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High-Frequency Risk Measures*

Denisa Banulescu[†], Gilbert Colletaz[‡], Christophe Hurlin[§] and Sessi Tokpavi[¶]

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Abstract

This paper proposes intraday High Frequency Risk (HFR) measures for market risk in the case of irregularly spaced high-frequency data. In this context, we distinguish three concepts of value-at-risk (VaR): the total VaR, the marginal (or per-time-unit) VaR, and the instantaneous VaR. Since the market risk is obviously related to the duration between two consecutive trades, these measures are completed with a duration risk measure, i.e., the time-at-risk (TaR). We propose a forecasting procedure for VaR and TaR for each trade or other market microstructure event. We perform a backtesting procedure specifically designed to assess the validity of the VaR and TaR forecasts on irregularly spaced data. The performance of the HFR measure is illustrated in an empirical application for two stocks (Bank of America and Microsoft) and an exchange-traded fund (ETF) based on Standard and Poor's (the S&P) 500 index. We show that the intraday HFR forecasts accurately capture the volatility and duration dynamics for these three assets.

Keywords: High Frequency Risk Measure, Value at Risk, Time at Risk, Backtesting.

J.E.L Classification : C22, C52, G28

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1 Introduction

This paper proposes intraday High Frequency Risk (HFR) measures for market risk in the case of irregularly spaced high-frequency data. We distinguish three concepts of Value-at-Risk (VaR): the *total* VaR, the *marginal* (or *per-time-unit*) VaR and the *instantaneous* VaR. The total VaR corresponds to the maximum expected loss for the time horizon of the next trade, whereas the marginal VaR refers to the time horizon of the next time-unit, which is generally defined in seconds. The third concept corresponds to the maximum expected loss for the time interval between t_i , and $t_i + \Delta$, for an infinitesimal time increase Δ , where t_i denotes the time of the last trade.

When considering tick-by-tick data, the market risk is obviously related to the duration between two consecutive trades (Diamond and Verrecchia, 1987; Easley and O'Hara, 1992). As a consequence, our market risk measures are completed with a duration risk measure. Here, we consider the Time-at-Risk (TaR), initially introduced by Ghysels, Gouriéroux and Jasiak (2004), which is defined as the minimum duration prior to the next trade with a given confidence level.

To our knowledge, only two papers derive intraday market risk models using tick-by-tick data. Giot (2005) relies on two approaches.¹ First, he proposes to re-sample the data along a pre-specified time grid, which yields temporally equidistant observations (10 or 30 minutes). Then, standard time-series models (RiskMetrics- or GARCH-type models) are used to forecast the conditional volatility and the VaR for equidistantly time-spaced returns. But, this approach neglects the irregular timing of trades. Second, Giot derives the returns volatility (and thus the VaR) from the conditional intensity process associated with the price duration, defined as the time necessary for a price of an asset to change by a given amount. Following Engle and Russell (1998), he considers an autoregressive conditional duration (ACD)-type model to describe the dynamics of duration and compute the irregularly spaced VaR for price events. Then, this VaR is rescaled at fixed-time intervals for backtesting purposes. More recently, Dionne, Duchesne and Pacurar (2009) propose an intraday VaR (IVaR), which is based on a UHF-GARCH-type model (Engle, 2000) and a Monte Carlo simulation approach, to infer VaR at any fixed-time horizon. However, in both studies the market risk measure is always rescaled at fixed time intervals because of the way it is constructed or for backtesting purposes. Alternatively, we propose to evaluate the market risk at each trade, while taking into account the irregular timing of transactions.

One advantage of our measures is that they can be extended to any type of market microstructure event by considering a subset of the trades with specific characteristics or marks. We can define a

¹We do not mention here the papers that propose intraday VaRs for regularly spaced returns (Coreneo and Veredas, 2012, etc.). Notice that Kozhan and Tham (2013) have recently proposed a measure for the execution risk in high-frequency arbitrage.

VaR and TaR for the transactions associated, for instance, with significant price changes (*i.e.*, price events) or with a minimum volume (*i.e.*, volume events). This feature differentiates our measure from the intraday VaR proposed by Giot (2005) that is only valid for price events.

The VaR and TaR are defined as quantiles of two conditional distributions: the distribution of the intraday returns and the distribution of durations. Since these variables are linked, many approaches can be used to model their joint dynamic (Engle, 2000; Engle and Russell, 1997, 1998; Ghysels and Jasiak, 1998; Gerhard and Hautsch, 2002; Grammig and Wellner, 2002; Meddahi, Renault and Werker, 2005). In this context, the definition of the conditioning information set and the exogeneity assumptions made for both processes are important. When forecasting the volatility (and thus the VaR) for the next trade, which is indexed by $i+1$, two solutions can be adopted. The first one assumes that the past prices and the other marks are known until the i^{th} trade (Ghysels and Jasiak, 1998). The second solution assumes that, in addition to this information, the duration between the i^{th} and the $i^{th} + 1$ trades is also known (Engle, 2000; Meddahi, Renault and Werker, 2005). This difference is important for risk management; the market risk is generally evaluated after each transaction for the time horizon of the future transaction and not one millisecond before its realization. Besides, the duration risk, as measured by the TaR, is not relevant if we assume that the duration before the next trade is known. Therefore, we consider only the information available at the current trade to compute our HFR forecasts, which is one of the primary differences of our approach from the IVaR of Dionne, Duchesne and Pacurar (2009).

We present a simple forecasting algorithm for the HFR measure based on an EACD model for the conditional duration process and a time-varying GARCH model (Ghysels and Jasiak, 1998) for the conditional volatility. The two models are estimated by QML. To compute then the intraday TaR and VaR, we propose a semi-parametric approach similar to that considered by Engle and Manzanelli (2001) in the day-to-day VaR perspective. No specific assumptions (except those required by the QML estimation method) are made regarding the conditional distributions of durations and returns. This forecasting algorithm is evaluated with a backtesting procedure specifically designed to assess the validity of the VaR and TaR forecasts obtained for each trade. In contrast to the previous studies, we do not rescale the VaR forecasts to fixed-time intervals to apply the typical backtesting procedures. The forecasts are evaluated at each transaction up to the time horizon of the next transaction, as it is generally done in the context of the high frequency risk management. We use three backtests compatible with irregularly spaced data: the LR test of Christoffersen (1998) based on a Markov-chain model, the duration-based test of Christoffersen and Pelletier (2004) and the GMM duration-based test proposed by Candelon et al. (2011).

In an empirical application, we consider three financial assets: Bank of America and Microsoft stocks and an exchange-traded fund (ETF) that tracks the Standard and Poor's (S&P) 500 index. The use of an ETF is justified by the increasing importance of these assets in the fund management industry.² For each asset, we compute a sequence of one-step-ahead forecasts for 1%-VaR and 1%-TaR based on the trade or price events recorded in September 2010. In all cases, the VaR and TaR forecasts accurately capture the volatility and duration dynamics. The VaR and TaR violations, which are defined as circumstances in which the ex-post tick-by-tick returns (durations) are smaller (higher) than the ex-ante VaR (TaR) forecasts, satisfy the unconditional coverage and independence assumptions (Christoffersen, 1998). The frequency of violations is always statistically not different from the level of risk, *i.e.*, 1% in our case. More importantly, these violations are not clustered, which indicates that TaR and VaR forecasts adjust rapidly to the changes observed in past durations and returns. Finally, we show that these backtesting results are valid throughout the day and the week, even if the EACD-GARCH model is not re-estimated. In addition, our findings are also robust to the choice of the split point used to separate the in-sample and out-of-sample periods (Hansen and Timmermann, 2012).

The remainder of the paper is organized as follows. In the first section, we define the HFR measures and introduce the method used to model the dynamics of returns and durations. The second section proposes the forecasting algorithm and the intraday periodicity adjustment. The third section presents the empirical application and the backtesting procedure, whereas the fourth section discusses marginal VaR and instantaneous VaR. The last section concludes and suggests further research.

2 High-frequency Risk Measures

The tick-by-tick data for a given stock are described by two variables: the time of the transactions and a vector of marked point processes. The latter variable contains, for example, the volume, the bid-ask spreads, the price of the contract observed at the time of the transaction. Consider a trade that occurred at time t_i at a log-price p_i , and denote z_i the corresponding vector of marks other than the price. Thus, the duration between two consecutive trades is defined as $x_i = t_i - t_{i-1}$ and the corresponding continuously compounded return is $r_i = p_i - p_{i-1}$. The information set available at time t_{i-1} is denoted by \mathcal{F}_{i-1} and includes all past durations and marked point processes: $\mathcal{F}_{i-1} = \{x_j, p_j, z_j, j \leq i-1\}$.

²At the end of August 2011, 2,982 ETFs worldwide were managing USD 1,348 bn, which represents 5.6% of the assets under the management of the fund management industry. Additionally, the total ETF turnover that occurs on-exchange via the electronic order book was 8.5% of the equity turnover (BlackRock, 2011).

In this framework, several complementary high frequency market risk measures can be defined. The first one is the (*total*) VaR:

Definition 1 For a shortfall probability $\alpha \in]0, 1[$, the (*total*) VaR is defined as:

$$\Pr(r_i < -VaR_i(\alpha) \mid \mathcal{F}_{i-1}) = \alpha, \quad (1)$$

Here, the VaR is defined as the maximum expected loss that will not be exceeded under normal conditions for a given confidence level $1 - \alpha$ at the time horizon of the *next trade*. Notice that VaR is defined in event time, not calendar time. As a consequence, this VaR is different from the intraday VaRs proposed by Giot (2005) and Dionne, Duchesne and Pacurar (2009), which are generally averaged for regularly time-spaced intervals for backtesting purposes.

Besides, the use of tick-by-tick data allows to define two additional VaR concepts: (i) the *per-time-unit* or *marginal* VaR and (ii) the *instantaneous* VaR.

Definition 2 For a shortfall probability $\alpha \in]0, 1[$, the *per-time-unit* α -VaR (or *marginal VaR*) is defined as follows:

$$\Pr\left(\frac{r_i}{x_i} < -VaR_i(\alpha; x_i) \mid \mathcal{F}_{i-1}\right) = \alpha. \quad (2)$$

The *instantaneous* α -VaR is defined for a constant time variation Δ :

$$\lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \Pr(r_{i-1}(\Delta) < -VaR_i(\alpha; \Delta) \mid \mathcal{F}_{i-1}) = \alpha, \quad (3)$$

where $r_{i-1}(\Delta) = p(t_{i-1} + \Delta) - p(t_{i-1})$ denotes the *instantaneous return* measured between t_{i-1} and $t_{i-1} + \Delta$.

The per-time-unit VaR corresponds to the maximum expected loss for the next time-unit, which is generally defined in seconds. The instantaneous VaR corresponds to the maximum expected loss for the time interval between t_{i-1} and $t_{i-1} + \Delta$, for an infinitesimal time increase Δ . These risk measures are particularly interesting in the context of high frequency risk management. They provide an assessment of the market risk scaled by the duration. Indeed, a (total) VaR of 0.2% associated with a transaction that will occur in ten seconds does not present the same risk for the investor as the same VaR associated with a transaction that will occur in ten minutes. On the contrary, the marginal and/or instantaneous VaRs allow to compare such situations.

Another advantage of the tick-by-tick data is that the risk measures can be extended to any type of market microstructure event. The previous definitions were proposed for a trade for which the occurrence time corresponds to the transaction arrival time. Alternatively, these HFR measures

can be considered for a subset of trades with specific characteristics or marks.³ For instance, we can define HFR measures for price events (Engle and Russell, 1997). The price duration represents the time necessary for the price of an asset to change by a given amount C , *i.e.*, $|p_i - p_{i-1}| \geq C$, where i is the index of the most recently selected process among the initial point processes. In this case, x_i and r_i denote the price duration and the log-return defined in price-event time, respectively, and the corresponding VaR is then *conditional* with respect to the condition $|r_i| \geq C$. The interpretation is the following: the conditional VaR is the maximum expected loss that will not be exceeded under normal conditions for a confidence level α at the time horizon of the next trade for which the log-price variation will exceed the threshold C . This concept is similar to the CoVaR (Conditional Value-at-Risk) proposed by Adrian and Brunnermeier (2011) in the context of systemic risk analysis. More generally, the HFR measures can be extended to any type of thinned point process and can be defined in price events, volume events (the time necessary for trading a given number of shares), etc. The only changes required are the interpretation of the HFR measures and the introduction of a conditioning event, denoted \mathbb{C}_i .

$$\Pr(r_i < -VaR_i(\alpha) \mid \mathcal{F}_{i-1}; \mathbb{C}_i) = \alpha, \quad (4)$$

$$\Pr\left(\frac{r_i}{x_i} < -VaR_i(\alpha; x_i) \mid \mathcal{F}_{i-1}; \mathbb{C}_i\right) = \alpha,$$

$$\lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \Pr(r_{i-1}(\Delta) < -VaR_i(\alpha; \Delta) \mid \mathcal{F}_{i-1}; \mathbb{C}_i) = \alpha, \quad (5)$$

where $\mathbb{C}_i = \emptyset$ in the case of trade events, $\mathbb{C}_i = \{|r_i| \geq C\}$ in the case of price events, etc.

For all these market risk measures (except for the instantaneous VaR for which the time increment is fixed), the duration x_i before the next trade or event is unknown given the information set \mathcal{F}_{i-1} . As a consequence, it is useful to define a risk measure for the durations. Here, we consider the TaR defined as the minimum duration before the next trade may occur for a confidence level of $1 - \alpha$ (Ghysels, Gouriéroux and Jasiak, 2004).⁴

Definition 3 *For a shortfall probability $\alpha \in]0, 1[$, the TaR is defined as:*

$$\Pr(x_i > TaR_i(\alpha) \mid \mathcal{F}_{i-1}; \mathbb{C}_i) = \alpha. \quad (6)$$

The TaR is sometimes interpreted as a "liquidity" risk measure since the length of intra-trade durations reveals the speed of activity. However, it does not allow to predict the potential price impact of liquidating a portfolio, which is accounted for instance by the so-called "Actual VaR"

³In the point process literature, retaining only the arrival times that are thought to carry special information is referred to a thinning the point process (Pacurar, 2008).

⁴The risk levels for TaR and VaR can differ. However, in the rest of this study, we will consider an identical level of risk α .

proposed by Giot and Grammig (2006). Besides, liquidity levels and systematic liquidity are priced in asset returns, and liquidity risk can thus be considered as incorporated into prices (market risk). Therefore, we will consider the TaR as a simple indicator of the duration risk faced by high frequency traders and not of the liquidity risk as a whole.

