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Anonymous Social Influence[☆]

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Abstract

We study a stochastic model of influence where agents have "yes" or "no" inclinations on some issue, and opinions may change due to mutual influence among the agents. Each agent independently aggregates the opinions of the other agents and possibly herself. We study influence processes modeled by ordered weighted averaging operators, which are anonymous: they only depend on how many agents share an opinion. For instance, this allows to study situations where the influence process is based on majorities, which are not covered by the classical approach of weighted averaging aggregation. We find a necessary and sufficient condition for convergence to consensus and characterize outcomes where the society ends up polarized. Our results can also be used to understand more general situations, where ordered weighted averages are only used to some extent. Furthermore, we apply our results to fuzzy linguistic quantifiers, i.e., expressions like "most" or "at least a few".

Keywords: Influence, anonymity, ordered weighted averaging operator, convergence, consensus, fuzzy linguistic quantifier
JEL: C7, D7, D85

1. Introduction

In the present work we study an important and widespread phenomenon which affects many aspects of human life – the phenomenon of *influence*. Being undoubtedly present, e.g., in economic, social and political behaviors, influence frequently appears as a dynamic

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process. Since social networks play a crucial role in the formation of opinions and the diffusion of information, it is not surprising that numerous scientific works investigate different dynamic models of influence in social networks.[1]

Grabisch and Rusinowska (2010, 2011) investigate a one-step deterministic model of influence, where agents have "yes" or "no" inclinations (beliefs) on a certain issue and their opinions may change due to mutual influence among the agents. Grabisch and Rusinowska (2013) extend it to a dynamic stochastic model based on aggregation functions, which determine how the agents update their opinions depending on the current opinions in the society. Each agent independently aggregates the opinions of the other agents and possibly herself. This aggregation determines the probability that "yes" is her updated opinion after one step of influence (and otherwise it is "no"). The other agents only observe this updated opinion. Since any aggregation function is allowed when updating the opinions, the framework covers numerous existing models of opinion formation. The only restrictions come from the definition of an aggregation function: unanimity of opinions persists (boundary conditions) and influence is positive (nondecreasingness). Grabisch and Rusinowska (2013) provide a general analysis of convergence in the aggregation model and find all terminal classes, which are sets of states the process will not leave once they have been reached. Such a class could only consist of one single state, e.g., the states where we have unanimity of opinions ("yes"- and "no"-consensus) or a state where the society is polarized, i.e., some group of agents finally says "yes" and the rest says "no".

Due to the generality of the model of influence based on arbitrary aggregation functions introduced in Grabisch and Rusinowska (2013), it would be difficult to obtain a deeper insight into some particular phenomena of influence by using this model. This is why the analysis of particular classes of aggregation functions and the exhaustive study of their properties are necessary for explaining many social and economic interactions. One of them concerns *anonymous social influence* which is particularly present in real-life situations. Internet, accompanying us in everyday life, intensifies enormously anonymous influence: when we need to decide which washing machine to buy, which hotel to reserve for our eagerly awaited holiday, we will certainly follow all anonymous customers and tourists that have expressed their positive opinion on the object of our interest. In the present paper we examine a particular way of aggregating the opinions and investigate influence processes modeled by *ordered weighted averaging operators (ordered weighted averages)*, commonly called *OWA operators* and introduced in Yager (1988), because they appear to be a very appropriate tool for modeling and analyzing anonymous social influence. Roughly speaking, OWA operators are similar to the ordinary weighted averages (weighted arithmetic means), with the essential difference that weights are not attached to agents, but to the *ranks* of the agents in the input vector. As a consequence, OWA operators are in general nonlinear, and include as particular cases the median, the minimum and the maximum, as well as the (unweighted) arithmetic mean.

We show that OWA operators are the only aggregation functions that are *anonymous*

[1]For an overview of the vast literature on influence we refer, e.g., to Jackson (2008).

in the sense that the aggregation does only depend on how many agents hold an opinion instead of which agents do so. Accordingly, we call a model *anonymous* if the transitions between states of the process do only depend on how many agents share an opinion. We show that the concept is consistent: if all agents use anonymous aggregation functions, then the model is anonymous. However, as we show by example, a model can be anonymous although agents do not use anonymous functions. In particular, anonymous models allow to study situations where the influence process is based on *majorities*, which means that agents say "yes" if some kind of majority holds this opinion.[2] These situations are not covered by the classical (commonly used) approach of weighted averaging aggregation.

In the main part, we consider models based on OWA operators. We discuss the different types of terminal classes and characterize terminal states, i.e., singleton terminal classes. The condition is simple: the OWA operators must be such that all opinions persist after mutual influence. In our main result, we find a necessary and sufficient condition for convergence to consensus. The condition says that there must be a certain number of agents such that if at least this number of agents says "yes", it is possible that after mutual influence more agents say "yes" and if less than that number of agents says "yes", it is possible that after mutual influence more agents say "no". In other words, we have a cascade that leads either to the "yes"- or "no"-consensus. Additionally, we also present an alternative characterization based on *influential coalitions*. We call a coalition influential on an agent if the latter follows (adopts) the opinion of this coalition – given all other agents hold the opposite opinion – with some probability.[3] Furthermore, we generalize the model based on OWA operators and allow agents to use a (convex) combination of OWA operators and general aggregation functions (*OWA-decomposable* aggregation functions). In particular, this allows us to combine OWA operators and ordinary weighted averaging operators. As a special case of this, we study models of mass psychology (also called herding behavior) in an example. We find that this model is equivalent to a convex combination of the majority influence model and a completely self-centered agent. We also study an example on *important agents* where agents trust some agents directly that are important for them and otherwise follow a majority model. Furthermore, we show that the sufficiency part of our main result still holds.[4]

As an application of our model we study *fuzzy linguistic quantifiers*, which were introduced in Zadeh (1983) and are also called *soft quantifiers*. Typical examples of such quantifiers are expressions like "almost all", "most", "many" or "at least a few"; see Yager and Kacprzyk (1997). For instance, an agent could say "yes" if "most of the agents say 'yes' ".[5] Yager (1988) has shown that for each quantifier we can find a unique corresponding OWA operator.[6] We find that if the agents use quantifiers that are similar in some sense,

[2]Examples are simple majorities as well as unanimity of opinions, among others.

[3]Note that although Grabisch and Rusinowska (2013) have already studied conditions for convergence to consensus and other terminal classes in the general model, our results are inherently different due to our restriction to anonymous aggregation functions.

[4]When applying the condition to the OWA operators in the convex combinations.

[5]Note that the formalization of such quantifiers is clearly to some extent ambiguous.

[6]With the only restriction that, due to our model, the quantifier needs to represent positive influence.

then they reach a consensus. Moreover, this result holds even if some agents deviate to quantifiers that are not similar in that sense. Loosely speaking, quantifiers are similar if their literal meanings are "close", e.g., "most" and "almost all". We also give examples to provide some intuition.

