have indeterminacy with a multiplicity of bubbly equilibria converging towards a steady state with oscillations. This result of indeterminacy is related to the existence of a kink on the indifference curves for $d^1 = d^2$. When $d^1 = d^2$, it is possible that different values of the price of the bubble are compatible with an equilibrium. Indeterminacy results from the RDU assumption, but is also related to the assumption of a linear production technology. It is possible to check that indeterminacy vanishes in the case of a Cobb-Douglas technology that eliminates the impact of uncertainty on dynamic inefficiency. Indeterminacy needs a high enough elasticity of substitution.

The existence of a multiplicity of equilibria with RDU preferences or

Choquet utility has been obtained in other frameworks: Tallon (1997) and Epstein and Wang (1994) also obtain this result in models of financial assets.

4 Conclusion

This paper has proposed a simple model that suggests that uncertainty associated with RDU preferences can extend the scope for the existence of rational financial bubbles. Pessimism favors the existence of deterministic bubbles, when optimism may promote the existence of stochastic bubbles. Moreover, associated with pessimism, the RDU assumption is a new cause of multiple bubbly equilibria.

It would be interesting to expand these first results into a more general framework. A first improvement would consist in introducing different production technologies subject to different shocks, with many states of the nature. Considering non-linear production technologies could also be an interesting generalization.

Another development would be to assume heterogeneous agents differing by their degree of pessimism or optimism.

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5 Appendixes

5.1 Appendix 1: proof of Proposition 1.

The proof uses the method developed by Homburg (1992) and De la Croix and Michel (2002). Some notations are introduced. h_t denotes a particular history from period 0 till t (the state σ_0 at period 0 is assumed to be known in $t = 0$): $h_t = (\sigma_0, \sigma_1, \sigma_2, ..., \sigma_t)$. H_t is the set of all possible t-period histories from period 0. $\#H_t = 2^t$ as σ_0 is known in 0. The applications τ and θ are defined such that, for an history $h_t = (\sigma_0, \sigma_1, \sigma_2, ..., \sigma_{t-1}, \sigma_t) \in H_t$, $\tau(h_t) = (\sigma_0, \sigma_1, \sigma_2, ..., \sigma_{t-1})$ and $\theta(h_t) = \sigma_t$.

The allocation corresponding to the competitive equilibrium is such that, for all $t \geq 0$: K_{-1} is given, $d_0 = R(\sigma_0)K_{-1}$, $c_t = \frac{w}{1+\beta} \equiv \bar{c}$, $K_t = \frac{\beta w}{1+\beta} \equiv \bar{K}$, $d_{t+1} = \frac{R(\sigma_{t+1})\beta w}{1+\beta}$.Using the following notation, $\bar{d}(\sigma_t) = \frac{R(\sigma_t)\beta w}{1+\beta}$, the corresponding *ex-ante* utility level is: $\bar{U} \equiv \ln(\bar{c}) + \beta\pi \ln\left(\bar{d}(1)\right) + \beta(1-\pi)\ln\left(\bar{d}(2)\right)$.

Assuming that this allocation is not interim Pareto optimal means that there exists another feasible allocation that, almost surely, gives a higher expected utility for all period t with a strict improvement on a set of states of positive measure. Formally, it means that it is possible to find an allocation $(\tilde{c}(h_t),\ \tilde{d}(h_t),\ \tilde{K}(h_t))_{h_t \in H_t,\ t \geq 0}$ such that $\forall t$

$$\begin{aligned} \tilde{c}(h_t) + \tilde{d}(h_t) + \tilde{K}(h_t) &= R(\theta(h_t))\tilde{K}(\tau(h_t)) + w \\ \tilde{K}(\tau(h_0)) &= K_{-1} \text{ (initial condition given)} \end{aligned}$$

(feasibility), and such that $\forall t$, $\forall h_t \in H_t$

$$\ln(\tilde{c}(h_t)) + \beta\pi \ln\left(\tilde{d}((h_t, 1))\right) + \beta(1-\pi)\ln\left(\tilde{d}((h_t, 2))\right) \geq \bar{U} \quad (41)$$

$$\tilde{d}(\sigma_0) \geq R(\sigma_0)K_0 \quad (42)$$

with a strict inequality for some h_{t_0}.