3 Methodology

The aim of the HFR measures is to quantify the risk on durations and the risk on price returns, while considering the irregular timing of trades or other types of market microstructure events. However, these risks are likely to be related. Several theoretical models and empirical studies in the microstructure literature report a link between market activity, which is measured by the time interval between two consecutive transactions, and price dynamics (Diamond and Verrecchia, 1987; Easley and O'Hara, 1992). In this section, we present the models used to describe the joint dynamic of durations (and TaR) and tick-by-tick returns (and VaR).

3.1 TaR and Durations

The TaR, is determined by the dynamics of durations. We assume that the duration dynamics are described by an autoregressive conditional duration (ACD) model (Engle and Russell, 1998). Let $\psi_i = \mathbb{E}(x_i | \mathcal{F}_{i-1})$ denote the expectation of the i^{th} duration conditional on the information set available at time t_{i-1} . We assume the following:

$$x_i = \psi_i v_i, \quad (7)$$

where v_i is an i.i.d. positive-valued process with a pdf denoted by $f_v(\cdot)$ such that $\mathbb{E}(v_i) = 1$. Various specifications (e.g., linear, log-linear, switching regime) can be considered here for the dynamics of the conditional duration ψ_i and for the density of v_i : each configuration corresponds to a specific ACD model (for a survey, see Pacurar, 2008 and Bauwens and Hautsch, 2009). In order to illustrate our HFR measures, we consider here the simplest and highly successful member of this family, i.e., the EACD(p, q) model, where E accounts for the exponential distribution of the errors:

$$\psi_i = w + \sum_{j=1}^p \alpha_j x_{i-j} + \sum_{j=1}^q \beta_j \psi_{i-j}, \quad (8)$$

$$f_v(v_i) = \exp(-v_i), \quad (9)$$

with $w > 0$, $\alpha_j \geq 0$ and $\beta_j \geq 0$ to ensure the positivity of ψ_i , as in ARCH-type models. Under these assumptions, the TaR is given by:⁵

$$TaR_i(\alpha) = -\psi_i \ln(\alpha). \quad (10)$$

⁵The quantile function of an exponential distribution with a rate parameter λ , is defined as $F_{\exp}^{-1}(p; \lambda) = -\ln(1-p)/\lambda$. In our case, $\lambda = 1$ and $p = 1 - \alpha$.

The exponential distribution implies a flat conditional hazard function, which is restrictive. An alternative to specifying another parametric distribution for v_i (e.g., Weibull, generalized gamma, Burr, etc.) is to view the EACD as a QML. A significant advantage of the exponential distribution is that it provides consistent quasi-maximum likelihood (QML) estimators for the ACD parameters.⁶ This property is particularly important for computing the TaR. Therefore, if the conditional duration is correctly specified, even if the conditional density (Equation 9) is misspecified, consistent QML estimates of ψ_i can be obtained when the true density (unknown) of v_i belongs to the linear exponential family (Gouriéroux, Montfort and Trognon, 1984). Under these assumptions, the TaR becomes:

$$TaR_i(\alpha) = \psi_i F_v^{-1}(1 - \alpha), \quad (11)$$

where $F_v^{-1}(p)$ denotes the p -quantile associated with the (unknown) distribution of v_i . This quantile can be estimated by a non-parametric method based on the standardized durations x_i/ψ_i .

The TaR can be computed for trade durations but also for any other thinned point process, such as price durations and volume durations. For thinned point processes, two solutions can be adopted. The first solution consists in estimating the conditional intensity for the trades and then deduces the intensity of the thinned point process. The relationship between both intensity functions depends on the probability of each trade being selected (Daley and Vere-Jones, 2007). For instance, for price durations, the relationship depends on the probability of observing a trade with a price change larger than the threshold C . The second approach is more direct (Engle and Russell, 1997, 1998). It estimates the conditional mean function of price (or volume) durations directly from the price (or volume) events. In the remainder of the study, we will use the second approach to compute the conditional TaR for a particular subset of trades.

3.2 Total VaR and Volatility

The (*total*) VaR depends on the total variance of the return process. For simplicity, let us consider the demeaned return process $r_{c,i} = r_i - \mathbb{E}(r_i | \mathcal{F}_{i-1})$ associated with the i^{th} trade and assume that:⁷

$$r_{c,i} = \sqrt{h_i} \varepsilon_i, \quad (12)$$

where $h_i = \mathbb{E}(r_{c,i}^2 | \mathcal{F}_{i-1})$ denotes the conditional variance of the tick-by-tick returns and ε_i is a strong white noise process with $\mathbb{E}(\varepsilon_i) = 0$ and $\mathbb{E}(\varepsilon_i^2) = 1$. If the distribution of ε_i belongs to

⁶More generally, Drost and Werker (2004) demonstrate that consistent estimates are obtained when the QML estimation is based on the standard gamma family of distributions, to which the exponential distribution also belongs.

⁷As in most empirical studies based on tick-by-tick data, we consider continuous prices and returns. However, intraday transaction prices often take only a limited number of different values because of certain institutional features with regard to price restrictions. An alternative consists of using an autoregressive conditional multinomial (ACM) model to account for the discreteness of transaction prices (Engle and Russell, 2005).

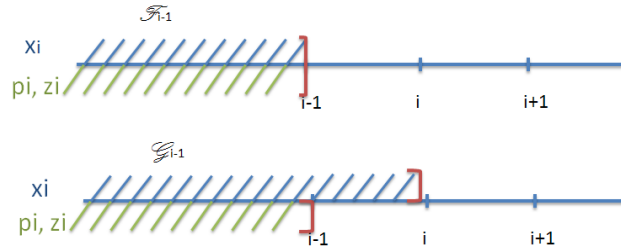
the location-scale family, the \mathcal{F}_{i-1} -conditional VaR can be defined as a linear function of the conditional volatility (Jorion, 2007):

$$VaR_{c,i}(\alpha) = -\sqrt{h_i} F_{\varepsilon}^{-1}(\alpha), \quad (13)$$

where $F_{\varepsilon}(\cdot)$ denotes the cdf of the innovation process ε_i .

Numerous approaches can be used to model the dynamics of h_i . Here, a key element is the conditioning information set. The conditioning information \mathcal{F}_{i-1} , which is used to define TaR and VaR, includes past returns, durations and other marks observed up to the event $i-1$, *i.e.*, $\mathcal{F}_{i-1} = \{x_j, p_j, z_j, j \leq i-1\}$. Consequently, conditionally on \mathcal{F}_{i-1} , the current duration x_i and return r_i are stochastic. Generally, a particular form of exogeneity is assumed, which substantially simplifies the estimation and forecasting procedures (Meddahi, Renault and Werker, 2005). For instance, Engle (2000), Meddahi, Renault and Werker (2005) and Dionne, Duchesne and Pacurar (2009) propose to model the conditional volatility by considering an information set \mathcal{G}_{i-1} that includes \mathcal{F}_{i-1} and the current duration x_i such that $\mathcal{G}_{i-1} = \{x_j, p_j, z_j, j \leq i-1; x_i\}$. In the context of intraday risk measures, Dionne, Duchesne and Pacurar (2009) use a UHF-EGARCH-type model (Engle, 2000) to define their \mathcal{G}_{i-1} -conditional IVaR. Therefore, the duration prior to the next trade is assumed to be known when evaluating the risk associated with the future price change. In contrast, the \mathcal{F}_{i-1} -conditional VaR defined in Equation (13) depends only on the information available at t_{i-1} , *i.e.*, the time of the last trade. This difference is important in the context of high frequency risk management (Figure 1). The use of \mathcal{G}_{i-1} (the bottom part of Figure 1) corresponds to a particular setting. In general, high-frequency traders and risk managers are interested in forecasting the market risk until the next event based on the information available at the time of the last event (the top part of Figure 1).

Figure 1: Information sets \mathcal{F}_{i-1} and \mathcal{G}_{i-1}



A solution consists in considering the ACD-GARCH model proposed by Ghysels and Jasiak (1998). Based on the temporal aggregation of GARCH process results of Drost and Nijman (1993) and

Drost and Werker (1996), Ghysels and Jasiak prove that the volatility model can be described by a time-varying coefficient GARCH model:

$$h_i = \omega_{i-1} + \alpha_{i-1}\varepsilon_{i-1}^2 + \beta_{i-1}h_{i-1}, \quad (14)$$

where the time-varying parameters $\omega_{i-1} = f_\omega(\omega, \alpha, \beta, \kappa, \psi_i)$, $\alpha_{i-1} = f_\alpha(\omega, \alpha, \beta, \kappa, \psi_i)$ and $\beta_{i-1} = f_\beta(\omega, \alpha, \beta, \kappa, \psi_i)$ are functions of the expected duration $\psi_i = \mathbb{E}(x_i|\mathcal{F}_{i-1})$, the kurtosis of the returns κ and three structural parameters denoted ω, α and β .⁸ These parameters correspond to those of a GARCH model defined on regularly time-spaced data with a frequency equal to the time scale, *i.e.* one second in our case. The time-varying parameters ω_{i-1} , α_{i-1} and β_{i-1} correspond to the parameters of the weak GARCH process associated with the cumulated per-second returns observed over a period of ψ_i seconds (for more details, see Drost and Nijman, 1993). The formal expressions of ω_{i-1} , α_{i-1} and β_{i-1} are provided in Appendix A.

The parameters of the EACD and the GARCH models can be estimated using a GMM procedure (Grammig and Wellner, 2002) or a two-step estimation method. The use of the two-step approach is based on the assumption that the spacing between transactions is weakly exogenous in the sense of Engle, Hendry and Richard (1983) with respect to the return process (Appendix A).

Notice that the VaR defined in Equation (13) can be forecast for each trade and for any other subset of transactions. Thus, the index i corresponds to the index of the most recently selected process among the initial point processes, and the information set includes in this case both \mathcal{F}_{i-1} and the appropriate conditioning event \mathbb{C}_i (for instance, $|r_i| \geq C$ in the case of a price event).

4 HFR Measure and Irregularly Spaced Data

The definition of the HFR measures requires a careful treatment of the intraday tick-by-tick data. In this section, we address two specific dimensions: (i) the use of the mid-quotes and/or transaction prices and (ii) the intraday periodicity adjustment. Then, we present the forecasting procedures for the (total) VaR and TaR.

4.1 Data

The relevance of our HFR measures is assessed on tick-by-tick data for two financial stocks (Microsoft (MSFT) and Bank of America (BAC)) and one ETF (SPY), provided by SPDR ETFs, that tracks the S&P 500 index. The data are extracted from the Trade and Quote (TAQ) database and include information on each trade and quote during the period from September 1 to October 29,

⁸Given the functional forms derived by Drost and Nijman (1993) for a weak GARCH representation, Ghysels and Jasiak (1998) consider a restriction of the form $\alpha_{i-1} + \beta_{i-1} = (\alpha + \beta)^{\psi_i}$.

2010 (42 trading days). The database consists of two parts. The trade database summarizes the trading process and contains the date and time stamp t_i for the i^{th} trade and additional marks, such as transaction log-prices (p_i) and volume. The quote database contains the quoting process and reports the date and time t_j of the j^{th} quote in addition to the bid (b_j) and ask (a_j) prices.

Because high-frequency data could contain incorrectly recorded elements, all of the series were cleaned prior to use by applying the set of baseline rules proposed by Brandorff-Nielsen et al. (2009). The two parts of the TAQ database were corrected and merged to adjust the trade prices using rules that include the bid-ask spread. Negative trade prices or volumes, zero bid and ask values were deleted. After merging the two databases, several simultaneous transactions were observed. Therefore, we considered only the first time stamp. The price and quote series are reported every trading day from 9:30 am to 4 pm Eastern Standard Time (EST), and the effect of the overnight or the opening auction is removed by deleting the opening trades. A total of 223,502 observations for the S&P 500, 78,265 for BAC and 432,626 for MSFT remained.

The HFR measures can be applied to either transaction prices or mid-quotes defined as $p_i = (b_i + a_i) / 2$. In the high-frequency volatility literature, mid-quotes are commonly used instead of transaction prices to avoid the bid-ask bounce exhibited by the trade process. For instance, Giot (2005) defines his IVaR for mid-quotes. However, Dionne, Duchesne and Pacurar (2009) argue that the use of mid-quotes may understate the true VaR. Because we are interested in estimating the VaR for real transactions, we also consider the transaction prices. Nevertheless, prices may be affected by various microstructure noises, and the corresponding returns are generally serially autocorrelated. To capture such microstructure effects, an ARMA(p, q) model is generally used for the tick-by-tick returns. Given the autocorrelation and partial autocorrelation functions of the returns, we use a simple AR(1) model for all of the assets and denote by $\mu_i = \mathbb{E}(r_i | \mathcal{F}_{i-1})$ the conditional mean of the returns with:

$$\mu_i = \theta + \rho r_{i-1}. \quad (15)$$

Therefore, the total VaR is expressed as a function of the VaR obtained for the demeaned process (Equation 13) as follows:

$$VaR_i(\alpha) = -\mu_i + VaR_{c,i}(\alpha). \quad (16)$$

Simple descriptive statistics on durations and returns are presented in Table 1. For each of the three series under analysis, we report descriptive statistics on the trade durations and returns as well as on the price durations and the corresponding returns. Price durations are defined for two stock-specific thresholds C . These thresholds are determined by analyzing the empirical distributions

of the absolute returns (Appendix B, Figure B1) and do not necessarily correspond to the price increments. The total number of observations decreases by approximately 50% and 80% between the entire sample and the samples of price events.

Among the three stocks, Microsoft is the most traded asset; the average trade duration is 12.55 seconds for BAC, 4.39 seconds for the S&P 500 and only 2.27 seconds for MSFT. For all of the assets, the average price duration is larger than the trade duration and increases with the threshold C ; the larger the price change is, the more significant is the time required for the price to change by this amount. As already observed in other studies, the trade and price durations for the three assets exhibit a positive autocorrelation, an overdispersion (*i.e.*, the standard deviation is greater than the mean) and a right-skeewed shape (Pacurar, 2008).⁹ The high values of the Ljung-Box Q-statistics obtained for 10 or 20 lags indicate the presence of ACD effects (duration clustering) at any reasonable level.