We terminate this section with a very brief overview of the related literature. One of the main differences between our work and the existing models on opinion formation lies in the way agents are assumed to aggregate the opinions. Except, e.g., Grabisch and Rusinowska (2013) many related works assume a convex combination as the way of aggregating opinions. Additionally, while we consider "yes"/"no" opinions, in some models of influence, like in the seminal model of opinion and consensus formation due to DeGroot (1974), the opinion of an agent is a number in $[0, 1]$. Moreover, in DeGroot (1974) every agent aggregates the opinions (beliefs) of other agents through an ordinary weighted average. The interaction among agents is captured by the social influence matrix. Several scholars have analyzed the DeGroot framework and proposed different variations of it, in which the updating of opinions can vary in time and along circumstances. However, most of the influence models usually assume a convex combination as the way of aggregating opinions. Golub and Jackson (2010) examine convergence of the social influence matrix and reaching a consensus, and the speed of convergence of beliefs, among other things. DeMarzo et al. (2003) consider a model where an agent may place more or less weight on her own belief over time. Another framework related to the DeGroot model is presented in Asavathiratham (2000) and López-Pintado and Watts (2008). Büchel et al. (2011) introduce a generalization of the DeGroot model by studying the transmission of cultural traits from one generation to the next one. Büchel et al. (2012) analyze an influence model in which agents may misrepresent their opinion in a conforming or counter-conforming way. Calvó-Armengol and Jackson (2009) study an overlapping-generations model in which agents, that represent some dynasties forming a community, take yes-no actions.

López-Pintado (2008, 2012) study the spreading of behavior in society and investigate the role of social influence therein. While these papers focus on the social network and use simple diffusion rules that are the same for all agents, we do not impose a network structure and allow for heterogeneous agents. Van den Brink and his co-authors study power measures in weighted directed networks, see, e.g., van den Brink and Gilles (2000); Borm et al. (2002). A different approach to influence, i.e., a method based on simulations, is presented in Mäs (2010). Morris (2000) analyzes the phenomenon of contagion which occurs if an action can spread from a finite set of individuals to the whole population.

Another stream of related literature concerns models of Bayesian and observational learning where agents observe choices over time and update their beliefs accordingly, see, e.g., Banerjee (1992), Ellison (1993), Bala and Goyal (1998, 2001), Gale and Kariv (2003) and Banerjee and Fudenberg (2004). This literature differs from the influence models mentioned above as in the latter the choices depend on the influence of others. Mueller-Frank (2010) considers continuous aggregation functions with a special property called "constricting" and studies convergence applied to non-Bayesian learning in social networks. Galeotti and Goyal (2009) model networks in terms of degree distributions and study influence strategies in the

presence of local interaction.

The literature on OWA operators comprises, in particular, applications to multi-criteria decision-making. Jiang and Eastman (2000), for instance, apply OWA operators to geographical multi-criteria evaluation, and Malczewski and Rinner (2005) present a fuzzy linguistic quantifier extension of OWA in geographical multi-criteria evaluation. Using ordered weighted averages in (social) networks is quite new, although some scholars have already initiated such an application; see Cornelis et al. (2010) who apply OWA operators to trust networks. To the best of our knowledge, ordered weighted averages have not been used to model social influence yet.

The remainder of the paper is organized as follows. In Section 2 we present the model and basic definitions. Section 3 introduces the notion of anonymity. Section 4 concerns the convergence analysis in the aggregation model with OWA operators. In Section 5 we apply our results on ordered weighted averages to fuzzy linguistic quantifiers. Section 6 contains some concluding remarks. The longer proofs of some of our results are presented in the appendix.

2. Model and Notation

Let $N := \{1, \ldots, n\}$, $n \geq 2$, be the set of agents that have to make a "yes" or "no" decision on some issue. Each agent $i \in N$ has an initial opinion $x_i \in \{0, 1\}$ (called *inclination*) on the issue, where "yes" is coded as 1. Let us denote by 1_S the characteristic vector of $S \subseteq N$, i.e., $(1_S)_j = 1$ if $j \in S$ and $(1_S)_j = 0$ otherwise. We can represent the vector of initial opinions by such a characteristic vector. We say that S is the initial *state* or *coalition* if 1_S is the vector of initial opinions. In other words, the initial state consists of the agents that have the inclination "yes". We sometimes denote a state $S = \{i, j, k\}$ simply by ijk and its cardinality or *size* by s. During the influence process, agents may change their opinion due to mutual influence among the agents. They update their opinion simultaneously at discrete time instants.

Definition 1 (**Aggregation function**). *An* n-place aggregation function *is any mapping* $A : \{0, 1\}^n \to [0, 1]$ *satisfying*

(i) $A(0, \ldots, 0) = 0, A(1, \ldots, 1) = 1$ *(boundary conditions) and*
(ii) *if* $x \leq x'$ *then* $A(x) \leq A(x')$ *(nondecreasingness).*

To each agent i we assign an aggregation function A_i that determines the way she reacts to the opinions of the other agents and herself.[7] Note that by using these functions we model positive influence only. Our aggregation model $\mathbf{A} = (A_1, \ldots, A_n)^T$ is stochastic,[8] the output $A_i(1_S) \in [0, 1]$ of agent i's aggregation function is her probability to say "yes" after one step

[7]Note that we use a modified version of aggregation functions by restricting the opinions to be from $\{0, 1\}$ instead of $[0, 1]$. We discuss this issue later on in Example 1.

[8]Superscript T denotes the transpose of a vector.

of influence when the current state is 1_S. The other agents do not know these probabilities, but they observe the realization of the updated opinions. Note that we do not explicitly model the realization of the updated opinions, which is for agent i a (biased) coin toss with probability $A_i(1_S)$ of "yes" and probability $1 - A_i(1_S)$ of "no". Therefore, we can represent the realized and observed opinions (after one step of influence) again by a state $S' \subseteq N$ such that $i \in S'$ with probability $A_i(1_S)$.

The aggregation functions our paper is mainly concerned with are *ordered weighted averaging operators* or simply *ordered weighted averages*. This class of aggregation functions was first introduced by Yager (1988).

Definition 2 (**Ordered weighted average**). *We say that an n-place aggregation function A is an* ordered weighted average $A = \mathsf{OWA}_w$ *with weight vector w, i.e., $0 \leq w_i \leq 1$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n w_i = 1$, if $A(x) = \sum_{i=1}^n w_i x_{(i)}$ for all $x \in \{0,1\}^n$, where $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}$ are the ordered components of x.*

The definition of an aggregation function ensures that the two consensus states – the "yes"-consensus $\{N\}$ where all agents say "yes" and the "no"-consensus $\{\emptyset\}$ where all agents say "no" – are fixed points of the aggregation model $\mathbf{A} = (A_1, \ldots, A_n)^T$. We call them *trivial terminal classes*. Before we go on, let us give an example of an ordered weighted average already presented in Grabisch and Rusinowska (2013), the *majority influence model*. Furthermore, we also use this example to argue why we do restrict opinions to be either "yes" or "no".

Example 1 (**Majority**). *A straightforward way of making a decision is based on majority voting. If the majority of the agents says "yes", then all agents agree to say "yes" after mutual influence and otherwise, they agree to say "no". We can model simple majorities as well as situations where more than half of the agents are needed to reach the "yes"-consensus. Let $m \in \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$. Then, the* majority *aggregation model is given by*

$$\mathsf{Maj}_i^{[m]}(x) := x_{(m)} \text{ for all } i \in N.$$

All agents use an ordered weighted average where $w_m = 1$. Obviously, the convergence to consensus is immediate.

To give some intuition for our restriction to opinions lying in $\{0,1\}$, note that in this example, allowing for opinions in $[0,1]$ means that the outcome only depends on $x_{(m)}$. And the only way to avoid this is the restriction to $\{0,1\}$.[9]

Furthermore, let us look at some examples apart from the majority model.

Example 2 (**Some ordered weighted averages**). *Consider some agent $i \in N = \{1, 2, \ldots, 5\}$ who uses an ordered weighted average, $A_i = \mathsf{OWA}_w$.*

[9]Note that also a change of the used ordered weighted average does not help, e.g., $\mathsf{Maj}_i^{[m]}(x) := \sum_{j=1}^m \frac{1}{m} x_{(j)}$ for all $i \in N$. The reason is that in this case, it is possible that an agent accepts with positive probability even if less than m agents have the inclination to accept with positive probability.