First, it is easy to check that the competitive solution $(\bar{c}, \bar{d}(1), \bar{d}(2))$ in period t can be obtained through the following program:

$$\max_{(c,d^1,d^2)} \ln(c) + \beta\pi \ln\left(d^1\right) + \beta(1-\pi)\ln\left(d^2\right)$$
$$\text{s.t. } w = c + \frac{\pi}{R_1}d^1 + \frac{(1-\pi)}{R_2}d^2$$

As a consequence of this property, (41) implies that, $\forall t$, $\forall h_t \in H_t$

$$\tilde{c}(h_t) + \frac{\pi}{R_1}\tilde{d}((h_t,1)) + \frac{(1-\pi)}{R_2}\tilde{d}((h_t,2)) \geq \bar{c} + \frac{\pi}{R_1}\bar{d}(1) + \frac{(1-\pi)}{R_2}\bar{d}(2)$$

with a strict inequality for some history h_{t_0}.

For a state σ_t, $\sigma_t \in \{1,2\}$, the function χ is defined as: $\chi(1) = \pi$ and $\chi(2) = 1-\pi$. For an history $h_t = (\sigma_0, \sigma_1, \sigma_2, ..., \sigma_{t-1}, \sigma_t) \in H_t$, $P(h_t)$ is defined as

$$P(h_t) = \prod_{i=1}^{t}\chi(\sigma_i) \Big/ \prod_{i=1}^{t}R(\sigma_i)$$

and $P(\sigma_0) = 1$. Therefore,

$$P(h_t) = \frac{\chi(\theta(h_t))}{R(\theta(h_t))}P(\tau(h_t))$$

For $T > t_0$, it is obtained that $\tilde{d}(\sigma_0) - d_0 +$

$$\sum_{t=0}^{T-1}\sum_{h_t\in H_t}P(h_t)\left[\tilde{c}(h_t) + \frac{\pi}{R_1}\tilde{d}((h_t,1)) + \frac{(1-\pi)}{R_2}\tilde{d}((h_t,2)) - \bar{c} - \frac{\pi}{R_1}\bar{d}(1) - \frac{(1-\pi)}{R_2}\bar{d}(2)\right] > 0$$

Rearranging the terms that depend on the same period, the following is obtained:

$$\sum_{t=0}^{T-1}\sum_{h_t\in H_t}P(h_t)\left[\tilde{c}(h_t) + \tilde{d}(h_t) - \bar{c} - \bar{d}(\theta(h_t))\right] + \sum_{h_T\in H_T}P(h_T)\left[\tilde{d}((h_T)) - \bar{d}(\theta(h_T))\right] > 0$$

From the feasibility constraints, it is obtained:

$$\sum_{t=0}^{T-1}\sum_{h_t\in H_t}P(h_t)\left[R(\theta(h_t))\tilde{K}(\tau(h_t)) - \tilde{K}(h_t) - R(\theta(h_t))\bar{K} + \bar{K}\right] +$$
$$\sum_{h_T\in H_T}P(h_T)\left[\tilde{d}((h_T)) - \bar{d}(\theta(h_T))\right] > 0$$

After simplifications, it is obtained:

$$\sum_{h_{T-1}\in H_{T-1}} P(h_{T-1})\left[-\tilde{K}(h_{T-1})+\bar{K}\right]+\sum_{h_T\in H_T} P(h_T)\left[\tilde{d}((h_T))-\bar{d}(\theta(h_T))\right]>0 \tag{43}$$

It is possible to write:

$$-\tilde{K}(h_{T-1})+\bar{K}=\frac{\pi}{R(1)}R(1)\left[-\tilde{K}(h_{T-1})+\bar{K}\right]+\frac{1-\pi}{R(2)}R(2)\left[-\tilde{K}(h_{T-1})+\bar{K}\right]$$

Replacing in (43) makes it possible to write:

$$\sum_{h_T\in H_T} P(h_T)\left[\tilde{d}(h_T)-R(\theta(h_T))\tilde{K}\left(\tau(h_T)\right)-\bar{d}(\theta(h_T))+R(\theta(h_T))\bar{K}\right]>0$$

Using the feasibility constraint leads to:

$$\sum_{h_T\in H_T} P(h_T)\left[\bar{c}+\bar{K}-\tilde{c}(h_T)-\tilde{K}\left(h_T\right)\right]>0$$

Finally, $\bar{c}+\bar{K}=w$ and it is obtained:

$$\left[\sum_{h_T\in H_T} P(h_T)\right]w>\sum_{h_T\in H_T} P(h_T)\left[\bar{c}+\bar{K}-\tilde{c}(h_T)-\tilde{K}\left(h_T\right)\right]>0$$

Introducing the notation

$$S_T=\sum_{h_T\in H_T} P(h_T)$$

it is straightforward to check that

$$S_T=\left(\frac{\pi}{R_1}+\frac{1-\pi}{R_2}\right)S_{T-1}$$

Therefore, if $\frac{\pi}{R_1}+\frac{1-\pi}{R_2}<1$, $\lim_{T\to\infty} S_T=0$ and the competitive equilibrium is interim Pareto-optimal.