The mean of the transaction returns is extremely small for all of the assets and much lower than its standard deviation. The return associated with the price event is not always larger (on average) than the trade return. However, its volatility is naturally higher. For all of the market microstructure events considered, BAC is the most volatile asset. The volatility of the BAC trade returns is nearly five times larger than that of the market index and two times that of MSFT. All of the trade return series display a kurtosis higher than that of a normal distribution. However, the leptokurticity decreases with the threshold C . The skewness is always positive for the S&P 500, negative for MSFT, and dependent on the threshold C for BAC. Finally, as previously mentioned, the returns are autocorrelated as the Q-statistics obtained for 10 or 20 lags result in the rejection of the null hypothesis of no autocorrelation. However, the values of the Q-statistics tend to decrease with the threshold C (except for MSFT).

4.2 Intraday Periodicity Adjustment

A salient feature of the tick-by-tick data is the intraday seasonal pattern. It is widely documented that high-frequency data exhibit a strong seasonality because of the progression of the market activity during the same day (intraday periodicity) and during each day of the week (interday periodicity). Generally, this pattern must be removed from the data prior to fitting the model to avoid distortions in the results. Consequently, we define the price duration variable as follows:

$$x_i = \bar{\omega}_d(t_i) \tilde{x}_i, \quad (17)$$

⁹The overdispersion suggests that the exponential distribution is not appropriate for the unconditional distribution of trade or price durations but not that conditional durations cannot be exponentially distributed.

with $\bar{\omega}_d(t_i)$ is the seasonal component and \tilde{x}_i is the seasonally adjusted duration. Many procedures have been proposed to estimate and adjust for the seasonal component (see Wood et al., 1985; Haris, 1986; Andersen and Bollerslev, 1997, 1998; Bauwens and Giot, 2000; Dufour and Engle, 2000; Drost and Werker, 2004; Hecq, Laurent and Palm, 2012). We follow the same approach as Dionne, Duchesne and Pacurar (2009) and Anatolyev and Shakin (2007). The deterministic component is identified as the expected trade duration conditional on the time-of-day and the day-of-week effects. For each day of the week, the seasonal factor is computed by averaging the durations over 30-minute intervals. Then, cubic splines are used to smooth the time-of-day effect and to extrapolate the latter effect for any time during the day.

Figure 2 displays the estimated intraday seasonal components for the duration series of the S&P 500, BAC and MSFT. The overall conclusion is that a day-of-week effect exists because the time-of-day component for Monday differs from the time-of-day component for Tuesday and so on throughout the week. The well-documented inverted-U shape (see, for example, Engle and Russell, 1998; Bauwens and Giot 2000; etc.) indicates that the market activity (resp. trade duration) is higher (resp. shorter) at the opening and closing of the trading day than around midday. To remove this seasonal component, we divide the raw data of observed price durations by the time-of-day effect and run the EACD model on the stochastic deseasonalized component \tilde{x}_i .

Returns must be also diurnally adjusted because volatility present also a daily-specific shape. Engle (2000) proposes to adjust the volatility *per-time-unit* by considering $r_i/\sqrt{x_i}$ and a spline procedure identical to that used for durations. However, this approach could result in a residual intraday seasonality in the *total* volatility. Therefore, we prefer to directly adjust the squared returns (instead of returns per-time-unit) for intraday periodicity. We assume that:

$$r_{c,i} = \sqrt{\bar{\omega}_r(t_i)} \tilde{r}_{c,i}, \quad (18)$$

where $\tilde{r}_{c,i}$ denotes the seasonally adjusted demeaned return and $\sqrt{\bar{\omega}_r(t_i)}$ is the corresponding seasonal component. As for the durations, for each day of the week, the seasonal factor is computed by averaging the corresponding values over 30-minute intervals and then using a cubic spline. The right panel of Figure 2 displays the estimated intraday seasonal components for volatility. The patterns are analogous to those found in previous studies. The seasonal component exhibits a U-shape, which indicates that the return volatility is generally lower in the middle of the day than at the beginning and end of the day.

4.3 Forecasting Algorithm

This section describes the algorithm required to forecast the total VaR and TaR measures. The last event of the in-sample period is n , and $n+1$ is the first event of the out-of-sample period. The procedure to forecast $TaR_{n+1}(\alpha)$ and $VaR_{n+1}(\alpha)$ is as follows:

- i) The parameters of the EACD-GARCH model are estimated (Appendix A) on deseasonalized duration and tick-by-tick demeaned return series, $\{\tilde{x}_i\}_{i=1}^n$ and $\{\tilde{r}_{ic}\}_{i=1}^n$.
- ii) We compute the one-step-ahead out-of-sample expected duration $\hat{\psi}_{n+1}$ and volatility \hat{h}_{n+1} for the seasonally adjusted series.
- iii) The quantiles of the innovation processes, ε_i and v_i , are estimated by the corresponding empirical quantiles of the in-sample series of standardized and deseasonalized returns $\{\hat{\varepsilon}_i\}_{i=1}^n$ and durations $\{\hat{v}_i\}_{i=1}^n$.

$$\hat{F}_\varepsilon^{-1}(\alpha) = \text{percentile}(\{\hat{\varepsilon}_i\}_{i=1}^n, 100\alpha), \quad (19)$$

$$\hat{F}_v^{-1}(1-\alpha) = \text{percentile}(\{\hat{v}_i\}_{i=1}^n, 100(1-\alpha)), \quad (20)$$

with $\hat{\varepsilon}_i = \tilde{r}_{c,i}/\hat{h}_i^{1/2}$ and $\hat{v}_i = \tilde{x}_i/\hat{\psi}_i$ for $i = 1, \dots, n$.

- iv) Finally, the VaR and TaR out-of-sample forecasts are given by:

$$TaR_{n+1}(\alpha) = \hat{F}_v^{-1}(1-\alpha) \hat{\psi}_{n+1} \bar{\omega}_d(t_{n+1}), \quad (21)$$

$$VaR_{n+1}(\alpha) = -\hat{\mu}_{n+1} - \hat{F}_\varepsilon^{-1}(\alpha) \hat{h}_{n+1}^{1/2} \bar{\omega}_r^{1/2}(t_{n+1}). \quad (22)$$

where $\bar{\omega}_z(t_{n+1})$ denotes the deterministic seasonal component (estimated in-sample) associated with time t_{n+1} , for $z = \{d, r\}$. Because the time of the next event t_{n+1} is unknown, a feasible estimator of VaR and TaR can be obtained by replacing $\bar{\omega}_z(t_{n+1})$ with $\bar{\omega}_z(t_n)$.

By replicating this procedure M times, we obtain a sequence of out-of-sample, one-step-ahead forecasts $\{VaR_{n+j}(\alpha)\}_{j=1}^M$ and $\{TaR_{n+j}(\alpha)\}_{j=1}^M$. The model is not re-estimated at each step, as in a standard rolling windows scheme. The parameters are estimated on the first n observations and then considered as fixed for $n+1, \dots, n+M$. For S&P 500 and BAC, the estimation period is September 1 to September 22, 2010 (15 trading days). For MSFT, because of the large number of observations, the estimation period ends on September 15. The estimation samples include 80,845 observations for S&P 500, 27,972 for BAC and 99,155 for MSFT.

5 Empirical Analysis

In this section, we apply the forecasting procedure described in the previous section to the three assets by considering the trade and the price events successively.

5.1 Estimation Results and HFR forecasts

Estimation Results: The estimation results for the various EACD-GARCH models are reported in Tables 2 and 3. For each asset, the two columns report the estimates for the trade and price events, respectively. The estimation results of the EACD are reported in Table 2. In all of the cases, the specification tests on durations (not reported) lead us to consider an EACD(2,2) model. All of the parameters are significant, and the sum of autoregressive parameters is generally close to one, which confirms the high persistence in durations. The model’s performance in capturing the latent structure of durations can be assessed by comparing the autocorrelations of the seasonally adjusted durations (cf. Table 1) with those of the standardized residuals $\hat{v}_i = \tilde{x}_i / \hat{\psi}_i$. In Table 2, we report the p-values of the Ljung-Box test Q-statistics based on 10 and 20 lags, respectively, for the series of standardized residuals. For all of the assets, the null hypothesis of no autocorrelation cannot be rejected at the 95% confidence level, which indicates that the models successfully remove the autocorrelation observed in the original durations.

In a second step, we use the sequence of predicted durations to evaluate the time-varying parameters ω_i , α_i and β_i of the GARCH(1,1) process. More precisely, we use the conditional raw durations, which are defined as $\hat{\psi}_{n+1} \bar{\omega}_d(t_n)$, because the GARCH parameters depend on the expected (and uncorrected) duration before the next trade (Drost and Nijman, 1993).¹⁰ The structural parameters ω , α and β are estimated by QML (Appendix A). The estimation results are reported in Table 3. These parameters can be interpreted as the parameters of a GARCH model based on regular high-frequency (one-second) data. Note that their estimated values are similar to those obtained for daily data; the ARCH parameter is much smaller than the GARCH parameter and their sum is close to unity. In contrast, for IBM, Ghysels and Jasiak (1998) obtain a relatively small value for the estimated GARCH parameter β , which is close to α . They explain this outcome by the fact that the drift ω_i depends directly on $\hat{\psi}_i$ and the expected duration absorbs all of the persistence. In our case, the persistence captured by ω_i seems to be insufficient to reproduce the persistence of volatility. From these estimates, we compute the duration-dependent parameters ω_i , α_i and β_i of the time-varying GARCH model (Equation 14) by applying the formulas of temporal aggregation of Drost and Nijman (1993). For example, for BAC the parameters α_i and β_i have means equal to 0.075 and 0.753, respectively, while their values are within the intervals [0.046; 0.076] and [0.343; 0.927]. Finally, this model does not remove all the serial dependence from the tick-by-tick standardized returns $\hat{\varepsilon}_i$, as shown by the Ljung-Box test Q-statistics. However, the remaining dependencies are much less important than those observed for the raw returns (Table 1).

¹⁰ Similar results (not reported) are obtained when we use the conditional seasonally adjusted duration $\hat{\psi}_{n+1}$.

Out-of-sample VaR and TaR Forecasts: Figure 3 displays the first 3,000 out-of-sample forecasts of the 1%-VaR and 1%-TaR for the S&P 500 and BAC trades on September 23, 2010 and the MSFT trades on September 16. These observations correspond to the time period between 9:30 am and 12:57 am for SPY, until 12:09 am for BAC and until 11:17 for MSFT. First, we observe that the TaR and VaR forecasts accurately capture the volatility and duration dynamics. For BAC, the minimum duration (over this period) before the next trade may occur (with a confidence level of 99%) ranges between 21.45 seconds and 4 minutes and 25 seconds (or 265 seconds). For the same period, the maximum expected loss that will not be exceeded (for a given confidence level of 99%) at the time horizon of the next trade ranges from 0.072% to 0.256%. Second, the frequencies of VaR violations (1% for S&P 500, 0.8% for BAC and 0.7% for MSFT) and TaR violations (1% for S&P 500, 0.9% for BAC and 0.6% for MSFT) are close to the nominal level of 1%, and, more importantly, they are not clustered. The same results are observed for the VaR and TaR forecasts on the price events (Figure C1, Appendix C).

Finally, we observe that the TaR and VaR are negatively correlated. When the market risk is low, the duration risk is important. However, the correlation between the two measures for BAC (-0.151) and MSFT (-0.111) is not high, as confirmed by the scatter plot in Figure 4. In contrast, the ETF correlation reaches -0.407 . This negative relationship confirms the previously mentioned theoretical findings. Notice that it is not necessarily caused by the model's structure. On the contrary, the use of a volatility measure based on the price intensity (Engle and Russell, 1998; Giot, 2005) necessarily implies a negative relationship between the volatility and the conditional duration.

So far the analysis compares the VaR with the returns but does not accurately reveal the ultra-high-frequency market risk. As previously mentioned, a VaR of 0.2% associated with a transaction that will occur in ten seconds does not present the same risk for the investor as the same VaR associated with a transaction that will occur in ten minutes. A first solution consists of comparing the ex-ante per-time-unit VaR, which is defined as $VaR_i(\alpha)/\psi_i$, with the ex-post returns standardized by the conditional duration (Figure 5). In the last section, we will complete this analysis by considering the marginal and instantaneous VaRs.

5.2 Backtesting

Traditionally, the quality of an economic variable forecast is assessed by comparing its *ex-post* realization with the *ex-ante* forecasted value. However, this approach is not suitable for VaR and TaR forecasts because the true quantiles of the corresponding distributions are not observable. Therefore, VaR assessment is generally based on the concept of violation (also known as hit or

exception). A violation is said to occur if the *ex-post* realization of the return (resp. duration) is more negative (resp. larger) than the *ex-ante* VaR (resp. TaR) forecast. Let us define two binary variables associated with the *ex-post* observation of an $\alpha\%$ -VaR violation and an $\alpha\%$ -TaR violation:

$$H_i(r) = \begin{cases} 1 & \text{if } r_i < -VaR_i(\alpha) \\ 0 & \text{otherwise} \end{cases}, \quad (23)$$

$$H_i(d) = \begin{cases} 1 & \text{if } x_i > TaR_i(\alpha) \\ 0 & \text{otherwise} \end{cases}. \quad (24)$$

As emphasized by Christoffersen (1998) and Berkowitz, Christoffersen and Pelletier (2011), VaR and TaR forecasts are valid if the violations satisfy the following two hypotheses:

- (i) The unconditional coverage (UC) hypothesis: the probability of an *ex-post* violation must be equal to the α coverage rate:

$$\Pr(H_i(z) = 1) = \mathbb{E}(H_i(z)) = \alpha, \quad \forall z \in \{r, d\}. \quad (25)$$

- (ii) The independence (IND) hypothesis: VaR and TaR violations observed at two different dates for the same coverage rate must be distributed independently. The variable $H_i(z)$ associated with a VaR or TaR violation for the i^{th} trade should be independent of the variable $H_{i-k}(z)$, $\forall k \neq 0$. Therefore, past violations should not influence current and future violations.