(i) *If* $w = \left(0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$, *then this agent will say "no" for sure if there is not even a simple majority in favour of the issue. Otherwise, she will say "yes" with a positive probability, which increases by* $\frac{1}{3}$ *with each additional agent being in favour of the issue.*

(ii) *If* $w = \left(\frac{1}{3}, \frac{2}{3}, 0, 0, 0\right)^T$, *then this agent will already say "yes" if only one agent does so and she will be in favour for sure whenever at least two agents say "yes". This could represent a situation where it is perfectly fine for the agent if only a few of the others are in favour of the issue.*

(iii) *If* $w = \left(\frac{1}{2}, 0, 0, 0, \frac{1}{2}\right)^T$, *then this agent will say "yes" with probability* $\frac{1}{2}$ *if neither all agents say "no" nor all agents say "yes". This could be interpreted as an agent who is indifferent and so decides randomly.*

We have already seen that there always exist the two trivial terminal classes. In general, a terminal class is defined as follows:

Definition 3 (**Terminal class**). *A* terminal class *is a collection of states* $\mathcal{C} \subseteq 2^N$ *that forms a strongly connected and closed component, i.e., for all* $S, T \in \mathcal{C}$, *there exists a path*[10] *from* S *to* T *and there is no path from* S *to* T *if* $S \in \mathcal{C}, T \notin \mathcal{C}$.

We can decompose the state space into disjoint terminal classes – also called absorbing classes – $\mathcal{C}_1, \ldots, \mathcal{C}_l \subseteq 2^N$, for some $l \geq 2$, and a set of transient states $\mathcal{T} = 2^N \backslash (\bigcup_{k=1}^{l} \mathcal{C}_k)$. Let us now define the notion of an influential agent (Grabisch and Rusinowska, 2013).

Definition 4 (**Influential agent**). (i) *An agent* $j \in N$ *is* "yes"-influential *on* $i \in N$ *if* $A_i(1_{\{j\}}) > 0$.

(ii) *An agent* $j \in N$ *is* "no"-influential *on* $i \in N$ *if* $A_i(1_{N\backslash\{j\}}) < 1$.

The idea is that j is "yes"-(or "no"-)influential on i if j's opinion to say "yes" (or "no") matters for i in the sense that there is a positive probability that i follows the opinion that is solely held by j. Analogously to influential agents, we can define influential coalitions (Grabisch and Rusinowska, 2013).

Definition 5 (**Influential coalition**). (i) *A nonempty coalition* $S \subseteq N$ *is* "yes"-influential *on* $i \in N$ *if* $A_i(1_S) > 0$.

(ii) *A nonempty coalition* $S \subseteq N$ *is* "no"-influential *on* $i \in N$ *if* $A_i(1_{N\backslash S}) < 1$.

Making the assumption that the probabilities of saying "yes" are independent among agents[11] and only depend on the current state, we can represent our aggregation model by a time-homogeneous Markov chain with transition matrix $\mathbf{B} = (b_{S,T})_{S,T\subseteq N}$, where

$$b_{S,T} = \Pi_{i\in T} A_i(1_S) \Pi_{i\notin T}(1 - A_i(1_S)).$$

[10] We say that there is a path from S to T if there is $K \in \mathbb{N}$ and states $S = S_1, S_2, \ldots, S_{K-1}, S_K = T$ such that $A_i(S_k) > 0$ for all $i \in S_{k+1}$ and $A_i(S_k) < 1$ otherwise, for all $k = 1, \ldots, K-1$.

[11] This assumption is not limitative, and correlated opinions may be considered as well. In the latter case, only the next equation giving $b_{S,T}$ will differ.

Hence, the states of this Markov chain are the states or coalitions of the agents that currently say "yes" in the influence process. Thus, $b_{S,T}$ denotes the probability, given the current state $S \subseteq N$, that the process is in state $T \subseteq N$ after one step of influence. Note that for each state S, the transition probabilities to states T are represented by a certain row of $\mathbf{B}$. Notice also that this Markov chain is neither irreducible nor recurrent since it has at least two terminal classes – also called communication classes in the language of Markov chains.

3. Anonymity

We establish the notions of *anonymous* aggregation functions and models. In what follows, we show that the notions of anonymity are consistent and that anonymous functions are characterized by OWA operators.

Definition 6 (**Anonymity**). (i) *We say that an n-place aggregation function A is* anonymous *if for all $x \in \{0,1\}^n$ and any permutation $\sigma : N \to N$, $A(x_1, \ldots, x_n) = A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.*

(ii) *Suppose $\mathbf{B}$ is obtained from an aggregation model with aggregation functions $A_1, \ldots, A_n$. We say that the model is* anonymous *if for all $s, t \in \{0, 1, \ldots, n\}$,*

$$\sum_{\substack{T \subseteq N: \\ |T|=t}} b_{S,T} = \sum_{\substack{T \subseteq N: \\ |T|=t}} b_{S',T} \text{ for all } S, S' \subseteq N \text{ of size } s.$$

For an agent using an anonymous aggregation function, only the size of the current coalition matters. Similarly, in models that satisfy anonymity, only the size of the current coalition matters for the further influence process. In other words, it matters how many agents share an opinion, but not which agents do so. Let us now confirm that our notions of anonymity are consistent in the sense that models where agents use anonymous functions are anonymous. Moreover, we characterize anonymous aggregation functions by ordered weighted averages.

Proposition 1. (i) *An aggregation model with anonymous aggregation functions $A_1, \ldots, A_n$ is anonymous.*

(ii) *An aggregation function A is anonymous if and only if it is an ordered weighted average.*

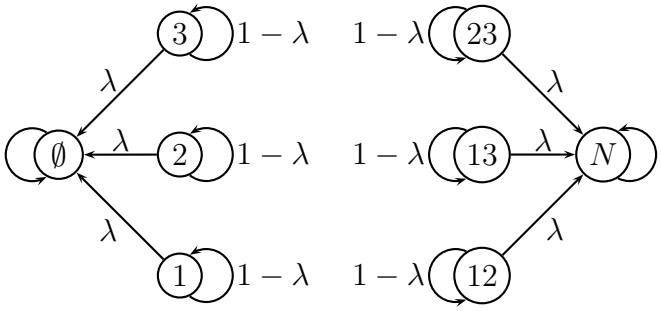
Proof. We omit the proof of (i) as well as the necessity part of (ii). For the sufficiency part, suppose that A is an anonymous aggregation function, i.e., for all $x \in \{0,1\}^n$ and any permutation $\sigma : N \to N$, $A(x_1, \ldots, x_n) = A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. This is equivalent to $A(1_S) = A(1_{S'})$ for all $S, S' \subseteq N$ such that $|S| = |S'|$. Hence, there exists $w \in \mathbb{R}^n$ such that $A(1_S) = \sum_{i \in N} w_i (1_S)_{(i)}$ for all $S \subseteq N$. It follows by the definition of aggregation functions, that $w_i \geq 0$ for all $i \in N$ (nondecreasingness) and $\sum_{i=1}^n w_i = 1$ (boundary condition), which finishes the proof. □

Note that the converse of the first part does not hold, a model can be anonymous although not all agents use anonymous aggregation functions as we now show by example. We study the phenomenon of *mass psychology*, also called herding behavior, considered in Grabisch and Rusinowska (2013).