5.2 Appendix 2: proof of Proposition 2.

The proof is adapted from the preceding one, replacing π by π_1. The competitive solution in period t can be obtained through the following program:

$$\max_{(c,d^1,d^2)} \ln(c)+\beta\pi_1\ln\left(d^1\right)+\beta(1-\pi_1)\ln\left(d^2\right)$$

$$\text{s.t. } w = c+\frac{\pi_1}{R_1}d^1+\frac{(1-\pi_1)}{R_2}d^2$$

As the agent is pessimistic, its preferences remain strictly convex. Therefore, the preceding reasoning can be used: a feasible allocation $(\tilde{c}(h_t), \tilde{d}(h_t), \tilde{K}(h_t))_{h_t \in H_t,\, t \geq 0}$ that interim Pareto-dominates the competitive equilibrium must satisfy $\forall t$, $\forall h_t \in H_t$

$$\tilde{c}(h_t) + \frac{\pi_1}{R_1}\tilde{d}((h_t,1)) + \frac{(1-\pi_1)}{R_2}\tilde{d}((h_t,2)) \geq \bar{c} + \frac{\pi_1}{R_1}\bar{d}(1) + \frac{(1-\pi_1)}{R_2}\bar{d}(2)$$

with a strict inequality for some history h_{t_0}. Thereafter, the proof is the same, replacing π by π_1.

5.3 Appendix 3: proof of Proposition 3.

The proof is adapted from Appendix 1. The competitive solution in period t can be obtained through the following program:

$$\begin{aligned} & \max_{(c,d^1,d^2)} \ln(c) + \beta\pi_1 \ln\left(d^1\right) + \beta(1-\pi_1)\ln\left(d^2\right) \\ \text{s.t. } w &= c + \frac{\pi_1}{R_1}d^1 + \frac{(1-\pi_1)}{R_2}d^2 \end{aligned}$$

But, as the agent is optimistic, its preferences are no more convex. More precisely, it is possible that the program

$$\begin{aligned} & \max_{(c,d^1,d^2)} \ln c + \beta E_\phi \ln(d) \\ \text{s.t. } w &= c + \frac{\pi_1}{R_1}d^1 + \frac{(1-\pi_1)}{R_2}d^2 \end{aligned} \tag{44}$$

has its optimal solution in the domain $d^1 < d^2$. In this case the solution results from the program

$$\begin{aligned} & \max_{(c,d^1,d^2)} \ln(c) + \beta\pi_2 \ln\left(d^1\right) + \beta(1-\pi_2)\ln\left(d^2\right) \\ \text{s.t. } w &= c + \frac{\pi_1}{R_1}d^1 + \frac{(1-\pi_1)}{R_2}d^2 \end{aligned}$$

The solution is:

$$c = \tfrac{w}{1+\beta} \qquad d^1 = \tfrac{\pi_2}{\pi_1}\tfrac{R_1\beta w}{1+\beta} \qquad d^2 = \tfrac{1-\pi_2}{1-\pi_1}\tfrac{R_2\beta w}{1+\beta}$$

It exists only if $d^1 < d^2$, which implies:

$$\frac{\pi_2}{\pi_1} R_1 < \frac{1-\pi_2}{1-\pi_1} R_2 \tag{45}$$

Moreover, it is the optimal solution only if the level of utility is higher than the level for the competitive equilibrium. This condition gives the inequality:

$$\begin{aligned} & \ln\left(\frac{w}{1+\beta}\right) + \beta\pi_2 \ln\left(\frac{\pi_2}{\pi_1}\frac{R_1\beta w}{1+\beta}\right) + \beta(1-\pi_2)\ln\left(\frac{1-\pi_2}{1-\pi_1}\frac{R_2\beta w}{1+\beta}\right) \\ > \; & \ln\left(\frac{w}{1+\beta}\right) + \beta\pi_1 \ln\left(\frac{R_1\beta w}{1+\beta}\right) + \beta(1-\pi_1)\ln\left(\frac{R_2\beta w}{1+\beta}\right) \end{aligned}$$

that can be simplified through:

$$\frac{R_2}{R_1} > \left[\frac{\pi_1^{\pi_2}(1-\pi_1)^{(1-\pi_2)}}{\pi_2^{\pi_2}(1-\pi_2)^{(1-\pi_2)}}\right]^{\frac{1}{\pi_1-\pi_2}}$$

