Dominance, dependence and interdependence in linear structures. A theoretical model and an application to the international trade flows

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Dominance, dependence and interdependence in linear structures
A theoretical model and an application to the international trade flows

Roland LANTNER[*] – Didier LEBERT[**]

Abstract
This paper aims to study the structure of international trade. It establishes, through a simple formalization of exchange coefficients, that many theorems can be proved on a function of the macroscopic structure (the determinant of the matrix). This determinant is the cornerstone of indicators to analyze the evolution of trade between countries and regions. The objective is to introduce new tools to rigorously measure the characteristics and effects of globalization. The structural analysis proposed in this way can be applied to many other areas.

Dominance, dépendance et interdépendance dans les structures linéaires
Modèle théorique et application aux flux du commerce international

Résumé
L'objet de cet article est d'étudier la structure du commerce international. A partir d'une formalisation simple des coefficients d'échange, il propose une série de théorèmes fondés sur les propriétés du déterminant matriciel. Ce déterminant est la pierre angulaire d'une série d'indicateurs permettant l'analyse du commerce entre les pays et les régions. Ces indicateurs nous conduisent à identifier les caractéristiques et les effets de la globalisation des échanges. Cette analyse structurale peut être appliquée à de nombreux autres domaines.

Keywords: Influence graph theory, interdependence, international trade

Mots-clés : Théorie des graphes d'influence, interdépendance, commerce international

JEL Classification: C65, C67, F14

AMS Classification: 05C25, 05C69, 94C15

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Introduction

Social scientists deal with quantified relationships networks between “activity centers” or “sectors” in an extensive meaning (for economists, industrial sectors, industrial groups, regions, countries and, for sociologists, individuals, social groups, etc.). In the graphs of influence, these sectors are represented by the vertices. The vertices are linked by physical exchange flows, or financial ones, information ones, power relationships, etc. In order to simplify the representation, we assume that the relations between sectors may be considered as linear ones (Leontief, 1951)[1]. There are two justifications for this approach:

- First, in some economic structures, the relationships between sectors have a long term stability (cf. e.g. de Mesnard and Dietzenbacher, 1995).
- Second, some indicators may be used to describe the static structure or to compare two states of an evolutive structure at two different times (cf. e.g. Lahr and Yang, 2011).

One of the main objectives of this paper is to define indicators in order to understand the general architecture of the structure: what is the dominance of one sector upon another one (Hurwicz, 1955, Leontief, 1986, Miller and Blair, 1986)? What is its degree of dominance on the whole structure (Perroux, 1948, 1973)? Is there any part of the structure in a dominant position, on another one or on some other ones (Aujac, 1960, Ponsard, 1969, 1972)? Are sectors more or less self-sufficient, interdependent, or/and dependent on some of them (Sonis and Hewings, 1998)? Is the structure more or less turned towards its outside (Goldwyn, 1960, Sonis and Hewings, 2001)?

This approach may be applied to various issues. For example, rectangular IO tables (Joyal, 1973), interregional relationships (Isard and Ostroff, 1960), capital movements (Lequeux, 2002), information flows (Gallo, 2006). It seems to be particularly relevant for the study of the international trade flows (Hewings et al., 2002, Lebert, 2010). Reichardt and White (2007), for example, use blockmodeling techniques to identify differentiated "roles" for countries participating in the international trade. The authors’ goal is to group countries with close structural characteristics and analyze hierarchical relationships between these sets. Barigozzi et al. (2009) investigate the degree of similarity between the trade structures for a

[1] About the structure of linear models, see for example Solow (1952).

hundred of agricultural and industrial goods. They assume that there exists "intrinsic correlations" between products. This assumption means that products with close characteristics are expected to have close international trade structures. These intrinsic correlations differ from "revealed correlations" which measure the empirical similarity of trade structures. Any movement of the revealed correlations is interpreted as the result of changes in the institutional organization of the international trade. Finally, Hidalgo et al. (2007) study the evolution of the historical positions of countries within an evolving product space. This product space is a representation of the distances between the structures of international trade flows for all products. Historical positioning of the countries in this space uses specific revealed comparative advantage indices. The authors insert these indices in a theoretical argument based on the notion of "capability" of countries: the complexity of the national productive structure determines the country’s potential to promote its own development.

A constraint that we set ourselves is to **combine quantitative and qualitative relationships**: a sector can strongly influence one or two others, while another can influence many of them more weakly. Below, we avoid to weight explicitly quantitative and qualitative aspects (Lantner, 1974, 2002[2], Gazon, 1976, Defourny et Thorbecke, 1984). In order to solve this problem, we prove and use some particular properties of the determinant of some matrices which lead us to an endogenous combination of quantities and qualities (in another way, cf. Maybee et al., 1989).

The first section of the paper is devoted to the way we represent the structure of exchanges. The second section focuses on the properties of the determinant of the matrix representing the structure; many important theorems are introduced. The third and fourth sections offer elements which expand the theoretical and operational scope of the approach (structural indicators, partition of the structure). The fifth section analyzes the structure of the international trade using the influence graph theory.

[2] See also Lantner (1972a, 1972b, 1976), Lantner and Carluer (2004).

1. Definition and representation of the structure

In order to give a very simple idea of the model, let us consider for instance a national productive structure (table of inter-industry trade). In this example, let the sales from sector *i* to sector *j* ($i, j \in [1, n]$) be denoted by $x_{ij} \in \mathbb{R}_+$. From the point of view of demand, it comes:

$$(S1)\ \ i, j \in [1, n]:\ \ X_i - \sum_{j=1}^{n} x_{ij} = Y_i$$

where X_i is the production of the i^{th} sector and Y_i the demand of the outside of the structure to this sector (given data).

Symmetrically, from a supply point of view:

$$(S2)\ \ \ i, j \in [1, n]:\ \ X_j - \sum_{i=1}^{n} x_{ij} = W_j$$

where W_j if the value added in the j^{th} sector.

Let the column vector of production (X_i) be denoted by X, the column vector of external demand (Y_i) be denoted by Y, and the matrix whose terms are $a_{ij} = x_{ij}/X_j$ (called "technical coefficients"; $\forall i, j$: $0 \leq a_{ij} \leq 1$ and $\sum_{i=1}^{n} a_{ij} \leq 1$) be denoted by a. From ($S1$) we draw ($S3$):

$$(S3)\ \ \ AX = Y$$

where $A = [I - a]$ is the Leontief matrix.

By construction of A, its inverse exists, and:

$$(S3')\ \ \ X = A^{-1}Y$$

In parallel, the terms $t_{ij} = x_{ij}/X_i$ (with: $\forall i,j$: $0 \leq t_{ij} \leq 1$ and $\sum_{j=1}^{n} t_{ij} \leq 1$) are called "trade coefficients". If the matrix whose terms are t_{ij} is denoted by *t* and the complementary matrix $[I - t]$ is denoted by *T*, we have:

$$(S4) \quad X^R T^T = W^R$$

where X^R is the row vector of X_i, W^R the row vector of W_i and T^T the transposed matrix of *T*.