Example 3 (**Mass psychology**). *Mass psychology or herding behavior means that if at least a certain number* $m \in \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$ *of agents share the same opinion, then these agents attract others, who had a different opinion before. We assume that an agent changes her opinion in this case with probability* $\lambda \in (0, 1)$. *In particular, we consider* $n = 3$ *agents and a threshold of* $m = 2$. *This means whenever only two agents are of the same opinion, the third one might change her opinion. This corresponds to the following* mass psychology *aggregation model:*

$$\mathsf{Mass}_i^{[2]}(x) = \lambda x_{(2)} + (1 - \lambda)x_i \text{ for all } i \in N.$$

Agents are "yes"- and "no"-influential on themselves and coalitions of size two or more are "yes"- and "no"-influential on all agents. The model gives the following digraph of the Markov chain:



The aggregation functions are not anonymous since agents consider their own opinion with weight $1 - \lambda > 0$. *However, the model turns out to be anonymous, there is no differentiation between different coalitions of the same size, as can be seen from the digraph.*

An immediate consequence of Proposition 1 is that models where agents use OWA operators are anonymous.

Corollary 1. *Aggregation models with aggregation functions* $A_i = \mathsf{OWA}_{w^i}, i \in N$, *are anonymous.*

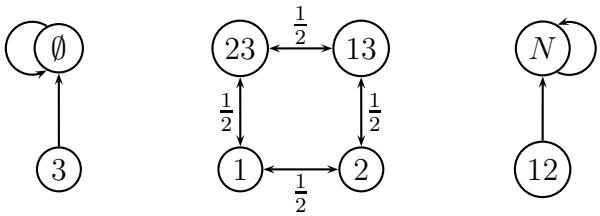
4. Convergence Analysis

In this section, we study the convergence of aggregation models where the influence process is determined by OWA operators, i.e., by anonymous aggregation functions. In Grabisch and Rusinowska (2013, Theorem 2), the authors show that there are three different types of terminal classes in the general model. To terminal classes of the first type, singletons $\{S\}$, $S \subseteq N$, we usually refer to as *terminal states*. They represent the two consensus states, $\{N\}$ and $\{\emptyset\}$, as well as situations where the society is eventually polarized: agents within the class say "yes", while the others say "no". Classes of the second type are called *cyclic terminal classes*, their states form a cycle of nonempty sets $\{S_1, \ldots, S_k\}$ of any length $2 \leq k \leq \binom{n}{\lfloor n/2 \rfloor}$ (and therefore they are periodic of period k) with the condition that all sets

are pairwise incomparable (by inclusion).[12] In other words, given the process has reached a state within such a class, the transition to the next state is deterministic. And the period of the class determines after how many steps a state is reached again.

Terminal classes of the third type are called *regular terminal classes*. They are collections $\mathcal{R}$ of nonempty sets with the property that $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_p$, where each subcollection $\mathcal{R}_j$ is an interval $\{S \in 2^N \mid S_j \subseteq S \subseteq S_j \cup K_j\}$, with $S_j \neq \emptyset, S_j \cup K_j \neq N$, and at least one K_j is nonempty.

Example 4 (**Regular terminal class**). *Consider an aggregation model with three agents and aggregation functions $A_1(x) = x_2, A_2(x) = x_1$ and $A_3(x) = (x_1 + x_2)/2$. Then, $\{\{1\},\{1,3\}\} \cup \{\{2\},\{2,3\}\}$ is a regular terminal class. The model gives the following digraph of the Markov chain:*



If such a class only consists of a single interval $\mathcal{R}_1 = \{S \in 2^N \mid S_1 \subseteq S \subseteq S_1 \cup K_1\}$, where $S_1 \neq \emptyset$ and $S_1 \cup K_1 \neq N$, then we can interpret this terminal class as a situation where agents in S_1 finally decided to say "yes" and agents outside $S_1 \cup K_1$ finally decided to say "no", while the agents in K_1 change their opinion non-deterministically forever. With more than one interval, the interpretation is more complex and depends on the transitions between the intervals. Reaching an interval $\mathcal{R}_j$ means that the process attains one of its states, i.e., the agents in S_j say "yes" for sure and with some probability, also some agents in K_j do so.

Our aim is to investigate conditions for these outcomes under anonymous influence. We also relax our setup and study the case where agents use ordered weighted averages only to some extent. Our results turn out to be – due to the restriction to anonymous aggregation functions – inherently different from those in the general model, see Grabisch and Rusinowska (2013). We first consider influential coalitions and discuss (non-trivial) terminal classes. In the following, we derive a characterization of convergence to consensus and finally provide a generalization of our setting.

Due to anonymity, it is not surprising that the influence of a coalition indeed solely depends on the number of individuals involved.

Proposition 2. *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N$.*

(i) *A coalition of size s, where $0 < s \leq n$, is "yes"-influential on $i \in N$ if and only if $\min\{k \in N \mid w^i_k > 0\} \leq s$.*

[12] Sets $S_1, \ldots, S_k \subseteq N$ are called pairwise incomparable (by inclusion) if for any distinct $S_i, S_j, i, j \in \{1, \ldots, k\}$, both $S_i \not\subseteq S_j$ and $S_i \not\supseteq S_j$.

(ii) *A coalition of size s, where $0 < s \leq n$, is "no"-influential on $i \in N$ if and only if* $\max\{k \in N \mid w_k^i > 0\} \geq n + 1 - s$.

Proof. Let $S \subseteq N$ have size $0 < s \leq n$ and be "yes"-influential on $i \in N$, i.e.,

$$A_i(1_S) = \sum_{k=1}^{s} w_k^i > 0 \Leftrightarrow \min\{k \in N \mid w_k^i > 0\} \leq s.$$

The second part is analogous. □

The result on influential agents follows immediately.

Corollary 2. *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N$. Then, all agents $j \in N$ are "yes"-("no"-)influential on $i \in N$ if and only if $w_1^i > 0$ ($w_n^i > 0$).*

Note that this means that either all agents are "yes"-(or "no"-)influential on some agent $i \in N$ or none. Next, we study non-trivial terminal classes. We characterize terminal states, i.e., states where the society is polarized (except for the trivial terminal states), and show that – due to anonymity – there cannot be a cycle.

Proposition 3. *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N$.*

(i) *A state $S \subseteq N$ of size s is a terminal state if and only if $\sum_{k=1}^{s} w_k^i = 1$ for all $i \in S$ and $\sum_{k=1}^{s} w_k^i = 0$ otherwise.*
(ii) *There does not exist any cycle.*

Proof. The first part is obvious. For the second part, assume that there is a cycle $\{S_1, \ldots, S_k\}$ of length $2 \leq k \leq \binom{n}{\lfloor n/2 \rfloor}$. This implies that there exists $l \in \{1, \ldots, k\}$ such that $s_l \leq s_{l+1}$, where $S_{k+1} \equiv S_1$. Thus,

$$\sum_{j=1}^{s_l} w_j^i = 1 \text{ for all } i \in S_{l+1}$$

and hence $S_{l+1} \subseteq S_{l+2}$, which is a contradiction to pairwise incomparability by inclusion, see Grabisch and Rusinowska (2013, Theorem 2). □

For regular terminal classes, note that an agent $i \in N$ such that $w_1^i = 1$ blocks a "no"-consensus and an agent $j \in N$ such that $w_n^j = 1$ blocks a "yes"-consensus – given that the process has not yet arrived at a consensus. Therefore, since there cannot be any cycle, these two conditions, while ensuring that there is no other terminal state, give us a regular terminal class with anonymous aggregation functions.