It is easy to check that this last condition is stronger than (45). Therefore, if the converse condition (13) is satisfied, the competitive equilibrium is the solution of the program (44). The reasoning of Appendix 1 can now be used: under (13), a feasible allocation $(\tilde{c}(h_t),\ \tilde{d}(h_t),\ \tilde{K}(h_t))_{h_t\in H_t,\ t\geq 0}$ that interim Pareto-dominates the competitive equilibrium must satisfy $\forall t$, $\forall h_t \in H_t$

$$\tilde{c}(h_t) + \frac{\pi_1}{R_1}\tilde{d}((h_t,1)) + \frac{(1-\pi_1)}{R_2}\tilde{d}((h_t,2)) \geq \bar{c} + \frac{\pi_1}{R_1}\bar{d}(1) + \frac{(1-\pi_1)}{R_2}\bar{d}(2)$$

with a strict inequality for some history h_{t_0}. Thereafter, the proof is the same, replacing π by π_1.

5.4 Appendix 4: proof of Corollary 1.

To prove the first part of corollary (1), a limited development of the right hand side of (13) is made, taking the ln:

$$\ln\left[\frac{\pi_1^{\pi_2}(1-\pi_1)^{(1-\pi_2)}}{\pi_2^{\pi_2}(1-\pi_2)^{(1-\pi_2)}}\right]^{\frac{1}{\pi_1-\pi_2}} = \frac{1}{\pi_1-\pi_2}\left[\pi_2 \ln\frac{\pi_1}{\pi_2} + (1-\pi_2)\ln\left(\frac{1-\pi_1}{1-\pi_2}\right)\right]$$

$$\sim \frac{1}{\pi_1-\pi_2}\left\{\pi_2\left[\frac{\pi_1-\pi_2}{\pi_2} - \frac{1}{2}\left(\frac{\pi_1-\pi_2}{\pi_2}\right)^2\right] + (1-\pi_2)\left[-\left(\frac{\pi_1-\pi_2}{1-\pi_2}\right) - \frac{1}{2}\left(\frac{\pi_1-\pi_2}{1-\pi_2}\right)^2\right]\right\}$$

$$= \frac{1}{\pi_1 - \pi_2}\left[-\frac{1}{2}(\pi_1 - \pi_2)^2\left(\frac{1}{\pi_2} + \frac{1}{1-\pi_2}\right)\right]$$
$$= -\frac{1}{2}(\pi_1 - \pi_2)\left(\frac{1}{\pi_2} + \frac{1}{1-\pi_2}\right)$$

This last expression tends to 0 when $\pi_1 - \pi_2 \to 0$. Therefore, condition (13) tends to $R_2/R_1 < 1$.

For proving the second part of the corollary, condition (13) can be written:

$$\frac{R_2}{R_1} < \left[\frac{(\pi^\rho)^{1-(1-\pi)^\rho}(1-\pi^\rho)^{(1-\pi)^\rho}}{(1-(1-\pi)^\rho)^{1-(1-\pi)^\rho}((1-\pi)^\rho)^{(1-\pi)^\rho}}\right]^{\frac{1}{\pi^\rho - 1 + (1-\pi)^\rho}}$$

For a given value of π, the right hand side of the inequality is an increasing function of ρ that maps $(0,1)$ to $(0,1)$. Therefore, (13) allows a lower bound on ρ to be defined.