Matrices *A* and *T* have the same diagonal coefficients, resulting from the sectoral auto-consumptions. By definition:

$$i \in [1, n]: \ell_i \equiv 1 - a_{ii} = 1 - t_{ii}$$

The **influence graph** of the exchange structure is a directed graph ("digraph") defined as follows (on graph theory, see for example Busacker and Saaty, 1965, Harary et al., 1966, Berge, 1970, Roy, 1970; on signal-flow graphs, see for example Coates, 1959, Chow and Cassignol, 1962):

1. Each country, sector, industrial group, social group, individual, etc. corresponds to a vertex i.
2. Each exchange x_{ij} is represented by an arc; all the arcs are oriented in the direction of the "dominant influence": either from the demand to the supply, or from the supply to the demand.
3. The arcs of the graph are valuated either by the technical coefficient a_{ij} or by the trade coefficient t_{ij}, and the "loops" (corresponding to the diagonal terms of the *A* or *T* matrices) are valuated by the coefficients $\ell_i = (1 - a_{ii}) = (1 - t_{ii})$.
4. To each vertex is associated a centrifugal or a centripetal arc of the graph linking the structure to its outside. The orientation of the arc is given by the direction of the 'dominant influence'. The weight of the arc is:

$$w_j \equiv W_j/X_j = \left(1 - \sum_{i=1}^{n} a_{ij}\right) \geq 0$$

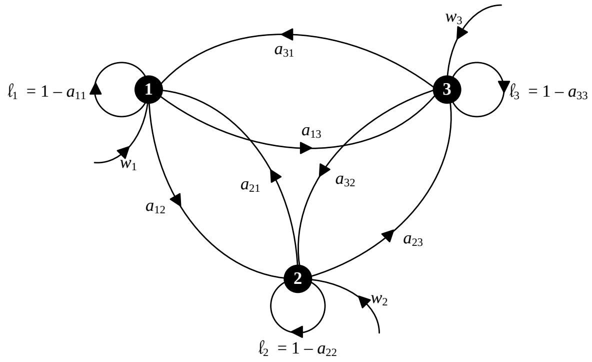
if the weights of the other arcs are the technical coefficients a_{ij}.

It is:

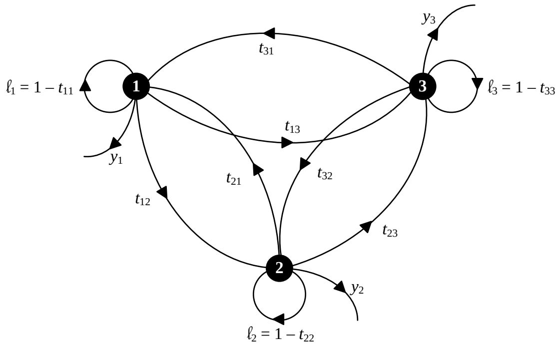
$$y_i \equiv Y_i / X_i = \left(1 - \sum_{j=1}^{n} t_{ij}\right) \geq 0$$

if the weights of the other arcs are the trade coefficients t_{ij}.

The *influence graph* with technical coefficients

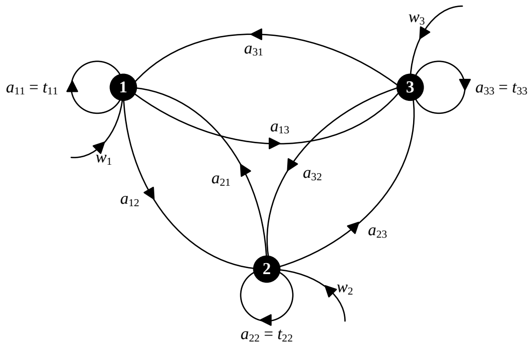


The *influence graph* with trade coefficients

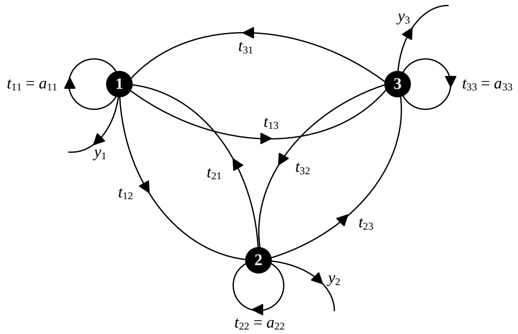


The difference with physical graphs lies in the coefficients of the loops

Graph of physical flows endowed with technical coefficients



Graph of physical flows endowed with trade coefficients



Representative matrices of the structures are respectively:

$$A = \begin{bmatrix} \ell_1 & -a_{12} & -a_{13} \\ -a_{21} & \ell_2 & -a_{23} \\ -a_{31} & -a_{32} & \ell_3 \end{bmatrix} \qquad T = \begin{bmatrix} \ell_1 & -t_{12} & -t_{13} \\ -t_{21} & \ell_2 & -t_{23} \\ -t_{31} & -t_{32} & \ell_3 \end{bmatrix}$$

The technical coefficients and the trade coefficients are two representations of the same structure of exchange. For example, if we want to compute the effect of ΔX_i on the total output X_i of the change ΔY_j in the final demand for sector *j*'s products, it is easier to use the matrix *A* (technical coefficients), which provides the "absolute global effect" (or "sensitivity") $\Delta X_i / \Delta Y_j$. Symmetrically, using T (trade coefficients) would provide the "relative global effect" (or "elasticity") $(\Delta X_i / X_i) / (\Delta Y_j / Y_j)$ (cf. appendix 3).

Why do we persist in using both graph theory and matrices to deal with such issues? The advantage of using graph theory is not "to see" (or even "to show" or "to draw") the arrangement of the structure of intersectoral flows. It is only to discover new properties of the architecture of the structure, to prove new theorems, and to find new structural indicators (autarky, dominance/dependence, interdependence).

Matrix calculation allows to compute the global effects of changes from the environment of the structure, but it can be convenient for an observer of the structure to know the main "direct" effects (carried on by the paths from vertex *j* to vertex *i* in the influence graph) and, for each one of those effects, the value of its amplification by circuits and feedbacks. Instead of being aware only of the final effects, the observer is able to understand the whole sequence of consequences of his actions and to know the way and, if we endow each arc with a transit time, the potential approximate schedule of those consequences. Graph theory and "micro-path analysis" are a way to build a strongly micro-founded meso-analysis of feedback effects in all kind of structures as described by relations (*S*1) and (*S*3), or (*S*2) and (*S*4). **But the most important reason of using graph theory is that, from a macro point of view, it can lead us to nice structural indicators such as circularities, triangularities, dominance/ dependence, interdependence, and autarkies.**

2. Properties of the determinant of the structure: some theorems

Let us now focus on the properties of the determinant *D*, which is the same for *A* and *T* (cf. appendix 2). One might think that the determinant is too complicated as a descriptor of the structure to be worth analyzing. Indeed, the determinant is a function of all the terms of the matrix and cannot be very simple. However, because its value is associated with the circuits in the influence graph, it becomes a meaningful indicator of the arrangement of the structure.

Theorem 1: Theorem of the loops and circuits.

Let us denote as “Hamiltonian” any set of circuits and loops of the **influence graph** where each vertex belongs once and only once to the set. Let us call those Hamiltonian sets “Hamiltonian Partial Graphs” (HPG) of the influence graph. The number of the circuits with two or more arcs in the h^{th} HPG is denoted by c_h. The “signed product” of the coefficients of all its loops and arcs is called the “value” of the h^{th} HPG and is denoted by $V_h \equiv (-1)^{c_h} v_h$, where v_h is the product of the (all positive) coefficients of the loops and circuits of the h^{th} HPG. Then:

$$D = \sum_h V_h$$

The determinant *D* is equal to the sum of the values of the HPG in the digraph representing the structure (“theorem of the loops and circuits”).

Proof: see appendix 1.

Theorem 2: Theorem of the circuit.

Each circuit in the influence graph leads to a decrease in the value of the determinant *D*.

Proof: Let us call C_j a circuit of the influence graph and $\mathcal{J}$ the set of vertices belonging to C_j. $\mathcal{J}^c$ is the set of all the other vertices which do not belong to C_j. The set of HPG can be divided in two subsets: the one which does not include the circuit C_j and the one which includes it. The contribution of the circuit C_j to the expression of the determinant *D* is carried on by all the HPG including circuit C_j and only by them. In order to assess the impact or effect of the circuit C_j on the value of the determinant *D*, we need to consider this subset of HPG only.

Let us take the circuit C_j out of all the HPG of this subset. Their remaining parts exactly coincides with the HPG of the subgraph $[\mathcal{J}^c]$, the vertices of which belong to $\mathcal{J}^c$. As the product of the coefficients of circuit C_j is denoted by π_j, the contribution of circuit C_j to the value of determinant D is equal[3] to $(-1)\pi_j m_j$, where m_j is the "multiplier" of the circuit C_j. This multiplier is exactly the sum of the value of all the HPG in the subgraph $[\mathcal{J}^c]$.

