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# Approximate Whittle Analysis of Fractional Cointegration and the Stock Market Synchronization Issue

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# Approximate Whittle Analysis of Fractional Cointegration and the Stock Market Synchronization Issue

Gilles de Truchis<sup>1</sup>

*Aix-Marseille University (Aix-Marseille School of Economics)*

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## Abstract

I consider a bivariate stationary fractional cointegration system and I propose a quasi-maximum likelihood estimator based on the Whittle analysis of the joint spectral density of the regressor and errors to estimate jointly all parameters of interest of the model: the long run coefficient and the long memory parameters of the regressor and errors. I lead a Monte Carlo experiment which reveals the good finite sample properties of this estimator, even when the parameter space is extended to the non-stationary regions. An application to the stock market synchronization is proposed to illustrate the empirical relevance of this estimator.

*Keywords:* Fractional cointegration, Frequency domain, Full-band estimator, Monte-Carlo simulation, Parametric estimation

*JEL:* C32, C15, C58, G15

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## 1. Introduction

Consider the following triangular bivariate fractional cointegration representation

$$(1 - L)^\gamma(y_t - \beta x_t) = \varepsilon_{1t}, \quad (1 - L)^\delta x_t = \varepsilon_{2t}, \quad t = 1, 2, \dots, n, \quad (1)$$

where  $\delta \in [0, 1/2)$ ,  $\gamma \in (-1/2, 1/2)$  and  $(1 - L)^\alpha$  is the fractional filter, further denoted  $\Delta^\alpha$  and defined by its binomial expansion

$$(1 - L)^\alpha = \sum_{k=0}^{+\infty} a_j(\alpha) L^j, \quad a_j(\alpha) = \frac{\Gamma(k - \alpha)}{\Gamma(k + 1)\Gamma(-\alpha)} L^k \quad (2)$$

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*Email address:* gilles.de-truchis@univ-amu.fr (Gilles de Truchis)

<sup>1</sup>Aix-Marseille School of Economics, Aix-Marseille University, Château La Farge, Route des Milles 13290 Aix-en-Provence. Tel.: 00 33 (0)4 42 93 59 60; fax: 00 33 (0)4 42 38 95 85.

$$\Gamma(z) = \int_0^{+\infty} t^{\tilde{z}-1} e^{-t} dt, \quad (3)$$

with  $L$  the lag operator. Following the general and seminal definition of the cointegration offered by [Granger \(1986\)](#), in equation (1),  $y_t$  is said cointegrated if the error term,  $v_t = y_t - \beta x_t$ , satisfies  $v_t \sim I(\gamma)$  with  $\gamma < \delta$  and  $x_t \sim I(\delta)$ . The estimation of equation (1) usually operates in two steps. The first step is to estimate the long run coefficient,  $\beta$ , whereas the second step is to estimate the order of integration of the residuals,  $\gamma$ . Many studies restrict their analysis to integer integration orders. In most cases,  $x_t$  is assumed to possess a unit root (*i.e.*  $\delta = 1$ ) and the traditional cointegration also imposes  $\gamma = 0$ . More recently, has emerged the idea that cointegration relationship can exist between stationary variables (that is  $\delta < 1/2$ ). Some studies have suggested adapted estimators for  $\beta$ . For instance, [Robinson \(1994\)](#) develops a consistent semi-parametric Narrow-Band Least Squares estimator (NBLS) of  $\beta$  that essentially performs the time domain Least Square Estimator (LSE) on a degenerating band of frequencies around the origin. Then, [Christensen and Nielsen \(2006\)](#) demonstrate the asymptotic normality of the NBLS when  $\delta + \gamma < 1/2$  and [Nielsen and Frederiksen \(2011\)](#) extend it to the weak fractional cointegration<sup>2</sup> (that is  $\delta - \gamma < 1/2$ ). In most studies, these estimators are combined with semi-parametric estimators of long memory (see for instance [Marinucci and Robinson \(2001\)](#), [Christensen and Nielsen \(2006\)](#) and [Nielsen and Frederiksen \(2011\)](#)). We can cite among other the well-known log-periodogram regression (LPE) of [Geweke and Porter-Hudak \(1983\)](#) but also, [Künsch \(1987\)](#), [Robinson \(1995\)](#) and [Andrews and Guggenberger \(2003\)](#). [Velasco \(2003\)](#) suggests to estimate simultaneously  $\delta$  and  $\gamma$  but requires a consistent estimator of  $\beta$ . This innovative approach of estimating jointly several parameters is extended to  $\beta, \delta$  and  $\gamma$  by [Nielsen \(2007\)](#). This consists in a local Whittle analysis of the joint spectral density of  $x_t$  and  $v_t$ . This methodology has the advantage of being invariant to the miss-specification of short-term dynamics but is only  $\sqrt{m}$ -consistent where  $m$  depends on the bandwidth selection. For practical purpose, the bandwidth selection is important because of the variance of the estimator that can increase. For instance, a too high bandwidth can deteriorate the variance if the process possess a short run dynamics because of the confusion in the frequency domain between the low and high frequencies. Simultaneously, [Hualde and Robinson \(2007\)](#) have developed a  $\sqrt{n}$ -consistent parametric estimator

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<sup>2</sup>The terms weak and strong fractional cointegration mean here the intensity with which deviations from the long-run equilibrium will disappear.

of weak fractional cointegration following the suggestion of [Robinson and Hualde \(2003\)](#) and exploiting an error correction form of (1). This estimator has the advantage of covering a wide range of integration orders, however, it requires a more complex optimization procedure than [Nielsen \(2007\)](#).