5.5 Appendix 5: proof of Lemma 1.

From equation (21), the dynamics can be written: $1 = D(\delta_t, \delta_{t+1})$, with

$$D(\delta_t, \delta_{t+1}) \equiv \frac{(1-\pi_1)R_1}{\delta_t R_1 - \delta_{t+1}} - \frac{\pi_1 R_2}{\delta_{t+1} - \delta_t R_2}$$

Firstly, it is shown that the dynamics is well defined. For a given value of δ_t, there exists a unique value of δ_{t+1} in $(\delta_t R_2, \delta_t R_1)$ such that $1 = D(\delta_t, \delta_{t+1})$. Indeed, the function $D(\delta_t, \delta_{t+1})$ is increasing on $(\delta_t R_2, \delta_t R_1)$, and

$$\lim_{\delta_{t+1} \to (\delta_t R_2)^+} D(\delta_t, \delta_{t+1}) = -\infty \qquad \lim_{\delta_{t+1} \to (\delta_t R_1)^-} D(\delta_t, \delta_{t+1}) = +\infty$$

Therefore, the dynamics is well defined. Moreover, as $D(\delta_t, \delta_{t+1})$ is a decreasing function of δ_t, the dynamics is monotonic.

Secondly, existence and stability of steady states are studied. After some calculations, the dynamics can also be written:

$$\delta_{t+1}^2 + \delta_{t+1}\left[(1-\pi_1)R_1 + \pi_1 R_2 - \delta_t(R_1 + R_2)\right] - R_1 R_2 \delta_t (1-\delta_t) = 0 \quad (46)$$

Two steady states may exist: 0 and

$$\delta = \frac{(1-\pi_1)R_1}{R_1 - 1} - \frac{\pi_1 R_2}{1 - R_2}$$

δ is positive iff $\frac{\pi_1}{R_1} + \frac{(1-\pi_1)}{R_2} > 1$. A simple derivation of (46) in $\delta_t = 0$ gives:

$$\left.\frac{d\delta_{t+1}}{d\delta_t}\right|_{\delta_t=0} = \frac{R_1 R_2}{(1-\pi_1) R_1 + \pi_1 R_2}$$

The condition $\left.\frac{d\delta_{t+1}}{d\delta_t}\right|_{\delta_t=0} < 1$ is equivalent to $\frac{\pi_1}{R_1} + \frac{(1-\pi_1)}{R_2} > 1$, which is required for the existence of a positive steady state δ.

The derivation of (46) in $\delta_t = \delta$ gives:

$$\left.\frac{d\delta_{t+1}}{d\delta_t}\right|_{\delta_t=\delta} = \frac{\delta(R_1 + R_2) + R_1 R_2 (1 - 2\delta)}{\delta + R_1 R_2 (1 - \delta)}$$

The inequality $\delta < 1$ is true as it is equivalent to (23). Therefore, the condition $\left.\frac{d\delta_{t+1}}{d\delta_t}\right|_{\delta_t=\delta} > 1$ is equivalent to $\delta(R_1 - 1)(1 - R_2) > 0$, which is true as $\delta > 0$ by (22).

Following these results and as the dynamics of (δ_t) is monotonic, if $\delta_0 < \delta$, δ_t converges toward 0; if $\delta_0 > \delta$, δ_t converges toward $+\infty$.

5.6 Appendix 6: proof of Proposition 8

In the case of stochastic bubbles, the intertemporal budget constraint is given by equation (27):

$$c_t + \frac{d^1_{t+1}}{R_1}\left(1 - R_2 \frac{p_t}{p_{t+1}}\right) + d^2_{t+1} \frac{p_t}{p_{t+1}} = w$$

For a steady state, the constraint becomes:

$$c + \frac{d^1}{R_1}(1 - R_2) + d^2 = w$$

that does not depend on the value of the bubble asset p and only depends on exogenous variables. To determine if the optimal choice of the consumer is obtained with $d^1 > d^2$ or $d^1 < d^2$, it is necessary to compare the indirect utilities associated with the two cases. It is straightforward to calculate that the optimal solution is associated with $d^1 < d^2$ iff:

$$\pi_2 \ln\left(\pi_2 \frac{R_1}{1 - R_2}\right) + (1-\pi_2)\ln(1-\pi_2) > \pi_1 \ln\left(\pi_1 \frac{R_1}{1 - R_2}\right) + (1-\pi_1)\ln(1-\pi_1)$$

The function $\zeta(\pi) = \pi \ln\left(\pi \frac{R_1}{1-R_2}\right) + (1-\pi)\ln(1-\pi)$ reaches it minimum value for $\pi = \frac{1-R_2}{R_1+1-R_2}$. Finally, the results of Proposition 8 are obtained.