When the UC and IND hypotheses are simultaneously valid, VaR and TaR forecasts are said to have a correct conditional coverage (CC), and the corresponding violation process is a martingale difference, with:

$$\mathbb{E}(H_i(z) - \alpha | \mathcal{F}_{t-1}) = 0, \quad \forall z \in \{r, d\}. \quad (26)$$

Several remarks are in order. First, one could also define a joint violation, *i.e.*, circumstances in which the return and duration exceed the VaR and TaR. However, because VaR violations and TaR violations might not be independent, the probability of observing a joint violation is not necessarily equal to α^2 . Therefore, we test the TaR and VaR forecasts independently. Second, the VaR forecasts are not rescaled at fixed-time intervals to apply usual backtesting procedures as in Giot (2005) or Dionne, Duchesne and Pacurar (2009). The summation of the tick-by-tick returns and the VaR forecasts over a fixed-time interval (for instance, five minutes) may obscure the potential misspecification of the VaR model. More fundamentally, our measures are designed to forecast the duration and market risks at the time horizon of the next trade. Consequently, their performance must be assessed for the same time horizon.

Here, we consider three backtests. The first one is the LR test proposed by Christoffersen (1998). This test is based on a Markov-chain model with two states (violation or no violation). The UC,

IND and CC assumptions are tested through parameter restrictions on the transition probability matrix. The UC test corresponds to the Kupiec’s test (1995) for the percentage of failures, which is also embedded in the regulatory requirements based on the backtesting of daily VaR models. The second test is the duration-based LR test proposed by Christoffersen and Pelletier (2004). This test exploits the duration between two consecutive violations.¹¹ Under the CC hypothesis, this duration follows a geometric distribution with parameter α . Exploiting this property, the authors consider a continuous exponential distribution and propose a LR test for the null hypothesis of CC. A lifetime distribution that nests the exponential distribution (for instance, a Weibull distribution) is specified under the alternative, and the memoryless property of the violations can be tested by means of parameter restrictions. The third test, proposed by Candelon et al. (2011), also exploits the duration between two consecutive violations. However, this test is based directly on the geometric distribution. The GMM framework, which was proposed by Bontemps and Meddahi (2012) to test for distributional assumptions, is applied to the VaR and TaR forecasts. The test statistic is a J-statistic based on the moments defined by the orthonormal polynomials associated with the geometric distribution.

Backtesting Results: Table 4 reports the results of the backtests for a 1%-VaR (Panel A) and a 1%-TaR (Panel B). These tests are based on a sequence of 1,000, 2,000 or 3,000 one-step-ahead forecasts obtained according to the forecasting procedure described in Section 4. For each test and asset (S&P 500, BAC or MSTF), we distinguish the UC, IND and CC null hypotheses and report the corresponding test p -values. Entries in italics denote a failure of the model at the 95% confidence level. Additionally, we report the empirical frequency of violations for each case.

The results indicate that the intraday VaR and TaR forecasts are satisfactory. First, the empirical frequencies of VaR and TaR violations are statistically not different from the 1% risk level for all of the assets. In all of the cases considered, the p -values associated with the UC tests exceed 5%. These p -values tend to decrease with sample size. For a frequency of violations close to, but different from the 1% level (for instance, 1.01%), the null hypothesis tends to be rejected more often when the sample size increases. Second, the VaR and TaR violations are not clustered, which indicates that the VaR and TaR forecasts accurately capture the volatility and duration dynamics. The IND tests indicate that the null hypothesis of independence between the current and past violations cannot be rejected. The only exception is observed in the VaR for MSFT when we consider a sample size of 3,000 observations. This independence property in the time dimension does not indicate that both violation processes are independent. We observe that the VaR and

¹¹Note that we must distinguish the duration between two consecutive violations from the trade duration used to compute the TaR. As a consequence, the first one can be used to backtest both the VaR and the TaR.

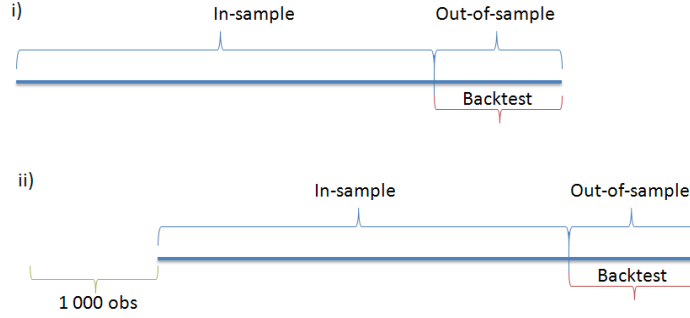
TaR violations occur simultaneously in fewer than 0.05% of the total number of violations for all of the assets. Extreme returns and extreme durations do not generally occur at the same time, which confirms the negative relationship between market risk and trading activity. Third, the CC tests do not allow us to reject the null hypothesis of a martingale difference both for the VaR and TaR violations, which supports the conclusion that the VaR and TaR forecasts are valid.

Because we consider irregularly spaced risk measures, backtesting a sample of 1,000 observations does not necessarily span the same period for BAC or MSFT (the least and the most traded stocks in our analysis). Additionally, one could be interested in testing the validity of the HFR measures over a fixed period. Table 5 reports the results of the backtests over a period of 30 minutes, one hour and four hours after the opening time on September 23, 2010 (for BAC and S&P 500) and on September 16, 2010 (for MSFT), respectively. For a period of 30 minutes, the backtesting sample includes 609 observations for S&P 500, 225 for BAC and 907 for MSFT. In the cases of both VaR and TaR, respectively, the null hypothesis of UC cannot be rejected at the 95% confidence level, which indicates that the frequency of violations is statistically not different from the level of 1%. Besides, these violations are not clustered. These results confirm that our measure accurately assesses the intraday market and duration risks.

Periodicity and Split Point: The previous backtesting procedure was conducted only for the first trading hours of the first forecasting day (September 23, 2010). To determine if the VaR and TaR forecasts remain valid for the next trading days, we propose a backtesting analysis based on a fixed rolling window of one hour over a period of seven days (September 23 to October 1) without re-estimation of the model. Figure 6 displays the p -values of Christoffersen's LR test (1998) for the frequency of violations (UC) of both 1%-TaR and 1%-VaR forecasts for the S&P 500 ETF. The results confirm the validity of the HFR forecasts. Regardless of the day or the hour considered over this period, the p -values are larger than 5%, which indicates that the null hypothesis of UC cannot be rejected for the VaR (upper panel of Figure 6) and TaR (middle panel of Figure 6) forecasts. For the first two trading days, the p -values exhibit an inverted-U shape; the probability is relatively small at the beginning and end of the day and larger at midday. This observation highlights the fact that the reliability of the VaR and TaR forecasts is less certain when the trading activity is particularly important and the returns are more volatile. It can also be explained by the number of observations used for each backtest (bottom panel of Figure 6). Because we consider a fixed period of one hour, the number of observations is more important at the beginning and end of the day. Consequently, even if the frequency of violations is constant over the day and assumed to be close to (but different from) the level of 1%, the inverted-U shape of the p -values reflects the increase in the power of the test with the number of observations.

Finally, our backtesting results may depend on how our sample is split into estimation and evaluation periods (Hansen and Timmermann, 2012). To assess the robustness of the results, we propose an analysis based on a rolling window in which the split point between the in-sample and out-of-sample periods moves by 1,000 observations at each iteration between September 22 and October 10. As shown in Figure 7, the rolling window scheme is chosen so that there is no overlapping of the backtesting sample. At each step, the EACD-GARCH parameters are estimated using the past 25,000 observations, and we compute 1,000 out-of-sample VaR and TaR forecasts for BAC. Figure 8 displays the corresponding p -values from Christoffersen's LR test for UC. The figure confirms the robustness of our results because whatever the split point used, the UC hypothesis cannot be rejected for a 95% confidence level.

Figure 7: Rolling Window Scheme



6 Marginal VaR and Instantaneous VaR

In the previous sections, we considered the total VaR, which is defined as the maximum expected loss that will not be exceeded under normal conditions for a $1 - \alpha$ confidence level at the time horizon of the next trade or microstructure event i . Alternatively, we propose here to compute two additional conditional VaR concepts for high frequency management: (i) the *per-time-unit* VaR or *marginal* VaR and (ii) the *instantaneous* VaR.

Marginal VaR The marginal VaR depends on the conditional marginal volatility or the conditional volatility per-time-unit. In the case of a trade event ($\mathbb{C}_i = \emptyset$), the marginal volatility (Gerhard and Hautsch, 2002) is defined as follows:

$$\sigma_i^2 = \mathbb{E}_{r,x} \left(\frac{r_{c,i}^2}{x_i} \middle| \mathcal{F}_{i-1} \right). \quad (27)$$

where $r_{c,i} = r_i - \mathbb{E}(r_i | \mathcal{F}_{i-1})$ denotes the demeaned return process. To our knowledge, no dynamic model has been proposed for the ratio $r_{c,i}^2/x_i$. For this reason, we consider here the information set $\mathcal{G}_{i-1} = \{x_j, p_j, z_j, j \leq i-1; x_i\}$ that includes the current duration x_i (Engle, 2000; Meddahi, Renault and Werker, 2005). Subsequently, the conditional volatility per-time-unit becomes:

$$\sigma_i^2 = \frac{1}{x_i} \mathbb{E}_r(r_{c,i}^2 | \mathcal{G}_{i-1}) = \frac{1}{x_i} \mathbb{V}(r_i | \mathcal{G}_{i-1}). \quad (28)$$

The corresponding \mathcal{G}_{i-1} -conditional marginal VaR for r_i/x_i can be defined as follows:

$$\begin{aligned} VaR_i(\alpha; x_i; \mathcal{G}_{i-1}) &= -\mathbb{E}_r\left(\frac{r_i}{x_i} \middle| \mathcal{G}_{i-1}\right) - F^{-1}(\alpha) \sqrt{\mathbb{V}_r\left(\frac{r_i}{x_i} \middle| \mathcal{G}_{i-1}\right)} \\ &= -\frac{\mu_i}{x_i} - F^{-1}(\alpha) \sqrt{\frac{1}{x_i^2} \mathbb{V}_r(r_i | \mathcal{G}_{i-1})} \\ &= -\frac{\mu_i}{x_i} - F^{-1}(\alpha) \frac{\sigma_i}{\sqrt{x_i}}. \end{aligned} \quad (29)$$

When the durations are seasonally adjusted, this expression becomes:

$$VaR_i(\alpha; x_i; \mathcal{G}_{i-1}) = -\frac{\mu_i}{x_i} - F_\varepsilon^{-1}(\alpha) \sigma_i \sqrt{\frac{\bar{\omega}_r(t_i)}{x_i}}, \quad (30)$$

where $F_\varepsilon^{-1}(\alpha)$ denotes the quantile of the standardized and adjusted returns $\varepsilon_i = \bar{\omega}_r(t_i) r_{c,i}/\sigma_i$. Modeling the \mathcal{G}_{i-1} -conditional marginal volatility enables the identification of the \mathcal{G}_{i-1} -conditional total volatility as $x_i \sigma_i^2$. This quantity differs from the \mathcal{F}_{i-1} -conditional total volatility h_i defined in the equation (12). Consequently, we can also derive a (\mathcal{G}_{i-1} -conditional) total VaR, denoted $VaR_i(\alpha; \mathcal{G}_{i-1})$, as:

$$VaR_i(\alpha; \mathcal{G}_{i-1}) = -\mu_i - \sigma_i \sqrt{x_i} F_\varepsilon^{-1}(\alpha) \sqrt{\bar{\omega}_r(t_i)}. \quad (31)$$

This VaR corresponds to the IVaR proposed by Dionne, Duchesne and Pacurar (2009). Notice that Dionne, Duchesne and Pacurar also use a simulation-based method to infer the VaR at any fixed-time horizon for backtesting purposes.

When given \mathcal{G}_{i-1} , various models for σ_i^2 can be considered. We propose to consider the simplest model, *i.e.*, the Ultra High-Frequency (UHF) GARCH model proposed by Engle (2000). The conditional volatility per-time-unit is then assumed to follow a simple GARCH(1,1) equation:

$$\sigma_i^2 = \omega + \alpha \left(\tilde{r}_{c,i-1} / \sqrt{\tilde{x}_{i-1}} \right)^2 + \beta \sigma_{i-1}^2 + \gamma \tilde{x}_i, \quad (32)$$

where $\tilde{r}_{c,i}$ and \tilde{x}_i represent the seasonally adjusted demeaned return and duration, respectively.¹²

¹²Other models could be used. Dionne, Duchesne and Pacurar (2009) consider an EGARCH-type of model in which the log-volatility depends on the current duration, past returns and past durations. However, Meddahi, Renault and Werker (2005) argue that it is better to model the variance per-time-unit instead of the total variance over the next event because total variances are primarily influenced by the associated duration. They propose an AR(1) model for σ_i^2 with time-varying parameters that depend on x_i . The time-varying feature of the parameters contrasts with the UHF-GARCH used by Engle (2000) who does not consider the effects of temporal aggregation on the model's parameters.

Instantaneous VaR Following the notations of Gerhard and Hautsch (2002), the instantaneous volatility of the price change is defined for a constant and infinitesimal time increase:¹³

$$\tilde{\sigma}_{i+\Delta}^2 = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbb{E} \left(r_{c,i-1}^2(\Delta) \mid \mathcal{F}_{i-1} \right), \quad (33)$$

where $r_{i-1}(\Delta) = p(t_{i-1} + \Delta) - p(t_{i-1})$ and $r_{c,i-1}(\Delta) = r_i(\Delta) - \mathbb{E}(r_i(\Delta) \mid \mathcal{F}_{i-1})$. The corresponding VaR for the instantaneous demeaned return process can be expressed as follows:

$$VaR_{c,i}(\alpha; \Delta) = -\tilde{\sigma}_{i+\Delta}^2 F_{\varepsilon(\Delta)}^{-1}(\alpha), \quad (34)$$

where $F_{\varepsilon(\Delta)}(\cdot)$ denotes the cdf of the innovations process associated with $r_i(\Delta)$. Giot (2005) uses a similar approach to compute an intraday VaR for price events. Because the innovations of the instantaneous returns cannot be identified, Giot substitutes $F_{\varepsilon(\Delta)}^{-1}(\alpha)$ with the empirical α -quantile of the standardized total returns $r_{c,i}/\tilde{\sigma}_{i+\Delta}$. Notice that the corresponding VaR is not exactly an instantaneous VaR as defined in the equation (34), because it combines the instantaneous volatility and the quantile of the innovations of the total returns.

The instantaneous volatility can be derived directly from the expected duration or the intensity. However, trade intensity may provide relatively little information on the price volatility. For instance, if the price fluctuates between two values every second, the asset exhibits a large trading intensity but no price change (Giot, 2001). Consequently, the instantaneous volatility is generally derived from the *price* durations (Engle and Russell, 1998), and the corresponding VaR is only reported for price events.