In this paper, I propose a one step estimator of fractional cointegration. It is a parametric approach that is sensible to miss-specification but that simulations suggest to be  $\sqrt{n}$ -consistent whereas than the one of [Nielsen \(2007\)](#) is only  $\sqrt{m}$ -consistent. It also contrasts with those of [Hualde and Robinson \(2007\)](#) and [Nielsen \(2007\)](#) because according to the Monte Carlo simulation, it can be applied to strong, weak and stationary cointegration cases. It is based on a frequency domain Whittle approximation of the likelihood further termed Whittle Quasi-Maximum Likelihood Estimator (Whittle QMLE). The remainder of the paper is laid out as follows. In the section 2 some generalities on the long memory are exposed and the Whittle QMLE is developed. In the section 3 the Monte Carlo simulation is described and performed. In the section 4, an application on the stock market synchronization is proposed to illustrate the empirical relevance of the Whittle QMLE. The section 5 concludes.

## 2. The stationary fractional cointegration model

### 2.1. The multivariate Whittle estimator of fractional cointegration

In this section, the model is developed with respect to the stationary regions of long memory parameters (that is  $\delta$  and  $\gamma$  are less than 1/2). In a more general form than the equation (2), the Gaussian process  $x_t$  (equivalently  $v_t$ , also assumed to be Gaussian) can be restated as follows

$$x_t = \sum_{j=0}^{\infty} \kappa(j; \delta) \varepsilon_{1t-j}, \quad \sum_{j=0}^{\infty} \kappa(j; \delta)^2 < \infty, \quad \kappa(0; \delta) = 1, \quad t \in \mathbb{Z} \quad (4)$$

for  $\kappa(\cdot; \delta) \in \mathbb{R}$ . In order to state the spectral density of  $x_t$ , let  $g(\lambda; \delta) = \sum_{j=0}^{\infty} \kappa(j; \delta) e^{ij\lambda}$  be the transfer function. Then, the spectral density of  $x_t$  is defined by,

$$f_x(\lambda; \delta) = \frac{\sigma^2}{2\pi} |g(\lambda; \delta)|^2, \quad \sigma^2 = \text{var}(\varepsilon_{2t}), \quad (5)$$

where  $|g(\cdot)|$  is the complex modulus of  $g(\cdot)$ . This results in that the well-known Whittle estimator of  $\delta$  is defined as  $\hat{\delta}_n = \arg \min_{\delta \in \mathcal{D}} \mathcal{Q}_n(\delta)$  with  $\mathcal{D}$  a compact subset of  $\mathbb{R}^3$  and

$$\mathcal{Q}_n(\delta) = -n \left[ \int_{-\pi}^{\pi} \log f_x(\lambda_j; \delta) d\lambda + \int_{-\pi}^{\pi} f_x(\lambda_j; \delta)^{-1} I_{xx}(\lambda_j) d\lambda \right], \quad (6)$$

where  $\int_{-\pi}^{\pi} \log f_x(\lambda_j; \delta) d\lambda < \infty$  is assumed, implying that  $x_t$  is non-deterministic. In equation (6),  $I_{xx}(\lambda_j)$  denotes the periodogram of  $x_t$  defined as

$$I_{xx}(\lambda_j) = w_x(\lambda_j) \bar{w}_x(\lambda_j), \quad w_x(\lambda_j) = \frac{1}{2\pi n} \sum_{t=1}^n x_t e^{it\lambda_j}, \quad (7)$$

with  $w_x(\lambda_j)$  the Fourier transform of  $x_t$  and  $\lambda_j = (2\pi j/n)$  the angular frequency. In the following, it will be preferred the discrete version of (6) that is

$$\mathcal{Q}_n(\delta) = - \sum_{j=1}^n \left[ \log f_x(\lambda_j; \delta) + f_x(\lambda_j; \delta)^{-1} I_{xx}(\lambda_j) \right] \quad (8)$$

Notice that in  $\mathcal{Q}_n(\delta)$  the zero frequency is left out of the summation, implying this estimator is invariant to the presence of nonzero mean. One can attempt to estimate step by step the equation (1) using a consistent estimator of  $\beta$ . However, it is more efficient to estimate jointly  $\beta$ ,  $\delta$  and  $\gamma$ . That is the reason why I consider a multivariate extension of the Whittle estimator described above.

Let  $\mathbf{w}_t$  be a bivariate Gaussian sequence such as  $\mathbf{w}_t = (v_t, x_t)'$  then let define  $\theta = (\theta_1, \theta_2)'$ ,  $\theta_1 = (\gamma, \delta)'$  and  $\theta_2 = (\beta)$ . Rewriting equation (4) in a multivariate framework, we obtain

$$\mathbf{w}_t = \sum_{j=0}^{\infty} K(j; \theta_1) \varepsilon_{1t-j}, \quad \sum_{j=0}^{\infty} \text{tr} K(j; \theta_1) \Sigma(\theta) K(j; \theta_1)' < \infty, \quad K(0; \delta) = I, \quad (9)$$

where elements of  $K$  are all real,  $t \in \mathbb{Z}$  and  $\Sigma$  is a nonsingular  $2 \times 2$  matrix. It follows the transfer matrix  $G(\lambda; \theta_1) = \sum_{j=0}^{\infty} K(j; \theta_1) e^{ij\lambda}$  and the joint spectral density function

