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2014.15

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Discounting, Risk and Inequality: A General Approach*

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March 2014

Abstract – The common practice consists in using a unique value of the discount rate for all public investments. Endorsing a social welfare approach to discounting, we show how different public investments should be discounted depending on: the risk on the return of the investment, the systematic risk on aggregate consumption, the distribution of gains and losses, and inequality. We also study the limit value of the discount rate for very long term investments, and the type of information that is needed about long-term scenarios in order to evaluate investments.

Résumé – Une seule et même valeur du taux d'escompte est généralement utilisée pour tous les investissements publics. En utilisant une définition du taux d'escompte fondée sur les fonctions de bien-être social, nous montrons comment ces investissements doivent être escomptés de façon différente selon les risques portant sur le taux de retour et la croissance agrégée, selon la distribution de gains et pertes et selon le niveau d'inégalité. Nous étudions également la valeur limite du taux d'escompte à long-terme et le type d'information sur les scénarios de long-terme nécessaire pour évaluer les investissements.

Keywords: Social discounting, risk, inequality.

Mots clés : Escompte social, risque, inégalité.

JEL Classification numbers: D63.

*This paper has benefited from the comments of S. Pacala and the audiences at a Princeton workshop and at the TIGER Conference in Toulouse. S. Zuber acknowledges the hospitality of the Princeton Institute for International and Regional Studies Research Community "Communicating Uncertainty: Sciences, Institutions and Ethics in the Politics of Global Climate Change". This research has been supported by the Chair on Welfare Economics and Social Justice at the Institute for Global Studies (FMSH - Paris), the Franco-Swedish Program on Economics and Philosophy (FMSH - Risksbankens Jubileumsfond), and the ANR project EquiRisk, Grant # ANR-12-INEG-0006-01.

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1 Introduction

Investments and policies having long term impacts are crucial for the development of the economy, and they often attract much attention in the public debate. One prominent example are of course mitigation policies aimed at preventing dramatic future climate change that may threaten the mere survival of many species, including humankind. The issue of climate policy has recently revealed that there is no agreement among economists about the appropriate welfare framework for evaluating such policies. The Stern review on the economics of climate change Stern (2006) has been heatedly debated on this ground (Nordhaus, 2007a,b; Weitzman, 2007; Dasgupta, 2008; Heal, 2007).

Although they reach very different conclusions, all these papers endorse the same basic welfare model, namely the Expected Discounted Utilitarian criterion

$$\sum_{t=0}^{\infty} e^{-\delta t} Eu(c_t), \tag{1}$$

where c_t is, to simplify, the consumption of the representative agent of generation t. This criterion yields the standard Ramsey equation (Ramsey, 1928): An investment from period 0 to period t that yields a sure rate of return r^* is worth doing, in the margin, if

$$u'(c_0) < e^{-\delta t} Eu'(c_t) \times e^{r^* t},$$

i.e., if

$$r^* > \delta - \frac{1}{t} \ln\left(\frac{Eu'(c_t)}{u'(c_0)}\right). \tag{2}$$

The debate around this welfare model has been mostly confined to a discussion of the parameters involved in Equation (2), in particular the rate of pure time preference δ and the elasticity of the marginal utility of consumption. One can roughly identify two positions: the 'ethical' or 'prescriptive' approach that recommends that ethical considerations should guide the choice of the parameters, and the descriptive approach arguing that the parameters should be chosen to match observed market rates. In the end, the choice between the two approaches is an ethical choice, in the sense that a normative justification of the welfare evaluation framework is called for.

Another issue has recently arisen, that seems to call into question the expected utility framework used in Equation (1), namely the question of catastrophic risks. In an influential paper, Weitzman (2009) has indeed presented a 'dismal theorem' conveying the idea that in the presence of catastrophic fat-tail risks, any investment for the future should be undertaken, whatever the value of the parameters of the Ramsey equation. More precisely, Weitzman shows that the threshold r^* in Equation (2) can then become infinitely negative. This conclusion has been much discussed (see Millner, 2013, for a recent discussion of the debates surrounding Weitzman's result).

The result by Weitzman (2009) is however representative of the profuse literature discussing the impact of risk on the social discount rate. In particular, it has been showed that the social discount rate is likely to be lower when there is a large risk on future growth (Weitzman, 1998; Gollier, 2002; Gollier and Weitzman, 2010). This kind of risk generally induces the 'Weitzman effect' (Weitzman, 1998) that the social discount rate should decrease with the time horizon. Another kind of risk is that on the rate of return of the investment. It generally yields the opposite 'Gollier effect' (Gollier, 2004) that the social discount rate increases with the time horizon. In general, both kind of risks co-exist, and they should be jointly studied (Gollier, 2012, contains a chapter on the issue but restricts attention to the discounted utilitarian approach).

In addition to risk, equity is another dimension that affects the social discount rate, even in the standard discounted utilitarian framework. The usual technique to deal with equity considerations in the literature on climate change policy has been to introduce equity weights putting greater emphasis on the damages affecting the poor than to damages affecting the rich. Early references include Azar and Sterner (1996), Azar (1999), Fankhauser Tol and Pearce (1997) and Pearce (2003), and reach ambiguous conclusions regarding the impact of equity considerations on the cost of carbon. Anthoff Hepburn and Tol (2009) is perhaps the more recent and complete study, and they find that equity considerations can significantly increase the social cost of carbon. These approaches do not directly incorporate equity considerations in the discount rate. Gollier (2010) is an attempt in that direction, which shows that equity considerations may yield an increase of the discount rate in the long run, when there is economic convergence, or even when inequalities are persistent. The paper however considers discount rates associated with a Utilitarian formula, where the costs and benefits of the investment are equally shared within generations. We want to consider a more general case allowing unequal sharing of the costs and benefits, and not restricted to the Utilitarian approach.

To sum up, the literature has identified some of the limitations of the basic welfare model used to derive social discounting. Several shortcomings however remain. First, the analysis is almost always confined to the standard Expected Discounted Utilitarian Model. The exceptions are papers considering non-expected utility models (Gollier, 2002, 2012; Gierlinger and Gollier, 2008; Traeger, 2009) and a few papers suggesting alternatives to Utilitarianism (Bommier and Zuber, 2008; Fleurbaey and Zuber, 2014; Zuber and Asheim, 2012). The papers abandoning the expected utility framework studied the impact of ambiguity aversion or preference for the timing of the resolution of uncertainty on the discount rate (Gollier, 2002, 2012; Gierlinger and Gollier, 2008; Traeger, 2009). In the present paper, we shall stick to the expected utility framework. The papers introducing alernative social welfare criteria generally limit themselves to some aspects of the risk, mainly the risk on the planning horizon. It is indeed a second shortcoming of the literature that the different issues (systematic risk, uncertainty of the returns, inequality) are treated separately. A last limitation of the literature is that it usually considers successive generations, often represented by a single agents, rather than considering overlapping generations.

In the present paper, we offer a general welfare framework to analyze the issues of risk, inequality, and variable population. Our analysis does not focus on a particular criterion but uncovers common features of all approaches. We allow both for a systematic risk that affects aggregate consumption and a risk on the returns of the policy, examining how their interactions affect social discounting. The main results of the paper are the following. First, we obtain general formulas for the discount rate, which decompose the main components related to growth, inequalities, and risks in consumption and in investment returns. Compared to the classical Ramsey formula, these formulas display important additional covariance terms. Second, we show that in the long run the key determinants of the discount rate are the situation of the worst-off individuals in the worst-case scenarios, as well as the maximum return on the investment. The third main lesson is that, in an OLG setting, the discount rate is a weighted average of market rates (individuals' own discount rates) and the social discount rate between different generations, implying that market rates are relevant for short-term investments but much less so in the long-run.

The difficult issues raised by the climate change problem also point to the limitations of the social discount rate. When there is a risk of an early extinction of the humanity, it is clear that the impacts of a policy on future consumption are not the only ones to be considered, and perhaps not the most important ones. The effects of the policy on the prospect of a catastrophe are also a key element, and we need tools to value changes in the probability of catastrophic events. These issues are discussed at the end of the paper.

Our paper is organized as follows. Section 2 introduces a general setting and proposes a definition of the social discount rate . Section 3 discusses how three aspects of the risk, the systematic risk on aggregate consumption, the risk on returns and the risk on the planning horizon, affect the social discount rate. Sections 4 tackles the issue of intra-generational inequalities in consumption and the distribution of costs and benefits. Section 5 derives an approximation formula for the social discount rate in the long run, showing that the key figures are the maximum return of the investment and the maximum net return for a poor-to-poor investment. Section 6 provides further extensions. First it considers an OLG economy where individuals live for several periods and shows how individuals' own discount rates enter the general formula for social discounting. Second it discusses the limitations of the social discount rate, in particular when policies affect the prospect of future catastrophes.

2 A general framework and the definition of the social discount rate

2.1 The framework

We let $\mathbb{N}_0$ denote the set of non-negative integers, $\mathbb{N}$ the set of positive integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+$ the set of non-negative real numbers. For a set X and any $n \in \mathbb{N}$, X^n is the n-fold Cartesian product of X.

We focus on evaluating distributions of consumption (or income) at the individual level across periods. An alternative c is a collection of consumption levels, one for each individual alive in the alternative. The set of potential individuals is $\mathbb{N}$, so that alternatives are elements of $\mathcal{C} = \bigcup_{N \subset \mathbb{N} \setminus \emptyset} \prod_{i \in N} \mathbb{R}_+$. We therefore consider a variable-population framework, in which the size of the population may vary from one alternative to another, depending on the subset of individuals alive in the alternative. For any $c \in \mathcal{C}$, we let $N(c)$ be the set of individuals alive in the alternative and $n(c) = |N(c)|$ be the number of individuals in the alternative.

We also need to know to which generation the people alive in an alternative belong. To do so, we assume that there exists a partition of $\mathbb{N}$ into subsets N^t containing the potential individuals of generation $t \in \mathbb{N}_0$. Hence, for each potential individual $i \in \mathbb{N}$, there exists a unique $t \in \mathbb{N}_0$ such that $i \in N^t$, meaning that individual i belongs to generation t. We will restrict attention to $C = \{c \in \mathcal{C} \mid N^0 \subset N(c)\}$, which means that all the members of the current generation are present in all the alternatives we consider. For any $c \in C$ and any $t \in \mathbb{N}_0$, we denote $N^t(c) = N^t \cap N(c)$ and $n^t(c) = |N^t(c)|$.

Uncertainty is described by $m \in \mathbb{N} \setminus \{1\}$ states of the world. The set of states of the world is $S = \{1, \cdots, m\}$. A prospect is a vector belonging to the set $\mathbf{C} = C^m$ with typical element $\mathbf{c} = (c_1, \cdots, c_m)$. For a prospect $\mathbf{c}$, c_s^i denotes the consumption of individual i in state of the world s, whenever $i \in N(c_s)$. To ease the exposition, when there is no ambiguity, we use the notation $N_s = N(c_s)$, $n_s = n(c_s)$, $N_s^t = N^t(c_s)$ and $n_s^t = n^t(c_s)$ for $\mathbf{c} \in \mathbf{C}$ and $s \in S$.

Let $\mathbf{P} = \left\{(p_1, \cdots, p_m) \in [0,1]^m \mid \sum_{s=1}^m p_s = 1\right\}$ denotes the closed $(m-1)$-simplex. A lottery is the combination of a probability vector $\mathbf{p} = (p_1, \cdots, p_m) \in \mathbf{P}$ with a prospect $\mathbf{c} = (c_1, \cdots, c_m) \in \mathbf{C}$. The set of lotteries is denoted

$$\mathbf{L} = \left\{(\mathbf{p}, \mathbf{c}) \in \mathbf{P} \times \mathbf{C}\right\}.$$

2.2 The social evaluation function

A social evaluation function F is a function $F : \mathbf{L} \to \mathbb{R}$ used to ranked lotteries. For any $(\mathbf{p}, \mathbf{c}), (\mathbf{p}', \mathbf{c}') \in \mathbf{L}$, $F(\mathbf{p}, \mathbf{c}) \geq F(\mathbf{p}', \mathbf{c}')$ means that the lottery $(\mathbf{p}, \mathbf{c})$ is deemed socially at least as good as the lottery $(\mathbf{p}', \mathbf{c}')$. We assume that the social evaluation function is an expected utility so that there exists a function $W : C \to \mathbb{R}$ such that:

$$F(\mathbf{p}, \mathbf{c}) = \sum_{s=1}^{m} p_s W(c_s). \tag{3}$$

Although we will mostly work with this general function, it will be useful to illustrate our results with a few more specific examples. When

$$W(c_s) = \sum_{i \in N_s} u(c_s^i), \tag{4}$$

we obtain the usual *Total Utilitarian* criterion. The criterion has been criticized for yielding the Repugnant Conclusion (Parfit, 1984) where, for any population with excellent lives, there is a population with lives barely worth living that is better, provided that the latter includes sufficiently many people.