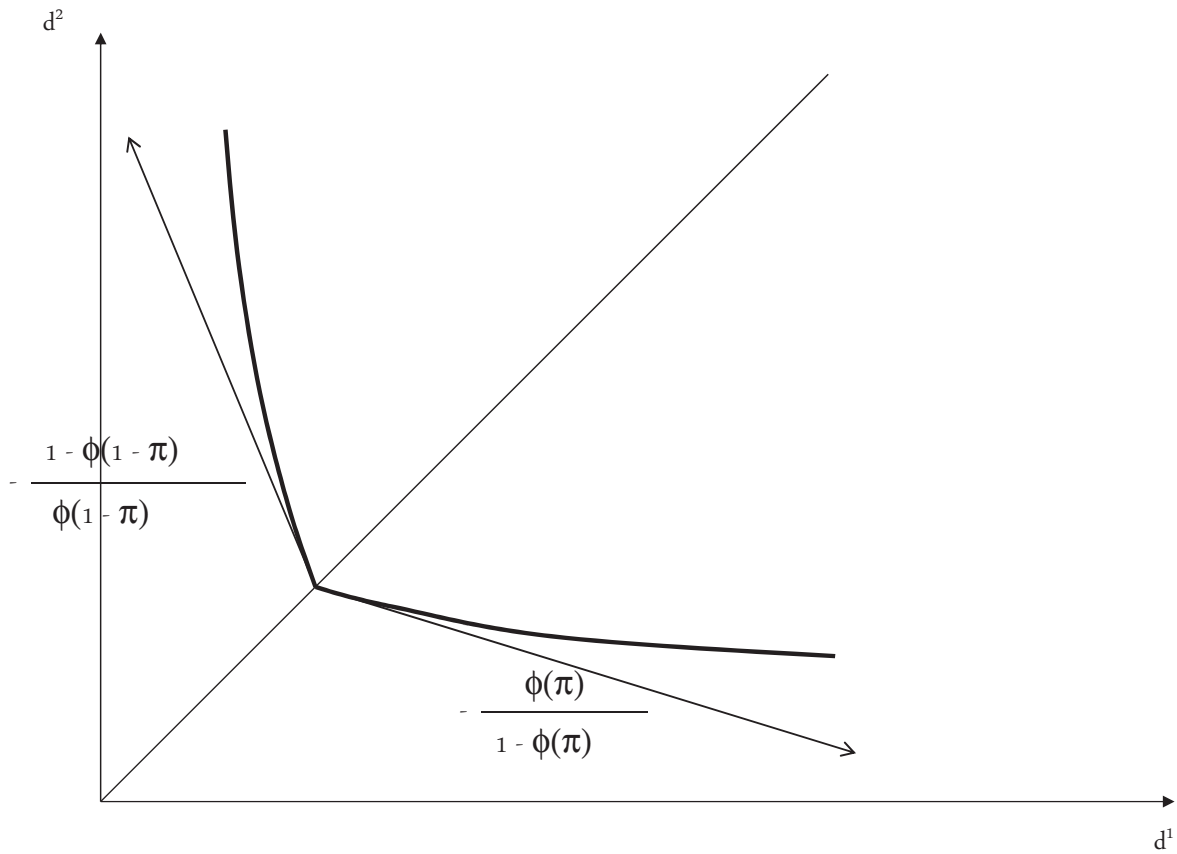

Fig 1: kink in the indifference curve of a pessimistic agent

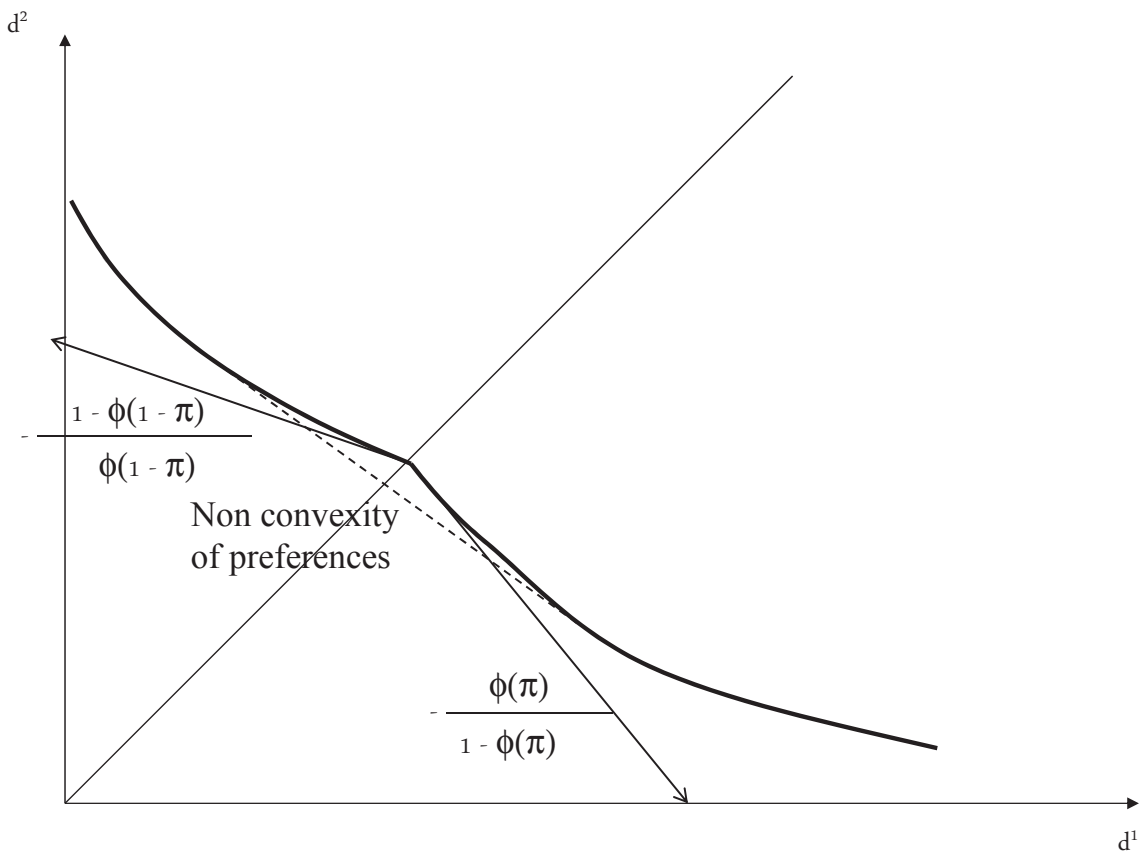