Example 5 (**Anonymous regular terminal class**). *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N = \{1, 2, 3\}$. Let agent 1 block a "no"-consensus and agent 3 block a "yes"-consensus, i.e., $w_1^1 = w_3^3 = 1$. Furthermore, choose $w_1^2 = w_3^2 = \frac{1}{2}$. Then, $\{\{1\}, \{1, 2\}\}$ is a regular terminal class. We have $\mathbf{A}(1_{\{1\}}) = \mathbf{A}(1_{\{1,2\}}) = (1 \ \frac{1}{2} \ 0)^T$.*

It is left to find conditions that avoid both non-trivial terminal states and regular terminal classes and hence ensure that the society ends up in a consensus. The following result characterizes the non-existence of non-trivial terminal classes. The idea is that – due to anonymity – for reaching a consensus, there must be some threshold such that whenever the size of the coalition is at least equal to this threshold, there is some probability that after mutual influence, more agents will say "yes". And whenever the size is below this threshold, there is some probability that after mutual influence, more agents will say "no".

Theorem 1. *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N$. Then, there are no other terminal classes than the trivial terminal classes if and only if there exists $\bar{k} \in \{1, \ldots, n\}$ such that both:*

(i) *For all $k = \bar{k}, \ldots, n-1$, there are distinct agents $i_1, \ldots, i_{k+1} \in N$ such that*

$$\sum_{j=1}^{k} w_j^{i_l} > 0 \text{ for all } l = 1, \ldots, k+1.$$

(ii) *For all $k = 1, \ldots, \bar{k}-1$, there are distinct agents $i_1, \ldots, i_{n-k+1} \in N$ such that*

$$\sum_{j=1}^{k} w_j^{i_l} < 1 \text{ for all } l = 1, \ldots, n-k+1.$$

The proof is in the appendix. Note that Theorem 1 implies a straightforward – but very strict – sufficient condition:

Remark 1. *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N$. Then, there are no other terminal classes than the trivial terminal classes if $w_1^i > 0$ for all $i \in N$ ($\bar{k} = 1$), or $w_n^i > 0$ for all $i \in N$ ($\bar{k} = n$).*

We get a more intuitive formulation of Theorem 1 by using influential coalitions.

Corollary 3. *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N$. Then, there are no other terminal classes than the trivial terminal classes if and only if there exists $\bar{k} \in \{1, \ldots, n\}$ such that both:*

(i) *For all $k = \bar{k}, \ldots, n-1$, there are $k+1$ distinct agents such that coalitions of size k are "yes"-influential on each of them.*
(ii) *For all $k = 1, \ldots, \bar{k}-1$, there are $n-k+1$ distinct agents such that coalitions of size $n-k$ are "no"-influential on each of them.*

In more general situations, the agents' behavior might only partially be determined by ordered weighted averages. We consider agents who use aggregation functions that are decomposable in the sense that they are (convex) combinations of ordered weighted averages and general aggregation functions.

Definition 7 (**OWA-decomposable aggregation function**). *We say that an n-place aggregation function A is* OWA_w-decomposable*, if there exists $\lambda \in (0,1]$ and an n-place aggregation function A' such that $A = \lambda\mathsf{OWA}_w + (1-\lambda)A'$.*

Such aggregation functions do exist since convex combinations of aggregation functions are again aggregation functions. Note that these functions are, in general, not anonymous any more, though. However, the mass psychology influence model presented in Section 3 – to which we will come back later on – is an example of an anonymous model that uses these decomposable aggregation functions. To provide some intuition for why these functions are useful, let us consider the class where ordered weighted averages are combined with weighted averages.[13]

Example 6 (**OWA-/WA-decomposable aggregation functions**). *Consider a convex combination of an ordered weighted average and a weighted average,*

$$A = \lambda\mathsf{OWA}_w + (1-\lambda)\mathsf{WA}_{w'},$$

where $\lambda \in (0,1)$ and w, w' are any weight vectors. This allows us to somehow combine our model with the classical model by DeGroot.[14] *We can interpret this as follows: to some extent λ, an agent updates her opinion anonymously to account, e.g., for majorities within her social group. But she might as well value her own opinion somehow – like in the mass psychology model – or some agents might be really important for her such that she wants to put also some weight directly on them, as we show in Example 8.*

As it turns out, the sufficiency part of Theorem 1 also holds if agents use such decomposable aggregation functions. If the ordered weighted average components of the decomposable functions fulfill the two conditions of Theorem 1, then the agents reach a consensus.[15]

Corollary 4. *Consider an aggregation model with OWA_{w^i}-decomposable aggregation functions $A_i, i \in N$. Then, there are no other terminal classes than the trivial terminal classes if there exists $\bar{k} \in \{1,\dots,n\}$ such that both:*

(i) *For all $k = \bar{k},\dots,n-1$, there are distinct agents $i_1,\dots,i_{k+1} \in N$ such that*

$$\sum_{j=1}^{k} w_j^{i_l} > 0 \text{ for all } l = 1,\dots,k+1.$$

(ii) *For all $k = 1,\dots,\bar{k}-1$, there are distinct agents $i_1,\dots,i_{n-k+1} \in N$ such that*

$$\sum_{j=1}^{k} w_j^{i_l} < 1 \text{ for all } l = 1,\dots,n-k+1.$$

[13] We say that an n-place aggregation function A is a *weighted average* $A = \mathsf{WA}_w$ with weight vector w, i.e., $0 \le w_i \le 1$ for $i = 1,\dots,n$ and $\sum_{i=1}^n w_i = 1$, if $A(x) = \sum_{i=1}^n w_i x_i$ for all $x \in \{0,1\}^n$.

[14] With the restriction that, differently to the DeGroot model, opinions are in {0,1}.

[15] It is clear that, in general, the necessity part does not hold since convergence to consensus may as well be (partly) ensured by the other component.

Let us finally apply the concept of decomposable aggregation functions to more specific examples. As it turns out, the example on mass psychology combines the majority influence model and a completely self-centered agent.

Example 7 (**Mass psychology, continued**). *We have seen in Example 3 that for parameters $n = 3$, $m = 2$ and $\lambda \in (0, 1)$, we get the following* mass psychology *aggregation model:*

$$\mathsf{Mass}_i^{[2]}(x) = \lambda x_{(2)} + (1 - \lambda)x_i \text{ for all } i \in N.$$

This aggregation function is OWA_w-decomposable with $w_2 = 1$ and by Corollary 4, taking $\bar{k} = 2$, we see that the group eventually reaches a consensus. This example is a particular case of Example 6 and furthermore, it is equivalent to a convex combination of the majority influence model and a completely self-centered agent:

$$\mathsf{Mass}_i^{[2]}(x) = \lambda \mathsf{Maj}_i^{[2]}(x) + (1 - \lambda)x_i \text{ for all } i \in N.$$

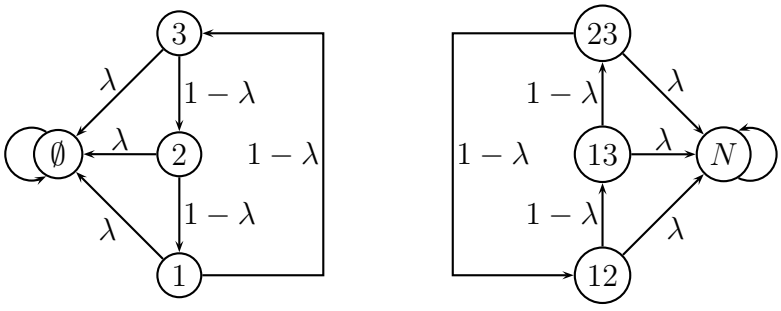
Hence, λ could be interpreted as a measure for how "democratically" – or, to put it the other way, "egoistically" – an agent behaves.

Finally, we study an example where agents use the majority influence model, but also put some weight directly on agents that are important for them. We study a case that turns out to be as well anonymous and furthermore, it is in some sense equivalent to the example on mass psychology.