Hence, from theorem 1 (of the loops and circuits), this multiplier m_j is equal to the determinant $D_{\mathcal{J}^c}$ of the submatrix corresponding to the subgraph $[\mathcal{J}^c]$. Therefore, the part of the value of the determinant D explained by the circuit C_j taken separately is $(-1)\pi_j D_{\mathcal{J}^c}$. As the product π_j is positive and the determinant of any subgraph of the influence graph is positive, the theorem of the circuit is proven: each circuit in the influence graph leads to a decrease in the value of the determinant D.

Theorem 3: Theorem of the flowers.

The minimal value of the determinant D is equal to the product of the proportion y_i of deliveries to final demand.

$$D \geq \prod_{i=1}^{n} y_i$$

Proof: Let us consider a particular case where every vertex is in autarky, except for relationships with final demand (see the next figure). The physical coefficients t_{ii} are maximal and the loops of the influence graph ℓ_i are minimal.

$$\ell_i = 1 - t_{ii} = y_i$$

There is a unique HPG made of the set of the n loops. The determinant D is given by:

$$D = \prod_{i=1}^{n} \ell_i = \prod_{i=1}^{n} y_i$$

Let notice that a symmetric proof could lead to the expression $D \geq \prod_{j=1}^{n} w_j$.

Now, suppose one adds an arc to the flower structure. Before the addition of the new arc we had:

[3] (- 1) because each circuit changes the parity of V_h.

$$\ell_1 = 1 - t_{11}$$

with $t_{11} = 1 - y_1$, giving $\ell_1 = y_1$. We have now:

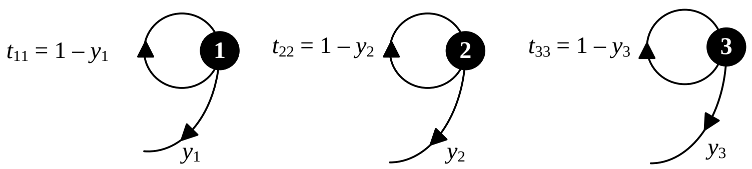
$$\ell_1' = 1 - t_{11}'$$

with $t_{11}' = 1 - y_1 - t_{12}$ because that is all that remains when you take away from the supply of vertex 1 not only y_1 (given) but t_{12} too. Therefore:

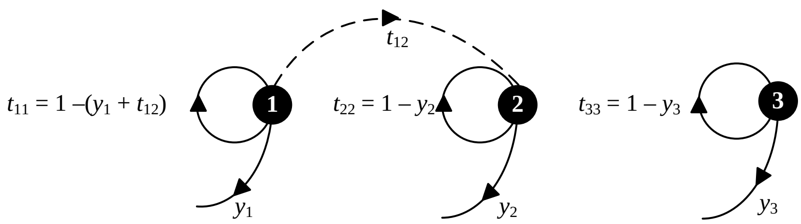
$$\ell_1' = 1 - t_{11}' = y_1 + t_{12} > y_1$$

and the value of D $(= \prod_{i=1}^{n} \ell_i)$ is increased. If one adds other arcs it will be the same, and the value of the determinant D will be increased.

Another proof of the same theorem is given below.



Structure of 'physical flows' with no inter-sectoral flow



Structure of 'physical flows' with an additional inter-sectoral flow t_{12}

Theorem 4: The triangular structure.

A HPH is a set of loops and circuits. A triangular structure has only one HPG made of the loops. As it is well known, its determinant D_T is given by the product of the diagonal terms:

$$D_T = \prod_{i=1}^{n} \ell_i$$

It follows that:

1. According to the theorem of the loops and circuits, for given values of y_i (or w_j) every change in the coefficients without creation of one or more HPG will not lead to a change in the value of the determinant D_T. According to the theorem of the circuit, every change creating one or more circuit(s) will lead to a decrease in the value of the determinant D_T. Therefore, $D_T = \prod_{i=1}^{n} \ell_i$ is an upper bound of the determinant D. The determinant D is smaller than or equal to the product of the coefficients of the loops in the influence graph:

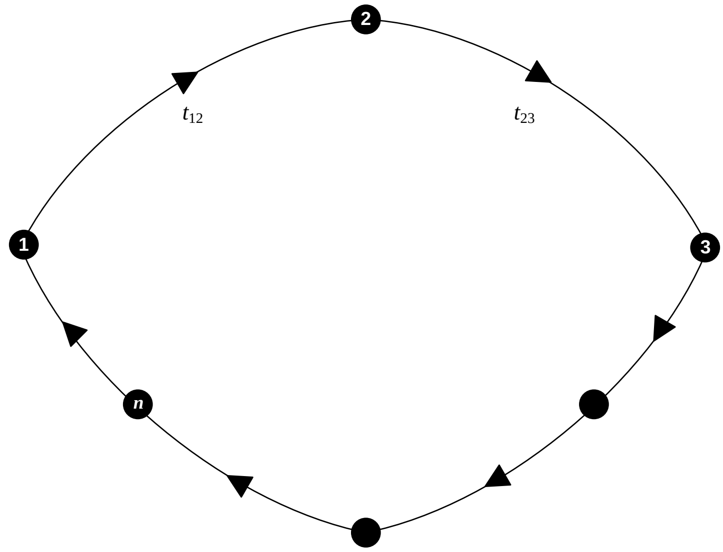
$$D \leq \prod_{i=1}^{n} \ell_i$$

2. In a triangular structure, there is at least one vertex sending deliveries only to itself and to the final demand. Let us assign the number 1 to this vertex. It comes $\ell_1 = 1 - t_{11} = y_1$, and as $D_T = \prod_{i=1}^{n} \ell_i$, $D_T = y_1 \prod_{i=2}^{n} \ell_i \leq y_1$ (given data). The determinant of a triangular matrix is smaller than the share of deliveries sent to final demand of the vertex that transmits nothing to the other vertices of the structure.

Theorem 5: theorem of the pure short-circuit.

Ceteris paribus, every pure short-circuit of an existing circuit leads to a decrease in the value of the determinant D.

Consider the following digraph.

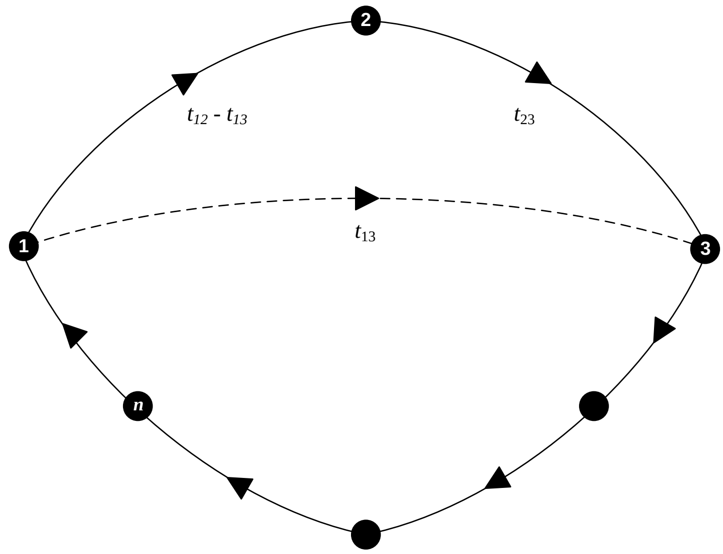


The determinant is equal to $D = \sum_h V_h \ (h \in [1, H])$. This formula can be decomposed in two parts:

– the HPG which contain all the circuits including the path [t_{12},t_{23}]. Let us denote their value by V_1;

– the HPG which do not contain t_{12} and t_{23}. Let us denote their value by V_2.

To create a pure oriented short-circuit, we have to change at least one coefficient. Let us introduce a short-circuit t_{13}. The arc t_{13} carries directly a part of the flow from the vertex 1 to the vertex 3. Therefore, the value of the arc t_{12} decreases by t_{13}.