Fig 2: kink in the indifference curve of an optimistic agent

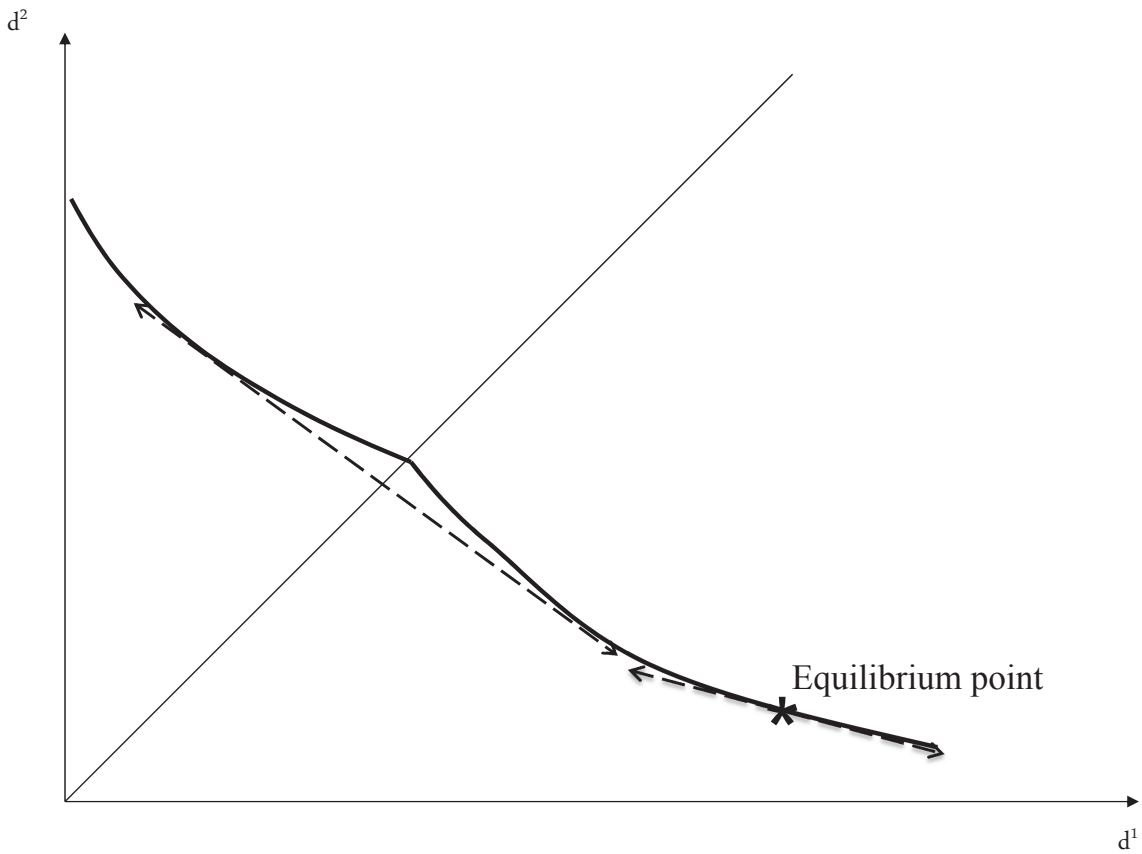

Fig 3: Pareto optimal equilibrium with optimism

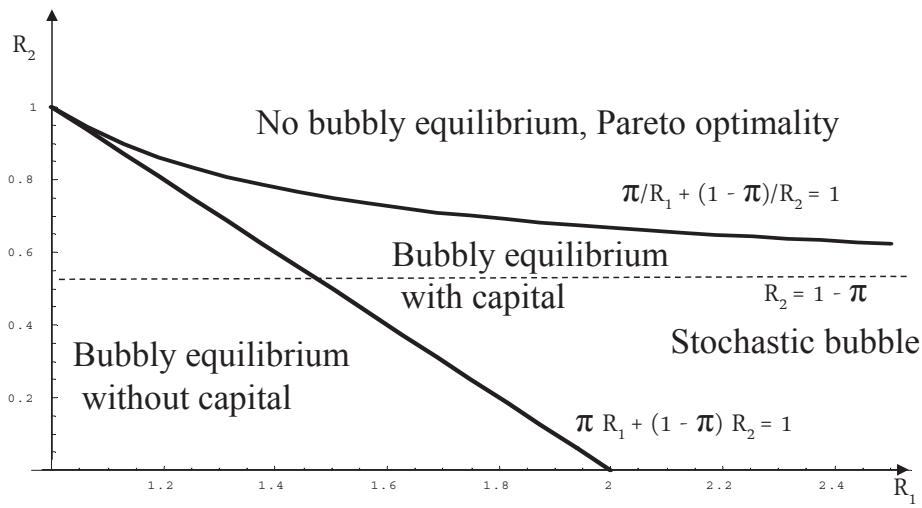