Several authors (Blackorby and Donaldson, 1984; Broome, 2004; Blackorby Bossert and Donaldson, 2005) have therefore proposed to use the *Critical-Level Utilitarian* criterion

$$W(c_s) = \sum_{i \in N_s} \Big(u(c_s^i) - \bar{u} \Big), \tag{5}$$

where $\bar{u}$ is the *critical-level* of utility. If $\bar{u} = 0$[1], we are back to the Total Utilitarian case, but if $\bar{u} > 0$ only lives with a high enough welfare level are worth adding to a population. One problem when $\bar{u} > 0$ is that we get the Very Sadistic Conclusion (Arrhenius, 2012) where, for any population with terrible lives not worth living, there is a population with good lives that is worse, provided that the latter includes sufficiently many people.

Yet another form is the *Equally Distributed Equivalent* (Fleurbaey, 2010), which can take the form:

$$W(c_s) = \phi^{-1} \left(\tfrac{1}{n_s} \sum_{i \in N_s} \phi\big(u(c_s^i)\big) \right), \tag{6}$$

where ϕ is an increasing and (weakly) concave function. When ϕ is affine, we obtain the *Average Utilitarian Criterion*, which does not satisfy the Negative Expansion Principle (Blackorby Bossert and Donaldson, 2005) that adding a life not worth living should decrease the value of a population. This is true of all Equally Distributed Equivalent evaluation functions whenever $\phi(0) = 0$.

More generally, one may want to consider the following class of *generalized additive* social

[1] $u = 0$ is known as the *neutral* utility level such that a life with higher welfare is worth living and a life with lower welfare is not.

evaluations functions:

$$W(c_s) = \Psi_{n_s}\left(\sum_{i\in N_s}\phi\left(u(c_s^i)\right)\right). \tag{7}$$

Although this last formulation is highly general, the purpose of this paper is not to endorse a specific social evaluation function. Several different forms have been proposed in the literature (see for instance: Ng, 1989; Hurka, 2000; Bommier and Zuber, 2008; Asheim and Zuber, 2014), and they all have normative drawbacks in terms of population ethics or social risk evaluation. Our purpose is to show a common structure of the discount rate for all these different normative approaches. The key role of the social evaluation function will be to determine the social marginal value of the consumption by a specific individual.

2.3 Defining the social discount rate

Following the usual approach to social discounting recalled in the introduction, the computation for our general evaluation function (3) goes as follows. Suppose that individual i in period 0 and state nature s invests \$$\varepsilon$, providing the return \$$B_t\varepsilon$ to an individual j living in period t and state s (hence $j \in N_s^t$).

Using the evaluation function (3), the welfare change induced by the investment at the margin is:

$$W\Big(\cdots, c_s^i - \varepsilon, \cdots, c_s^j + B_t\varepsilon, \cdots\Big) - W(c_s).$$

When ε is sufficiently small, this welfare change is approximately equal to:

$$dF = -\frac{\partial W(c_s)}{\partial c_s^i}\varepsilon + B_t\varepsilon\frac{\partial W(c_s)}{\partial c_s^j}$$

which can be expressed in present-value money as

$$-\varepsilon + B_t\varepsilon\frac{\frac{\partial W(c_s)}{\partial c_s^j}}{\frac{\partial W(c_s)}{\partial c_s^i}} = -\varepsilon + B_t\varepsilon\frac{1}{(1+r^*)^t} \tag{8}$$

Equation (8) implicitly defines the discount rate r^* to be used for the evaluation of this specific investment, given that $j \in N_s^t$. Observe that the investment is worth doing if and only if $B_t \geq (1+r^*)^t$, which shows that the discount rate can equivalently be defined as the minimal rate of return that makes the investment valuable.

This discount rate concerns investments involving only two specific individuals, with a specific time difference, in a specific state of the world. We shall therefore name it the *state-specific person-to-person* discount rate.

Denote $W_s^i = \partial W(c_s)/\partial c_s^i$, the social priority of individual's i consumption in state s. We have the following definition.

Definition 1 *For all* $(\mathbf{p}, \mathbf{c}) \in \mathbf{L}$*, for all* $s \in S$*, and for all* $i \in N^0, j \in N_s^t$*, the* state-specific person-to-person discount rate *from person* i *of generation* 0 *to person* j *of generation* t *in state* s*, denoted* $\delta_s^{i,j}$*, is defined by:*

$$\ln\left(1+\delta_s^{i,j}\right) = \tfrac{1}{t}\ln\left(\tfrac{W_s^i}{W_s^j}\right). \tag{9}$$

In the rest of this paper, we will identify $\ln\left(1+\delta_s^{i,j}\right)$ and $\delta_s^{i,j}$, and work with the continuous-time formula e^{rt} rather than the discrete-time formula $(1+r)^t$. This simplifies the notations and does not change anything of substance.

Realistic investments do not involve only two individuals in a specific state of the world. In addition, the returns of the investment are rarely certain, but may vary depending on the state of the world. Let r_s denote the rate of return in state s. The periot t certainty equivalent rate of return is then defined as the solution r to

$$e^{rt} = \sum_{s=1}^{m} p_s e^{r_s t}, \tag{10}$$

and the relative period t return in state s is defined as

$$\theta_s^t = \tfrac{e^{r_s t}}{e^{rt}}. \tag{11}$$

Consider an $\$\varepsilon$ investment that all individuals from the current period 0 make together, and which is shared between them using the (state-independent) sharing rule $(\sigma^i)_{i\in N^0}$, such that $\sum_{i\in N^0}\sigma^i = 1$. The aggregate return is shared in period t and state s by individuals $j \in N_s^t$ using the sharing rule $(\sigma_s^j)_{j\in N_s^t}$, such that $\sum_{j\in N_s^t}\sigma_s^j = 1$. Individual j in state s and period t therefore receives $\sigma_s^j\theta_s^t e^{rt}\varepsilon$.

If we consider an investment with certainty equivalent rate of return r, using the evaluation function (3) and assuming that ε is small enough, the welfare change induced by the investment is

$$dF = -\sum_{s=1}^{m} p_s \sum_{i\in N^0} \sigma^i W_s^i \varepsilon + \sum_{s=1}^{m} p_s \sum_{j\in N_s^t} \sigma_s^j \theta_s^t e^{rt} W_s^j \varepsilon. \tag{12}$$

We want to define a risk-and-equity adjusted social discount rate δ, suitable for discounting the *expected future aggregate returns of the investment.* Following the methodology used for state-specific person-to-person discount rates, this risk-and-equity adjusted social discount rate δ is the rate r such that the welfare change in Equation (12) is $dF = 0$. This yields the following definition.

Definition 2 *The period* t risk-and-equity adjusted social discount rate δ *for a project with relative returns* $(\theta_s^t)_{s\in S}$ *and sharing rules* $(\sigma^i)_{i\in N^0}$ *and* $(\sigma_s^j)_{j\in N_s^t, s\in S}$ *is:*

$$\delta = \frac{1}{t}\ln\left(\frac{\sum_{s=1}^{m} p_s \sum_{i\in N^0}\sigma^i W_s^i}{\sum_{s=1}^{m} p_s \sum_{j\in N_s^t}\sigma_s^j \theta_s^t W_s^j}\right). \tag{13}$$

The fact that the discount rate depends on characteristics of each particular investment to be evaluated (shares σ^i, σ_s^j, relative return θ_s^t) should not lead to the confusion that this formula computes an internal rate of return of the investment, i.e., the discount rate that would render this investment a matter of social indifference. Investments with the same $\left(\sigma^i, \sigma_s^j, \theta_s^t\right)$ parameters but different expected aggregate returns r should be evaluated with the same social discount rate given by (13).

We can express the risk-and-equity adjusted social discount rate as a generalized mean of the state-dependent person-to-person discount rates. To do so, let $w_s^i = W_s^i / \left(\sum_{s=1}^{m} p_s W_s^i\right)$ denote the social priority of individual i of generation 0 in state s relative to her expected priority.

Proposition 1 *For any* $(\mathbf{p}, \mathbf{c}) \in \mathbf{L}$, *the period* t risk-and-equity adjusted social discount rate δ *for a project with relative returns* $(\theta_s^t)_{s\in\{1,\cdots,m\}}$ *and sharing rules* $(\sigma^i)_{i\in N^0}$ *and* $(\sigma_s^j)_{j\in N_s^t, s\in S}$ *is given by the formula:*

$$\delta = \frac{1}{t}\ln\left(\sum_{i\in N^0}\sigma^i\left(\sum_{s=1}^{m} p_s w_s^i \theta_s^t \sum_{j\in N_s^t}\sigma_s^j e^{-\delta_s^{i,j} t}\right)^{-1}\right) \tag{14}$$

Proof. From Equation (13), we have:

$$\begin{aligned}
\delta &= \tfrac{1}{t}\ln\left(\frac{\sum_{s=1}^{m} p_s \sum_{i\in N^0} \sigma^i W_s^i}{\sum_{s=1}^{m} p_s \sum_{j\in N_s^t} \sigma_s^j \theta_s^t W_s^j}\right) \\
&= \tfrac{1}{t}\ln\left(\sum_{i\in N^0} \sigma^i \frac{\sum_{s=1}^{m} p_s W_s^i}{\sum_{s=1}^{m} p_s \sum_{j\in N_s^t} \sigma_s^j \theta_s^t W_s^j}\right) \\
&= \tfrac{1}{t}\ln\left(\sum_{i\in N^0} \sigma^i \left(\sum_{s=1}^{m} p_s \theta_s^t \sum_{j\in N_s^t} \sigma_s^j \frac{e^{-\delta_s^{i,j} t} W_s^i}{\sum_{s=1}^{m} p_s W_s^i}\right)^{-1}\right) \\
&= \tfrac{1}{t}\ln\left(\sum_{i\in N^0} \sigma^i \left(\sum_{s=1}^{m} p_s w_s^i \theta_s^t \sum_{j\in N_s^t} \sigma_s^j e^{-\delta_s^{i,j} t}\right)^{-1}\right),
\end{aligned}$$

the penultimate step using the fact that, by Definition 1, $W_s^j = e^{-\delta_s^{i,j} t} W_s^i$. ■

If w_s^i and θ_s^t are not uniform across states of the world, the formula in Proposition 1 is not just a generalized mean of the $\delta_s^{i,j}$. When all $\delta_s^{i,j}$ are identical ($= \delta^*$) across individuals and states, the formula becomes

$$\delta = \delta^* + \tfrac{1}{t}\ln\left(\sum_{i\in N^0} \sigma^i \left(\sum_{s=1}^{m} p_s w_s^i \theta_s^t\right)^{-1}\right),$$

so that the discount rate is lower than δ^* if the investment has greater returns in the states in which investors (or, equivalently, beneficiaries, since a constant $\delta_s^{i,j}$ means that their priorities are perfectly correlated) have greater priority, as measured by w_s^i. In the next sections we provide a more transparent analysis of this formula.

In all the above formulas, a key variable is the social priority of individuals' consumptions. The normative choice of function W will be crucial in determining this value.

For instance, in the well-known Total Utilitarian case defined in Equation (4), this value is given by the marginal utility of consumption $W_s^i = u'(c_s^i)$. This is also the case for the Critical-Level Utilitarian criterion (5), so that the critical level does not affect social discounting in that case.[2] In both cases, $w_s^i = u'(c_s^i)/(\sum_{s'=1}^{m} p_{s'} u'(c_{s'}^i))$, so that $w_s^i = 1$ when there is no uncertainty about first period consumption, which further simplifies Equation (14).

[2]The critical level may however affect the choice of the optimal policy, as shown in Millner (2013).