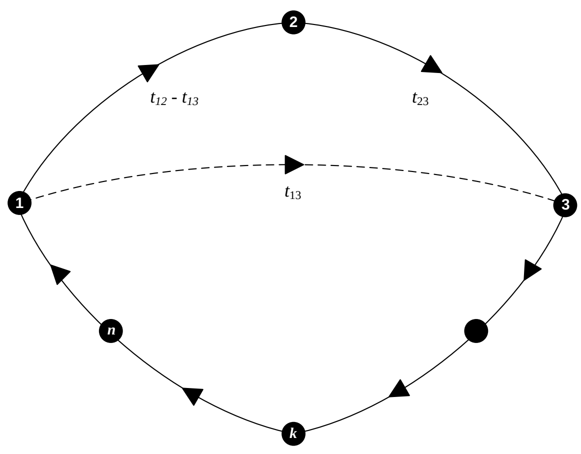
By definition, V_2 is not affected by the change in the coefficient t_{12}. Regarding V_1, the circuits from the relevant HPG can be decomposed into two parts:

- circuits containing the path $[(t_{12} - t_{13}), t_{23}]$;
- (the same) circuits containing the arc t_{13}.

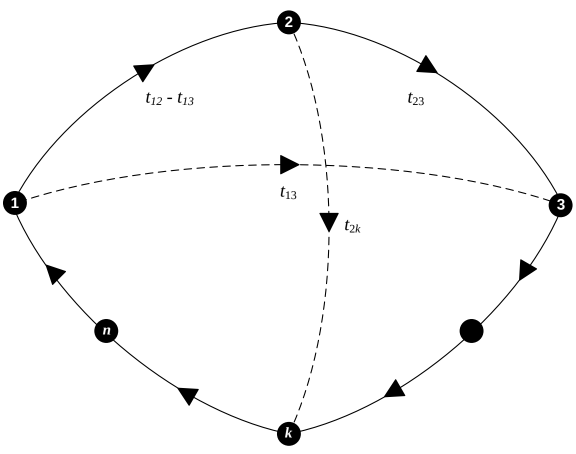
According to the theorem of the circuit, each circuit in the influence graph leads to a decrease in the value of the determinant. Let us analyze the variation ΔD of the determinant D as a result of the creation of the short circuit t_{13}. The multiplier of initial arcs including the path $[t_{12}, t_{23}]$ can be denoted by K. Therefore, V_1 can be written $Kt_{12}t_{23}$. Taking into account the short circuit t_{13}, this value changes as follow: $K[(t_{12} - t_{13})t_{23} + t_{23}]$. The weight of every circuit is negative. Hence:

$$\Delta D = (-1)K[t_{13}(1 - t_{23})]$$

which is negative. This result comes from the upper "weight" of the circuits including the arc t_{13} compared to the circuits including the path $[(t_{12} - t_{13}), t_{23}]$.



t_{13} is a "pure" short circuit



t_{13} is not a "pure" short circuit

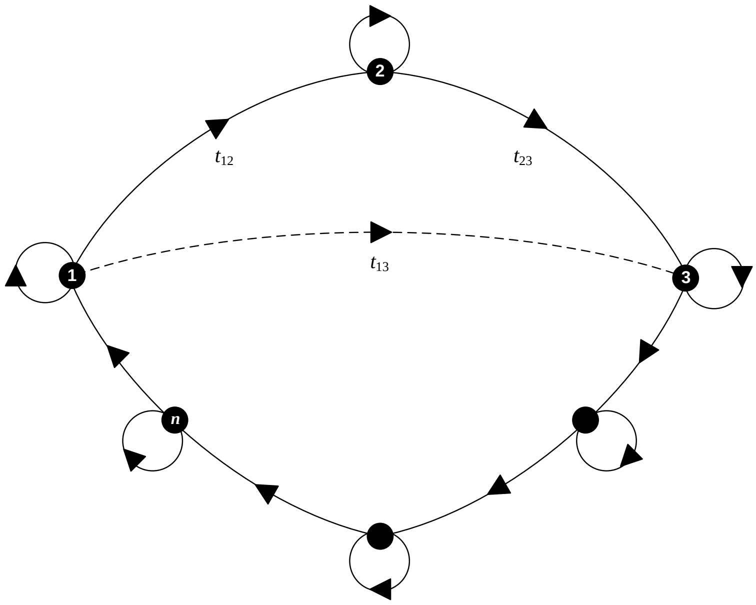
Another proof for the theorem of the flowers:

It will be sufficient to remark that a loop is always a pure short circuit. As a pure short circuit leads to a decrease in the value of the determinant and as it is not possible to get shorter circuit than the loops, the theorem is proven.

Theorem 6: The upper bound of the determinant.

With proportions y_i ($i \in [1, n]$) sent to final demand, the upper bound of the determinant D of an I-O matrix is a simple function of the proportion y_i:

$$D \leq 1 - \prod_{i=1}^{n}(1 - y_i).$$



Proof: Consider the case of an exchange structure with only one simple circuit and n loops. It follows that $t_{ij} = 1 - y_i$. Since auto-consumptions do not exist in this structure, $t_{ii} = 0$. Hence:

$$\prod_{i=1}^{n} \ell_i = \prod_{i=1}^{n}(1 - t_{ii}) = 1.$$

The definition of the determinant is given by the theorem 1 (“loops and circuits”). Let us call it D'. Since there are only two HPG, it immediately derives:

$$D' = \prod\nolimits_{i=1}^{n} \ell_i - \prod\nolimits_{i=1}^{n} (1 - y_i) = 1 - \prod\nolimits_{i=1}^{n} (1 - y_i)$$

Any change of a coefficient in accordance with $1 - \sum_{j=1}^{n} t_{ij} = y_i$ (given data) would create at least one arc which would be a pure short-circuit in the structure and lead to a smaller value of the determinant. Thus, the determinant D' is a maximal value of the determinant *D*.

3. Structural indicators

Taking into account the point of view of Leontief, and according to the previous theorems, we try to define simple values, or indicators, which are not "mathematical measures", of: interdependence (on this issue, see Malinvaud, 1955), autarky, and dominance / dependence.

According to the theorem of the loops and circuits, the difference the product of loops $\prod_{i=1}^{n} \ell_i$ and the determinant *D* derives exclusively from circuits, which are carrying on circular influence, that means interdependence between vertices. Thus, interdependence is smaller than or equal to:

$$\prod\nolimits_{i=1}^{n} \ell_i - D$$

As proven in the previous theorems, the higher the circularities, the lower the determinant. Trees, or triangularities, carry on dominance/dependence. Following the intuition, triangularities and circularities are complementary. On the other hand, the determinant is always superior to a minimum value (cf. theorem of the flowers). That is why the difference $D - \prod_{i=1}^{n} y_i$ which is equal to $D - D_{\min}$, could be a nice indicator of dominance/dependence.

In fact, there is a double scale:

- The first one gives us indicators depending only on the given data (y_i or w_j): $D_{\min} = \prod_{i=1}^{n} y_i$ or $D_{\max} = 1 - \prod_{i=1}^{n}(1 - y_i)$ for example.
- The second one takes into account one of the elements of organization of the structure: the degrees of autarky (self-consumption a_{ii} or self-supply t_{ii} with $a_{ii} = t_{ii}$) of the different sectors or vertices of the influence graph.