$$f_{\mathbf{w}}(\lambda, \theta) = \frac{1}{2\pi} G(\lambda; \theta_1)^{-1} \Sigma(\theta) (G(\lambda; \theta_1)^{-1})^*, \quad (10)$$

where the superscript asterisk denotes the conjugate transpose. Similarly to the univariate case, the multivariate Whittle estimator of  $\theta$  is defined as  $\hat{\theta} = \arg \min_{\theta \in \mathcal{D}} \mathcal{Q}_n(\theta)$  with  $\mathcal{D}$  a compact subset of  $\mathbb{R}^2$  and

$$\mathcal{Q}_n(\theta) = - \sum_{j=1}^n \left[ \log \det f_{\mathbf{w}}(\lambda_j; \theta_1) + \text{tr } f_{\mathbf{w}}(\lambda_j; \theta_1)^{-1} I_{\mathbf{w}\mathbf{w}}(\lambda_j; \theta_2) \right] \quad (11)$$

With respect to the periodogram in the multivariate case, we have

$$I_{\mathbf{w}\mathbf{w}}(\lambda_j; \theta_2) = w_{\mathbf{w}}(\lambda_j; \theta_2) w_{\mathbf{w}}(\lambda_j; \theta_2)^*, \quad w_{\mathbf{w}}(\lambda_j; \theta_2) = (2\pi n)^{-1} \sum_{t=1}^n \begin{pmatrix} v_t \\ x_t \end{pmatrix} e^{it\lambda_j} \quad (12)$$

It follows that

$$I_{\mathbf{w}\mathbf{w}}(\lambda_j; \theta_2) = \begin{pmatrix} I_{yy}(\lambda_j) - 2\beta I_{xy}(\lambda_j) + \beta^2 I_{xx}(\lambda_j) & I_{xy}(\lambda_j) - \beta I_{xx}(\lambda_j) \\ I_{xy}(\lambda_j) - \beta I_{xx}(\lambda_j) & I_{xx}(\lambda_j) \end{pmatrix}, \quad (13)$$

with  $I_{ab}(\lambda_j)$  the cross-periodogram of  $a_t$  and  $b_t$  defined by

$$I_{ab}(\lambda_j) = w_a(\lambda_j) \bar{w}_b(\lambda_j) \quad (14)$$

## 2.2. The fractionally integrated VAR(p) model of cointegration

Now, I develop the model when  $(v_t, x_t)'$  follows a vector autoregressive fractionally integrated (VARFI) process. In this case, let the system of equations in (1) be

$$\begin{pmatrix} \varphi_{11}(L) & \varphi_{12}(L) \\ \varphi_{21}(L) & \varphi_{22}(L) \end{pmatrix} \begin{pmatrix} (1-L)^\gamma \\ (1-L)^\delta \end{pmatrix} \begin{pmatrix} v_t \\ x_t \end{pmatrix} = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \quad (15)$$

$$\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \sim i.i.d N \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right] \quad (16)$$

with  $\varphi_{ij}(L)$  the lag polynomial of order  $p$ , where  $\varphi_{ij}(L) = \sum_{k=1}^p 1 - \phi_{ij}(k)L^k$  and has all its eigenvalues less than 1 in modulus. Let  $G(\lambda; \theta_1, \phi)$  be the transfer function of (15) and defined as

$$G(\lambda; \theta_1) = \sum_{j=0}^{\infty} K(j; \theta_1) e^{ij\lambda} A(j; \phi) e^{ij\lambda} = \sum_{j=0}^{\infty} \Lambda(e^{ij\lambda}) \Phi(e^{ij\lambda}), \quad (17)$$

where  $\phi = \phi_{ij}(k)$  for  $\{i, j\} = \{1, 2\}$  and  $k = \{1, \dots, p\}$ . Thus, the autocorrelated model has a spectral density expressed by

$$f_w(\lambda, \theta, \phi) = \frac{1}{2\pi} \Lambda(e^{jj\lambda})^{-1} \Phi(e^{jj\lambda})^{-1} \Sigma(\theta, \phi) (\Phi(e^{jj\lambda})^{-1})^* (\Lambda(e^{jj\lambda})^{-1})^* \quad (18)$$

### 3. Finite sample investigation

In this section, I propose a Monte Carlo simulation in order to investigate the finite sample performance of the multivariate Whittle estimator of fractional cointegration. The study of asymptotic properties will be the subject of a forthcoming paper. For now, the consistency of the multivariate Whittle estimator of long-range dependent process is provided by [Hosoya \(1997\)](#) in the stationary case. However, this experiment is not confined to the stationary regions of  $\delta$  and  $\gamma$  because I anticipate that the multivariate Whittle estimator is still consistent when  $\delta \in [0, 1/2) \cup (1/2, 1]$  and  $\gamma \in (-1/2, 1/2) \cup (1/2, 1]$ . A similar conjecture is done by [Shao \(2010\)](#) in the univariate case. It is interesting to investigate the finite sample properties of this estimator in these cases because some empirical stylized fact imply the stationary, the weak or the strong (that is  $\delta - \gamma > 1/2$ ) fractional cointegration cases. For instance, volatility data are often found to be stationarity while macroeconomics data often possess unit root.