This simple identity between the marginal utility of consumption and social priority is no longer the case for the more general EDE criterion (6). Indeed, then,

$$W_s^i = u'(c_s^i)\ \phi'\big(u(c_s^i)\big)\ \Big(n_s\ \phi'\big(EDE(c_s)\big)\Big)^{-1}, \tag{15}$$

where $EDE(c_s) = \phi^{-1}\left(\frac{1}{n_s}\sum_{i\in N_s}\phi\big(u(c_s^i)\big)\right)$. This expression shows that the social priority of an individual may depend on at least three factors in addition to the marginal utility of consumption: a) inequality aversion represented by an additional equity weight to ensure welfare (rather than consumption) equality; b) population size; c) global welfare in the whole population (including all present and future people).

3 Risk on the returns, systematic risk and population risk

The literature studying the impact of risk on the social discount rate generally distinguishes two forms of the risk. One is the systematic risk that affects the baseline scenario (aggregate consumption path) under consideration. The other is the risk on the returns of the investment. A third impact has been studied by a smaller strand of the literature: the risk on the existence of future generations. It is briefly tackled in the Stern report (Stern, 2006, Appendix to Chapter 1), following the seminal contribution by Dasgupta and Heal (1979). Bommier and Zuber (2008), Asheim and Zuber (2014), and Fleurbaey and Zuber (2014) are recents contributions addressing this kind of risk.

In this section, we show how the three aspects of risk enter the formula for the social discount rate in our more general framework. More precisely, we separate three terms. A first term is the probability of the existence of future generations. A second term is the weighted social priority (weighted by the sharing rule), which represents the systematic risk: unfavorable states of the world have a higher social priority. The third term involves the returns of the investment.

To describe how these three terms enter the formula for the social discount rate, we need to introduce some additional notation. For two variables x_s and y_s, expected value and covariance are denoted $E(x)=\sum_{s=1}^m p_s x_s$ and $cov(x_s,y_s)=\sum_{s=1}^m p_s(x_s-E(x))(y_s-E(y))$. For any $j\in\mathbb{N}$, her probability of existence is $p^j=\sum_{s:j\in N_s}p_s$, and the expected value and covariance conditional on her existence are $E^j(x)=\sum_{s:j\in N_s}\frac{p_s}{p^j}x_s$ and $cov^j(x,y)=$

$\sum_{s:j\in N_s} \frac{p_s}{p^j}(x_s - E^j(x))(y_s - E^j(y))$.

Proposition 2 *For any* $(\mathbf{p}, \mathbf{c}) \in \mathbf{L}$*, the period* t *risk-and-equity adjusted social discount rate* δ *for a project with relative returns* $(\theta_s^t)_{s\in S}$ *and sharing rules* $(\sigma^i)_{i\in N^0}$ *and* $(\sigma_s^j)_{j\in N_s^t, s\in S}$ *is given by the formula:*

$$\delta = \frac{1}{t}\ln\left(\sum_{i\in N^0}\sigma^i\left(\sum_{j\in N^t} p^j \frac{E^j(\sigma^j W^j)}{E(W^i)} E^j(\theta^t)\left(1 + cov^j\left(\frac{\theta^t}{E^j(\theta^t)}, \frac{\sigma^j W^j}{E^j(\sigma^j W^j)}\right)\right)\right)^{-1}\right) \tag{16}$$

Proof. From Equation (13), we have:

$$\begin{aligned}
\delta &= \frac{1}{t}\ln\left(\frac{\sum_{s=1}^m p_s \sum_{i\in N^0}\sigma^i W_s^i}{\sum_{s=1}^m p_s \sum_{j\in N_s^t}\sigma_s^j\theta_s^t W_s^j}\right) \\
&= \frac{1}{t}\ln\left(\sum_{i\in N^0}\sigma^i\left(\frac{1}{E(W^i)}\sum_{s=1}^m p_s \sum_{j\in N_s^t}\sigma_s^j\theta_s^t W_s^j\right)^{-1}\right) \\
&= \frac{1}{t}\ln\left(\sum_{i\in N^0}\sigma^i\left(\sum_{j\in N^t}\frac{1}{E(W^i)}\sum_{s:j\in N_s} p_s\theta_s^t\sigma_s^j W_s^j\right)^{-1}\right) \\
&= \frac{1}{t}\ln\left(\sum_{i\in N^0}\sigma^i\left(\sum_{j\in N^t} p^j E^j(\theta^t)\frac{E^j(\sigma^j W^j)}{E(W^i)}\sum_{s:j\in N_s}\frac{p_s}{p^j}\frac{\theta_s^t}{E^j(\theta^t)}\frac{\sigma_s^j W_s^j}{E^j(\sigma^j W^j)}\right)^{-1}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{t}\ln\left(\sum_{i\in N^0}\sigma^i\left(\sum_{j\in N^t} p^j E^j(\theta^t)\frac{E^j(\sigma^j W^j)}{E(W^i)} E^j\left(\frac{\theta^t}{E^j(\theta^t)}\frac{\sigma^j W^j}{E^j(\sigma^j W^j)}\right)\right)^{-1}\right) \\
&= \frac{1}{t}\ln\left(\sum_{i\in N^0}\sigma^i\left(\sum_{j\in N^t} p^j \frac{E^j(\sigma^j W^j)}{E(W^i)} E^j(\theta^t)\left(1 + cov^j\left(\frac{\theta^t}{E^j(\theta^t)}, \frac{\sigma^j W^j}{E^j(\sigma^j W^j)}\right)\right)\right)^{-1}\right),
\end{aligned}$$

where the last step invokes $E^j(xy) = E^j(x)E^j(y) + cov^j(x, y)$. ■

It is possible to distinguish three terms in formula (16). First the probability of the existence of the future individual j, p^j. Second, the relative priority of individual j weighted by her share of the benefits, $\frac{E^j(\sigma^j W^j)}{E(W^i)}$. This term implicitly captures the systematic risk, which will determine whether future people are more or less well-off. It may also include the risk on population, for the relative priority of individual j may depend on population size, as exemplified in Equation (15). Third, a term involving the risk on the returns,

$E^j(\theta^t)\left(1 + cov^j\left(\frac{\theta^t}{E^j(\theta^t)}, \frac{\sigma^j W^j}{E^j(\sigma^j W^j)}\right)\right)$, which itself is decomposed into a term of association between j's existence and the relative return, $E^j(\theta^t)$, and a term of association between j's social priority and the relative return.

If one abstracts from the issue of the existence of future generations, formula (16) is related to existing formulas in the case where there is both a systematic risk on aggregate consumption and a risk on the returns. For instance, with a Discounted Utilitarian evaluation function with only one individual per generation, Gollier (2012, Chap. 12) proposes to evaluate future cash flows by first computing a certainty equivalent of the risky cash flow, and then discounting it using a discount rate for risk-free investment. To follow his notation, considering the criterion

$$\sum_{t=0}^{\infty} e^{-\delta t} u\left(c^t\right),$$

and the risky cash flow is a random variable B_s^t, and the future cash flows should be evaluated using the formula $e^{-rt}F^t$, where $r = \delta - \frac{1}{t}\ln\left(\frac{Eu'(c^t)}{Eu'(c^0)}\right)$ and $F^t = \frac{EB^t u'(c^t)}{Eu'(c^t)}$.

If one would like to discount the cash flows instead of the certainty equivalent cash flows, and following Gollier's line of argument, one could also use the formula $e^{-\tilde{r}t}EB^t$, where $\tilde{r} = \delta - \frac{1}{t}\ln\left(\frac{Eu'(c^t)}{Eu'(c^0)}\right) - \frac{1}{t}\ln\left(\frac{E\theta^t u'(c^t)}{Eu'(c^t)}\right)$ and $\theta_s^t = \frac{B_s^t}{EB^t}$. Realizing that $u'(c_s^i)$ is the same as W_s^i in the Utilitarian special case, it is clear that formula (16) extends Gollier's formula. The difference is therefore that our formula is used to directly discount expected cash flows, rather than the certainty-equivalent of the cash-flow.

It is possible to follow this route to define a risk-adjusted person-to-person discount rate.

Definition 3 *For any* $(\mathbf{p}, \mathbf{c}) \in \mathbf{L}$*, and for any* $i \in N^0, j \in N_s^t$*, the* risk-adjusted person-to-person discount rate *from person* i *to person* j*, denoted* $\widetilde{\delta}^{i,j}$*, is:*

$$\widetilde{\delta}^{i,j} = -\frac{1}{t}\ln\left(p^j\right) - \frac{1}{t}\ln\left(\frac{E^j W^j}{E^i W^i}\right) - \frac{1}{t}\ln\left(1 + cov^j\left(\theta^t, \frac{W^j}{E^j W^j}\right)\right). \tag{17}$$

When there is no uncertainty about the composition of the population and the sharing rule in period t, the risk-and-equity adjusted social discount rate can be written as a generalized mean of the risk-adjusted person to person discount rates:

Corollary 1 *Assume that* $(\mathbf{p}, \mathbf{c}) \in \mathbf{L}$ *is such that, for all* $s \in S$*,* $N_s^t \neq \varnothing$*,* $N_s^t = N_a^t$ *and* $\sigma_s^j = \sigma_a^j$ *for all* $j \in N_a^t$*. The period* t *risk-and-equity adjusted social discount rate* δ *is given* *by the formula:*

$$\delta = \frac{1}{t} \ln \left(\sum_{i \in N^0} \sigma^i \left(\sum_{j \in N_a^t} \sigma_a^j e^{-\widetilde{\delta}^{i,j} t} \right)^{-1} \right) \tag{18}$$

This formula generalizes equations (21) and (24) in Fleurbaey and Zuber (2013).

One key factor in Equation (16) is the covariance between the relative return and the relative priority of future people in different states of the world. A positive covariance will decrease the discount rate, and therefore increase the value of future costs and benefits. To understand what sign we can expect for this term, we now discuss a simple example.

Example 1: Returns are proportional to the growth rate of consumption. A common assumption in the capital asset pricing model literature is that the cash flow of the investment project and aggregate income are positively stochastically dependent (See for instance Gollier, 2012, Chap. 12). Let us therefore assume that the returns of the investment project and the growth rate of consumption are related by the equation:

$$r = \alpha + \beta g + \epsilon$$

where r is the rate of return of the investment, g is the growth rate of consumption and ϵ is a random variable, which is independent of growth. The parameter β is know in the finance literature as the "beta" of the project, which is an important feature of standard capital asset pricing formulae. It is know to be related to the covariance between the systematic risk on consumption and the risk on returns.

To obtain simple and clear-cut results, we assume a very persistent growth pattern. The consumption growth rate g is determined once for all at the initial period, and it is supposed to be a Gaussian random variable with mean μ_g and standard deviation σ_g. We also assume that ϵ is a Gaussian random variable with mean μ_e and standard deviation σ_e, and which is independent of g.[3] Hence consumption in period t is $c_t = e^{gt} c_0$ and the cash flows from the investment are $B_s^t = e^{rt} I_0 = e^{(\alpha+\beta g+\epsilon)t} I_0$ where I_0 is the investment in the initial period. The relative return in period t is $\theta^t = \frac{B_s^t}{EB_s^t}$.

[3] It is straightforward to extend our results to the case were ϵ is not Gaussian and may not be persistent (a different random variable would be drawn each period). Since this plays no role in the analysis, we do not consider such complications.

Throughout the example, we assume that there are N identical individuals in each generation and no uncertainty on the planning horizon (all generations exist until period $T > t$). We consider two social evaluation functions: the Total Utilitarian criterion (4) and the Equally Distributed Equivalent criterion (6).