For given values of the y_i, the coefficients of the loops ℓ_i for this graph may vary between y_i (for $t_{ii} = 1 - y_i$) and 1 (for $t_{ii} = 0$). It follows:

$$D_{\min} \leq \prod_{i=1}^{n} \ell_i \leq 1.$$

We suggest to take into account:

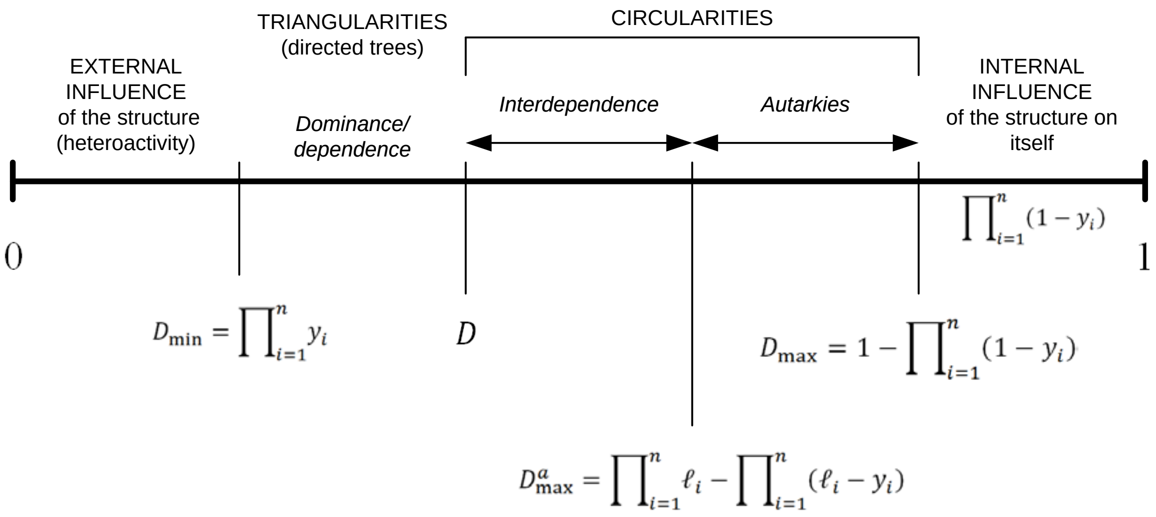
– First: the given data y_i (or w_j).
– Second: the intra-consumption or autarkies $t_{ii} = a_{ii}$.
– Third: the other arcs and circuits.

This way, the impact of autarkies may be assessed. At the end of the second step, the structure of the flowers corresponding to $D_{\min}$ will not have changed. But the large circuit corresponding to $D_{\max}$ will include a certain number of loops whose values will be different from 1 in the influence graph. If those values are considered as structural data, the coefficient of each arc from vertex i to vertex $(i + 1)$ will be reduced to the value $1 - a_{ii} - y_i = \ell_i - y_i$. That means that the value of $D_{\max}$ without any autarky, $D_{\max} = 1 - \prod_{i=1}^{n}(1 - y_i)$, will be $D_{\max}^{a} = \prod_{i=1}^{n} \ell_i - \prod_{i=1}^{n}(\ell_i - y_i)$ with $\ell_i \neq 1$ for some i.

The difference between $D_{\max}$ and $D_{\max}^{a}$ appears as a good global indicator of autarky[4].

[4] For very different approaches, see for example Guccione, Gillen, Blair and Miller (1988) and Rose and Casler (1996).

Table of structural indicators



We will often use indifferently the terms “circularity” or “interdependence”. That is pertinent as long as autarkies are nil or loops clearly separated from circuits of two arcs or more.

4. Partition of the structure

Structural indicators previously defined provide a representation of the global properties of the structure. We can try to explain the theorems and the indicators proposed above by connecting them to the local properties of the substructures making up the overall structure[5].

Theorem 7: Theorem of the connected HPGs.

If we call “connected HPGs” the HPGs including at least one circuit linking two parts of a complete partition P of the structure, and if the values of such HPGs are denoted by V_c, the sum of the values of those connected HPGs is negative or nil. Thus, these are the connected HPGs that lower the value of the determinant:

$$\sum_c V_c \leq 0.$$

[5] Gillen and Guccione (1990) deal with this kind of problem in a very different way.

Proof: Let us introduce a complete partition P of the structure, the parts being indexed by p. Let us divide the HPGs of the structure into two categories: the "disjoint" (d) HPGs, having no circuit reaching vertices of different parts of the partition P, and the "connected" (c) HPGs having at least one circuit linking vertices of two (or more) parts of the partition P.

Taking into account the theorem of the loops and the circuits, the determinant may be written $D = \sum_h V_h = \sum_d V_d + \sum_c V_c$. According to the theorem of the loops and the circuits again, the sum $\sum_d V_d$ is the product of the determinant D_p of the submatrices defined by the partition P. The relation becomes:

$$D = \prod_p D_p + \sum_c V_c.$$

If we take into account the non-diagonal parts of the global matrix, either we add no circuit, and thus no HPG:

$$\sum_c V_c = 0, \text{ and we have } D = \prod_p D_p,$$

or we add at least a circuit, and following the theorem of the circuit, $D < \prod_p D_p$. It comes:

$$\sum_c V_c \leq 0.$$

Corollary of the theorem 7.

In whatever way a square productive matrix is completely divided into square submatrices, its determinant is always smaller than or equal to the product of the determinants of these submatrices:

$$D \leq \prod_p D_p.$$

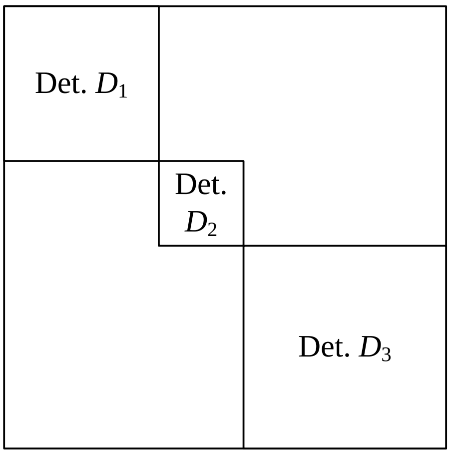
Theorem 8: A measure of partial interdependences.

The measure of the linkage, or interdependence, between submatrices is given by the difference between the product of the determinants of the submatrices of a complete partition P and the determinant of the matrix of the overall structure:

$$I = \prod_p D_p - D.$$

Indeed, without any circuit linking vertices of two parts or more of the partition P, $\sum_c V_c = 0$.

Hence, the term $-\sum_c V_c \geq 0$ constitutes a consistent measure of the **linkage**, or **interdependence**, between the submatrices of the partition P.



The determinant D of the large matrix is smaller than the product $D_1 \times D_2 \times D_3$ except if there is no circuit linking the vertices of two or more of the three submatrices defined by the partition. Therefore, $I_{1-2-3} = D_1 \times D_2 \times D_3 - D$

The theorem 8 is true for any part (submatrix) as well as for the whole matrix. The higher the level of disaggregation, the greater the number of disjoint HPGs and, hence, the greater the linkage (or interdependence) between parts.

Theorem 9: Different measures of interdependence between parts of the structure.

To each complete partition P of the structure corresponds a specific value of interdependence between the parts of P.

Proof (recurrence relation): The variables X_i can be aggregated into subset s. According to the theorem 8 on the measure of interdependence:

$$D_s = \prod_{i \in s} \ell_i - I_s.$$

The subsets s can be aggregated into parts p. It comes:

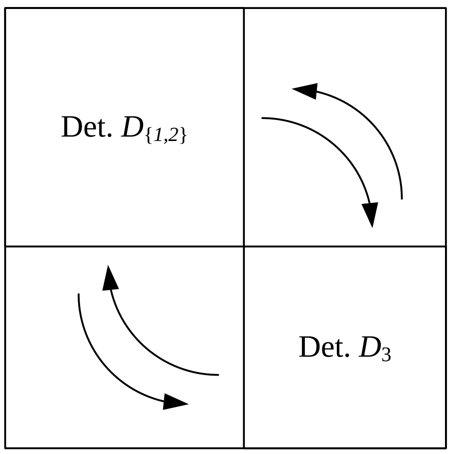
$$D_p = \prod_{s \in p} D_s - I_p.$$

The determinant $D_{\mathcal{P}}$ of the matrix whose elements are the parts p is:

$$D_{\mathcal{P}} = \prod_{p \in \mathcal{P}} D_p - I_{\mathcal{P}} = \prod_p \left\{ \prod_s \left[\prod_i \ell_i - I_s \right] - I_p \right\} - I_{\mathcal{P}}$$

The interdependence between the parts p divided into subsets s made of variables i can be written:

$$I_{\mathcal{P}} = \prod_p \left\{ \prod_s \left[\prod_i \ell_i - I_s \right] - I_p \right\} - D_{\mathcal{P}}$$



The aggregation of two parts 1 and 2 of the previous partition leads to:

$$I_{\{1,2\}-3} = D_{\{1,2\}} \times D_3 - D$$

It is possible to increase the level of disaggregation: as many partitions, as many values of interdependence between the parts defined.