Example 8 (**Important agents**). *Although agents might follow somehow a majority influence model, there might still be some important agents, e.g., very good friends or agents with an excellent reputation, whom they would like to trust directly as well. In particular, we consider $n = 3$ agents and that each agent follows to some extent $\lambda \in (0, 1)$ the simple majority model. Moreover, for each agent, the agent with the next higher index has a relative importance of $1 - \lambda$ for her.*[16] *This corresponds to the following* important agents *aggregation model:*

$$\mathsf{Imp}_i^{[2;i+1]}(x) = \lambda \mathsf{Maj}_i^{[2]}(x) + (1 - \lambda)x_{i+1} \text{ for all } i \in N.$$

Agent $i + 1$ is "yes"- and "no"-influential on agent i for all $i \in N$ and coalitions of size two or more are "yes"- and "no"-influential on all agents. The model gives the following digraph of the Markov chain:



[16] We consider $4 \equiv 1$.

From the digraph, we can see that the model is anonymous although the aggregation functions are not.[17] *Furthermore, when abstracting from the identity of the agents, i.e., considering only the size of a state, this digraph is identical to the one of the mass psychology example. Therefore, we can say that the two models are* anonymously equivalent*: starting in a state of size one or two, both models stay within the set of states of the same size with probability* $1-\lambda$ *and converge to the "no"- or "yes"-consensus, respectively, with probability* λ*.*

We also analyze the *speed of convergence* to terminal classes as well as the *probabilities of convergence* to different classes in the general model studied by Grabisch and Rusinowska (2013) and in our case of anonymous models. For the latter, we find that the computational demand reduces a lot compared to the general case. For details on this we refer to the working paper (Förster et al. (2012)).

5. Applications to fuzzy linguistic quantifiers

Instead of being sharp edged, e.g., as in the majority model, the threshold of an agent initially saying "no" for changing her opinion might be rather "soft". For instance, she could change her opinion if "*most* of the agents say 'yes' ". This is called a *soft majority* and phrases like "most" or "many" are so-called *fuzzy linguistic quantifiers*. Furthermore, *soft minorities* are also possible, e.g., "*at least a few* of the agents say 'yes' ". Our aim is to apply our findings on ordered weighted averages to fuzzy linguistic quantifiers. Mathematically, we define the latter by a function which maps the agents' proportion that says "yes" to the degree to which the quantifier is satisfied, see Zadeh (1983).

Definition 8 (**Fuzzy linguistic quantifier**)**.** *A* fuzzy linguistic quantifier $\mathcal{Q}$ *is defined by a nondecreasing function*

$$\mu_{\mathcal{Q}} : [0,1] \to [0,1] \text{ such that } \mu_{\mathcal{Q}}(0) = 0 \text{ and } \mu_{\mathcal{Q}}(1) = 1.$$

Furthermore, we say that the quantifier is regular *if the function is strictly increasing on some interval* $(\underline{c}, \bar{c}) \subseteq [0,1]$ *and otherwise constant.*

Fuzzy linguistic quantifiers like "most" are ambiguous in the sense that it is not clear how to define them exactly mathematically. For example, one could well discuss which proportion of the agents should say "yes" for the quantifier "most" to be fully satisfied. Nevertheless, let us give some typical examples, see Yager and Kacprzyk (1997).

Example 9 (**Typical quantifiers**)**.** *We define*

(i) $\mathcal{Q}_{aa}$ = *"almost all" by*

$$\mu_{\mathcal{Q}_{aa}}(x) := \begin{cases} 1, & \text{if } x \geq \frac{9}{10} \\ \frac{5}{2}x - \frac{5}{4}, & \text{if } \frac{1}{2} < x < \frac{9}{10} \\ 0, & \text{otherwise} \end{cases},$$

[17] Note that this is a consequence of our choice of important agents. For most choices, the model would not be anonymous, e.g., if two agents would be important for each other and one of them would as well be important for the third one.

(ii) $\mathcal{Q}_{mo}$ = *"most" by*

$$\mu_{\mathcal{Q}_{mo}}(x) := \begin{cases} 1, & \text{if } x \geq \frac{4}{5} \\ \frac{5}{2}x - 1, & \text{if } \frac{2}{5} < x < \frac{4}{5} \\ 0, & \text{otherwise} \end{cases},$$

(iii) $\mathcal{Q}_{ma}$ = *"many" by*

$$\mu_{\mathcal{Q}_{ma}}(x) := \begin{cases} 1, & \text{if } x \geq \frac{3}{5} \\ \frac{5}{2}x - \frac{1}{2}, & \text{if } \frac{1}{5} < x < \frac{3}{5} \\ 0, & \text{otherwise} \end{cases},$$

(iv) $\mathcal{Q}_{af}$ = *"at least a few" by*

$$\mu_{\mathcal{Q}_{af}}(x) := \begin{cases} 1, & \text{if } x \geq \frac{3}{10} \\ \frac{10}{3}x, & \text{otherwise} \end{cases}.$$

Note that these quantifiers are regular. For every quantifier, there exists a corresponding ordered weighted average in the sense that the latter represents the quantifier.[18] We can find its weights as follows.

Lemma 1 (Yager, 1988)**.** *Let $\mathcal{Q}$ be a fuzzy linguistic quantifier defined by $\mu_{\mathcal{Q}}$. Then, the weights of its corresponding ordered weighted average $\mathsf{OWA}_{\mathcal{Q}}$ are given by*

$$w_k = \mu_{\mathcal{Q}}\left(\frac{k}{n}\right) - \mu_{\mathcal{Q}}\left(\frac{k-1}{n}\right), \text{ for } k = 1, \ldots, n.$$

In other words, the weights w_k of the corresponding ordered weighted average are equal to the increase of $\mu_{\mathcal{Q}}$ between $(k-1)/n$ and k/n, i.e., since $\mu_{\mathcal{Q}}$ is nondecreasing, all weights are nonnegative and by the boundary conditions, it is ensured that they sum up to one. We are now in the position to apply our results to regular quantifiers. We find that if all agents use such a quantifier, then under some similarity condition, the group will finally reach a consensus. This condition says that there must be a common point where all the fuzzy quantifiers are strictly increasing. This implies that there is a common non-zero weight of the corresponding OWA operators, which turns out to be sufficient to satisfy the condition of Theorem 1. Moreover, we show that the result still holds if some agents deviate to a quantifier that is not similar in that sense. In the following, we denote the quantifier of an agent i by $\mathcal{Q}^i$.

Proposition 4. *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{\mathcal{Q}^i}, i \in N$.*

(i) *If $\mathcal{Q}^i$ is regular for all $i \in N$ and $\cap_{i \in N}(\underline{c}_i, \bar{c}_i) \neq \emptyset$, then there are no other terminal classes than the trivial terminal classes.*

[18] Note that this is due to our definition. The conditions in Definition 8 ensure that there exists such an ordered weighted average. In general, one can define quantifiers also by other functions, cf. Zadeh (1983).

(ii) *Suppose* $\min_{i\in N} \underline{c}_i > 0$*, then the result in* (i) *still holds if less than* $\lceil \bar{c}_d n \rceil$ *agents deviate to a regular quantifier* $\mathcal{Q}_d$ *such that* $\bar{c}_d < \min_{i\in N} \underline{c}_i$*.*

(iii) *Suppose* $\max_{i\in N} \bar{c}_i < 1$*, then the result in* (i) *still holds if less than* $\lceil (1 - \underline{c}_d)n \rceil$ *agents deviate to a regular quantifier* $\mathcal{Q}_d$ *such that* $\max_{i\in N} \bar{c}_i < \underline{c}_d$*.*

The proof is in the appendix. Note that this result can be generalized such that the deviating agents might also use different quantifiers. We can also characterize terminal states in a model where agents use regular quantifiers. We find that S is a terminal state if and only if the quantifiers of the agents in S are already fully satisfied at s/n, while the quantifiers of the other agents are not satisfied at all at this point.