The Utilitarian case: Assume that u is a power function $u(c) = c^{1-\eta}/(1-\eta)$. Then, for $j \in N^t$: $W_s^j = (c_s^j)^{-\eta}$ and

$$\tfrac{E^j W^j}{E^i W^i} = E e^{-\eta g t}.$$

The random variable $-\eta g t$ is normally distributed with mean $-\eta \mu_g t$ and variance $\eta^2 \sigma_g^2 t^2$. It known that $Ee^X = e^{(E(X)+0.5V(X))}$ when X is normally distributed with mean $E(X)$ and variance $V(X)$, so that[4]

$$-\tfrac{1}{t} \ln \left(\tfrac{E^j W^j}{E^i W^i} \right) = \eta \mu_g - 0.5 \eta^2 \sigma_g^2 t.$$

Similarly, we have $EB_s^t = e^{rt} I_0 = e^{(\alpha+\beta\mu_g+\mu_e)t+0.5(\beta^2\sigma_g^2+\sigma_e^2)t^2} I_0$, because the random variable rt is normally distributed with mean $(\alpha + \beta\mu_g + \mu_e)t$ and variance $(\beta^2\sigma_g^2 + \sigma_e^2)t^2$. Hence

$$E\left(\theta^t \tfrac{W^j}{E(W^j)}\right) = E\left(\tfrac{B_s^t}{E(R_s^t)} \tfrac{W^j}{E(W^j)}\right) = \frac{Ee^{rt} I_0 e^{-\eta g t} c_0^{-\eta}}{E(B_s^t)E(W^j)} = \frac{Ee^{(\alpha+(\beta-\eta)g+\varepsilon)t}}{e^{(\alpha+\beta\mu_g+\mu_e)t+0.5(\beta^2\sigma_g^2+\sigma_e^2)t^2} e^{-\eta\mu_g t+0.5\eta^2\sigma_g^2 t^2}}.$$

Given our assumptions $(\alpha + (\beta - \eta)g + \epsilon)t$ is normally distributed with mean $(\alpha + (\beta - \eta)\mu_g + \mu_e)t$ and variance $((\beta-\eta)^2\sigma_g^2 + \sigma_e^2)t^2$, so that,

$$E\left(\theta^t \tfrac{W^j}{E(W^j)}\right) = \frac{e^{(\alpha+(\beta-\eta)\mu_g+\mu_e)t+0.5((\beta-\eta)^2\sigma_g^2+\sigma_e^2)t^2}}{e^{(\alpha+\beta\mu_g+\mu_e)t+0.5(\beta^2\sigma_g^2+\sigma_e^2)t^2} e^{-\eta\mu_g t+0.5\eta^2\sigma_g^2 t^2}},$$

and

$$-\tfrac{1}{t} \ln\left(1 + cov^j\left(\theta^t, \tfrac{W^j}{E^j W^j}\right)\right) = -\tfrac{1}{t} \ln\left(E\left(\theta^t \tfrac{W^j}{E^j W^j}\right)\right) = \beta\eta\sigma_g^2 t.$$

The risk on the returns of the investment is positively associated with the systematic risk on consumption growth, which increases the social discount rate.

The Equally Distributed Equivalent case: Consider the Equally Distributed Equivalent

[4]We therefore obtain a term structure of the risk-free discount rate, which is decreasing with time. This is not surprising given the strong persistence in the growth process. Our example is extreme in the sense that the discount rate tends to minus infinity when times goes towards infinity. We do not claim that the underlying growth process is realistic: it is used to facilitate the exposition of our example.

criterion (6), and assume that u is a power function, i.e., $u(c) = c^{1-\eta}/(1-\eta)$, where $0 < \eta < 1$, and $\phi(v) = \ln(v)$. Then we have, for $j \in N^t$:

$$W_s^j = (T+1)^{-1}(c_s^j)^{-1} \exp\left(\frac{1-\eta}{T+1}\sum_{\tau=0}^{T} \ln(c_s^\tau)\right),$$

where $c_s^\tau = c_s^k$ for $k \in N^\tau$.

Hence, if $j \in N^t$, $W_s^j = (T+1)^{-1}c_0^{-1}e^{-gt}\exp(\frac{1-\eta}{T+1}\sum_{\tau=0}^T g\tau)$, and, using $\sum_{\tau=0}^T \tau = T(T+1)/2$,

$$\begin{aligned}
\frac{E^jW^j}{E^iW^i} &= \frac{E(T+1)^{-1}c_0^{-1}e^{\left(\frac{(1-\eta)T}{2}-t\right)g}}{E(T+1)^{-1}c_0^{-1}e^{\left(\frac{(1-\eta)T}{2}\right)g}} \\
&= \frac{e^{\left(\frac{(1-\eta)T}{2}-t\right)\mu_g+0.5\left(\frac{(1-\eta)T}{2}-t\right)^2\sigma_g^2}}{e^{\left(\frac{(1-\eta)T}{2}\right)\mu_g+0.5\left(\frac{(1-\eta)T}{2}\right)^2\sigma_g^2}},
\end{aligned}$$

because the random variable $\left(\frac{(1-\eta)T}{2}-t\right)g$ is normally distributed with mean $\left(\frac{(1-\eta)T}{2}-t\right)\mu_g$ and variance $\left(\frac{(1-\eta)T}{2}-t\right)^2\sigma_g^2$ and the random variable $\left(\frac{(1-\eta)T}{2}\right)g$ is normally distributed with mean $\left(\frac{(1-\eta)T}{2}\right)\mu_g$ and variance $\left(\frac{(1-\eta)T}{2}\right)^2\sigma_g^2$. One therefore obtains

$$-\frac{1}{t}\ln\left(\frac{E^jW^j}{E^iW^i}\right) = \mu_g - 0.5\sigma_g^2 t + \frac{(1-\eta)T}{2}\sigma_g^2.$$

As before, we can compute

$$\begin{aligned}
E\left(\theta^t\frac{W^j}{E(W^j)}\right) = E\left(\frac{B_s^t}{E(B_s^t)}\frac{W^j}{E(W^j)}\right) &= \frac{Ee^{rt}I_0(T+1)^{-1}c_0^{-1}e^{\left(\frac{(1-\eta)T}{2}-t\right)g}}{E(B_s^t)E(W^j)} \\
&= \frac{Ee^{(\alpha+\epsilon)t+\left(\beta t+\left(\frac{(1-\eta)T}{2}-t\right)\right)g}}{e^{(\alpha+\beta\mu_g+\mu_e)t+0.5(\beta^2\sigma_g^2+\sigma_e^2)t^2}e^{\left(\frac{(1-\eta)T}{2}-t\right)\mu_g+0.5\left(\frac{(1-\eta)T}{2}-t\right)^2\sigma_g^2}}.
\end{aligned}$$

The random variable $(\alpha+\epsilon)t+\left(\beta t+\left(\frac{(1-\eta)T}{2}-t\right)\right)g$ is normally distributed with mean $\alpha t+\mu_e t+\left(\beta+\left(\frac{(1-\eta)T}{2}-t\right)\right)\mu_g$ and variance $\sigma_e^2t^2+\left(\beta t+\left(\frac{(1-\eta)T}{2}-t\right)\right)^2\sigma_g^2$, which implies that:

$$E\left(\theta^t \frac{W^j}{E(W^j)}\right) = \frac{e^{(\alpha+\beta\mu_g+\mu_e)t+\left(\frac{(1-\eta)T}{2}-t\right)\mu_g+0.5(\beta^2\sigma_g^2+\sigma_e^2)t^2+0.5\left(\frac{(1-\eta)T}{2}-t\right)^2\sigma_g^2+\beta\left(\frac{(1-\eta)T}{2}-t\right)t^2\sigma_g^2}}{e^{(\alpha+\beta\mu_g+\mu_e)t+0.5(\beta^2\sigma_g^2+\sigma_e^2)t^2}e^{\left(\frac{(1-\eta)T}{2}-t\right)\mu_g+0.5\left(\frac{(1-\eta)T}{2}-t\right)^2\sigma_g^2}},$$

and

$$-\frac{1}{t}\ln\left(1+cov^j\left(\theta^t, \frac{W^j}{EW^j}\right)\right) = -\frac{1}{t}\ln\left(E\left(\theta^t\frac{W^j}{E(W^j)}\right)\right) = -\beta\left(\frac{(1-\eta)T}{2}-t\right)\sigma_g^2 t.$$

The key result here is that, when $\frac{(1-\eta)T}{2} > t$ that is when the planning horizon is sufficiently large compared to the date at which the cash flows are discounted, we can obtain that the covariance term as a negative impact on social discounting. This is in contrast with the result obtained for the Utilitarian case.

The reason for this different result is that individuals in period t have a higher W^j in the favorable scenarios (when growth is large) than in the bad scenarios. Since returns are also higher in favorable scenarios, the covariance between social priority and the risk on the returns of the investment. The reason why social priority is high in good state is because it does not depend only on marginal utility but also on the relative ranking of individuals in *their state of the world*. When growth is positive, generation t is relatively less well-off when there are many future generations. On the contrary, generation t is relatively better-off when all future generations will be poorer in the bad scenario.

The systematic risk on aggregate consumption and the risk on the returns are well-known and studied in the literature on social discounting (and in the related finance literature on valuing uncertain projects). This is not the case of the risk on population size which has only received limited attention yet. The main contributions addressing the issue limit themselves to noticing that the hazard rate of the risk can be used to pinpoint the value of a parameter known as the rate of pure time preference (Dasgupta and Heal, 1979; Stern, 2006).

As is clear from Equation (16), the relative probability of the existence of future generations is indeed one aspect of the risk on the population size that would enter the discounting formula for any social evaluation function. But it is not the only way population size matters if one goes beyond the standard Total Utilitarian case, as is demonstrated by the following example.

Example 2: The risk on population size. Assume that there is no risk on consumption or the returns of the investment. We consider a case in which people of a given generation t are all equal and consume c^t. The only risk is on the existence of the future generations. Each period, with probability p the world survives to the next period, and with probability $1-p$ the human species (and any species relevant for welfare) disappears.[5]

We also consider that potential population (i.e. absent the extinction risk) grows at a given exponential rate n. Hence denoting $n^t = |N^t|$, we have $n^{t+1} = (1+n)n^t$. With probability $(1-p)p^t$, the population size is therefore exactly $\sum_{\tau=1}^{\tau} n^0(1+n)^t = n^0\frac{(1+n)^{t+1}-1}{n} = (1+n)^t\left(\frac{n^0(1+n)}{n} - \frac{n^0}{n(1+n)^t}\right)$. Generation t exists with probability p^t.

Consider first the Total Utilitarian criterion (4). Then, for any $j \in N^t$: $E^jW_s^j = u'\left(c^t\right)$ and, using formulas (17) and (18) we obtain:

$$\delta = -\ln p - \frac{1}{t}\ln\left(\frac{u'\left(c^t\right)}{u'\left(c^0\right)}\right),$$

where only the hazard rate of the extinction risk enters the discounting formula.

If we now turn to the Average Utilitarian criterion (i.e. criterion (6) with ϕ an affine function), we obtain that for any $j \in N^t$:

$$\begin{aligned} E^jW_s^j &= u'\left(c^t\right)\sum_{\tau=t}^{\infty}(1-p)p^{\tau-t}(1+n)^{-\tau}\left(\frac{n^0(1+n)}{n} - \frac{n^0}{n(1+n)^\tau}\right)^{-1} \\ &= u'\left(c^t\right)(1+n)^{-t}\sum_{\tau=0}^{\infty}(1-p)p^{\tau}(1+n)^{-\tau}\left(\frac{n^0(1+n)}{n} - \frac{n^0}{n(1+n)^{\tau+t}}\right)^{-1}. \end{aligned}$$

Using formulas (17) and (18) we obtain:

$$\widetilde{\delta}^{i,j} = -\ln p + \ln(1+n) - \frac{1}{t}\ln\left(\frac{u'\left(c^t\right)}{u'\left(c^0\right)}\right) - \frac{1}{t}\ln\left(\frac{\sum_{\tau=0}^{\infty}p^\tau(1+n)^{-\tau}\left(\frac{n^0(1+n)}{n} - \frac{n^0}{n(1+n)^{\tau+t}}\right)^{-1}}{\sum_{\tau=0}^{\infty}p^\tau(1+n)^{-\tau}\left(\frac{n^0(1+n)}{n} - \frac{n^0}{n(1+n)^{\tau}}\right)^{-1}}\right).$$

[5]Note that we consider a countably infinite number of states of the world. All our formulas can be extended to that case.

Neglecting the last term, which is small for large values of t,[6] it seems that adopting an Average Utilitarian view, rather than a Total Utilitarian view, implies adjusting the discount rate for population growth. In fact, compared to the Total Utilitarian formula, we have to add up the population growth rate in the Average Utilitarian formula, which can significantly alter the discount rate.

The example shows that important choices concerning population ethics (i.e., the criterion used to assess populations of different sizes) may deeply alter the social discount rate and therefore policy recommendations. This is not a debate that can remain unaddressed.