Theorem 10: The “general” interdependence between the variables of a linear system of equations.

According to the theorem 9 addressing the measures of interdependence, the “general” interdependence between all the variables X_i of the system of linear equations representing a productive structure is given by the difference between the product of the diagonal terms of the matrix and the determinant of this matrix:

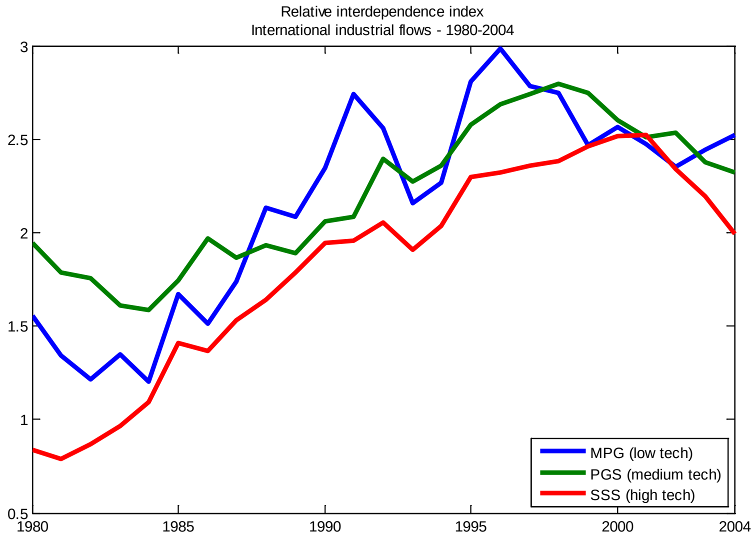
$$GI = \prod_i \ell_i - D.$$

5. Application: structure and dynamics of the international trade

To illustrate the operational dimension of the influence graph theory, we consider the international trade flows of manufactured goods during the period 1980-2004. We use the database *TradeProd* from CEPII for 28 manufactured goods, data in thousands of current US dollars (Mayer et al., 2008). We merge data for ex-socialist countries and isolate the giant component for all the 25 years; the total structure contains 171 countries. We also merge products flows according to their technological intensity: low (PGS), medium (MPG), and high (SSS). By construction, the autarkies are absent and, hence, $\prod_i \ell_i = 1$, and the external flows are all included in the trade imbalances such that $\prod_i y_i = 0$ (cf. Lebert et al., 2009).

The first issue results from an application of the theorems 1 to 6. We define a “relative interdependence index” (*RII*) which corresponds to the ratio between the structural interdependence index ($= 1 - D$) and the structural dependence index ($= D$). This relative index is used in order to quantify the phenomenon of trade globalization.

$$RII = \frac{1 - D}{D}$$



First, we can see that the value of *RII* is increasing over the period. This growth is stronger for products with the highest technological content. Second, from the second half of the 1980s, interdependence dominates all other forms of structural relationships between countries (i.e. dependence), regardless the kind of products.

The value of *RII* is decomposable by vertices (countries). The theorem 7 identifies the intensity of circularities that link the substructures, regardless of how these substructures are defined. We consider here the special case where each country is compared to the rest of the world. The intensity of the circularities between a vertex and the rest of the graph is quantified by the diagonal cofactors of the matrix. These sub-determinants are necessarily greater than, or equal to, the value of the matrix determinant. The greater the gap between these two indicators, the higher the (weighted) circularities inside the whole structure intermediated by the country; from this point of view, the index we present here can be named "betweenness centrality of countries". It represents the strategic weight of the country in the international trade for manufactured goods. The contribution of a vertex *i* to the value of *RII* corresponds to the ratio:

$$BC_i = \frac{RII - RII_{-i}}{RII} = \frac{D_{-i} - D}{D_{-i}(1 - D)}$$

where RII_{-i} is the value of *RII* for the subgraph which does not contain the vertex *i*, and D_{-i} is the determinant of the correspondent submatrix.

Betweenness centrality of countries

Rank	1980 Global	1980 PGS	1980 MPG	1980 SSS	2004 Global	2004 PGS	2004 MPG	2004 SSS
1	'DEU'	'DEU'	'USA'	'DEU'	'USA'	'DEU'	'DEU'	'USA'
2	'USA'	'FRA'	'DEU'	'FRA'	'DEU'	'USA'	'USA'	'DEU'
3	'FRA'	'USA'	'FRA'	'USA'	'FRA'	'FRA'	'FRA'	'CHN'
4	'GBR'	'NLD'	'CAN'	'GBR'	'ITA'	'ITA'	'ITA'	'JPN'
5	'NLD'	'BLX'	'BLX'	'NLD'	'GBR'	'BLX'	'ESP'	'FRA'
6	'ITA'	'ITA'	'GBR'	'ITA'	'JPN'	'NLD'	'CAN'	'GBR'
7	'BLX'	'GBR'	'ITA'	'BLX'	'CHN'	'ESP'	'BLX'	'SGP'
8	'CAN'	'JPN'	'NLD'	'JPN'	'BLX'	'GBR'	'GBR'	'MEX'
9	'JPN'	'CAN'	'SWE'	'CHE'	'NLD'	'CAN'	'MEX'	'NLD'
10	'SWE'	'AUT'	'JPN'	'SWE'	'CAN'	'MEX'	'NLD'	'ITA'
11	'CHE'	'HKG'	'CHE'	'CAN'	'MEX'	'AUT'	'CHN'	'BLX'
12	'DNK'	'SWE'	'AUT'	'DNK'	'ESP'	'CHN'	'JPN'	'KOR'
13	'AUT'	'CHE'	'NOR'	'AUT'	'SGP'	'SWE'	'AUT'	'TWN'
14	'FIN'	'DNK'	'FIN'	'IRL'	'KOR'	'CSK'	'SWE'	'MYS'
15	'HKG'	'IRL'	'ESP'	'NOR'	'AUT'	'DNK'	'CSK'	'CAN'
16	'ESP'	'CHN'	'DNK'	'FIN'	'CHE'	'POL'	'POL'	'HKG'
17	'SGP'	'NOR'	'ZAF'	'ESP'	'TWN'	'PRT'	'RUS'	'CHE'
18	'IRL'	'AUS'	'IRL'	'SGP'	'SWE'	'HKG'	'KOR'	'ESP'
19	'NOR'	'ARG'	'AUS'	'KOR'	'CSK'	'RUS'	'CHE'	'SWE'
20	'RUS'	'SGP'	'POL'	'HKG'	'HKG'	'IRL'	'HKG'	'AUT'

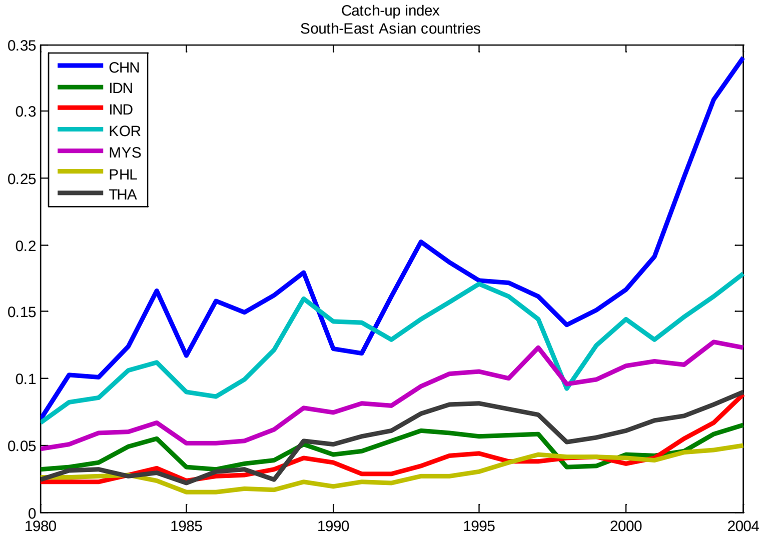
Legend: ranking of countries according to the BC_i index for 1980 and 2004. Colors represent different geographical areas (emerging countries).