Let $N(t)$ be a counting variable equal to the total number of price events that have occurred by time t . The conditional price intensity function is defined as follows:

$$\lambda(t_i \mid \mathcal{F}_{i-1}; \mathbb{C}_i) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \Pr(N(t_i + \Delta) > N(t_i) \mid \mathcal{F}_{i-1}; \mathbb{C}_i), \quad (35)$$

where $\mathbb{C}_i = \{|r_i| \geq C\}$. As shown by Engle and Russell, the instantaneous volatility can be expressed as a linear function of the conditional intensity:¹⁴

$$\tilde{\sigma}_{i+\Delta}^2 = \lambda(t_i \mid \mathcal{F}_{i-1}; \mathbb{C}_i) C^2. \quad (36)$$

If an EACD model is applied to price durations, this expression becomes:

$$\tilde{\sigma}_{i+\Delta}^2 = \frac{C^2}{\psi_i}. \quad (37)$$

¹³By considering arithmetic returns, Engle and Russell (1998) define the instantaneous volatility as follows:

$$\tilde{\sigma}_i^2 = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbb{E}_i((p_{i+\Delta} - p_i)/p_i)^2 = \frac{1}{p_i^2} \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbb{E}_i(p_{i+\Delta} - p_i)^2$$

This expression resembles the expression obtained for logarithmic returns and log prices (except for the term p_i^2).

¹⁴For arithmetic returns, Engle and Russell (1998) obtain a similar expression with $\tilde{\sigma}_{i+\Delta}^2 = \lambda(t_i \mid \mathcal{F}_{i-1}; \mathbb{C}_i) C^2/p_{i-1}^2$.

The intuition is the following. If the price change (observed over a time interval Δ) is smaller than the threshold C , we assume that there is no price change. In fact, the change is unobservable because the thinned point process considers only the trades for which $|r_i| \geq C$. In other cases, Engle and Russell assume that the instantaneous price change is precisely equal to C . The price change can then be viewed as a binomial process; the probability of observing a return equal to C is equal to $\lambda(t_i | \mathcal{F}_{i-1}; \mathbb{C}_i)$, otherwise there is no change. So, the expression (36) simply corresponds to the variance of this binomial process.

Figure 9 shows the three types of 1%-VaR (*i.e.*, the total, the instantaneous and the time per-unit) for the S&P 500 for the period between September 23 at 9:30 am and September 24 at 10:00 am. For comparison, all of the VaRs are computed for price events with a threshold C equal to 0.0075%. The total VaR is logically larger than the per-time-unit VaR, which is generally larger than the instantaneous VaR. The total VaR measures the maximum expected loss that will be exceeded for a risk level of 1% at the time horizon of the next price change, whereas the marginal VaR is defined for the time horizon of one second. Because the average price duration is equal to 8.8 seconds (Table 1), the total VaR is approximately eight times larger than the marginal VaR. The instantaneous VaR, defined for an infinitesimal time increase, should always be smaller than the marginal VaR. Here, the instantaneous VaR is based on the empirical quantile of the standardized per-time-unit returns $r_{c,i}/(x_i\tilde{\sigma}_{i+\Delta})$, and not on the standardized returns $r_{c,i}/\tilde{\sigma}_{i+\Delta}$, as in Giot (2005). However, the difference between the instantaneous and marginal VaRs is smaller than it would be if we could identify the innovation of the instantaneous return process. Beyond their values, which are not directly comparable, the three types of VaR exhibit strong correlations, even using different computation methods (UHF-GARCH versus EACD). The correlation between the marginal and the total VaR is equal to 0.866. The correlation coefficients between the two other combinations of VaR measures are slightly smaller but still high (0.714 for the correlation between the instantaneous VaR and the total VaR and 0.711 between the instantaneous VaR and time per-unit VaR). This result confirms the robustness of our risk measures.

7 Conclusion

In this paper, we propose three intraday high frequency market risk measures for irregularly spaced data: the total VaR, the marginal VaR and the instantaneous VaR. These measures are completed with a duration risk measure, *i.e.* the TaR, that gives an evaluation of the risk associated to the duration between the current trade and the next one. These measures are computed and evaluated at each transaction for the time horizon of the next trade. In particular, the VaR is hence not rescaled at fixed-time intervals for backtesting purposes. One advantage of our approach

is that it can be extended to any type of market microstructure event (e.g., price events or volume events) in contrast to the IVaR proposed by Giot (2005), which is relevant only for price events. Additionally, our forecasting procedure complies with high frequency risk management practices. When forecasting the VaR for the next trade, we do not assume that the duration prior to this trade is known, in contrast to Dionne, Duchesne and Pacurar (2009). Thus, TaR and VaR are estimated with a model in which the parameters of the volatility model depend on the expected durations. We propose a backtesting procedure specifically designed to assess the validity of the VaR and TaR forecasts for each microstructure event. We test for the frequency of VaR and TaR violations and for the independence of these violations.

The HFR measure is empirically applied to two stocks (BAC and MSFT) and an ETF based on the S&P 500 index. We prove that the VaR and TaR accurately capture the volatility and duration dynamics of these three assets. The backtests show that the VaR and TaR forecasts are valid because their violations occur with a frequency close to the level of risk and they are not clustered. The VaR and TaR out-of-sample forecasts remain valid throughout the day and the week, although the model is not re-estimated. In addition, these results are robust with respect to the choice of the split point between in-sample and out-of-sample periods.

Interesting further research projects are conceivable. A natural development consists in considering a multivariate framework to compute the risk measure on a portfolio. However, Russell (1999) and Hautsch (2004, 2012) show the difficulty of using ACD-type models in the multivariate framework. The difficulty mainly arises when information, such as quote updates and depth changes, arrives within a trade duration. They advocate modelling point processes from the intensity framework.

A Appendix A: EACD-GARCH model

The ACD-GARCH, proposed by Ghysels and Jasiak (1998), is a time varying coefficient GARCH model, where the durations between transactions determine the dynamic of the parameters. The parameter behavior is described by the temporal aggregation formulas of a weak GARCH proposed by Drost and Nijman (1993) and Drost and Werker (1996). Let us denote $x_i = t_i - t_{i-1}$ the duration between two consecutive trade times and $\psi_i = \mathbb{E}(x_i | \mathcal{F}_{i-1})$ the conditional expected duration given the information set $\mathcal{F}_{i-1} = \{x_j, p_j, z_j\}_{j=1}^{i-1}$ (see section 3.1). Using the temporal aggregation formula of Drost and Nijman (1993) for the case of a weak GARCH process and a flow variable, Ghysels and Jasiak (1998) propose an ACD-GARCH model for the conditional volatility h_i of returns defined as follows:

$$h_i = \omega_{i-1} + \alpha_{i-1} r_{i-1}^2 + \beta_{i-1} h_{i-1}, \quad (38)$$

where the parameters ω_{i-1} , α_{i-1} , β_{i-1} are functions of the expected duration ψ_i :

$$\omega_{i-1} = \psi_i \omega \frac{1 - (\alpha + \beta)^{\psi_i}}{1 - (\alpha + \beta)}, \quad (39)$$

$$\alpha_{i-1} = (\alpha + \beta)^{\psi_i} - \beta_{i-1}, \quad (40)$$

and $|\beta_{i-1}| < 1$ is the solution of the quadratic equation:

$$\frac{\beta_{i-1}}{(1 + \beta_{i-1}^2)} = \frac{a(\alpha, \beta, \kappa, \psi_i)(\alpha + \beta)^{\psi_i} - b(\alpha, \beta, \psi_i)}{a(\alpha, \beta, \kappa, \psi_i)\left(1 + (\alpha + \beta)^{2\psi_i}\right) - 2b(\alpha, \beta, \psi_i)}, \quad (41)$$

$$\begin{aligned} a(\alpha, \beta, \kappa, \psi_i) &= \psi_i (1 - \beta)^2 + 2\psi_i (\psi_i - 1) \frac{(1 - \alpha - \beta)^2 (1 - (\alpha + \beta)^2 + \alpha^2)}{(\kappa - 1)(1 - (\alpha + \beta)^2)} \\ &\quad + \frac{4c(\alpha, \beta, \psi_i)}{1 - (\alpha + \beta)^2}, \end{aligned} \quad (42)$$

$$b(\alpha, \beta, \psi_i) = \frac{\{\alpha - \beta\alpha(\alpha + \beta)\} (1 - (\alpha + \beta)^{2\psi_i})}{1 - (\alpha + \beta)^2}, \quad (43)$$

$$c(\alpha, \beta, \psi_i) = \left(\psi_i (1 - \alpha - \beta) - 1 + (\alpha + \beta)^{\psi_i} \right) \left(\alpha (1 - (\alpha + \beta)^2) + \alpha^2 (\alpha + \beta) \right). \quad (44)$$

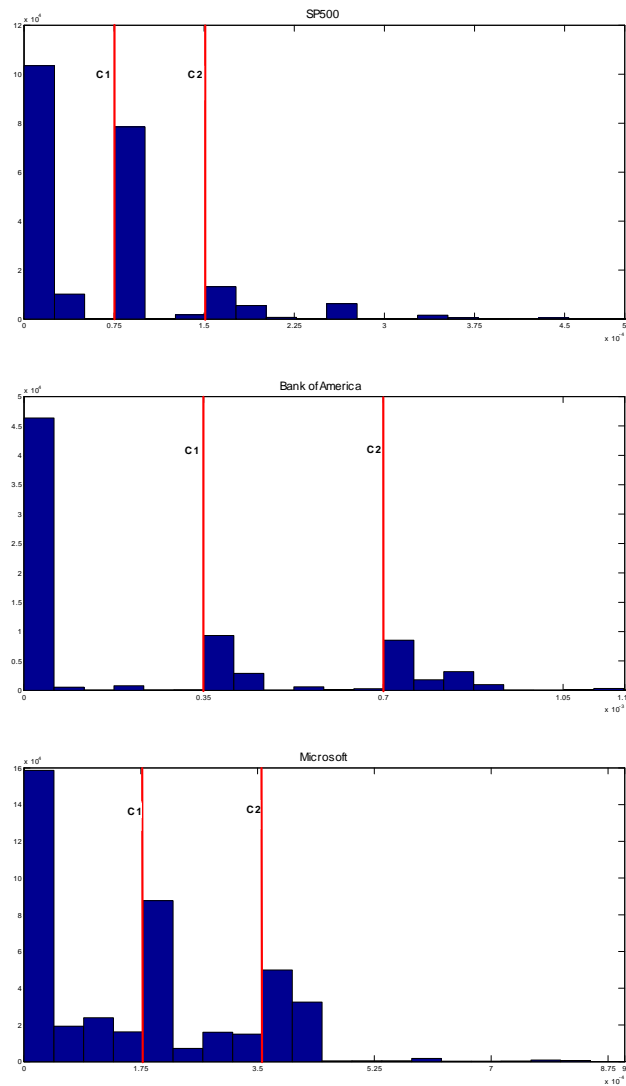
where κ denotes the kurtosis of the returns. The parameters ω , α , β correspond to the GARCH model defined for the returns sampled at a regular frequency of one second. The time varying parameters ω_{i-1} , α_{i-1} , β_{i-1} correspond to the weak GARCH process obtained for the aggregated returns over a period of ψ_i seconds.

Let $\theta = (\theta^d, \theta^r)$ be the complete parameter vector, with θ^d the subvector of parameters of the ACD model and θ^r a subvector containing the parameters of the conditional mean and variance for the

unequally time-spaced returns. As shown by Ghysels and Jasiak (1998), there are many ways to estimate the ACD-GARCH model. In this article we adopt a sequential procedure by estimating in a first step an ACD(p, q) model as in Engle and Russell (1996), and by using in a second step the sequence of expected duration estimates $\psi_i(\hat{\theta}^d)$ to estimate the weak GARCH model along the lines of Ghysels and Jasiak (1998).

B Appendix B: Empirical distribution of absolute returns

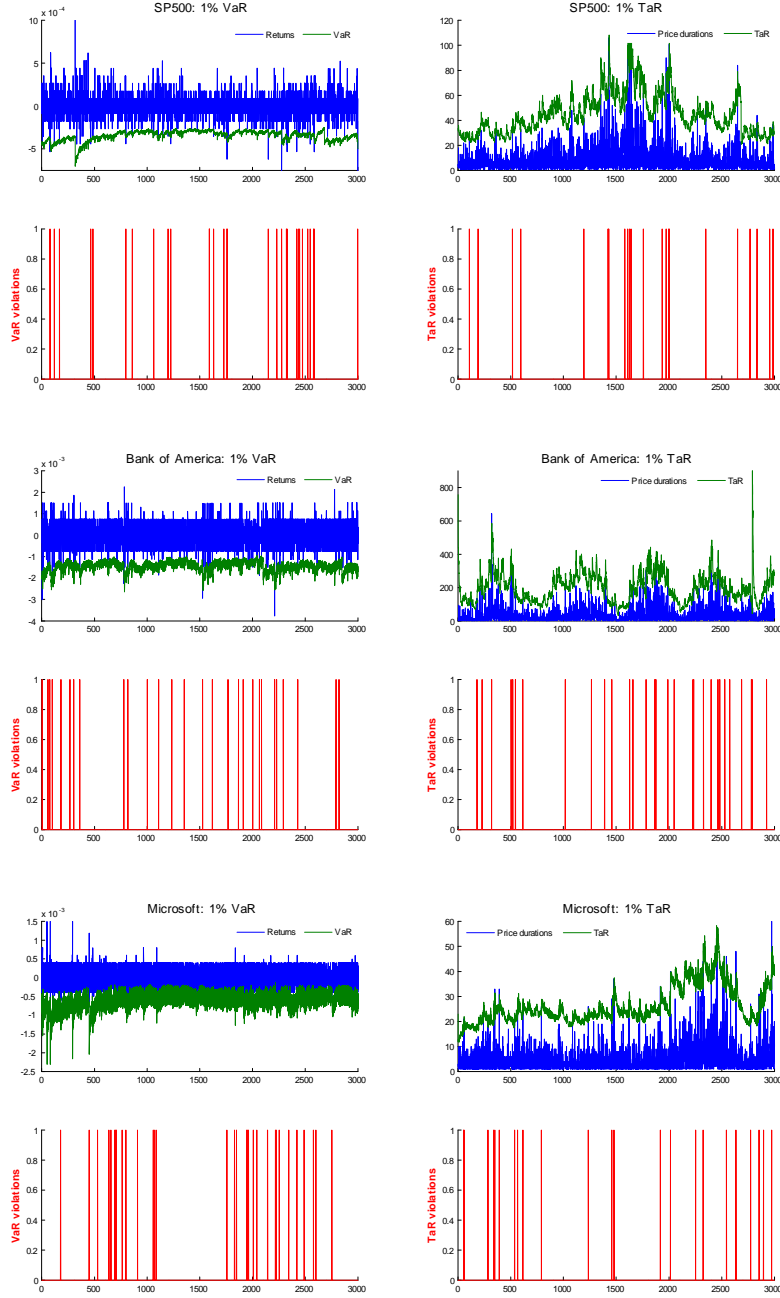
Figure B1: Empirical distribution of absolute returns



Note: This figure reports the empirical distribution of absolute returns for each of the three stocks under analysis. The histogram is computed with 100 classes, but to better observe the price changes we report here a zoom on the first 20 classes. The red vertical lines mark the thresholds C_1 and C_2 used to define price events.