Table 1: Model A -  $n = 256 - \phi_{11} = 0$

n = 256	$\hat{\theta}$	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
$\delta = 1.0$	$\gamma$	0		0,4		0,6		1,0	
	$\hat{\beta}$	0.000	0.012	-0.003	0.040	0.000	0.067	-0.007	0.263
	$\hat{\gamma}$	-0.027	0.068	-0.124	0.155	-0.091	0.133	-0.044	0.103
	$\hat{\delta}$	-0.034	0.086	-0.036	0.090	-0.037	0.094	-0.029	0.086
$\delta = 0.8$	$\gamma$	0		0,4		0,6		0,8	
	$\hat{\beta}$	0.000	0.024	-0.003	0.061	0.002	0.087	-0.001	0.127
	$\hat{\gamma}$	-0.026	0.068	-0.117	0.149	-0.084	0.128	-0.049	0.112
	$\hat{\delta}$	-0.076	0.121	-0.079	0.125	-0.080	0.129	-0.069	0.117
$\delta = 0.6$	$\gamma$	0		0,2		0,4		0,6	
	$\hat{\beta}$	0.000	0.040	-0.003	0.057	0.001	0.080	0.000	0.094
	$\hat{\gamma}$	-0.023	0.067	-0.105	0.130	-0.109	0.142	-0.082	0.129
	$\hat{\delta}$	-0.076	0.121	-0.079	0.125	-0.080	0.129	-0.069	0.117
$\delta = 0.4$	$\gamma$	0		0,2		0,3		0,4	
	$\hat{\beta}$	0.000	0.057	0.000	0.070	0.002	0.083	-0.001	0.084
	$\hat{\gamma}$	-0.021	0.066	-0.100	0.125	-0.109	0.137	-0.109	0.143
	$\hat{\delta}$	-0.106	0.137	-0.106	0.140	-0.107	0.142	-0.097	0.132

The simulations are conducted in the following model considering some regular restrictions:



$$\begin{pmatrix} \Delta^\gamma v_t \\ \Delta^\delta x_t \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} \Delta^\gamma v_{t-1} \\ \Delta^\delta x_{t-1} \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}, \quad (19)$$

where  $\phi_{ij}(k), k = 1$  is denoted  $\phi_{ij}$ . According to equation (17),  $\Lambda(e^{ij\lambda})$  is simply

$$\Lambda(e^{ij\lambda}) = \begin{pmatrix} |1 - e^{ij\lambda}|^\gamma & 0 \\ 0 & |1 - e^{ij\lambda}|^\delta \end{pmatrix}, \quad (20)$$

and  $\phi_{ij}$  enters the transfer matrix through

$$\Phi(e^{ij\lambda}) = I - \begin{pmatrix} \phi_{11}e^{ij\lambda} & \phi_{12}e^{ij\lambda} \\ \phi_{21}e^{ij\lambda} & \phi_{22}e^{ij\lambda} \end{pmatrix} \quad (21)$$

From the system of equations in (19) I set up two models. For each model, I generate 1000 artificial series for sample size  $n = \{256; 512\}$ . I also constrain  $\phi_{12} = \phi_{21} = \phi_{22} = 0$  to preserve the weak exogeneity of  $x_t$ .

$$\begin{aligned} \text{i) Model A: } & \begin{cases} (1-L)^\delta x_t = \varepsilon_{2t} \\ (1-L)^\gamma v_t = \varepsilon_{1t} \end{cases}, \quad \rho = \text{corr}(\varepsilon_{1t}, \varepsilon_{2t}) = 0; \\ \text{ii) Model B: } & \begin{cases} (1-L)^\delta x_t = \varepsilon_{2t} \\ (1-0.5L)(1-L)^\gamma v_t = \varepsilon_{1t} \end{cases}, \quad \rho = \text{corr}(\varepsilon_{1t}, \varepsilon_{2t}) = 0; \end{aligned}$$

$y_t$  is generated from (1) with  $\beta = 1$ . The case of the stationary fractional cointegration is explored for  $\delta = 0.4$  and  $\gamma = \{0.0, 0.2, 0.3, 0.4\}$  where  $\gamma = 0.4$  corresponds to spurious fractional regression. The case of the weak fractional cointegration is investigated when  $\delta = \{0.6, 0.8, 1.0\}$  and  $\delta - \gamma \leq 1/2$ . Alternatively, the case of the strong fractional cointegration is considered when  $\delta = \{0.6, 0.8, 1.0\}$  and  $\delta - \gamma > 1/2$ . Finally, the standard cointegration occurs when  $\delta = 1$  and  $\gamma = 0$ . When  $\delta = \gamma = 1$ , the regression is spurious. For each simulation, I report the bias and the root mean squared error (RMSE), defined by  $\frac{1}{T} \sum_{i=1}^T (E[\widehat{\theta}_i - \theta]) \triangleq \text{var}(\widehat{d}) + (\text{Bias}[\widehat{d}|d])^2$ . All computations are performed using RATS 8.01<sup>3</sup>.

The model A and B are consistent with the model presented in section 2. Nonetheless, it increases in complexity with B because of the presence of short run dynamics. Conversely, the model A is very simple and describes a multivariate fractional white noise. I also perform an

<sup>3</sup>RATS codes and unreported results are available upon request.

unreported investigation of model A and B in presence of nonzero correlation between  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$ . Consequently,  $\Sigma$  is no longer block-diagonal and one can anticipate some complication in this case. Notably, the model cannot identify separately  $\rho$  and  $\beta$  and estimate  $\beta + \rho$ .