We observe first that the number of emerging countries in the top 20 is increasing during the period, from various parts of the world. Second, these countries are technologically specialized: Eastern European countries on low and medium tech goods, South-East Asian countries on high tech goods.

The "catching-up of emerging countries" phenomenon in international trade can be supported by the normalized ratio:

$$CI_i = \frac{BC_i}{\max_i(BC) - \min_i(BC)}$$

Catch-up index



Legend: evolving values of CI_i for south-east Asian countries.

All of these Asian countries are in a dynamics of catching up, particularly strong for China, South Korea and Malaysia, smaller for the others. As we can see in this figure, the depressive effect of the Asian crisis of 1997-1998 on the weight of these economies in the world trade is easily detectable.

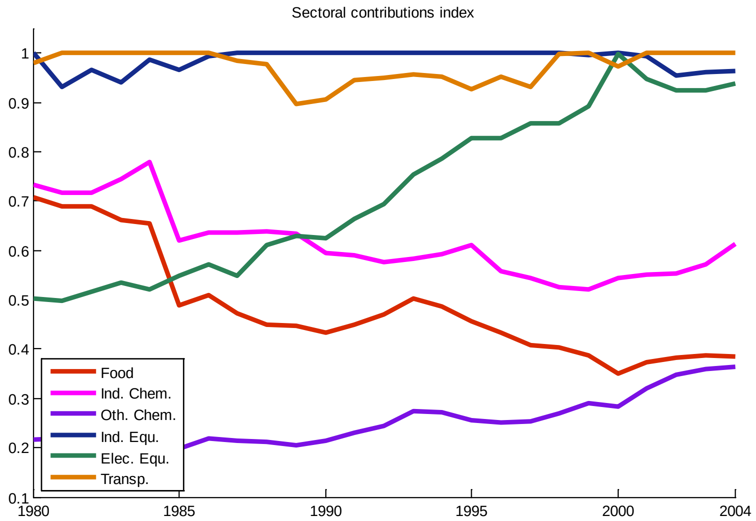
The value of *RII* is also decomposable by flows. If we consider, as in this example, a multiplex structure (i.e. a structure in which the vertices are linked by multiple arcs, here by the flows of the different industrial goods traded internationally), the measurement of *RII* on the partial multigraph omitting the flows of a particular good ("layer") captures the contribution of this good to the value of the structural determinant. This contribution can be empirically assessed by the ratio:

$$CT_j = \frac{RII - RII^{-j}}{RII} = \frac{D^{-j} - D}{D^{-j}(1 - D)}$$

where RII^{-j} is the value of interdependence of the multigraph not including the layer j, and D^{-j} is the determinant of the same multigraph. This indicator can be interpreted as a measure of betweenness centrality of goods in international trade.

To identify the sectoral trajectories of catching up, we normalize the *CT* value between 0 and 1. Industrial equipment and transports are the leading sectors in the international trade. Overall, the secondary high-tech sectors (pharmaceuticals, computer equipment) increase their relative importance.

Sectoral contributions index

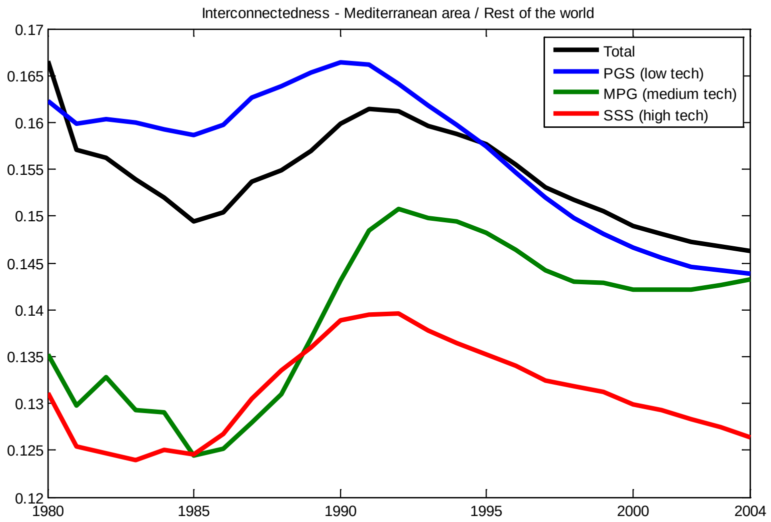


Legend: normalized CT_j values for main internationally traded products.

Following theorems 7 through 10 (on the partition of the structure), we can isolate any group of countries in international trade. The group we choose here consists of countries that share a border with the Mediterranean Sea: Albania, Algeria, Cyprus, Egypt, Spain, France, Gibraltar, Greece, Israel, Italy, Lebanon, Libya, Morocco, Malta, Syria, Tunisia, Turkey and ex-Yugoslavia. We measure the interconnectedness between this group of countries (*med*) and the rest of the world (*RW*) using:

$$I_{med-RW} = D_{med} \times D_{RW} - D$$

Interconnectedness of the Mediterranean area



Legend: interconnectedness of Mediterranean countries relatively to the rest of the world (I_{med-RW}).

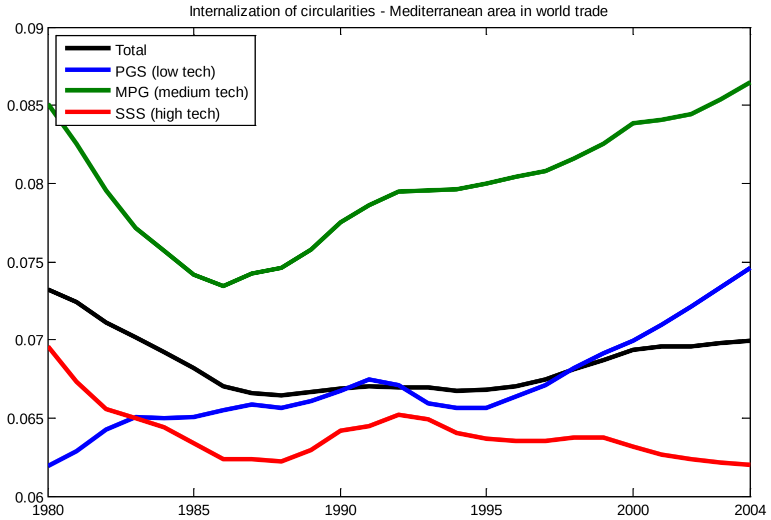
The second half of the 1980s is characterized by a significant jump of circularities between the two groups of countries, but it has tended to decline since the early 1990s. This trend reversal is primarily driven by low tech and high tech products. At the single products level, the linkage with the rest of the world is intense for food, leather, and textiles (low tech), rubber, iron / steel and transport (medium tech), chemistry and, dramatically since the mid-1990s, pharmaceuticals (high tech).

The last index measures the circularities of the whole structure concentrated into the Mediterranean area. It is constructed by comparing the intra-Mediterranean trade when the X_i and X_j of the whole structure are used as deflator ("open structure", *o*) to the same trade when the X_i and X_j are computed from the sole intra-Mediterranean trade ("closed structure", *c*). The determinant of the sub-structure is higher in the second case (i.e. the circularities are lower), and the more the circularities are concentrated into the area, the more the *IC* ratio is close to unity. Ultimately, if all of the circularities of the whole structure are internalized in the group of Mediterranean countries, the value of *IC* becomes equal to 1.

$$IC_{med} = \frac{1 - D_{med}^{c}}{1 - D_{med}^{o}}$$

where $1 - D^c_{med}$ is the interdependence of the "closed" Mediterranean area, and $1 - D^o_{med}$ the one of the "open" Mediterranean sub-structure.

Internalization of circularities – Mediterranean area



Legend: value of IC_{med} for different kinds of products

The value of *IC* has slightly declined during the first half of the 1980s. Since then, this value remains constant. While the concentration of circularities in the low tech and medium tech categories increases significantly, it decreases tendentiously for the high tech category: the intra-Mediterranean trade is, relatively, less and less based on the most demanded products at the international level. Within each category, the situations of the different products are specific. For example, in the low tech category, the leadership of the furniture sector had been replaced by that of the leather sector: the *IC* ratio for this last product has a value of 0.15 at the end of the period.