C Appendix C: HFR forecasts for price events

Figure C1: VaR and TaR out-of-sample forecasts (price events)



Note: The left panel of each figure displays the forecasts of the intraday 1%-VaR, the ex-post returns, as well as the corresponding violations (in red). The right panel displays the intraday 1%-TaR, the durations and the corresponding violations. The analysis is performed for price events. The out-of-sample period starts on September 16, 2010 at 9:30 am for Microsoft and on September 23, 2010 at 9:30 am for S&P 500 and Bank of America.

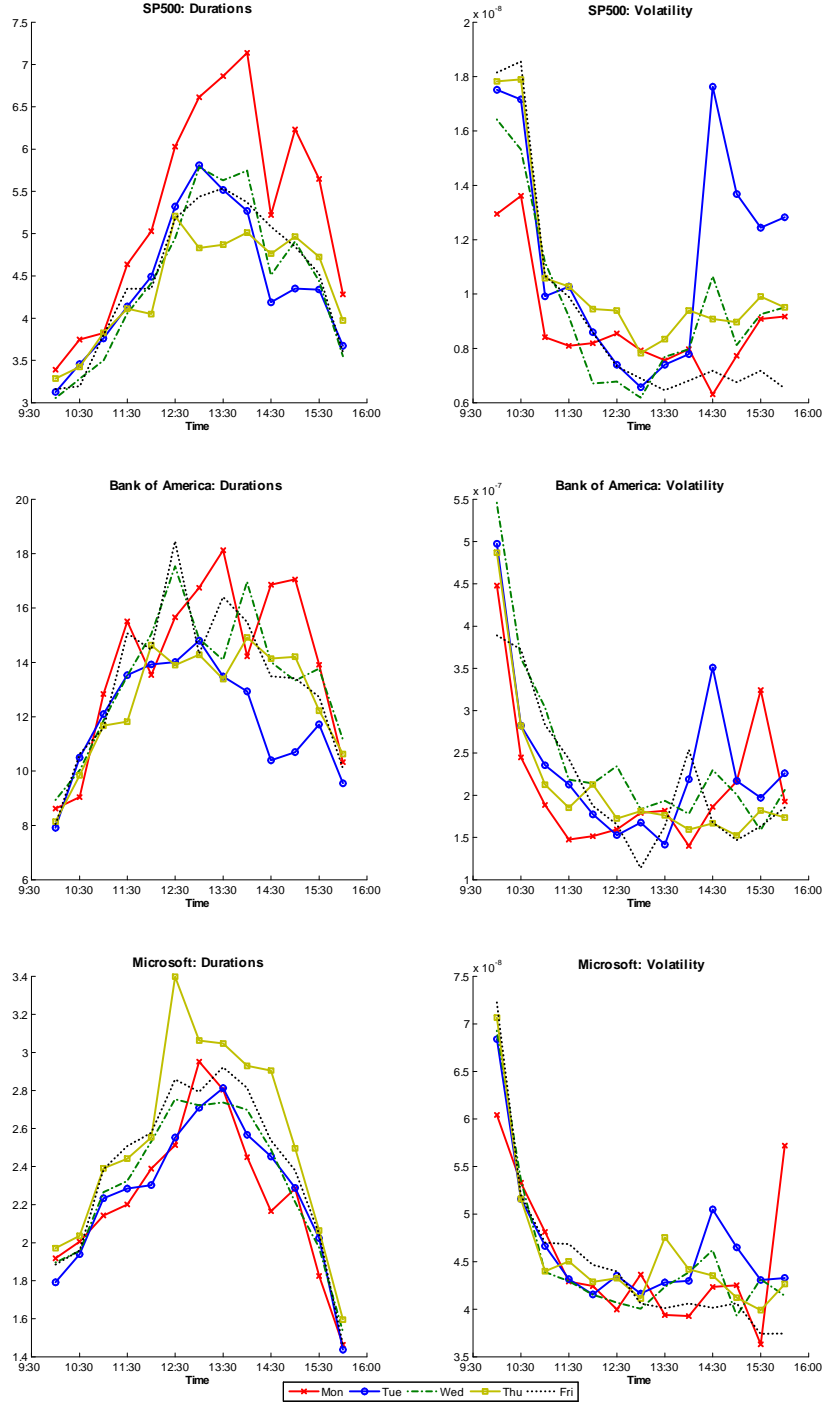
References

- [1] Adrian, T., Brunnermeier, M., 2011. CoVaR. Working Paper, Princeton University and Federal Reserve Bank of New York.
- [2] Ahlstedt, J., Villysson, J. 2012. High-frequency trading, Working Paper.
- [3] Anatolyev, S., Shakin, D., 2007. Trade Intensity in the Russian Stock Market: Dynamics, Distribution and Determinants. *Applied Financial Economics* 17, 87-104.
- [4] Andersen, T., Bollerslev, T., 1997. Intraday Periodicity and Volatility Persistence in Financial Markets, *Journal of Empirical Finance*, 4, 115-158.
- [5] Andersen, T., Bollerslev, T., 1998. Deutsche Mark-Dollar Volatility: Intraday Activity Patterns, Macroeconomic Announcements, and Longer Run Dependencies, *Journal of Finance*, 53, 219-265.
- [6] Barndorff-Nielsen, O., Hansen, P.R., Lunde, A., Shephard, N., 2009. Realized Kernels in Practice: Trades and Quotes, *Econometrics Journal*, 12, 1-32.
- [7] Bauwens, L., Giot, P., 2000. The Logarithmic ACD Model: an Application to the Bid-Ask Quote Process of Three NYSE Stocks, *Annales d'Economie et de Statistique*, 60, 11-149.
- [8] Bauwens, L., Hautsch, N., 2009. Modelling Financial High Frequency Data Using Point Processes, in *Handbook of Financial Time Series*, Springer, 953-979.
- [9] Berkowitz, J., Christoffersen, P., Pelletier, D., 2011. Evaluating Value-at-Risk Models with Desk-Level Data, *Management Science*, 57, 2213-2227.
- [10] Biais, B., 2011. High-frequency trading, European Institute of Financial Regulation.
- [11] BlackRock. 2011. ETF Landscape – Industry Highlights – End Q3 2011.
- [12] Bontemps, C., Meddahi, N., 2012. Testing Distributional Assumptions: A GMM Approach, *Journal of Applied Econometrics*, 27 (6), 978-1012.
- [13] Candelon, B., Colletaz, G., Hurlin, C., Tokpavi, S., 2011. Backtesting Value-at-Risk: A GMM Duration-based Test, *Journal of Financial Econometrics*, 9 (2), 314-343.
- [14] Christoffersen, P. F., 1998. Evaluating Interval Forecasts, *International Economic Review*, 39, 841-862.
- [15] Christoffersen, P. F., Pelletier, D., 2004. Backtesting Value-at-Risk: A Duration-Based Approach, *Journal of Financial Econometrics*, 2 (1), 84-108.
- [16] Coroneo L., Veredas, D., 2012. A Simple Two-Component Model for the Distribution of Intraday Returns, the *European Journal of Finance*, 18 (9), 775-797.
- [17] Daley D.J., Vere-Jones D., 2007. *An Introduction to the Theory of Point Processes: General Theory and Structure*, Springer.
- [18] Diamond, D. W., Verrecchia, R. E., 1987. Constraints on Short-selling and Asset Price Adjustment to Private Information, *Journal of Financial Economics* 18, 277-311.
- [19] Dionne, G., Duchesne P., Pacurar, M., 2009. Intraday Value at Risk (IVaR) Using Tick-by-Tick Data with Application to the Toronto Stock Exchange, *Journal of Empirical Finance*, 16, 777-792.

- [20] Drost, F.C., Nijman, T.E., 1993. Temporal Aggregation of GARCH Processes, *Econometrica*, 61, 909-927.
- [21] Drost, F.C., Werker, B.J.M., 1996. Closing the GARCH gap: Continuous Time GARCH Modelling, *Journal of Econometrics*, 74, 31-57.
- [22] Drost, F.C., Werker, B.J.M., 2004. Semiparametric Duration Models, *Journal of Business and Economic Statistics*, 22, 40-50.
- [23] Dufour A, Engle R., 2000. Time and the Price Impact of a Trade, *Journal of Finance*, 6, 2467-2498.
- [24] Easley, D., O'Hara, M., 1992. Time and the Process of Security Price Adjustment, *Journal of Finance* 47, 577-605.
- [25] Engle, R.F., 2000. The Econometrics of Ultra-high-frequency Data, *Econometrica*, 1, 1-22.
- [26] Engle R.F., Hendry D., Richard J.F., 1983. Exogeneity, *Econometrica*, 51 (2), 277-304.
- [27] Engle, R. F., Manganelli, S., 2004. CAViaR: Conditional Autoregressive Value-at-Risk by Regression Quantiles, *Journal of Business and Economic Statistics*, 22, 367-381.
- [28] Engle, R. F., Russell, J. R., 1997. Forecasting the Frequency of Changes in Quoted Foreign Exchange Prices with Autoregressive Conditional Duration Models, *Journal of Empirical Finance*, 4, 187-212.
- [29] Engle, R. F., Russell, J. R., 1998. Autoregressive Conditional Duration: A New Model for Irregularly Spaced Transaction Data, *Econometrica*, 66, 1127-1162.
- [30] Engle, R. F., Russell, J. R., 2005. A Discrete-State Continuous-Time Model of Financial Transactions Prices and Times: The Autoregressive Conditional Multinomial - Autoregressive Conditional Duration Model, *Journal of Business and Economic Statistics*, 23, 166-180.
- [31] Gerhard F., Hautsch N., 2002. Volatility Estimation on the Basis of Price Intensities, *Journal of Empirical Finance*, 9, 57-89.
- [32] Ghysels, E., Gouriéroux C. and J. Jasiak, 2004. Stochastic Volatility Duration Models, *Journal of Econometrics*, 119, 413-433.
- [33] Ghysels, E., Jasiak, 1998. GARCH for Irregularly Spaced Financial Data: the ACD-GARCH model, *Studies in Nonlinear Dynamics and Econometrics*, 2, 133-149.
- [34] Giot, P., 2001. Time Transformations, Intraday Data, and Volatility Models, *Journal of Computational Finance*, 4, 31-62.
- [35] Giot, P., 2005. Market Risk Models for Intraday Data, *European Journal of Finance*, 4, 309-324.
- [36] Giot, P., Grammig, J., 2006. How Large is Liquidity Risk in an Automated Auction Market, *Empirical Economics*, 30, 867-887.
- [37] Gomber, P., Arndt, B., Lutat, M., Uhle, T., 2013. High-Frequency Trading, Working Paper.
- [38] Gouriéroux, C., Montfort, A., Trognon, A., 1984. Pseudo-Maximum Likelihood Methods: Theory", *Econometrica*, 52, 681-700.
- [39] Grammig, J., Wellner, M., 2002. Modeling the Interdependence of Volatility and Inter-Transaction Duration Processes, *Journal of Econometrics* 106, 369-400.