4 Inequality

The distribution of costs and benefits is one of the key issues in the theory of cost-benefit analysis. The mainstream approach to the problem consists in assuming an efficient distribution of the costs and benefits (or an implicit redistribution compensating those treated unfairly), so that equity considerations can be dispensed with.

If one does not want to assume away the equity issue, the standard technique in cost-benefit analysis consists in introducing "equity weights." These are weights on the costs and benefits depending on individuals' relative welfare: the less well-off receive higher weights. The technique has been used in the case of climate change to adjust the social cost of carbon for equity considerations.

We propose a technique to incorporate equity considerations directly in the discount rate. We first highlight the importance of the covariance between the individual's relative social priority and her share of costs and benefits. To do so, we need to introduce some additional notation. For $j \in N_s^t$, $\bar{w}_s^j = W_s^j / (\frac{1}{n_s^t} \sum_{k \in N_s^t} W_s^k)$ is the relative social priority of j with respect to individuals in the same generation and state of the world. We also define $Cov_{pop}^t(x^t, y^t) = \frac{1}{n_s^t} \sum_{j \in N_s^t} \left(x^j - \frac{1}{n_s^t} \sum_{k \in N_s^t} x^k \right) \left(y^j - \frac{1}{n_s^t} \sum_{k \in N_s^t} y^k \right)$.

Proposition 3 *For any $(\mathbf{p}, \mathbf{c}) \in \mathbf{L}$, the period t risk-and-equity adjusted social discount rate*

[6] $\left(\frac{n^0(1+n)}{n} - \frac{n^0}{n(1+n)^\tau} \right) < \left(\frac{n^0(1+n)}{n} - \frac{n^0}{n(1+n)^{t+\tau}} \right) < \frac{n^0(1+n)}{n}$ and $n^0 < \left(\frac{n^0(1+n)}{n} - \frac{n^0}{n(1+n)^\tau} \right)$, so that

$$\frac{1}{t} \ln\left(\frac{n}{1+n}\right) = \frac{1}{t} \ln \left(\frac{\sum_{\tau=0}^{\infty} p^\tau (1+n)^{-\tau} \left(\frac{n^0(1+n)}{n} \right)^{-1}}{\sum_{\tau=0}^{\infty} p^\tau (1+n)^{-\tau} \left(n^0 \right)^{-1}} \right) < \frac{1}{t} \ln \left(\frac{\sum_{\tau=0}^{\infty} p^\tau (1+n)^{-\tau} \left(\frac{n^0(1+n)}{n} - \frac{n^0}{n(1+n)^{\tau+t}} \right)^{-1}}{\sum_{\tau=0}^{\infty} p^\tau (1+n)^{-\tau} \left(\frac{n_0(1+n)}{n} - \frac{n^0}{n(1+n)^{\tau}} \right)^{-1}} \right) < 0.$$

δ for a project with relative returns $(\theta_s^t)_{s\in S}$ and sharing rules $(\sigma^i)_{i\in N^0}$ and $(\sigma_s^j)_{j\in N_s^t, s\in S}$ is given by the formula:

$$\delta = \frac{1}{t}\ln\left(\frac{\sum_{s=1}^m p_s\left(\frac{1}{n^0}\sum_{i\in N^0} W_s^i\right)\left[1+n^0 Cov_{pop}^0\left(\sigma_s^0,\bar{w}_s^0\right)\right]}{\sum_{s=1}^m p_s\theta_s^t\left(\frac{1}{n_s^t}\sum_{j\in N_s^t} W_s^j\right)\left[1+n_s^t Cov_{pop}^t\left(\sigma_s^t,\bar{w}_s^t\right)\right]}\right) \tag{19}$$

Proof. From Equation (13), we have:

$$\begin{aligned}
\delta &= \frac{1}{t}\ln\left(\frac{\sum_{s=1}^m p_s\sum_{i\in N^0}\sigma^i W_s^i}{\sum_{s=1}^m p_s\sum_{j\in N_s^t}\sigma_s^j\theta_s^t W_s^j}\right)\\
&= \frac{1}{t}\ln\left(\frac{\sum_{s=1}^m p_s\left(\frac{1}{n^0}\sum_{i\in N^0} W_s^i\right)\left(\sum_{i\in N^0}\sigma^i\bar{w}_s^i\right)}{\sum_{s=1}^m p_s\theta_s^t\left(\frac{1}{n_s^t}\sum_{j\in N_s^t} W_s^j\right)\left(\sum_{j\in N_s^t}\sigma_s^j\bar{w}_s^j\right)}\right)\\
&= \frac{1}{t}\ln\left(\frac{\sum_{s=1}^m p_s\left(\frac{1}{n^0}\sum_{i\in N^0} W_s^i\right)\left[1+n^0 Cov_{pop}^0\left(\sigma_s^0,\bar{w}_s^0\right)\right]}{\sum_{s=1}^m p_s\theta_s^t\left(\frac{1}{n_s^t}\sum_{j\in N_s^t} W_s^j\right)\left[1+n_s^t Cov_{pop}^t\left(\sigma_s^t,\bar{w}_s^t\right)\right]}\right),
\end{aligned}$$

where the last step uses the fact that $b_k = \sum_k a_k\bar{b} + \sum_k\left(a_k-\bar{a}\right)\left(b_k-\bar{b}\right)$, letting $\bar{a}$ and $\bar{b}$ denote the average values. Note that $\sum_{i\in N^0}\sigma_s^i = \sum_{j\in N_s^t}\sigma_s^j = 1$ and $\frac{1}{n^0}\sum_{i\in N^0}\bar{w}_s^i = \frac{1}{n_s^t}\sum_{j\in N_s^t}\bar{w}_s^j = 1$. ■

Formula (19) introduces the terms $Cov_{pop}^0\left(\sigma_s^0,\bar{w}_s^0\right)$ and $Cov_{pop}^t\left(\sigma_s^t,\bar{w}_s^t\right)$ to take into account the equity in the distribution of costs and benefits. When costs are born by the poor today, this will therefore tend to increase the discount rate: the investment is less valuable because it involves increasing inequality today. But if the benefits are received by the poor tomorrow, this will tend to decrease the discount rate: the investment is more valuable because it will reduce inequality tomorrow.

One aspect of equity is not highlighted in formula (19): the level of inequality today and tomorrow. It is implicitly taken into account in the average social priority of people in generation t and state s. For most social evaluation functions, the expression $\frac{1}{n_s^t}\sum_{j\in N_s^t} W_s^j$ is increasing with inequality. For instance, in the Utilitarian case, if $u(c) = \frac{1}{1-\eta}c^{1-\eta}$ and $\eta > 0$, one obtains

$$\sum_i u'\left(c^i\right) = \sum_i\left(c^i\right)^{-\eta},$$

an expression that decreases when a progressive transfer is made between two agents. Therefore formula (19) shows that increasing inequality in the future tends to lower the discount rate. Intuitively, a more unequal future deserves more investment from the present generation, because, other things equal, more people will be in great need of the benefits of the investment. Note that this feature is consistent with the finding by Gollier (2010) that economic convergence increases the social discount rate. Similarly, if we were not to use a power utility function, we could obtain the finding by Gollier (2010) that the mere existence of inequalities (unchanged with time) increases the discount rate when the function u exhibits decreasing and convex relative risk aversion.

The ratio $\frac{1}{n_s^t}\sum_{j\in N_s^j} W_s^j / \frac{1}{n^0}\sum_{i\in N^0} W_s^i$ is not simply a measure of the evolution of inequality because it also incorporates consumption growth (which decreases priority when it is positive). Separating the two aspects is possible with many social evaluation functions, as shown by the following example.

Example 3: The role of intragenerational inequality. Consider the case of a generalized additive social evaluation (7), with power functions $u(c) = \frac{1}{1-\eta}c^{1-\eta}$, for $0 < \eta < 1$, and $\phi(u) = \frac{1}{1-\gamma}u^{1-\gamma}$, for $\gamma > 0$.[7] Denote $\upsilon = \eta + (1-\eta)\gamma$ and let $\bar{c}^0$ and $\bar{c}_s^t$ denote the average consumptions in generations 0 and t, respectively. In that case,

$$W_s^j = (c_s^j)^{-\upsilon} \times \Psi'_{n(c_s)}\left(\sum_{i\in N_s} \frac{\left(c_s^i\right)^{1-\upsilon}}{1-\upsilon}\right),$$

so that,

$$\frac{1}{n_s^t}\sum_{j\in N_s^t} W_s^j = (\bar{c}_s^t)^{-\upsilon} \times \left(\frac{1}{n_s^t}\sum_{j\in N_s^t}\left(\frac{c_s^j}{\bar{c}_s^t}\right)^{-\upsilon}\right) \times \Psi'_{n_s}\left(\sum_{i\in N_s} \frac{\left(c_s^i\right)^{1-\upsilon}}{1-\upsilon}\right),$$

The term

$$Eq_s^t = \left(\frac{1}{n_s^t}\sum_{j\in N_s^t}\left(\frac{c_s^j}{\bar{c}_s^t}\right)^{-\upsilon}\right)^{-1/\upsilon}$$

is a measure of equality, which is equal to 1 when the situation is perfectly equal and tends to 0 when one individual consumes everything.

[7] The same reasoning would hold for $u(c) = \frac{1}{1-\eta}c^{1-\eta}$, for $\eta > 1$, and $\phi(u) = \frac{1}{(1+\gamma)(1-\eta)}(-u)^{1+\gamma}$, for $\gamma > 0$.

In the simple situation where there is no uncertainty, one obtains

$$
\begin{aligned}
\delta &= \frac{1}{t}\ln\left(\frac{\left(\frac{1}{n^0}\sum_{i\in N^0} W^i\right)\left[1+n^0 Cov^0_{pop}\left(\sigma^0,\bar{w}^0\right)\right]}{\left(\frac{1}{n^t(c)}\sum_{j\in N^t(c)} W^j\right)\left[1+n^t Cov^t_{pop}\left(\sigma^t,\bar{w}^t\right)\right]}\right)\\
&= \upsilon\times\frac{1}{t}\ln\left(\frac{\bar{c}^t}{\bar{c}^0}\right)+\upsilon\times\frac{1}{t}\ln\left(\frac{Eq^t}{Eq^0}\right)-\frac{1}{t}\ln\left(\frac{1+n^t Cov^t_{pop}\left(\sigma^t,\bar{w}^t\right)}{1+n^0 Cov^0_{pop}\left(\sigma^0,\bar{w}^0\right)}\right)
\end{aligned}
$$

In this formula, the discount rate can therefore be decomposed in three terms, which all represent one aspect of equity. The first term is related to the growth of average consumption and accounts for intergenerational equity. The second term is related to the change in equality within generations, and therefore accounts for intragenerational equity. The last term measures the evolution of the covariance between relative welfare and the shares in cost and benefits; it accounts for the equity in the distribution of costs and benefits.

5 Long-term discounting and the information needed to evaluate climate policies

The formulas that we have presented above require a lot of information about the distribution of costs and benefits, the distribution of consumption within generations and the probability of the different states of the world. A key question is to determine which information must be known with good accuracy to evaluate very long-term impacts of policies. The results of this section show that, fortunately, only a small fraction of the information is relevant in the very long term.

The following Proposition states that the most important information, for the computation of the social discount rate, is the maximum possible return of the investment, and the maximum possible return net of a specific person-to-person discount. We can dispense with the distribution shares of impacts (and of contributions) and with the probabilities of the different states.

Proposition 4 *Consider* $(\mathbf{p},\mathbf{c})\in\mathbf{L}$ *such that for all* $t\in\mathbb{N}$*, for all* $s\in S$*, for all* $j\in N^t_s$*,* $\sigma^j_s\geq\sigma_0$ *for some* $\sigma_0>0$*. When* $t\to\infty$*, the period* t *risk-and-equity adjusted social discount* *rate δ satisfies the formula:*

$$\delta = \max_{s\in S} r_s - \min_{i\in N^0} \max_{s\in S, j\in N_s^t} \left(r_s - \delta_s^{i,j}\right) + O(1/t) \tag{20}$$

Proof. See Appendix A. ■

In Equation (20), the expression $O(1/t)$ means that there is $A > 0$ such that

$$\left|\delta - \left(\max_{s\in S} r_s - \min_{i\in N^0} \max_{s\in S, j\in N_s^t} \left(r_s - \delta_s^{i,j}\right)\right)\right| < A/t$$

for t high enough. Therefore this approximation result is compatible with the fact that the distribution of r_s and $\delta_s^{i,j}$ may vary with time (and even diverge).