Fig 4: existence of bubbly equilibria with respect to R_1 and R_2

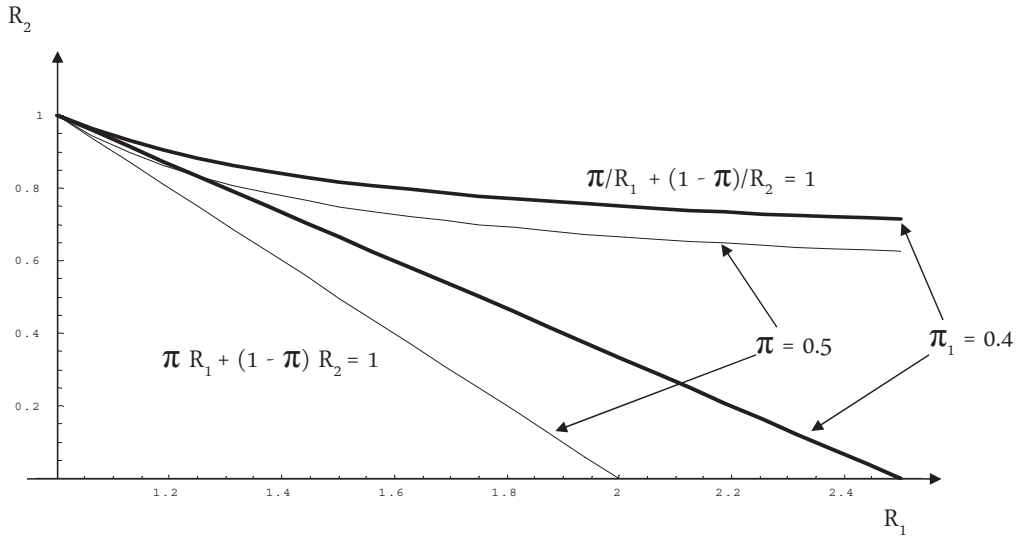

Fig 5: effect of pessimism on the existence of bubbly equilibria