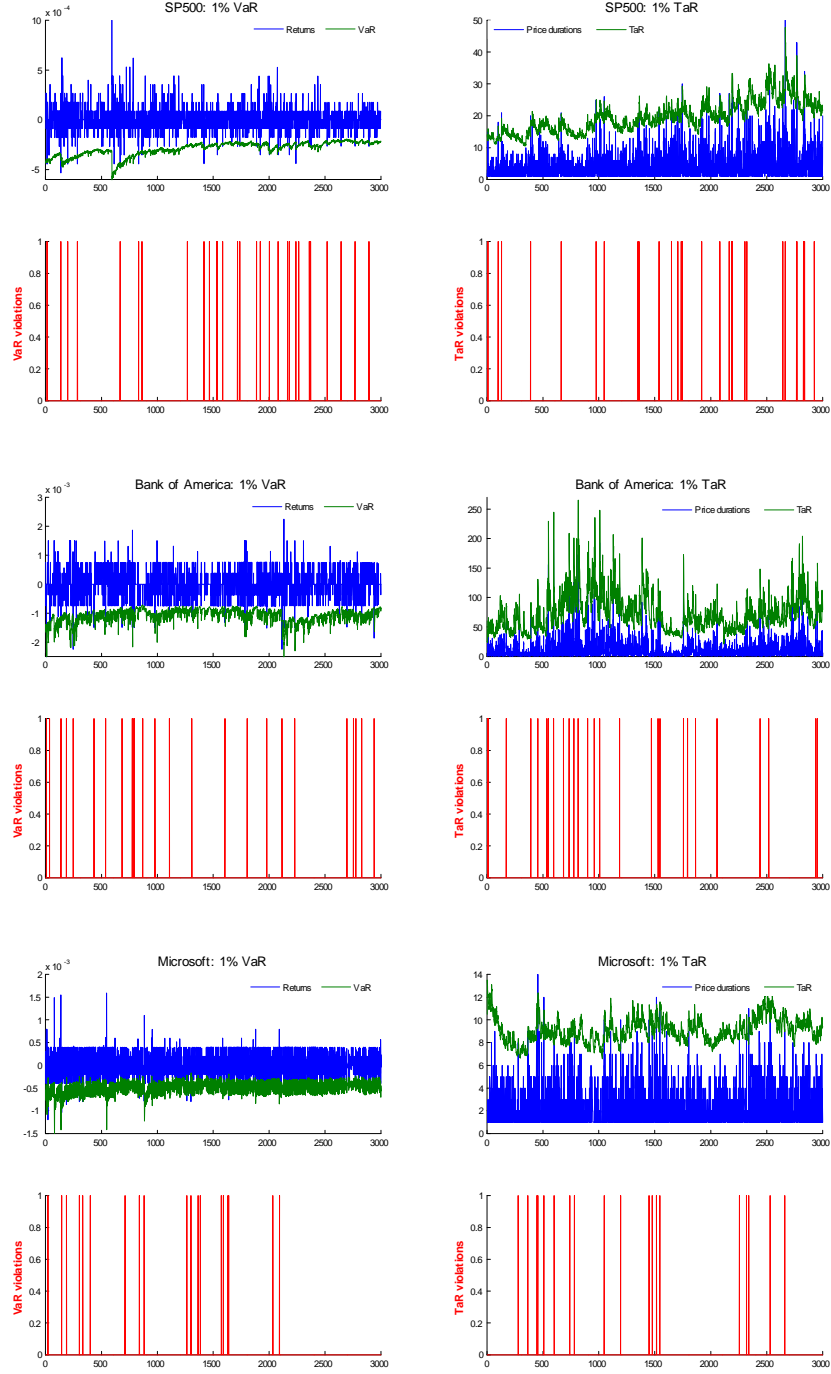
- [40] Grant, J., 2010. High-frequency Trading: Up against a Bandsaw, Financial Times. Sept. 10, 2010.
- [41] Hansen, P.R., Timmermann, A., 2012. Choice of Sample Split in Out-of-Sample Forecast Evaluation, Working Paper.
- [42] Harris, L., 1986. A Transaction Data Study of Weekly and Intradaily Patterns in Stock Returns, Journal of Financial Economics, 16, 99-117.
- [43] Hautsch, N. 2004. Modelling Irregularly Spaced Financial Data Theory and Practice of Dynamic Duration Models, Lecture Notes in Economics and Mathematical Systems, 539, Springer, Berlin.
- [44] Hautsch, N. 2012. Econometrics of Financial High-Frequency Data, Springer, Berlin.
- [45] Hecq, A., Laurent, S., Palm, F. 2012. Common Intraday Periodicity, Journal of Financial Econometrics, 10, 325-353.
- [46] Jorion, P., 2007. Value at Risk: The New Benchmark for Managing Financial Risk, McGraw-Hill, 3rd Edition.
- [47] Kozhan R., Tham W.W., 2013. Execution Risk in High-Frequency Arbitrage. Management Science, 58, 2131-2149.
- [48] Kupiec, P., 1995. Techniques for Verifying the Accuracy of Risk Measurement Models, Journal of Derivatives, 3, 73-84.
- [49] Pacurar, M., 2008. Autoregressive Conditional Duration (ACD) Models in Finance: A Survey of the Theoretical and Empirical Literature, Journal of Economic Surveys, 22. 711-751.
- [50] Meddahi, N., Renault, E., Werker, B., 2005. GARCH and Irregularly Spaced Data, Economics Letters, 90, 200-204.
- [51] Russell, J. R., 1999. Econometric modelling of multivariate irregularly-spaced high-frequency data, University of Chicago, Graduate School of Business working paper series.
- [52] Tse, Y., Yang, T.T., 2012. Estimation of High-Frequency Volatility: An Autoregressive Conditional Duration Approach, Journal of Business & Economic Statistics, 30(4), 2012.
- [53] Wood, R.A., McInish, T.H., Ord, J.K., 1985. An Investigation of Transaction Data for NYSE Stocks, Journal of Finance, 25, 723-739.

Figure 2: Intraday periodicity



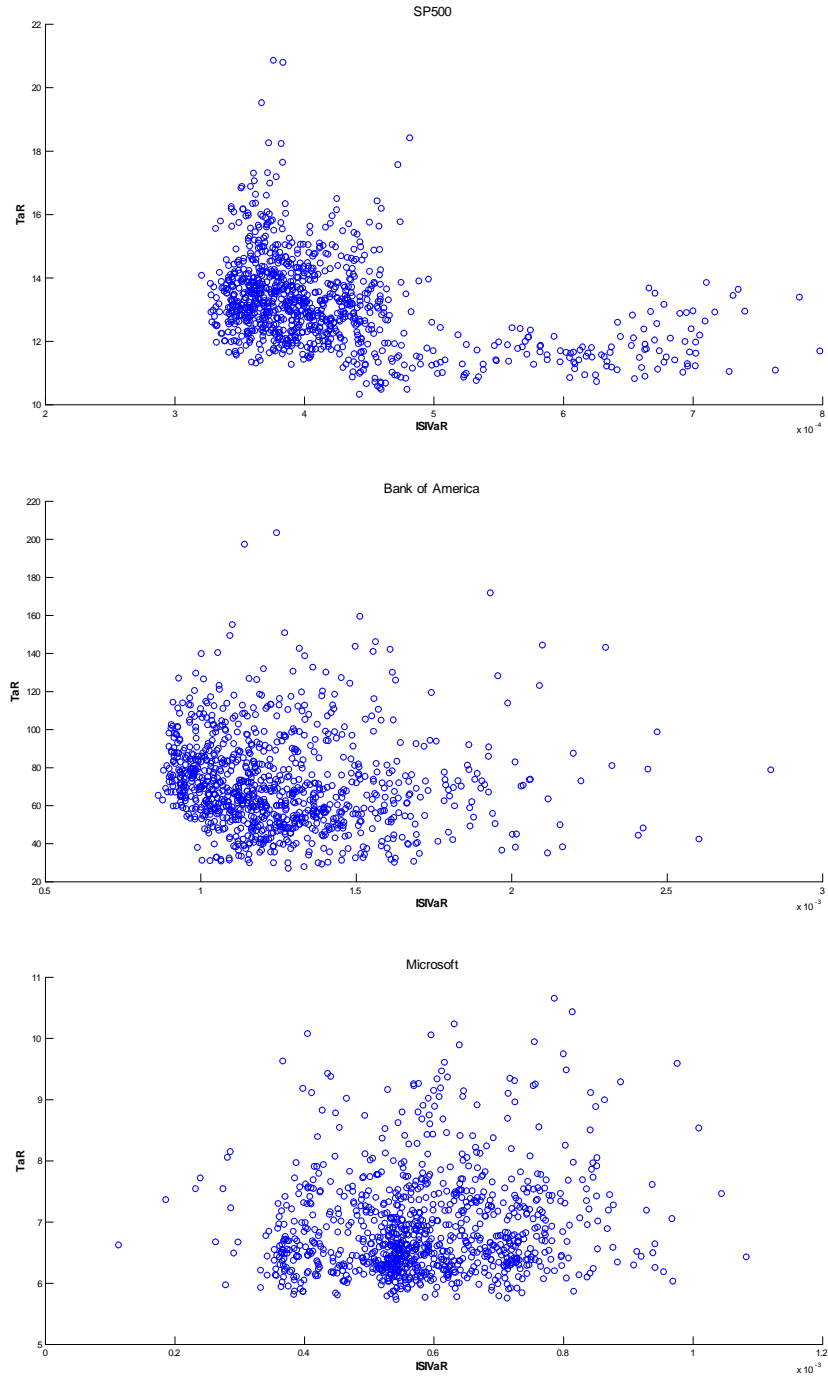
Note: This figure displays the deterministic seasonal component for S&P500, Bank of America and Microsoft. It emphasizes the presence of intraday periodicity in average durations and returns across each trading day of the week.

Figure 3: VaR and TaR out-of-sample forecasts



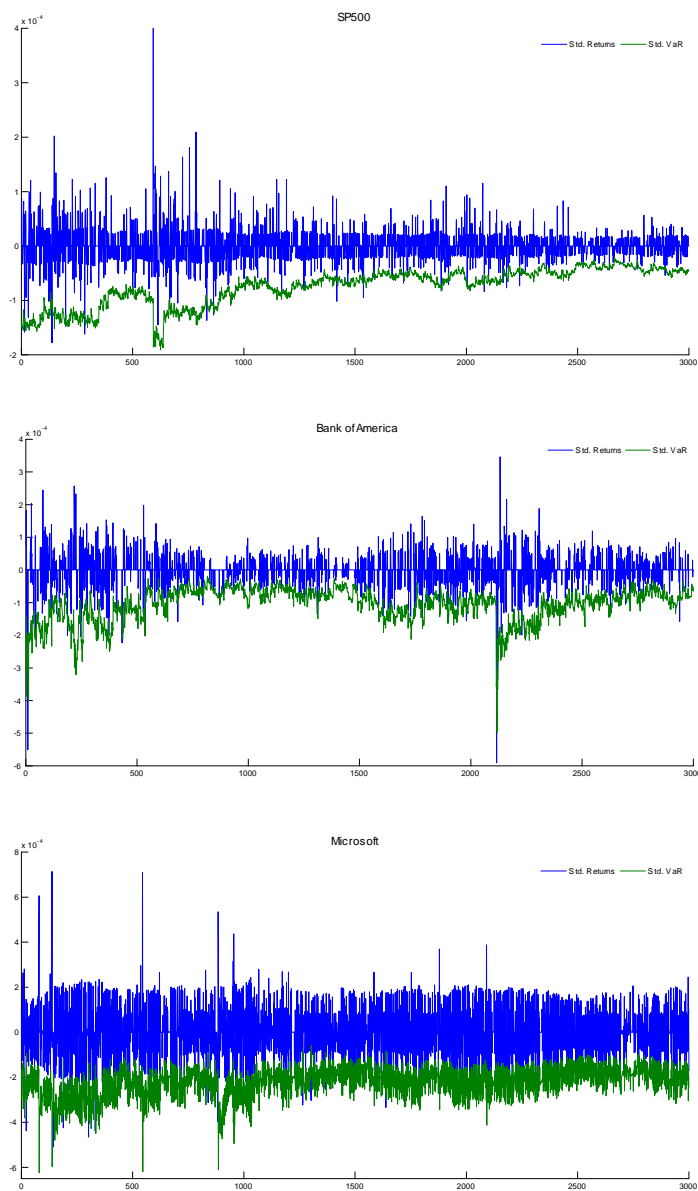
Note: The left panel of each figure displays the forecasts of the intraday 1%-VaR, the ex-post returns, as well as the corresponding violations (in red). The right panel displays the intraday 1%-TaR, the durations and the corresponding violations. The analysis is performed for trade events. The out-of-sample period starts on September 16, 2010 at 9:30 am for Microsoft and on September 23, 2010 at 9:30 am for S&P 500 and Bank of America.

Figure 4: Relationship between intraday VaR and TaR



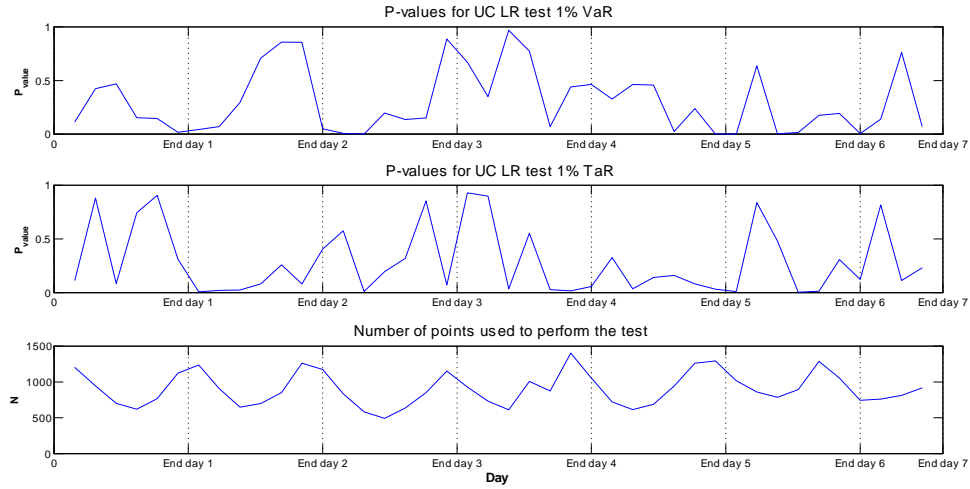
Note: This scatter plot illustrates the relation between the intraday 1%-VaR and 1%-TaR forecasts for each of the three assets under analysis. The sample period ranges from September, 23, 2010 to October 29, 2010 for S&P 500 and Bank of America and from September 16 to October 29, 2010 for Microsoft.

Figure 5: Per-time unit VaR forecasts



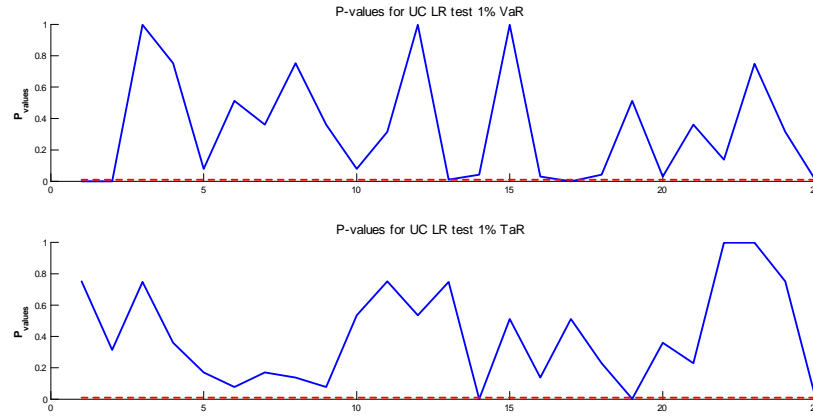
Note: This figure compares the ex-ante per-time-unit VaR to the ex-post returns standardized by the conditional duration. The out-of-sample period starts on September 16, 2010 at 9:30 am for Microsoft and on September 23, 2010 at 9:30 am for S&P 500 and Bank of America.