Conclusion

Theoretically, the analysis of structures raises two main problems:

- Combining qualitative and topologic data on the one hand, and quantitative data on the other hand.
- Seeing some details and substructures (nearsightedness) in their context without losing the overview of the overall structure (farsightedness) at the same time.

In this paper and in its application to international trade flows we have tried to keep far away from these two risks and to have a new look on the theory of structures and some of their possible applications.

This paper can naturally lead us to some extensions. Theoretically, we are working on the theory of amplification and scheduled effects of a signal in a graph of influence. Empirically, the model seems to fit perfectly the new analysis of international trade in value added (by country, by large region such as EU, by economic activity, etc.). The rankings and the measures of dominance, dependence and interdependence will be strongly affected by the paradigmatic change made by OECD and WTO in the representation of international trade flows.

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Appendices

Appendix 1: The theorem of the loops and circuits

By definition, the determinant *D* of a matrix $[d_{ij}]$ is given by the sum of all the products of *n* terms of the matrix belonging to different rows and different columns. These products are multiplied by $(-1)^I$ where *I* denotes the number of inversions of the permutation of the index of the columns *j* when the factors are put in such a way that the index of the rows *i* come into the natural order, from 1 to *n*.

The determinant can be written $D = \sum(-1)^I\, d_{1a_1}d_{2a_2}\cdots d_{na_n}$, where *I* is the number of inversion of the permutation $(a_1, a_2, \cdots, a_n)$.

For each substitution $S = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$, a Hamiltonian Partial Graph (HPG) can be associated.

It is easy to prove that when you draw the arcs $1 \to a_1$, $2 \to a_2$, …, $n \to a_n$, you find a set of loops and circuits which is a HPG (each vertex of the graph being reached once and only once).

$(-1)^I$ gives the sign of the substitution *S*. This substitution *S* can be considered as a product of *T* transpositions. It is well known that $(-1)^I = (-1)^T$.

If one examines successively the terms of the development of the determinant *D* containing zero, one, two, …, till (*n* – 1) transpositions, one will find successively the unique HPG of the *n* loops, then all the HPG with only one circuit of two arcs, …, until the Hamiltonian circuits of *n* arcs (if they exist).

It becomes easy to prove that:

$$D = \sum(-1)^I\, d_{1a_1}d_{2a_2}\cdots d_{na_n} = \sum_{h=1}^{H} V_h$$

where *h* is the index of the *h*th existing HPG defined as follows:

$$V_h = (-1)^{c_h} v_h$$

with c_h the number of circuits of two or more arcs, and v_h the value of the product of the all positive coefficients of the loops and arcs belonging to the *h*th HPG.

The equality:

$$D = \sum_{h=1}^{H} V_h$$

will be called the theorem of the loops and circuits.

It is very useful, not in order to compute the determinant, but to understand the meaning of its lower and upper bounds, of its variations due to the circularities of the structures, etc. A lot of our issues and of our results are connected with this theorem.

Appendix 2: Matrix *A* and matrix *T* have the same determinant

The initial linear system is:

$$(S1) \;\; i, j \in [1, n]: \;\; X_i - \sum_{j=1}^{n} x_{ij} = Y_i$$

The first diagonal terms of *A* and *T* are the same:

$$\ell_i = 1 - \frac{x_{ii}}{X_i} = a_{ii} = t_{ii}$$

The other terms are:

$$i \neq j,\ i,j \in [1,n]:\ \ a_{ij} = -\frac{x_{ij}}{X_j} \text{ and } t_{ij} = -\frac{x_{ij}}{X_i}$$

Starting with matrix $\mathcal{X}$ whose diagonal terms are $(X_i - x_{ii})$ and the other terms are $\left(-x_{ij}\right)$, we get easily:

$$|A| = \frac{|\mathcal{X}|}{\prod_{j=1}^{n} X_j}$$

and:

$$|T| = \frac{|\mathcal{X}|}{\prod_{i=1}^{n} X_i}$$

which proves that:

$$|A| = |T| = D$$

(by definition). We will use both *A* and *T*.

Appendix 3: Absolute and relative influence ("sensitivities" and "elasticities")

By definition, the **absolute influence** of a variation of the given data ΔY_j (demand) on the production X_i is equal to the "sensitivity" $s_{(j)i} = \frac{\Delta X_i}{\Delta Y_j}$. If it is required, the sensitivity of X_i when the production X_j varies can be denoted $s_{ji} = \frac{\Delta X_i}{\Delta X_j}$. As a_{ij} is defined by $-\frac{x_{ij}}{X_j}$ and the diagonal terms of the matrix *A* are equal to $(1 - a_{ii})$, we have:

$$AX = Y \text{ and } X = A^{-1}Y$$

The influence graph theory or Cramer formula gives the value of $s_{(j)i}$.

On the other hand, we can define the **relative influence** of a variation ΔY_j on the value of production X_i by the "elasticity":

$$e_{(j)i} = \frac{\Delta X_i / X_i}{\Delta Y_j / Y_j}$$

Let P_i and $\mathcal{D}_j$ respectively denote the relative variation $\Delta X_i / X_i$ of production *i* and the relative variation $\Delta Y_j / Y_j$ of demand *j*. If it is required, the elasticity of X_i when the production X_j varies can be denoted $e_{ji} = \frac{\Delta X_i / X_i}{\Delta X_j / X_j} = \frac{P_i}{P_j}$.

Derived from the system $(S1)$, the system of variation can be written:

$$(S4) \;\; i, j \in [1, n]: \;\; \Delta X_i - \sum_{j=1}^{n} \Delta x_{ij} = \Delta Y_i$$

As it is assumed that technical coefficients a_{ij} remains unchanged:

$$\frac{\Delta x_{ij}}{\Delta X_j} = \frac{x_{ij}}{X_j} = a_{ij}$$

it comes:

$$\frac{\Delta x_{ij}}{x_{ij}} = \frac{\Delta X_j}{X_j} = P_j$$

Hence:

$$\Delta x_{ij} = x_{ij} P_j$$

With the definition given to the term t_{ij}:

$$\Delta x_{ij} = t_{ij} X_i P_j$$

Let us divide the general relation of the system of variations $(S4)$ by X_i:

$$P_i - \sum_{j=1}^{n} t_{ij} P_j = \frac{\Delta Y_i}{Y_i} \frac{Y_i}{X_i} = y_i \mathcal{D}_i$$

(by definition of y_i). Using matrices, it comes with column vectors P and $\mathcal{D}^*$ where:

$$\mathcal{D}^* \equiv \begin{pmatrix} y_1 \mathcal{D}_1 \\ \vdots \\ y_n \mathcal{D}_n \end{pmatrix}$$

$$TP = \mathcal{D}^*, \text{ or } P = T^{-1} \mathcal{D}^*$$

The relative influences, or elasticities, $e_{(j)i}$ are given by the influence graph theory or by the Cramer formula.

Appendix 4: Leontief and Ghosh

The Leontief matrix is A, with $a_{ij} = \frac{X_{ij}}{X_j}$. It is generally linked with the computation of the effects of the variation of the vector of demands (ΔY_j) on the vector of production (ΔX_i).

Assuming that demand is dominant, it comes: $AX = Y$, or $X = A^{-1}Y$. Economists often call "sensitivities" the quotient $\Delta X_i / \Delta Y_j$, and "elasticities" the "relative" quotient $\frac{\Delta X_i / X_i}{\Delta Y_j / Y_j}$.

There is no "Ghosh matrix". The name of Ghosh is associated with the hypothesis that supply is dominant (Ghosh, 1960).

Concerning computational point of view, it seems easier to use the technical coefficient a_{ij} to get the "absolute influences" or "sensitivities" $\Delta X_i / \Delta Y_j$, and the trade coefficients t_{ij} to get the "relative influences" or elasticities.

The arcs of the influence graph may be directed as well from demand to supply or from supply to demand, but those two orientations must not be mixed in the same graph.