Table 2: Model A -  $n = 512$  -  $\phi_{11} = 0$

n = 512	$\hat{\theta}$	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
$\delta = 1.0$	$\gamma$		0		0,4		0,6		1,0
	$\hat{\beta}$	0.000	0.007	0.000	0.025	0.000	0.047	-0.002	0.246
	$\hat{\gamma}$	-0.010	0.029	-0.085	0.094	-0.071	0.084	-0.050	0.064
	$\hat{\delta}$	-0.047	0.058	-0.048	0.060	-0.045	0.058	-0.045	0.057
$\delta = 0.8$	$\gamma$		0		0,4		0,6		0,8
	$\hat{\beta}$	0.000	0.014	0.000	0.037	0.000	0.050	-0.001	0.072
	$\hat{\gamma}$	-0.009	0.028	-0.082	0.091	-0.067	0.081	-0.047	0.066
	$\hat{\delta}$	-0.044	0.062	-0.047	0.065	-0.043	0.063	-0.041	0.061
$\delta = 0.6$	$\gamma$		0		0,2		0,4		0,6
	$\hat{\beta}$	0.000	0.024	0.000	0.035	0.000	0.043	-0.001	0.044
	$\hat{\gamma}$	-0.008	0.028	-0.069	0.077	-0.079	0.089	-0.068	0.081
	$\hat{\delta}$	-0.066	0.079	-0.069	0.082	-0.066	0.079	-0.064	0.077
$\delta = 0.4$	$\gamma$		0		0,2		0,3		0,4
	$\hat{\beta}$	0.001	0.035	0.001	0.040	0.000	0.043	-0.001	0.041
	$\hat{\gamma}$	-0.007	0.027	-0.067	0.075	-0.077	0.086	-0.080	0.090
	$\hat{\delta}$	-0.080	0.088	-0.081	0.091	-0.079	0.088	-0.077	0.087

As regards the stationary fractional cointegration, the bias and the RMSE of  $\beta$  are very low for all sample size. Results of  $\delta$  and  $\gamma$  are particularly subject to higher bias and variances when  $\gamma$  is close to  $1/2$ . Overall, the bias and the variance decrease when the sample size increases. Compared with the results of Nielsen (2007) and Hualde and Robinson (2007), these results are slightly more biased but display lower variance. When  $\delta$  is greater than  $1/2$ , the assumption  $\delta \in [0, 1/2)$  is violated. However, results are suggesting that the Whittle QMLE of fractional cointegration is still consistent for some non-stationary regions of  $\delta$  and  $\gamma$  including the unit root case. When  $\delta = \{0.6, 0.8\}$  the bias is still very low but the RMSE increases when  $\delta = \gamma$ . No significant differences appear between the cases of strong ( $\delta - \gamma > 1/2$ ) and weak ( $\delta - \gamma \leq 1/2$ ) fractional cointegration. Concerning the long memory parameters, results are similar to the stationary case. Since Hualde and Robinson (2007) essentially investigate the finite sample properties of their estimator on  $\beta$ , results cannot be compared. Nielsen (2007) does not explore the non-stationary cases and similarly, results cannot be confronted in this case. I give a special attention to the results when  $\delta = 1$  because this case is the most investigated in the literature.

Note that in this case, the results are not significantly different. Moreover, in the case of the spurious regression, the Whittle QMLE does not seem to have an additional bias. This seems to confirm the conjecture that the estimator is still consistent when  $\delta = 1$ .

Table 3: Model B -  $n = 256 - \phi_{11} = 0.4$

n = 256	$\hat{\theta}$	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
$\delta = 1.0$	$\gamma$	0		0,4		0,6		1,0	
	$\hat{\beta}$	0.001	0.020	0.002	0.064	0.001	0.106	-0.006	0.438
	$\hat{\gamma}$	-0.098	0.172	-0.236	0.404	-0.349	0.564	-0.319	0.641
	$\hat{\delta}$	-0.098	0.200	-0.109	0.232	-0.095	0.225	-0.162	0.382
$\delta = 0.8$	$\gamma$	0		0,4		0,6		0,8	
	$\hat{\beta}$	0.001	0.037	-0.001	0.081	-0.001	0.103	0.001	0.152
	$\hat{\gamma}$	-0.088	0.165	-0.179	0.343	-0.197	0.411	-0.214	0.483
	$\hat{\delta}$	-0.084	0.195	-0.092	0.221	-0.106	0.268	-0.113	0.280
$\delta = 0.6$	$\gamma$	0		0,2		0,4		0,6	
	$\hat{\beta}$	-0.001	0.055	-0.003	0.074	-0.002	0.081	-0.001	0.089
	$\hat{\gamma}$	-0.074	0.167	-0.116	0.237	-0.139	0.287	-0.164	0.364
	$\hat{\delta}$	-0.096	0.193	-0.106	0.204	-0.101	0.197	-0.116	0.228
$\delta = 0.4$	$\gamma$	0		0,2		0,3		0,4	
	$\hat{\beta}$	-0.003	0.066	-0.003	0.074	-0.003	0.075	-0.002	0.076
	$\hat{\gamma}$	-0.064	0.151	-0.108	0.220	-0.112	0.247	-0.122	0.275
	$\hat{\delta}$	-0.124	0.184	-0.121	0.183	-0.122	0.183	-0.122	0.184

The Model B is increasing in complexity. Again, it is difficult to compare the results with the literature but clearly the short-run dynamics negatively impact the bias and the variance. Observe that the bias in the estimate of  $\gamma$  is less pronounced in the case of stationary fractional cointegration, that is  $\delta = 0.4$  and  $\gamma \leq 0.4$  in this simulation. While the Whittle QMLE poorly estimates the model B when the sample size is  $n = 256$ , the bias considerably decreases when the sample size increase. Notice that the Whittle QMLE seems inconsistent in the classical spurious regression case, that is  $\delta = 1.0$  and  $\gamma = 1.0$ . Globally, the bias concerning  $\hat{\gamma}$  increases when  $\delta = 1.0$ . Unreported experiment investigates the case of a non-null correlation between  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$ , termed  $\rho$ . In this case,  $\Sigma$  is no-longer block-diagonal and one can anticipate some complication in this case. Regarding to this specification, the estimates of the integration orders are moderately impacted but the model cannot identify separately  $\rho$  and  $\beta$  and estimate  $\beta + \rho$ .