Proposition 5. *Consider an aggregation model with aggregation functions* $A_i = \mathsf{OWA}_{\mathcal{Q}^i}, i \in N$*. If* $\mathcal{Q}^i$ *is regular for all* $i \in N$*, then a state* $S \subseteq N$ *of size* s *is a terminal state if and only if*

$$\max_{i\in S} \bar{c}_i \leq \frac{s}{n} \leq \min_{i\in N\setminus S} \underline{c}_i.$$

Proof. Suppose $S \subseteq N$ of size s is a terminal state. By Proposition 3, we know that this is equivalent to

$$\begin{aligned}
&\sum_{k=1}^{s} w_k^i = 1 \text{ for all } i \in S \text{ and } \sum_{k=1}^{s} w_k^i = 0 \text{ otherwise} \\
\Leftrightarrow &\mu_{\mathcal{Q}^i}(s/n) = 1 \text{ for all } i \in S \text{ and } \mu_{\mathcal{Q}^i}(s/n) = 0 \text{ otherwise} \\
\Leftrightarrow &\max_{i\in S} \bar{c}_i \leq \frac{s}{n} \leq \min_{i\in N\setminus S} \underline{c}_i.
\end{aligned}$$

□

To provide some intuition, let us come back to Example 9 and look at the implications our findings have on the quantifiers defined therein.

Example 10 (**Typical quantifiers, continued**)**.** *Consider an aggregation model with aggregation functions* $A_i = \mathsf{OWA}_{\mathcal{Q}^i}, i \in N$*.*

(i) *If* $\mathcal{Q}^i \in \{\mathcal{Q}_{aa}, \mathcal{Q}_{mo}, \mathcal{Q}_{ma}\}$ *for all* $i \in N$*, then there are no other terminal classes than the trivial terminal classes. The result still holds if less than* $\lceil \frac{3}{10} n \rceil$ *agents deviate to* $\mathcal{Q}_{af}$*.*

(ii) *If* $\mathcal{Q}^i \in \{\mathcal{Q}_{ma}, \mathcal{Q}_{af}\}$ *for all* $i \in N$*, then there are no other terminal classes than the trivial terminal classes. The result still holds if less than* $\lceil \frac{1}{2} n \rceil$ *agents deviate, each of them either to* $\mathcal{Q}_{aa}$ *or* $\mathcal{Q}_{mo}$*.*

(iii) *A state* $S \subseteq N$ *of size* s *is a terminal state if* $\mathcal{Q}^i = \mathcal{Q}_{af}$ *for all* $i \in S$*,* $\mathcal{Q}^i = \mathcal{Q}_{aa}$ *(*$\mathcal{Q}^i \in \{\mathcal{Q}_{aa}, \mathcal{Q}_{mo}\}$*) otherwise and* $\frac{3}{10} \leq \frac{s}{n} \leq \frac{1}{2}$ $(\leq \frac{2}{5})$*.*

6. Conclusion

We study a stochastic model of influence where agents aggregate opinions using *OWA operators*, which are the only *anonymous* aggregation functions. As one would expect, an aggregation model is *anonymous* if all agents use these functions. However, our example on *mass psychology* shows that a model can be anonymous although agents do not use anonymous functions.

In the main part of the paper, we characterize *influential coalitions*, show that *cyclic terminal classes* cannot exist due to anonymity and characterize *terminal states*. Our main result provides a necessary and sufficient condition for *convergence to consensus*. It turns out that we can express this condition in terms of influential coalitions. Due to our restriction to anonymous functions, these results are inherently different to those obtained in the general case by Grabisch and Rusinowska (2013). We also extend our model to *decomposable* aggregation functions. In particular, this allows to combine OWA operators with the classical approach of ordinary weighted averages. This class of decomposed functions comprises our example on mass psychology: it is equivalent to a convex combination of the majority influence model and a completely self-centered agent. We also study an example on *important agents* and show that in some cases, this model is anonymous as well and, additionally, *anonymously equivalent* to the example on mass psychology. Moreover, it turns out that our previous condition on convergence to consensus is still sufficient in this generalized setting.

Furthermore, we apply our results to *fuzzy linguistic quantifiers* and show that if agents use in some sense similar quantifiers and not too many agents deviate from these quantifiers, the society will eventually reach a consensus.

These results rely on the fact that for each quantifier, we can find a unique corresponding ordered weighted average (Lemma 1), which allows to apply our results on OWA operators. Note that these corresponding ordered weighted averages clearly depend on the number of agents in the society. Therefore, we can see a quantifier as well as a more general definition of an OWA operator (usually called an *extended OWA operator*; see Grabisch et al., 2009), which does not anymore require a fixed number of agents. In other words, assigning to each agent such an extended OWA operator allows to vary the number of agents n in the society.

Appendix A.

Appendix A.1. Proof of Theorem 1

First, suppose that there exists $\bar{k} \in \{1, \dots, n\}$ such that (i) and (ii) hold. Let us take any coalition $S \subsetneq N$ of size $s \geq \bar{k}$ and show that it is possible to reach the "yes"-consensus, which implies that S is not part of a terminal class. By choice of S, it is sufficient to show that there is a positive probability that after mutual influence, the size of the coalition has strictly increased. That is, it is sufficient to show that there exists a coalition $S' \subseteq N$ of size $s' > s$, such that $A_i(1_S) > 0$ for all agents $i \in S'$. Set $k := s$, then by condition (i), there are distinct agents $i_1, \dots, i_{k+1} \in N$ such that

$$A_{i_l}(1_S) = \sum_{j=1}^{k} w_j^{i_l} > 0 \text{ for all } l = 1, \dots, k+1,$$

i.e., setting $S' := \{i_1, \ldots, i_{k+1}\}$ finishes this part. Analogously, we can show by condition (ii) that for any nonempty $S \subseteq N$ of size $s < \bar{k}$ it is possible to reach the "no"-consensus. Hence, there are only the trivial terminal classes.

Now, suppose to the contrary that for all $\bar{k} \in \{1, \ldots, n\}$ either (i) or (ii) does not hold. Note that in order to establish that there exists a non-trivial terminal class, it is sufficient to show that there are $k_*, k^* \in \{1, \ldots, n-1\}, k_* \leq k^*$, such that for all $S \subseteq N$ of size $s = k_*$,

$$A_i(1_S) < 1 \text{ for at most } n - k_* \text{ distinct agents } i \in N \qquad (C_*[k_*])$$

and for all $S \subseteq N$ of size $s = k^*$,

$$A_i(1_S) > 0 \text{ for at most } k^* \text{ distinct agents } i \in N. \qquad (C^*[k^*])$$

Indeed, condition $C_*[k_*]$ says that it is not possible to reach a coalition with less than k_* agents starting from a coalition with at least k_* agents. Similarly, condition $C^*[k^*]$ says that it is not possible to reach a coalition with more than k^* agents starting from a coalition with at most k^* agents.[19] Therefore, it is not possible to reach the trivial terminal states from any coalition S of size $k_* \leq s \leq k^*$, which proves the existence of a non-trivial terminal class.