Obviously, when t goes to infinity, the discount rate converges not just to the formula but to the limit of the formula if there is one. For instance, if the growth rate of consumption tends to zero in the very long run, the person-to-person discount rate $\delta_s^{i,j}$ tends to zero for all i, j, and the limit discount rate is zero.

In order to understand which individuals $i \in N^0$ and $j \in N^t$ are relevant in the computation of $\min_{i\in N^0} \max_{s\in S} \max_{j\in N_s^t} \left(r_s - \delta_s^{i,j}\right)$, first observe that for a given i and s, the greatest $r_s - \delta_s^{i,j}$ is obtained for j having the lowest $\delta_s^{i,j}$, i.e., the greatest W_s^j. That will be the most disadvantaged individual in state s.

Now, assume that for every given s and j, the maximum value of $\delta_s^{i,j}$ is obtained for the same individual i, i.e., for every s the same i has the greatest W_s^i. Although in general the individual i with the greatest W_s^i might depend on s, it is quite natural, when there is no risk on generation 0, that no such dependence occurs. For instance, when the social evaluation function takes the generalized additive form (7), one simply has

$$\delta_s^{i,j} = \frac{1}{t} \ln\left(\frac{u'(c^i)\phi' \circ u(c^i)}{u'(c_s^j)\phi' \circ u(c_s^j)}\right),$$

so that the individual i with the greatest W_s^i is the individual with the lowest consumption. When the same individual i has the greatest W_s^i for all s (and obviously for all j as well), one has

$$\min_{i\in N^0} \max_{s\in S, j\in N_s^t} \left(r_s - \delta_s^{i,j}\right) = \max_{s\in S}\left(r_s - \max_{i\in N^0} \min_{j\in N_s^t} \delta_s^{i,j}\right)$$

To compute the long-term discount rate, one can then proceed as follows. For any given $s \in S$, focus on the worst-off-to-worst-off discount rate, and compute a net return on the investment using this discount rate —the net return being the difference between the return and the discount rate. Then pick the state $s \in S$ in which this net return is maximal. The difference between the maximum return and the maximum net return is the social discount rate.

In case the highest net return $r_s - \delta_{ijs}$ is obtained in the state s^* where r_s is greatest, the final formula is simply

$$\max_{i \in N^0} \min_{j \in N^t_{s^*}} \delta^{i,j}_{s^*}.$$

This is so when high returns occur in states where the worst-off of the future generations live in the deepest poverty, a situation that may be the case for climate change. If climate change affects growth and the climate damages are higher in high-temperature scenarios, an investment to reduce climate damages may be more profitable in bad states where future generations are poor.

But if greatest net return $r_s - \delta^{i,j}_s$ is obtained for a low r_s, because the returns are correlated with the well-being of j, then the discount rate is greater than $\max_{i \in N^0} \min_{j \in N^t_s} \delta^{i,j}_s$ for any state. This means that investments that pay when the future worst-off are especially badly-off should be evaluated with a lower discount rate than investments paying when the future worst-off are less badly-off. As an extreme example, consider a case in which there is no inequality within generations and in every state the discount rate is an increasing function of the rate of return, with a coefficient greater than one (as in a Ramsey formula with elasticity of utility greater than one, and a growth rate equal to the rate of return on investments). Then the maximum of $r_s - \delta^{i,j}_s$ is obtained for the lowest r_s, and the discount rate is then equal to $\max_{s \in S} r_s - \min_{s \in S} r_s + \min_{s \in S} \delta^{i,j}_s$, where i and j are any representative agents of their generation.

Formula (20) also provides results which are reminiscent of Gollier's (2004) and Weitzman's (1998) analyses of the long-run discount rate, in special cases. When the rate of return on the investment is, in every state of the world, equal to the discount rate (e.g., because of a market equilibrium condition, as in Gollier and Weitzman (2010), or when there is no inequality within generations, the rate of return is the same as the growth rate, and utility is logarithmic), then the second term vanishes and the discount rate on expected returns is, in the limit, the maximum possible rate of return, as in Gollier (2004). In contrast, if there is no uncertainty about the rate of return, then r_s disappears from the formula, and in absence of inequality within generations, it simplifies into $\min_{s\in S} \delta_s^{i,j}$, where i and j are any representative agents of their generation. This is similar to Weitzman's (1998) perspective on the issue.[8]

Although Equation (20) sheds light on what matters in the very long run, it does not imply that the approximation formula can be applied without precautions. In particular, the approximation for the discount rate may be reasonable only in the very long run, as the convergence of the brackets to zero may be slow in some cases. The error on the discount rate made by using the approximation formula is bounded by A/t, for

$$A = \max \left\{ \left| \ln \left(\min_{i\in N^0} \sigma^i \right) + \ln \left(\min_{s\in S} p_s \right) \right|, \sum_{i\in N^0} \frac{\sigma^i/\sigma_0}{\min_{s\in S} p_s w_s^i} \right\},$$

which varies like $(1/t)\ln 10^k \approx 2.3k/t$, where k may be as high as 12 (in particular, σ_0 may be of the order of magnitude of a billionth, and the lowest p_s of the order of a percentage point). One then needs to go beyond $t = 3000$ in order to make the error go below one percentage point. The large uncertainty about the impact of our current actions on such remote times may render the approximation useless, as the relevant consequences for decision-making may be in the medium-term, before the approximation is correct. On the other hand, the following example shows that one can find cases for which the convergence happens within a few hundred years.

Example 4: The long run discount rate. Assume that we use a Total Utilitarian social welfare function, and that u is a power function $u(c) = -c^{-1}$.

Consider a society composed of two groups of people: the rich and the poor. The population is constant, the two groups have the same size in all periods, and costs and

[8]Gollier's and Weitzman's arguments were not based on a social welfare function. Gollier (2004) noted that, when $t \to \infty$, the expected future net value of an investment is driven by the greatest rate of return, whereas Weitzman (1998) observed that the expected present value of an investment is driven by the lowest discount rate when there is uncertainty about growth. Neither the expected future value nor the expected present value are generally the relevant criterion in our social welfare function approach.

benefits are shared on a per capita basis. In period 0, the rich consume five times as much as the poor.

The society faces the following risk. With probability 0.9 (a good state of nature), the consumption of the rich grows at a 1.5% rate, and the consumption of the poor grows at a 1.3% rate. With probability 0.1 (a bad state), the consumption stays constant for both the rich and the poor.

We consider three kinds of investments. A first kind of investment (Investment 1) only yields a return in the bad state of the world. A second kind of investment (Investment 2) yields a higher return in the bad state, but still has a positive return in the good state: the difference between the two rates of return is 1%. A last kind of investment (Investment 3) yields the same return in the two states.

Time	$\min_{i,j,s} \delta_s^{i,j}$	$\max_{i,j,s} \delta_s^{i,j}$	$\bar{\delta}$	δ_1	δ_2	δ_3
50	-0.064	0.094	0.000	0.000	-0.019	0.021
100	-0.032	0.062	0.000	0.000	-0.012	0.018
200	-0.016	0.046	0.000	0.000	-0.003	0.011
500	-0.006	0.036	0.000	0.000	0.000	0.004

Table 1: Long-run convergence of the social discount rate

Table 1 reports the maximum and minimum person-to-person discount rates, the approximate formula $\bar{\delta} = \max_{s \in S} r_s - \min_{i \in N^0} \max_{s \in S, j \in N_s^t} \left(r_s - \delta_s^{i,j} \right)$, and the correct discount rate $\delta_k, k = 1, 2, 3$, for these different kinds of investments for different time horizons.

In all cases presented above, the error of Approximation (20) is less than 0.5 percentage points after 500 years. The approximation is almost immediately correct for Investment 1, where only the bad state of the world matters. For the other two investments, the error is around 2 percentage point for a 100 years horizon, and 1 percentage point for a 200 years horizon. So, while the mistake can be substantial for shorter term impacts, it is limited for impacts at reasonably long horizon.

Approximation (20) is relevant for the computation of the social discount rate. But the important term for the computation of the net present value of an investment is $e^{(r-\delta)t}$, for which this approximation result is not very useful, as errors on the discount rate get compounded with t. Let us therefore examine how one can approximate $e^{(r-\delta)t}$ for large values of t. To do so, we introduce the following condition.

Condition (C) For $(\mathbf{p},\mathbf{c})\in\mathbf{L}$, define the following sets:

$$\begin{aligned}
J^i_{s,t} &= \left\{ j\in N^t_s \mid r_s-\delta^{i,j}_s = \max_{j'\in N^t_s}\left(r_s-\delta^{i,j'}_s\right)\right\},\\
S^i_t &= \left\{ s\in S \mid \max_{j\in N^t_s}\left(r_s-\delta^{i,j}_s\right) = \max_{s'\in S, j'\in N^t_s}\left(r_{s'}-\delta^{i,j'}_{s'}\right)\right\},\\
I_t &= \left\{ i\in N^0 \mid \max_{s\in S, j\in N^t_s}\left(r_s-\delta^{i,j}_s\right) = \min_{i'\in N^0}\max_{s'\in S, j'\in N^t_s}\left(r_{s'}-\delta^{i',j'}_{s'}\right)\right\}.
\end{aligned}$$

A lottery $(\mathbf{p},\mathbf{c})\in\mathbf{L}$ satisfies Condition (C) if there exists $\alpha>0$ such that:

$$\begin{aligned}
\min_{i\notin I_t}\max_{s\in S, j\in N^t_s}\left(r_s-\delta^{i,j}_s\right)-\min_{i\in N^0}\max_{s\in S, j\in N^t_s}\left(r_s-\delta^{i,j}_s\right) &> \alpha \qquad \text{for all } t\in\mathbb{N},\\
\max_{s\in S, j\in N^t_s}\left(r_s-\delta^{i,j}_s\right)-\max_{s\notin S^i_t, j\in N^t_s}\left(r_s-\delta^{i,j}_s\right) &> \alpha \qquad \text{for all } t\in\mathbb{N}, i\in N^0,\\
\max_{j\in N^t_s}\left(r_s-\delta^{i,j}_s\right)-\max_{j\notin J^i_{s,t}}\left(r_s-\delta^{i,j}_s\right) &> \alpha \qquad \text{for all } t\in\mathbb{N}, i\in N^0, s\in S.
\end{aligned}$$

We say that a function $f(t)$ is $o(1/t)$ if for all $A>0$, $|f(t)|<A/t$ for t high enough. Following a line of reasoning similar to the one used in Proposition 4, we obtain the following result.

Proposition 5 *Consider a lottery $(\mathbf{p},\mathbf{c})\in\mathbf{L}$ satisfying Condition (C). Assume also that there exists $\sigma_0>0$ such that $\sigma^j_s\geq\sigma_0$ for all $t\in\mathbb{N}$, all $s\in S$, all $j\in N^t_s$. Then, when $t\to\infty$, the present value of the return at t satisfies the formula:*

$$e^{(r-\delta)t} = \left(1+o(1/t)\right)\left(\sum_{i\in I_t}\sigma^i\left(\sum_{s\in S^i_t}p_s w^i_s\sum_{j\in J^i_{s,t}}\sigma^j_s\right)^{-1}\right)^{-1} e^{\left(\min_{i\in N^0}\max_{s\in S, j\in N^t_s}\left(r_s-\delta^{i,j}_s\right)\right)t}.$$

Proof. See Appendix B. ■

What is interesting about this second result is that it restores a role for shares and probabilities in the approximation. But it also confirms the importance of the expression

$$\min_{i\in N^0}\max_{s\in S, j\in N^t_s}\left(r_s-\delta^{i,j}_s\right).$$

It should be clear that this analysis leads us very far from the usual practice of taking discount rates based on average situations and ordinary market conditions.

6 Extensions

6.1 Overlapping generations

Until now, we have considered that individuals lived for only one period, or alternatively that the utility of an individual in different periods is considered separately in the social evaluation. Of course, individuals live for several periods, and it arguably makes sense normatively that the social evaluation takes into account lifetime utility (Broome 2004). Considering people living for several periods implies considering an overlapping generation framework. It will also reintroduce and clarify the role of individuals' consumption discount rate in the social evaluation.