Table 4: Model B -  $n = 512$  -  $\phi_{11} = 0.4$ 

n = 512	$\hat{\theta}$	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
$\delta = 1.0$	$\gamma$	0		0,4		0,6		1,0	
	$\hat{\beta}$	0.000	0.010	0.000	0.037	0.000	0.065	0.009	0.371
	$\hat{\gamma}$	-0.012	0.161	-0.119	0.461	-0.071	0.403	-0.057	0.433
	$\hat{\delta}$	-0.017	0.102	-0.020	0.077	-0.013	0.117	-0.028	0.219
$\delta = 0.8$	$\gamma$	0		0,4		0,6		0,8	
	$\hat{\beta}$	0.000	0.021	0.000	0.043	0.002	0.050	0.003	0.085
	$\hat{\gamma}$	-0.008	0.163	-0.063	0.367	-0.047	0.354	-0.018	0.341
	$\hat{\delta}$	-0.002	0.086	-0.001	0.126	-0.002	0.123	-0.003	0.142
$\delta = 0.6$	$\gamma$	0		0,2		0,4		0,6	
	$\hat{\beta}$	0.000	0.032	0.001	0.039	0.001	0.041	0.001	0.041
	$\hat{\gamma}$	0.000	0.148	-0.031	0.280	-0.052	0.336	-0.009	0.267
	$\hat{\delta}$	-0.011	0.097	-0.010	0.099	-0.009	0.079	-0.011	0.118
$\delta = 0.4$	$\gamma$	0		0,2		0,3		0,4	
	$\hat{\beta}$	0.001	0.039	0.001	0.040	0.001	0.039	0.001	0.038
	$\hat{\gamma}$	0.005	0.147	-0.015	0.245	-0.026	0.278	-0.024	0.283
	$\hat{\delta}$	-0.022	0.144	-0.023	0.148	-0.023	0.146	-0.014	0.101

#### 4. Empirical illustration

In this section, I investigate the question of the stock market synchronization in order to illustrate the empirical relevance of the Whittle QMLE. This issue has found a greater interest in the last decades with the creation of the Economic and Monetary Union (see for instance [Adjaout and Danthine \(2004\)](#) and [Gilmore and McManus \(2003\)](#)) and has many implications in terms of portfolio diversification. Indeed, the stock market synchronization is closely related to monetary integration because of the reduction in the exchange rate volatility that promotes economic trade and financial integration. Consequently, it can be expected that the creation of the EMU has increased the stock market synchronization of the member countries. The question can be asked concerning the non-member European countries but also for some non-European countries because of the raise in trade intensity, international shocks and portfolios diversification. A convenient way to test for stock market synchronization is to explore the presence of comovement ([Wälti \(2011\)](#)) or long run correlation between stock market returns or volatility. [Gilmore and McManus \(2002\)](#), [Gilmore and McManus \(2003\)](#) and [Égert and Kočenda \(2007\)](#) investigate the stock market synchronization issue in the long-run through the traditional cointegration but find no strong evidence of long-term correlation. In this paper I bring the new perspective of the

fractional cointegration. In contrast to Wälti (2011) that leads a panel analysis and examines the synchronization causes, this study is more interested by the question of the existence of the synchronization phenomenon in a bivariate context. A similar methodology is used by Nielsen and Frederiksen (2011) in order to analyze how the volatility of a particular stock depends on the volatilities of two stock market indexes.

In the following, I consider the squared returns of six national stock indexes: CAC 40, IBEX 35, SMI 20, FTSE 100, S&P 500, NIKKEI 225. Weekly average data are used from January 1994 to December 2011 and expressed in local currency<sup>4</sup>. The weekly frequency is preferred here to avoid the issue of non-synchronous trading hours encountered with daily data and because of the lack of information provided by monthly data. Concerning the monetary integration, obviously the French index and the Hispanic index are strongly related while the Switzerland index and the British index represent a lower degree of integration. The American and Japanese stock indexes are investigated although they are not European countries because of the interest of these countries in terms of portfolio diversification. Several empirical studies point out that the financial integration can vary over time (Cappiello et al. (2006), Égert and Kočenda (2011)) and has notably been impacted by the creation of the euro. Thus we also consider sub-periods, before and after the creation of the euro.

I estimate the equation (1) for each bilateral relation and I test the lag structure against the hypothesis of a vector fractional white noise process. The  $VARFI(1, \delta, \gamma)$  process hypothesis can never be rejected. The star tags do not identify the significance of the result at 10 percent but a stationary fractional cointegration relationship.