Figure 6: Fixed period backtesting



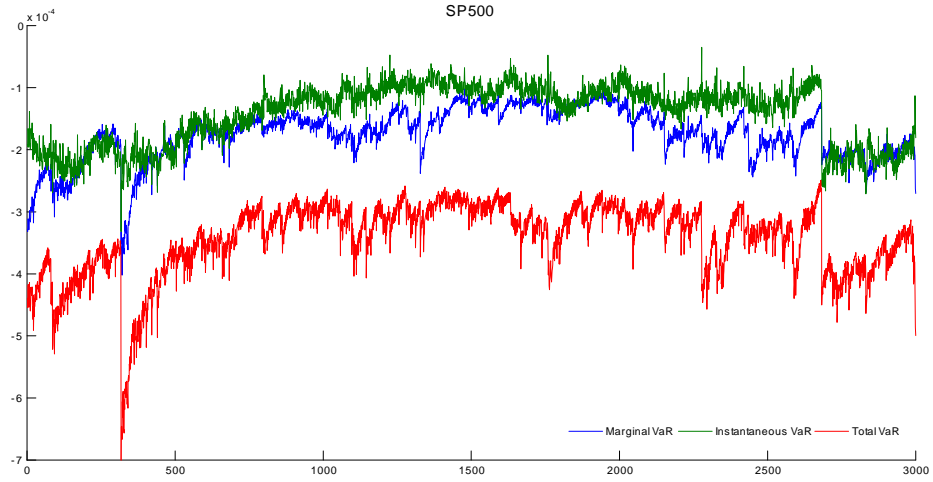
Note: The first two figures report the p -values of the Christoffersen's LR test (1998) for unconditional coverage (UC) for the S&P 500 1%-VaR and 1%-TaR forecasts.. The backtesting procedure is performed with a rolling window of one hour during seven days (September, 23 to October, 1st). The last figure displays the corresponding number of observations used to perform the aforementioned tests.

Figure 8: Rolling window backtesting



Note: This figure presents the p -values values of the Christoffersen's LR test (1998) of unconditional coverage (UC) for the BAC 1%-VaR and 1%-TaR forecasts. The analysis is performed by using at each iteration a rolling window of 1,000 observations for both the estimation and backtesting samples (with no overlapping for the latter).

Figure 9: Total, Instantaneous and Marginal 1%-VaR



Note: This figure illustrates the total, the marginal and the instantaneous 1%-VaR for SP500. The analysis is performed for price events and a threshold $C=0.0075\%$. The out-of-sample period starts on September 16, 2010 at 9:30 am.

Table 1: Descriptive statistics

Panel A: Durations									
C	S&P500			BAC			MSFT		
	Trade	0.0075%	0.015%	Trade	0.035%	0.07%	Trade	0.0175%	0.035%
Mean	4.396	8.870	24.707	12.552	32.283	50.378	2.271	4.535	8.828
Std.Dev	4.706	10.226	30.112	17.638	42.377	67.220	2.067	5.039	15.601
Overdisp	1.070	1.153	1.219	1.405	1.313	1.334	0.910	1.111	1.767
Skewness	3.125	3.488	3.798	3.859	3.645	3.807	3.116	3.607	16.134
Kurtosis	19.504	26.477	30.911	27.324	27.948	28.802	19.642	32.978	592.208
Q(10)	19,038	14,633	13,939	8,232	5,015	4,920	55,613	12,651	3,770
Q(20)	31,748	26,083	24,525	11,375	8,441	8,131	98,958	21,951	4,603
N	223,501	110,774	39,756	78,265	30,423	19,484	432,626	216,641	111,272
Panel B: Returns									
Mean	2.2E-07	4.3E-07	1.2E-06	-2.8E-06	-7.2E-06	-1.1E-05	6.3E-08	1.2E-07	2.3E-07
Std.Dev	1.03E-04	1.46E-04	2.36E-04	4.94E-04	7.92E-04	9.75E-04	2.35E-04	3.28E-04	4.22E-04
Skewness	0.031	0.017	0.006	-0.042	-0.010	0.004	-0.004	-0.005	-0.004
Kurtosis	14.944	7.448	3.308	9.815	3.826	2.717	4.627	2.468	1.824
Q(10)	845	696	39	1,461	964	582	72,621	172,101	143,114
Q(20)	850	702	45	1,466	977	593	72,630	172,661	155,022

Note: This table reports some descriptive statistics for the three assets under analysis, namely S&P500, Bank of America (BAC) and Microsoft (MSFT), both for durations (Panel A) and returns (Panel B) series. $Q(\cdot)$ denotes the Ljung-Box Q-statistics computed with 10 or 20 lags. For each asset we consider the duration and return series for trade events (first column) and price events (second and third columns) defined by the threshold C . N denotes the number of observations. The sample period ranges from September 1st to October 29, 2010.

Table 2: Estimates of EACD models

C	S&P500		BAC		MSFT	
	Trade	0.0075%	Trade	0.035%	Trade	0.0175%
ω	0.002 (9.944)	0.008 (6.453)	0.008 (10.042)	0.088 (39.331)	0.000 (8.831)	0.014 (691.270)
α_1	0.062 (437.221)	0.069 (120.233)	0.154 (140.059)	0.076 (33.239)	0.051 (2,917.927)	-0.003 (-8.110)
α_2	-0.056 (-1,388.710)	-0.050 (-158.719)	-0.131 (-435.223)	0.095 (145.334)	-0.049 (-654.938)	0.050 (272.490)
β_1	1.663 (8,640.545)	1.518 (1,083.607)	1.484 (584.150)	0.039 (2.996)	1.830 (93,255.556)	0.132 (23.277)
β_2	-0.671 (-4,329.126)	-0.544 (-1,521.715)	-0.515 (-598.255)	0.711 (77.015)	-0.832 (-8,772.861)	0.808 (145.954)
Q(10)	0.819	0.925	0.063	0.994	0.737	0.105
Q(20)	0.983	0.950	0.214	0.584	0.917	0.167

Note: This table contains the parameter estimates of the EACD models for SP&500, Bank of America (BAC) and Microsoft (MSFT). Corresponding t-statistics are in parentheses. The estimation sample covers the first three weeks of our sample (September 1st to September 22, 2010), including 80,845 trade durations for SP500, 27,972 for BAC and 99,155 for MSFT. $Q(\cdot)$ denotes the p -values of the Ljung-Box Q-statistics for serial correlation computed with 10 and 20 lags for the standardized durations. The model is estimated both for trade durations (first column) and price durations (second column) defined for a threshold C .

Table 3: Estimates of GARCH models

C	S&P500		BAC		MSFT	
	Trade	0.0075%	Trade	0.035%	Trade	0.0175%
ω	2.90E-06 (0.001)	2.27E-06 (0.000)	1.17E-04 (0.020)	6.29E-06 (0.001)	2.81E-05 (0.009)	1.02E-05 (0.002)
α	0.005 (1.484)	0.009 (1.451)	0.034 (5.636)	0.034 (2.752)	0.019 (5.833)	0.022 (3.358)
β	0.992 (282.171)	0.989 (156.343)	0.951 (159.130)	0.963 (78.980)	0.974 (306.724)	0.974 (151.694)
Mean ω_i	0.011	0.020	0.182	0.121	0.018	0.023
Range ω_i	[0.005; 0.032]	[0.007; 0.077]	[0.028; 0.686]	[0.020; 0.424]	[0.008; 0.041]	[0.007; 0.077]
Mean α_i	0.014	0.023	0.075	0.061	0.026	0.046
Range α_i	[0.009; 0.022]	[0.016; 0.031]	[0.046; 0.077]	[0.045; 0.063]	[0.019; 0.033]	[0.028; 0.068]
Mean β_i	0.976	0.957	0.753	0.822	0.957	0.933
Range β_i	[0.947; 0.987]	[0.894; 0.977]	[0.344; 0.927]	[0.547; 0.935]	[0.927; 0.973]	[0.860; 0.965]
Q(10)	45	47	65	24	5,836	468
Q(20)	49	54	70	32	5,842	486

Note: This table contains the parameter estimates of the ACD-GARCH models for S&P500, Bank of America (BAC) and Microsoft (MSFT). Corresponding t-statistics are in parentheses. The estimation sample covers the first three weeks of our sample (September 1st to September 22, 2010), including 80,845 trade durations for SP500, 27,972 for BAC and 99,155 for MSFT. We report the mean and the average of the time varying parameters ω_i , α_i and β_i . $Q(\cdot)$ denotes the Ljung-Box Q-statistics for serial correlation computed with 10 and 20 lags for the standardized returns. The model is estimated both for trade durations (first column) and price durations (second column) defined for a threshold C .

Table 4: Backtesting of HFR out-of-sample forecasts

Panel A: VaR backtesting									
S&P500									
Sample size	1,000			2,000			3,000		
%Hits	0.007			0.009			0.010		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.315	0.575	0.753	0.490	0.681	0.589	0.855	0.741	0.452
LR duration test		0.231	0.302		0.560	0.581		0.432	0.214
GMM duration test	0.476	0.823	0.881	0.648	0.937	0.972	0.945	0.754	0.516
BAC									
Sample size	1,000			2,000			3,000		
%Hits	0.013			0.009			0.008		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.360	0.554	0.558	0.649	0.766	0.567	0.346	0.520	0.517
LR duration test		0.259	0.127		0.178	0.084		0.162	0.125
GMM duration test	0.276	0.522	0.440	0.810	0.561	0.346	0.451	0.485	0.406
MSFT									
Sample size	1,000			2,000			3,000		
%Hits	0.011			0.010			0.007		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.752	0.016	0.004	0.822	<i>0.041</i>	<i>0.012</i>	0.081	<i>0.006</i>	<i>0.008</i>
LR duration test		0.706	0.404		0.242	0.105		<i>0.006</i>	<i>0.012</i>
GMM duration test	0.574	0.905	0.954	0.963	0.193	0.079	0.094	<i>0.002</i>	<i>0.015</i>
Panel B: TaR backtesting									
S&P500									
Sample size	1,000			2,000			3,000		
%Hits	0.006			0.008			0.010		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.171	0.377	0.788	0.241	0.448	0.634	0.855	0.556	0.286
LR duration test		0.097	0.205		0.296	0.514		0.637	0.381
GMM duration test	0.207	0.103	0.737	0.331	0.288	0.696	0.944	0.351	0.190
BAC									
Sample size	1,000			2,000			3,000		
%Hits	0.014			0.011			0.009		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.229	0.398	0.528	0.657	0.709	0.484	0.577	0.670	0.484
LR duration test		0.224	0.141		0.445	0.210		0.634	0.551
GMM duration test	0.206	0.440	0.581	0.545	0.820	0.696	0.663	0.870	0.780
MSFT									
Sample size	1,000			2,000			3,000		
%Hits	0.008			0.007			0.006		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.512	0.756	0.719	0.155	0.329	0.657	<i>0.031</i>	0.085	0.623
LR duration test		0.606	0.970		0.240	0.808		0.057	0.735
GMM duration test	0.673	0.703	0.495	0.180	0.431	0.929	<i>0.027</i>	<i>0.027</i>	0.885

Note: This table reports the p -values of the backtesting tests of intraday 1%-VaR (Panel A) and 1%-TaR (Panel B) for the three assets under analysis. Italic entries indicate a failure of the model at the 95% confidence level. We test for the unconditional coverage (UC) hypothesis, the independence of violations (IND) and for the conditional coverage (CC). Three tests are considered: Christoffersen's LR test (1998) based on a Markov chain model, Christoffersen and Pelletier's duration-based test (2004) and the GMM-based test proposed by Candelon et al. (2011). %Hits denotes the empirical frequency of violations obtained for a sample size of 1,000, 2,000 and 3,000 observations.

Table 5: Backtesting of HFR out-of-sample forecasts (fixed time periods)

Panel A: VaR backtesting									
S&P500									
Sample period	30 min (608 obs)			1h (1,122 obs)			4h (3,294 obs)		
%Hits	0.007			0.006			0.010		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.3679	0.6493	0.8178	0.1746	0.3807	0.7668	0.9902	0.716	0.4137
LR duration test		0.348	0.654		0.172	0.436		0.220	0.083
GMM duration test	0.591	0.911	0.931	0.264	0.474	0.871	0.808	0.549	0.286
BAC									
Sample period	30 min (224 obs)			1h (470 obs)			4h (1,162 obs)		
%Hits	0.018			0.013			0.012		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.284	0.523	0.702	0.560	0.781	0.693	0.495	0.668	0.559
LR duration test		0.240	0.106		0.336	0.142		0.193	0.077
GMM duration test	0.109	0.257	0.680	0.387	0.717	0.853	0.315	0.410	0.204
MSFT									
Sample period	30 min (906 obs)			1h (1,744 obs)			4h (6,101 obs)		
%Hits	0.012			0.011			0.009		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.529	0.017	0.005	0.709	0.050	0.016	0.292	0.000	0.000
LR duration test		0.725	0.459		0.526	0.260		<i>0.003</i>	<i>0.001</i>
GMM duration test	0.370	0.576	0.551	0.584	0.478	0.268	0.343	<i>0.004</i>	<i>0.008</i>
Panel B: TaR backtesting									
S&P500									
Sample period	30 min (608 obs)			1h (1,122 obs)			4h (3,294 obs)		
%Hits	0.007			0.006			0.010		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.3679	0.6493	0.8178	0.1746	0.3807	0.7668	0.9902	0.6369	0.3422
LR duration test		0.344	0.635		0.099	0.192		0.744	0.453
GMM duration test	0.548	0.603	0.883	0.250	0.281	0.826	0.815	0.300	0.131
BAC									
Sample period	30 min (224 obs)			1h (470 obs)			4h (1,162 obs)		
%Hits	0.013			0.011			0.013		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.623	0.850	0.775	0.887	0.938	0.743	0.339	0.520	0.531
LR duration test		0.977	0.881		0.900	0.748		0.350	0.200
GMM duration test	0.425	0.368	0.190	0.510	0.571	0.244	0.271	0.518	0.473
MSFT									
Sample period	30 min (906 obs)			1h (1,744 obs)			4h (6,101 obs)		
%Hits	0.009			0.008			0.008		
Test	UC	CC	IND	UC	CC	IND	UC	CC	IND
LR test	0.720	0.873	0.706	0.393	0.619	0.634	0.110	0.187	0.373
LR duration test		0.740	0.754		0.496	0.684		0.220	0.841
GMM duration test	0.881	0.646	0.381	0.542	0.590	0.544	0.119	0.165	0.779

Note: This table reports the p -values for the backtesting tests of intraday 1%-VaR (Panel A) and 1%TaR (Panel B) for the three assets under analysis. Italic entries indicate a failure of the model at the 95% confidence level. The backtesting procedure is applied over a fixed period of time (30 minutes, 1 hour, and 4 hours, respectively). The corresponding number of observations is reported for each asset. We test for the unconditional coverage (UC) hypothesis, the independence of violations (IND) and for the conditional coverage (CC) hypothesis. Three tests are considered: Christoffersen's LR test (1998) based on a Markov chain model, Christoffersen and Pelletier's duration-based test (2004) and the GMM-based test proposed by Candelon et al. (2011). %Hits denotes the empirical frequency of violations.