Let now $\bar{k} = 1$. Then, clearly condition (ii) is satisfied and thus condition (i) cannot be satisfied by assumption. Hence, there exists $k^* \in \{1, \ldots, n-1\}$ such that there are at most k^* distinct agents $i_1, \ldots, i_{k^*}$ such that

$$\sum_{j=1}^{k^*} w_j^{i_l} > 0 \text{ for } l = 1, \ldots, k^*.$$

This implies that condition (i) is not satisfied for $\bar{k} = 1, \ldots, k^*$. If $k^* \geq 2$ and additionally condition (ii) was not satisfied for some $\bar{k} \in \{2, \ldots, k^*\}$, we were done since then there would exist $k_* \in \{1, \ldots, k^* - 1\}$ such that there are at most $n - k_*$ distinct agents $i_1, \ldots, i_{n-k_*}$ such that

$$\sum_{j=1}^{k_*} w_j^{i_l} < 1 \text{ for } l = 1, \ldots, n - k_*,$$

i.e., $(C_*[k_*])$ and $(C^*[k^*])$ were satisfied for $k_* \leq k^*$. Therefore, suppose w.l.o.g. that condition (ii) is satisfied for all $\bar{k} = 1, \ldots, k^*$. $(*)$

For $\bar{k} = n$, clearly condition (i) is satisfied and thus condition (ii) cannot be satisfied. Hence, using $(*)$, there exists $k_* \in \{k^*, \ldots, n-1\}$ such that there are at most $n - k_*$ distinct agents $i_1, \ldots, i_{n-k_*}$ such that

$$\sum_{j=1}^{k_*} w_j^{i_l} < 1 \text{ for } l = 1, \ldots, n - k_*,$$

i.e., $(C_*[k_*])$ and $(C^*[k^*])$ are satisfied. We now proceed by case distinction:

[19] Note that monotonicity of the aggregation function implies that $(C_*[k_*])$ also holds if we replace S by a coalition $S' \subseteq N$ of size $s' > k_*$. Analogously for $(C^*[k^*])$.

(1) If $k_* = k^*$, then we are done.
(2) If $k_* > k^*$, then let $\bar{k} = k_*$. By assumption, either (i) or (ii) does not hold.
   (2.1) If (i) does not hold, then there exists $k^{**} \in \{k_*, \dots, n-1\}$ such that there are at most k^{**} distinct agents $i_1, \dots, i_{k^{**}}$ such that

$$\sum_{j=1}^{k^{**}} w_j^{i_l} > 0 \text{ for } l = 1, \dots, k^{**},$$

   i.e. $(C_*[k_*])$ and $(C^*[k^{**}])$ are satisfied for $k_* \leq k^{**}$ and hence we are done.
   (2.2) If (ii) does not hold, then, using $(*)$, there exists $k_{**} \in \{k^*, \dots, k_* - 1\}$ such that there are at most $n - k_{**}$ distinct agents $i_1, \dots, i_{n-k_{**}}$ such that

$$\sum_{j=1}^{k_{**}} w_j^{i_l} < 1 \text{ for } l = 1, \dots, n - k_{**},$$

   i.e., $(C_*[k_{**}])$ is satisfied. If $k_{**} = k^*$, then we are done, otherwise we can repeat this procedure using k_{**} instead of k_*.

Since $k_{**} \lneq k_*$, we find $k_{**} = k^*$ after a finite number of repetitions, which finishes the proof.

Appendix A.2. Proof of Proposition 4

(i) By assumption, there exists $c \in \cap_{i \in N}(\underline{c}_i, \bar{c}_i)$. Let us define $\bar{k} := \min\{k \in \mathbb{N} \mid \frac{k}{n} > c\}$, then clearly $\frac{\bar{k}-1}{n} \leq c$. We show that conditions (i) and (ii) of Theorem 1 are satisfied for $\bar{k}$. Since for all $i \in N$, $\mu_{\mathcal{Q}^i}$ is nondecreasing and, in particular, strictly increasing on the open ball $B_\epsilon(c)$ around c for some $\epsilon > 0$, we get by Lemma 1 that

$$w_{\bar{k}}^i = \mu_{\mathcal{Q}}\left(\frac{\bar{k}}{n}\right) - \mu_{\mathcal{Q}}\left(\frac{\bar{k}-1}{n}\right) \geq \mu_{\mathcal{Q}}\left(\frac{\bar{k}}{n}\right) - \mu_{\mathcal{Q}}(c) > 0 \text{ for all } i \in N.$$

This implies that for all $k = \bar{k}, \dots, n-1$,

$$\sum_{j=1}^{k} w_j^i \geq w_{\bar{k}}^i > 0 \text{ for all } i \in N$$

and for all $k = 1, \dots, \bar{k} - 1$,

$$\sum_{j=1}^{k} w_j^i \leq \sum_{j \neq \bar{k}} w_j^i = 1 - w_{\bar{k}}^i < 1 \text{ for all } i \in N,$$

i.e., (i) and (ii) of Theorem 1 are satisfied for $\bar{k}$, which finishes the first part.

(ii) Suppose $\min_{i\in N} \underline{c}_i > 0$ and denote by $D \subseteq N$ the set of agents that deviate to the quantifier $\mathcal{Q}_d$. Similar to the first part, there exists $c \in \cap_{i\in N\setminus D}(\underline{c}_i, \bar{c}_i)$ and we can define $\bar{k} := \min\{k \in \mathbb{N} \mid \frac{k}{n} > c\}$. This implies that for all $k = \bar{k}, \dots, n-1$,

$$\sum_{j=1}^{k} w_j^i > 0 \text{ for all } i \in N\setminus D \qquad (*)$$

and for all $k = 1, \dots, \bar{k} - 1$,

$$\sum_{j=1}^{k} w_j^i < 1 \text{ for all } i \in N\setminus D. \qquad (**)$$

Furthermore, we have by assumption $\mu_{\mathcal{Q}_d}(\bar{k}/n) = 1$, which implies $w_j^i = 0$ for all $j = \bar{k}+1, \dots, n$ and $i \in D$. Thus, for all $k = \bar{k}, \dots, n-1$

$$\sum_{j=1}^{k} w_j^i = \sum_{j=1}^{\bar{k}} w_j^i = 1 > 0 \text{ for all } i \in D,$$

i.e., in combination with $(*)$, condition (i) of Theorem 1 is satisfied for $\bar{k}$. It is left to check condition (ii). Define for $i \in D$,

$$\tilde{k} := \max\{k \in \mathbb{N} \mid w_k^i > 0\} = \min\{k \in \mathbb{N} \mid k/n \geq \bar{c}_d\} \leq \bar{k}.$$

Hence, for $k = 1, \dots, \tilde{k} - 1$,

$$\sum_{j=1}^{k} w_j^i < 1 \text{ for all } i \in D.$$

If $\tilde{k} = \bar{k}$, condition (ii) is – in combination with $(**)$ – satisfied for $\bar{k}$ and any $D \subseteq N$. Otherwise, we have $\tilde{k} < \bar{k}$ and then, for $k = \tilde{k}, \dots, \bar{k} - 1$,

$$\sum_{j=1}^{k} w_j^i = 1 \text{ for all } i \in D.$$

This implies in combination with $(**)$ that condition (ii) is only satisfied if $\max_{k=\tilde{k},\dots,\bar{k}-1}(n-k+1) = n - \tilde{k} + 1$ agents do not deviate, i.e.,

$$|D| \leq n - (n - \tilde{k} + 1) = \tilde{k} - 1 \;\Leftrightarrow\; |D| \lneq \tilde{k} \;\Leftrightarrow\; |D| \lneq \lceil \bar{c}_d n \rceil.$$

Thus, (i) and (ii) of Theorem 1 are satisfied for $\bar{k}$ if $|D| \lneq \lceil \bar{c}_d n \rceil$, which finishes the proof.

(iii) Analogous to the second part.

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