In this section, we slightly alter the framework to introduce overlapping generations.[9] We assume that individuals live for A periods.[10] Hence, alternatives are now elements of $\mathcal{C} = \bigcup_{N \subset \mathbb{N} \setminus \emptyset} \prod_{i \in N} \mathbb{R}_+^A$. For any $c \in \mathcal{C}$, as before, we let $N(c)$ be the set of individuals alive in the alternative and $n(c) = |N(c)|$ be the number of individuals in the alternative. For any $c \in C$ and $i \in N(c)$, $c^i = (c^{i1}, \cdots, c^{iA})$ is the consumption stream of individual i in alternative c. For any $a \in \{1, \cdots, A\}$, c^{ia} is therefore the consumption of individual i at age a.

We assume that there exists a partition of $\mathbb{N}$ into subsets N^t containing the potential individuals of generation $t \in \mathbb{N}_0 \cup \{1-A, \cdots, -1\}$. The 'generation' of an individual is the first period of his existence.[11] Hence, individual $i \in N^t$ lives in periods $t, \cdots, t+A-1$ when he exists, and we let $G(i) = t$ denote his generation's name. For $\tau = G(i), \cdots, G(i)+A-1$, we also let $a^{i\tau} = \tau + 1 - G(i)$ denote the age of individual i in period τ. We restrict attention to $C = \{c \in \mathcal{C} \mid \cup_{\tau=1-A}^{0} N^\tau \subset N(c)\}$. The currently existing people are present in

[9] Our results therefore complement those by Dasgupta and Maskin presented in Dasgupta (2012, Section 6). In his main results, Dasgupta (2012) considers a dynasty of successive (non overlapping) generations, where the current generation uses a utilitarian criterion. The non overlapping structure may induce cycles in the discount rates, while the undiscounted utilitarian criterion implies no pure time discounting. Our framework is more general in the sense that we do not commit to Utilitarian ethics (while we do not exclude it). Our overlapping structure would in general not generate cycles in the social discount rate.

[10] Extension to the case where individuals have different lengths of life is straightforward but much more involved notationally.

[11] That is why we have to include generations $\{1-A, \cdots, -1\}$ who still live in period 0.

all alternatives.

We also need to know which individuals live in any given period in a given alternative. We therefore change the notation $N^t(c)$, and denote $N^t(c) = \left(\cup_{\tau=t-A+1}^{t} N^t\right) \cap N(c)$. $N^t(c)$ is therefore the set of all individuals living in period t in alternative c. Accordingly, we denote $n^t(c) = |N^t(c)|$ the number of all individuals living in period t in alternative c. All other pieces of notation remain the same.

We also have to modify the specification of the social welfare function. We assume that for each potential individual $i \in \mathbb{N}$ there exists a utility function u^i which is used in the social evaluation to assess his welfare given a specific consumption stream c^i. Therefore, we add the restriction (satisfied by all specific criteria introduced in Section 2.2) that, for any $c_s \in C$:

$$W(c_s) = F_{n_s}\left(\left(u^i(c_s^i)\right)_{i \in N_s}\right).$$

Denoting $u_s^{ia} = \partial u^i(c_s^i)/\partial c_s^{ia}$, $F_s^i = \partial F/\partial u_s^i$ and $W_s^{it} = \partial W(c_s)/\partial c_s^{ia^{it}}$, we obtain:

$$W_s^{it} = u^{ia^{it}} F_s^i = \frac{u_s^{ia^{it}}}{u_s^{i1}} u_s^{i1} F_s^i.$$

The ratio $u_s^{ia^{it}}/u_s^{i1}$ is the marginal rate of substitution between consumption at age a^{it} and consumption at age 1 for individual i. Hence $d_s^{ia} = \ln\left(u_s^{i1}/u_s^{ia}\right)/(a-1)$ is (approximatively) the consumption discount rate at age $a \neq 1$ for individual i along a consumption path c_s^i (we let $d_s^{i1} = 0$).

We are now able to rewrite the state-specific person-to-person discount rate from person i living in period 0 to person j living in period t in state s, that we denote $\delta_s^{i,jt}$.[12] It is simply

$$\begin{aligned}
\delta_s^{i,jt} &= \frac{1}{t}\ln\left(\frac{W_s^{i0}}{W_s^{jt}}\right) = \frac{1}{t}\ln\left(\frac{e^{-d_s^{ia^{i0}}(a^{i0}-1)} u_s^{i1} F_s^i}{e^{-d_s^{ja^{jt}}(a^{jt}-1)} u_s^{j1} F_s^j}\right) \\
&= \frac{d_s^{ja^{jt}}(a^{jt}-1) - d_s^{ia^{i0}}(a^{i0}-1)}{t} + \frac{t+a^{i0}-a^{jt}}{t}\frac{1}{G(j)-G(i)}\ln\left(\frac{u_s^{i1}F_s^{i0}}{u_s^{jt}F_s^{jt}}\right) \\
&= \frac{d_s^{ja^{jt}}(a^{jt}-1) - d_s^{ia^{i0}}(a^{i0}-1)}{t} + \frac{t+a^{i0}-a^{jt}}{t} D_s^{i,j}. \qquad (21)
\end{aligned}$$

Naming $D_s^{i,j} = \frac{1}{G(j)-G(i)}\ln\left(\frac{u_s^{i1}F_s^{i0}}{u_s^{jt}F_s^{jt}}\right)$ the 'state-specific person-to-person birth discount

[12]Because individuals live for more than one period, we now need to specify the period t of the future consumption.

rate from person i to person j', we see that this term is prominent in the long-run. Indeed, if consumption discount rates d_s^{ia} are bounded for all relevant alternatives, the first term in Equation (21) tends to 0 when t goes to infinity. Similarly, the term $\frac{t+a^{i0}-a^{jt}}{t}$ goes to one. This means that, in the long-run, the ages of the persons involved in the investment and their consumption discount rates do not play much role. Only state-specific person-to-person birth discount rates will matter.

On the other hand, for shorter term investments, we need to take into account individuals' consumption discount rates. A simple example is of course when $i = j$, that is when an individual makes an investment for his on sake. In that case, the second term in Equation (21) disappears (by definition, $D_s^{i,i} = 0$). Only the individual consumption discount rate matters for this kind of investment and the society does not interfere. The individual consumption discount rate is typically the kind of investment we observe in financial markets. The information on market rates is therefore very relevant to the social discount rate for short term investments. But they do not provide much guidance for long term investments.

Finally, remark that all our results extend to the OLG case, in particular Propositions 4 and 5. It suffices to replace the value of $\delta_s^{i,j}$ by the expression for $\delta_s^{i,jt}$ displayed in Equation (21).

6.2 Conclusion: Beyond the discount rate

Standard cost-benefit analysis considers the impact of policy on consumption. Consumption can be interpreted in a rather comprehensive way, including non-market goods and public goods such as biodiversity and ecosystem services (although this implies the difficult task of assigning a value to those goods). One ominous aspect of climate change, however, is that it may threaten livelihood on earth, and therefore the mere existence of many future generations (not only human, but also for other species).

It may therefore be the case that policy affects not only consumption, but also the prospects that we face concerning the future. We may be able to change the probability of future catastrophic events. The technique used to evaluate changes in probabilities, when there is a risk on the existence or longevity of an individual, consists in computing the 'value of a statistical life' (VSL). We can extend the methodology in the case of risks on the existence of future generations (the idea was suggested, but used in a very different way by

Weitzman, 2009).

Suppose that the goal of a policy is to shift probability from state s to state s' by a small amount δ. What cost ε can be imposed on individual i who is living now, for the sake of implementing this policy?[13] The variation in social welfare is equal to

$$dF = \delta\left(W(c_{s'}) - W(c_s)\right) - \sum_{s=1}^{m} p_s W_s^i \varepsilon,$$

and it is equal to zero when

$$\frac{\varepsilon}{\delta} = \frac{W(c_{s'}) - W(c_s)}{\sum_{s=1}^{m} p_s W_s^i}. \tag{22}$$

The right-hand-side of the above expression defines a concept similar to VSL, that we can name the *social value of risk reduction*. It determines how much the society is ready to pay for a small reduction of the risk (represented by scenario s realizing).

It turns out that the right-hand side of Equation (22) can be reformulated using the person-to-person discount rates, when the two states s and s' are not too different, and in particular when the same population $N_s = N_{s'}$ lives in the two states. In that case,

$$\begin{aligned}
\frac{W(c_{s'}) - W(c_s)}{\sum_{s=1}^{m} p_s W_s^i} &\approx \frac{\sum_{j\in N_s} W_s^j \left(c_{s'}^j - c_s^j\right)}{\sum_{s=1}^{m} p_s W_s^i} \\
&= \frac{W_s^i}{\sum_{s=1}^{m} p_s W_s^i} \sum_{j\in N_s} \frac{W_s^j}{W_s^i}\left(c_{s'}^j - c_s^j\right) \\
&= w_s^i \sum_t \sum_{j\in N_s^t} e^{-\delta_s^{i,j} t}\left(c_{s'}^j - c_s^j\right).
\end{aligned}$$

In this case, the social value of risk reduction is proportional to the discounted sum of the consumption gains for all people in all generations. This expression is intuitive: it is equal to the social willingness to make i pay for a 100% probability shift from s to s'.

In the case of climate change though, there are reasons to think that the marginal analysis used in the approximation will not hold. First, we may want to shift probability from a catastrophic scenario in which people are all deprived to a very different scenario in which people are able to enjoy much better lives. Second, we would like to shift probability from a catastrophic scenario in which few generations exist to a scenario in which many more

[13] Like before, the cost of the policy could be shared between different individuals of the current generation, involving additional equity consideration. Here we leave these complications aside for the sake of simplicity.

generations exist. This involves comparing populations of different sizes, and therefore making hard ethical decisions concerning the value of additional lives.

Consider for instance the critical level utilitarian criterion (5). Using this criterion, so that $W_s^i = u'(c_s^i)$, and assuming that $N_s \subset N_{s'}$, we obtain:

$$\begin{aligned}\frac{W(c_{s'}) - W(c_s)}{\sum_{s=1}^m p_s W_s^i} &= \sum_{j \in N_s} \frac{u(c_{s'}^j) - u(c_s^j)}{u'(c_s^i)} + \sum_{j \in N_{s'} \setminus N_s} \frac{u(c_{s'}^j) - \bar{u}}{u'(c_s^i)} \\ &\approx \sum_t \left(\sum_{j \in N_s^t} e^{-\delta_s^{i,j} t} \left(c_{s'}^j - c_s^j \right) + \sum_{j \in N_{s'}^t \setminus N_s^t} \frac{u(c_{s'}^j) - \bar{u}}{u'(c_s^i)} \right).\end{aligned}$$

The new expression crucially depends on the critical level $\bar{u}$. If we set high standards for the future, it may be the case that the second term in the expression is negative, because we are not able to both increase the survival of humanity and keep high standards of living. If on the contrary $\bar{u}$ is rather low, the social value of catastrophic risk reduction may be very high, suggesting an additional (and perhaps more powerful) reason why climate policies are socially valuable.

In any case, the reasoning suggests that we must go beyond traditional cost-benefit analysis valuing future aggregate consumption benefits using a social discount rate. Policies devoted to mitigating the risk of climate change typically modify the probabilities of various scenarios and change the level of consumption for various subgroups of the population in the scenarios. Hence, trying to convert the costs and benefits of such a policy into monetary amounts and comparing the corresponding total rate of return to a single macroeconomic benchmark in the form of a discount rate may be at best quite roundabout and at worst very misleading.

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Appendix

Appendix A: Proof of Proposition 4.

From Equation (14), we have:

$$\begin{aligned}\delta &= \tfrac{1}{t}\ln\left(\sum_{i\in N^0}\sigma^i\left(\sum_{s=1}^m p_s w_s^i\theta_s^t\sum_{j\in N_s^t}\sigma_s^j e^{-\delta_s^{i,j}t}\right)^{-1}\right)\\ &= \tfrac{1}{t}\ln\left(\sum_{s=1}^m p_s e^{r_s t}\right)+\tfrac{1}{t}\ln\left(\sum_{i\in N^0}\sigma^i\left(\sum_{s=1}^m p_s\sum_{j\in N_s^t}\sigma_s^j w_s^i e^{(r_s-\delta_s^{i,j})t}\right)^{-1}\right)\end{aligned}$$

The first term $\frac{1}{t}\ln\left(\sum_{s=1}^m p_s e^{r_s t}\right)=\ln\left(\left(\sum_{s=1}^m p_s e^{r_s t}\right)^{1/t}\right)$ tends to $\ln(e^{\max_s r_s})=\max_s r_s$ when t tends to infinity, provided the returns $(r_s)_{s\in S}$ do not depend on time. However, one has to take account of the fact that the distribution of r_s may change with t.