The full sample estimation reveals several long-run relationship (table 5). Concerning the two European countries, the dimension reduction occurs with the IBEX 35 and the CAC 40 suggesting a market synchronization. Similar relations appear between the FTSE 100 and the CAC 40, the IBEX 35 and the FTSE and the SMI and the CAC although the United Kingdom and the Switzerland are not in the euro zone. In most of cases there is no evidence of cointegration on bilateral basis between European countries and non European countries, except for the SMI 20 and the S&P500 and the IBEX 35 and S&P500.

The first sub-sample covers the period before the creation of the euro (table 6). During this period, the monetary integration was insured by a basket of the currencies of the European

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<sup>4</sup>Data are collected from <http://fr.finance.yahoo.com/>.

Table 5: Estimating for bilateral stationary fractional cointegration for the period 1994-2011 ( $n = 934$ )

$y_t$	CAC			FTSE			NIKKEI		
$x_t$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$
CAC				0.452*	0.236*	0.137*	0.090	0.236	0.129
				(0.027)	(0.032)	(0.039)	(0.033)	(0.035)	(0.041)
FTSE	0.459	0.231	0.745				0.114	0.231	0.127
	(0.027)	(0.035)	(0.039)				(0.033)	(0.038)	(0.042)
NIKKEI	0.098	0.147	0.226	0.108	0.147	0.219			
	(0.033)	(0.040)	(0.033)	(0.032)	(0.038)	(0.039)			
S&P	0.287	0.220	0.208	0.269	0.220	0.180	0.077	0.220	0.133
	(0.034)	(0.035)	(0.033)	(0.032)	(0.034)	(0.039)	(0.032)	(0.034)	(0.042)
IBEX	0.392	0.191	0.170	0.340	0.191	0.170	0.100	0.191	0.134
	(0.033)	(0.035)	(0.032)	(0.029)	(0.033)	(0.038)	(0.033)	(0.034)	(0.041)
SMI	0.341	0.179	0.178	0.325	-0.780	0.158	0.082	0.179	0.137
	(0.031)	(0.035)	(0.032)	(0.029)	(0.015)	(0.037)	(0.031)	(0.035)	(0.039)

  

$y_t$	S&P			IBEX			SMI		
$x_t$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$
CAC	0.261	0.236	0.194	0.365*	0.236*	0.101*	0.355*	0.236*	0.099*
	(0.032)	(0.031)	(0.033)	(0.028)	(0.032)	(0.036)	(0.031)	(0.030)	(0.039)
FTSE	0.279	0.231	0.177	0.347*	0.231*	0.120*	0.371*	0.231*	0.098*
	(0.034)	(0.038)	(0.035)	(0.031)	(0.037)	(0.035)	(0.033)	(0.037)	(0.039)
NIKKEI	0.105	0.147	-0.373	0.104	0.147	0.182	0.085	0.147	0.173
	(0.035)	(0.039)	(0.014)	(0.033)	(0.040)	(0.033)	(0.035)	(0.039)	(0.037)
S&P				0.205*	0.220*	0.149*	0.228*	0.220*	0.134*
				(0.034)	(0.034)	(0.036)	(0.035)	(0.035)	(0.037)
IBEX	0.213	0.191	0.869				0.346	0.191	0.128
	(0.032)	(0.035)	(0.045)				(0.033)	(0.034)	(0.038)
SMI	0.209	0.179	0.185	0.313	0.179	0.139			
	(0.031)	(0.034)	(0.034)	(0.031)	(0.036)	(0.035)			

Community member states: the European Currency Unit (ECU). This is supported by one cointegration relationship between the CAC 40 and the IBEX 35 implying that the CAC 40 squared returns' depended on the IBEX 35 in the long run. A few Similar long-term correlations appear between the SMI 20 and the FTSE 100 and the SMI 20 and the IBEX 35.

In order to appreciate the impact of the creation of the euro, I consider now the second sub-sample that covers the period 1999-2011 (table 7). Clearly there is more cointegration relationships in comparison with the first sub-period but the impact is moderated. As in previous estimates, the IBEX 35 and the CAC 40 are correlated in the long run. The IBEX index is also cointegrated with the FTSE 100 and the S&P 500. A similar cointegration relationship can be observed between the SMI 20 and the CAC40, the FTSE and the S&P 500. In comparison with the ECU system, there is less benefit for US investors to invest in these stock index, in terms of portfolio diversification, since the creation of the euro. Indeed, the S&P 500 is cointegrated with

Table 6: Estimating for bilateral stationary fractional cointegration for the period 1994-1999 ( $n = 261$ )