Let $\bar{r}=\max_{s\in S} r_s$ and denote

$$X(t)=\tfrac{1}{t}\ln\left(\sum_{s=1}^m p_s e^{r_s t}\right)-\bar{r}=\tfrac{1}{t}\ln\left(\sum_{s=1}^m p_s e^{(r_s-\bar{r})t}\right).$$

One has $e^{(r_s-\bar{r})t}\leq 1$ for all $s\in S$, so that $X(t)\leq 0$. On the other hand,

$$\sum_{s=1}^m p_s e^{(r_s-\bar{r})t}=\sum_{s:r_s=\bar{r}}p_s+\sum_{s:r_s\neq\bar{r}}p_s e^{(r_s-\bar{r})t}\geq\min_{s\in S}p_s.$$

Therefore

$$\tfrac{1}{t}\ln\left(\min_{s\in S}p_s\right)\leq X(t)\leq 0.$$

Denote $r_i=\max_{s,j}(r_s-\delta_s^{i,j})$ and $\tilde{r}=\min_{i\in N}r^i$. If the $(r_s-\delta_s^{i,j})_{i,j,s}$ were independent of t, for a given $i \in N^0$ we would have that

$$\left(\sum_{s=1}^{m} p_s \sum_{j \in N_s^t} \sigma_s^j w_s^i e^{(r_s - \delta_s^{i,j})t}\right)^{1/t}$$

tends to e^{r_i}.

Here again, we must take into account that the $(r_s - \delta_s^{i,j})_{i,j,s}$ may change with t. To do so, we let

$$\begin{aligned} Y(t) &= \tfrac{1}{t} \ln \left(\sum_{i \in N^0} \sigma^i \left(\sum_{s=1}^{m} p_s \sum_{j \in N_s^t} \sigma_s^j w_s^i e^{(r_s - \delta_s^{i,j})t} \right)^{-1} \right) + \tilde{r} \\ &= \tfrac{1}{t} \ln \left(\sum_{i \in N^0} \sigma^i \frac{e^{(\tilde{r} - r_i)t}}{\sum_{s=1}^{m} p_s w_s^i \sum_{j \in N_s^t} \sigma_s^j e^{\left(r_s - \delta_s^{i,j} - r_i\right)t}} \right). \end{aligned}$$

One has $e^{(\tilde{r}-r_i)t} \leq 1$, $\sum_{i \in N^0} \sigma_i e^{(\tilde{r}-r_i)t} \geq \min_i \sigma_i$ and

$$\min_{s \in S, j \in N_s^t} p_s w_s^i \sigma_s^j \leq \sum_{s=1}^{m} p_s w_s^i \sum_{j \in N_s^t} \sigma_s^j e^{\left(r_s - \delta_s^{i,j} - r_i\right)t} \leq \sum_{s=1}^{m} p_s w_s^i \sum_{j \in N_s^t} \sigma_s^j = 1,$$

so that:

$$\tfrac{1}{t} \ln \left(\min_{i \in N^0} \sigma^i \right) \leq \tfrac{1}{t} \ln \left(\sum_{i \in N^0} \sigma_i e^{(\tilde{r}-r_i)t} \right) \leq Y(t) \leq \tfrac{1}{t} \ln \left(\sum_{i \in N^0} \frac{\sigma^i}{\min_{s,j} p_s w_s^i \sigma_s^j} \right) \leq \tfrac{1}{t} \ln \left(\sum_{i \in N^0} \frac{\sigma^i / \sigma_0}{\min_{s \in S} p_s w_s^i} \right).$$

Letting $O(1/t) = X(t) + Y(t)$, we obtain

$$\delta = \max_{s \in S} r_s - \min_{i \in N^0} \max_{s \in S, j \in N_s^t} \left(r_s - \delta_s^{i,j} \right) + O(1/t)$$

and

$$\tfrac{1}{t} \left(\ln \left(\min_{i \in N^0} \sigma^i \right) + \ln \left(\min_{s \in S} p_s \right) \right) \leq O(1/t) \leq \tfrac{1}{t} \ln \left(\sum_{i \in N^0} \frac{\sigma^i / \sigma_0}{\min_{s \in S} p_s w_s^i} \right).$$

Appendix B: Proof of Proposition 5.

One has

$$e^{(r-\delta)t} = \left(\sum_{i\in N^0}\sigma^i\left(\sum_{s=1}^m p_s w_s^i \sum_{j\in N_s^t}\sigma_s^j e^{(r_s-\delta_s^{i,j})t}\right)^{-1}\right)^{-1}$$

Therefore,

$$\begin{aligned}\left(\frac{e^{(r-\delta)t}}{e^{\min_i \max_{s,j}(r_s-\delta_s^{i,j})t}}\right)^{-1} \quad = \quad & \sum_{i\in I_t}\sigma^i \frac{1}{\sum_{s=1}^m p_s w_s^i \sum_{j\in N_s^t}\sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t}} \\ & + \sum_{i\notin I_t}\sigma^i \frac{e^{\left(\min_{i\in N^0}\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)-\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t}}{\sum_{s=1}^m p_s w_s^i \sum_{j\in N_s^t}\sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t}}.\end{aligned}$$

For every $i \notin I_t$, one has

$$\begin{aligned} e^{\left(\min_{i\in N^0}\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)-\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t} &< e^{-\alpha t}, \\ \sum_{s=1}^m p_s w_s^i \sum_{j\in N_s^t}\sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t} &> \min_{s\in S} p_s w_s^i \sigma_0, \end{aligned}$$

so that

$$0 < \sum_{i\notin I_t}\sigma^i \frac{e^{\left(\min_{i\in N^0}\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)-\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t}}{\sum_{s=1}^m p_s w_s^i \sum_{j\in N_s^t}\sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t}} < \frac{e^{-\alpha t}/\sigma_0}{\min_{i\in N^0,s\in S} p_s w_s^i}.$$

Let us examine the first term,

$$\sum_{i\in I_t}\sigma^i\left(\sum_{s=1}^m p_s w_s^i \sum_{j\in N_s^t}\sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t}\right)^{-1}.$$

The denominator reads

$$\sum_{s\in S_t^i} p_s w_s^i \sum_{j\in J_{s,t}^i}\sigma_s^j + \sum_{s\in S_t^i} p_s w_s^i \sum_{j\notin J_{s,t}^i}\sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t} + \sum_{s\notin S_t^i} p_s w_s^i \sum_{j\in N_s^t}\sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{s\in S,j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t}.$$

One has (recall that $\sum_{s\in S} p_s w_s^i = 1$):

$$
\begin{aligned}
0 &< \sum_{s\in S_t^i} p_s w_s^i \sum_{j\notin J_{s,t}^i} \sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t} < e^{-\alpha t},\\
0 &< \sum_{s\notin S_t^i} p_s w_s^i \sum_{j\in N_s^t} \sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{s\in S, j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t} < e^{-\alpha t}.
\end{aligned}
$$

One therefore obtains:

$$
\begin{aligned}
& \sum_{i\in I_t} \sigma^i \left(\sum_{s\in S_t^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j + 2e^{-\alpha t}\right)^{-1}\\
< & \sum_{i\in I_t} \sigma^i \left(\sum_{s=1}^{m} p_s w_s^i \sum_{j\in N_s^t} \sigma_s^j e^{\left(r_s-\delta_s^{i,j}-\max_{s\in S, j\in N_s^t}\left(r_s-\delta_s^{i,j}\right)\right)t}\right)^{-1}\\
< & \sum_{i\in I_t} \sigma^i \left(\sum_{s\in S^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j\right)^{-1}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
& \sum_{i\in I_t} \sigma^i \left(\sum_{s\in S^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j + 2e^{-\alpha t}\right)^{-1} - \sum_{i\in I_t} \sigma^i \left(\sum_{s\in S_t^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j\right)^{-1}\\
< & \left(\frac{e^{(r-\delta)t}}{e^{\min_i \max_{s,j}(r_s-\delta_s^{i,j})t}}\right)^{-1} - \sum_{i\in I_t} \sigma^i \left(\sum_{s\in S_t^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j\right)^{-1}\\
< & \frac{e^{-\alpha t}}{\sigma_0 \min_{i\in N^0, s\in S} p_s w_s^i}.
\end{aligned}
$$

Using the fact that $\frac{1}{1+a} > 1-a$ for $a>0$, one has

$$
\begin{aligned}
\sum_{i\in I_t} \sigma^i \left(\sum_{s\in S_t^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j + 2e^{-\alpha t}\right)^{-1} &= \sum_{i\in I_t} \sigma^i \left(\sum_{s\in S_t^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j\right)^{-1} \frac{1}{1+\frac{2e^{-\alpha t}}{\sum_{s\in S_t^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j}}\\
&> \sum_{i\in I_t} \sigma^i \left(\sum_{s\in S_t^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j\right)^{-1} \left(1-\frac{2e^{-\alpha t}}{\sum_{s\in S_t^i} p_s w_s^i \sum_{j\in J_{s,t}^i} \sigma_s^j}\right),
\end{aligned}
$$

so that

$$
\begin{aligned}
& \sum_{i\in I_t}\sigma^i\left(\sum_{s\in S_t^i}p_s w_s^i\sum_{j\in J_{s,t}^i}\sigma_s^j+2e^{-\alpha t}\right)^{-1}-\sum_{i\in I_t}\sigma^i\left(\sum_{s\in S_t^i}p_s w_s^i\sum_{j\in J_{s,t}^i}\sigma_s^j\right)^{-1} \\
> \quad & -2e^{-\alpha t}\sum_{i\in I_t}\sigma^i\left(\sum_{s\in S_t^i}p_s w_s^i\sum_{j\in J_{s,t}^i}\sigma_s^j\right)^{-2} \\
> \quad & -2e^{-\alpha t}\left(\sigma_0\min_{i\in N^0,s\in S}p_s w_s^i\right)^{-2}.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
& -2e^{-\alpha t}\left(\sigma_0\min_{i\in N^0,s\in S}p_s w_s^i\right)^{-2} \\
< \quad & \left(\frac{e^{(r-\delta)t}}{e^{\min_i\max_{s,j}(r_s-\delta_s^{i,j})t}}\right)^{-1}-\sum_{i\in I_t}\sigma^i\left(\sum_{s\in S_t^i}p_s w_s^i\sum_{j\in J_{s,t}^i}\sigma_s^j\right)^{-1} \\
< \quad & \frac{e^{-\alpha t}}{\sigma_0\min_{i\in N^0,s\in S}p_s w_s^i}.
\end{aligned}
$$

This inequality has the form

$$
-ae^{-\alpha t}<\frac{y}{x}-z<be^{-\alpha t},
$$

for $z=\sum_{i\in I_t}\sigma^i\left(\sum_{s\in S_t^i}p_s w_s^i\sum_{j\in J_{s,t}^i}\sigma_s^j\right)^{-1}$. Note that $\min_{i\in N^0}\sigma^i\le\sum_{i\in I_t}\sigma^i<z<b=\left(\sigma_0\min_{i\in N^0,s\in S}p_s w_s^i\right)^{-1}$.

It implies, for t large enough,

$$
\frac{1}{1+\frac{b}{z}e^{-\alpha t}}<\frac{x}{y/z}<\frac{1}{1-\frac{a}{z}e^{-\alpha t}}<\frac{1}{1-\frac{a}{\min_{i\in N^0}\sigma^i}e^{-\alpha t}}
$$

and therefore, for t great enough so that $\frac{a}{\min_{i\in N^0}\sigma^i}e^{-\alpha t}<1/2$,

$$
1-\frac{b}{z}e^{-\alpha t}<\frac{x}{y/z}<1+2\frac{a}{\min_{i\in I}\sigma^i}e^{-\alpha t}
$$

An expression $f(t)$ satisfying

$$-\frac{b}{z}e^{-\alpha t} < f\left(t\right) < 2\frac{a}{\min_{i \in I} \sigma^i}e^{-\alpha t}$$

is $o\left(1/t\right)$.