$y_i$	CAC			FTSE			NIKKEI		
$x_i$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$
CAC				0.287 (0.057)	0.150 (0.061)	0.163 (0.076)	0.088 (0.057)	0.150 (0.060)	0.254 (0.079)
FTSE	0.310 (0.059)	0.197 (0.071)	0.099 (0.066)				0.063 (0.060)	0.197 (0.069)	0.258 (0.079)
NIKKEI	0.090 (0.066)	0.253 (0.076)	0.147 (0.061)	0.071 (0.066)	0.253 (0.076)	0.201 (0.067)			
S&P	-0.014 (0.066)	0.235 (0.069)	0.152 (0.061)	0.029 (0.065)	0.235 (0.069)	0.193 (0.069)	0.091 (0.058)	0.235 (0.068)	0.253 (0.074)
IBEX	0.317* (0.061)	0.188* (0.069)	0.077* (0.069)	0.260 (0.062)	0.188 (0.067)	0.165 (0.075)	0.039 (0.058)	0.188 (0.067)	0.253 (0.076)
SMI	0.157 (0.056)	0.150 (0.070)	0.108 (0.063)	0.281 (0.051)	0.150 (0.073)	0.143 (0.065)	0.017 (0.053)	0.150 (0.074)	0.253 (0.078)
$y_i$	S&P			IBEX			SMI		
$x_i$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$
CAC	-0.073 (0.063)	0.150 (0.060)	0.239 (0.070)	0.315 (0.059)	0.150 (0.058)	0.128 (0.069)	0.199 (0.068)	0.150 (0.062)	0.102 (0.078)
FTSE	0.091 (0.059)	0.197 (0.068)	0.730 (0.071)	0.275 (0.062)	0.197 (0.068)	0.155 (0.074)	0.375* (0.071)	0.197* (0.068)	0.071* (0.081)
NIKKEI	0.114 (0.063)	0.253 (0.076)	0.806 (0.076)	0.027 (0.067)	0.253 (0.076)	0.014 (0.035)	0.033 (0.075)	0.253 (0.077)	0.149 (0.074)
S&P				-0.073 (0.066)	0.235 (0.065)	0.200 (0.069)	-0.021 (0.073)	0.235 (0.069)	0.154 (0.074)
IBEX	0.031 (0.060)	0.188 (0.075)	0.774 (0.075)				0.356* (0.070)	0.188* (0.067)	0.065* (0.083)
SMI	-0.018 (0.058)	0.150 (0.076)	0.237 (0.071)	0.278 (0.054)	0.150 (0.074)	0.132 (0.073)			

the CAC 40 and the FTSE 100. as the CAC 40 and the FTSE 100.

To summarize, the creation of the euro has seemingly increased the stock markets synchronization of the countries considered but moderately. This has led to a greater monetary integration than the ECU system. This implies an impact in terms of portfolios diversification and results can help to clarify the investors horizons.

## 5. Final comments

I considered a parametric Whittle multivariate approach of a bivariate system of fractional cointegration. I suggested to jointly estimate all parameters of interest of the model and possibly others nuisance parameters. A Monte Carlo experiment is proposed and confirms the good finite sample properties of this estimator in the stationary regions of the residuals and the regressor (that is  $\delta$  and  $\gamma$  are less than  $1/2$ ). This also reveals that the Whittle QMLE can be extended to the

Table 7: Estimating for bilateral stationary fractional cointegration for the period 1999-2011 ( $n = 674$ )

$y_t$	CAC			FTSE			NIKKEI		
$x_t$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$
CAC				0.512*	0.254*	0.111*	0.099	0.254	0.083
				(0.032)	(0.039)	(0.050)	(0.039)	(0.040)	(0.050)
FTSE	0.556	0.241	0.167				0.144	0.241	0.071
	(0.035)	(0.045)	(0.041)				(0.040)	(0.047)	(0.050)
NIKKEI	0.089	0.113	0.244	0.118	0.113	0.221			
	(0.040)	(0.048)	(0.039)	(0.039)	(0.045)	(0.045)			
S&P	0.390	0.209	0.219	0.352	0.209	0.157	0.078	0.209	0.094
	(0.038)	(0.041)	(0.039)	(0.037)	(0.039)	(0.048)	(0.039)	(0.040)	(0.048)
IBEX	0.416	0.192	0.190	0.371	0.192	0.163	0.129	0.192	0.087
	(0.038)	(0.039)	(0.042)	(0.038)	(0.041)	(0.044)	(0.041)	(0.040)	(0.047)
SMI	0.419	0.195	0.193	0.348	0.195	0.152	0.115	0.195	0.090
	(0.036)	(0.039)	(0.041)	(0.039)	(0.042)	(0.046)	(0.039)	(0.044)	(0.048)

  

$y_t$	S&P			IBEX			SMI		
$x_t$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\gamma}$
CAC	0.370*	0.254*	0.160*	0.387*	0.254*	0.090*	0.409*	0.254*	0.106*
	(0.036)	(0.038)	(0.039)	(0.034)	(0.040)	(0.041)	(0.035)	(0.040)	(0.049)
FTSE	0.368*	0.241*	0.131*	0.376*	0.241*	0.107*	0.373*	0.241*	0.107*
	(0.038)	(0.046)	(0.042)	(0.035)	(0.042)	(0.036)	(0.039)	(0.046)	(0.045)
NIKKEI	0.065	0.113	0.201	0.121	0.113	0.180	0.099	0.113	0.187
	(0.039)	(0.048)	(0.043)	(0.038)	(0.042)	(0.039)	(0.038)	(0.045)	(0.039)
S&P				0.293*	0.209*	0.129*	0.314*	0.209*	0.123*
				(0.034)	(0.039)	(0.039)	(0.038)	(0.041)	(0.044)
IBEX	0.303	0.192	0.153				0.344	0.192	0.150
	(0.038)	(0.040)	(0.040)				(0.037)	(0.039)	(0.042)
SMI	0.309	0.195	0.143	0.330	0.195	0.142			
	(0.037)	(0.044)	(0.042)	(0.035)	(0.044)	(0.039)			

analysis of the weak and strong fractional cointegration cases. These results slice with other fractional cointegration estimators that concentrate on specific cases. A concise empirical analysis of the question of the stock market synchronization is presented. It points out that the stationary fractional cointegration model is appropriated to investigate this question and highlights several long-term relationship between different stock market indexes. Conclusions in terms of portfolio diversification can be deducted.

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