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To cite this version:

Elyès Jouini, Clotilde Napp. Arbitrage and investment opportunities. Finance and Stochastics, 2001, 5 (3), pp.305-325. halshs-00778381

HAL Id: halshs-00778381

https://shs.hal.science/halshs-00778381

Submitted on 19 Jan 2013

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Arbitrage and Investment Opportunities

Elyès Jouini* Clotilde Napp†

September 99

Abstract

We consider a model in which all investment opportunities are described in terms of cash flows. We don't assume that there is a numéraire, the time horizon is not supposed to be finite, the investment opportunities are not specifically related to the buying and selling of securities on a financial market. In this quite general framework, we show that the assumption of no-arbitrage is essentially equivalent to the existence of a "discount process" under which the "net present value" of any investment is nonpositive. Since most market imperfections, such as short sale constraints, convex cone constraints, proportional transaction costs, no borrowing or different borrowing and lending rates, etc., can fit in our model for a specific set of investments, we then obtain a characterization of the no-arbitrage condition in these imperfect models, from which it is easy to derive pricing formulae for contingent claims.

Compared with existing results, our approach allows to consider markets with no numéraire or with a numéraire that is subject to constraints. Besides, we introduce a notion of no-free lunch which is less restrictive than the usual one. Last, we characterize the assumption of no-arbitrage (or no-free lunch) for more general investment opportunities, which enables us to consider investments that are not necessarily related to a market model and, more interestingly, to generalize the results obtained for imperfect markets and to obtain them all in a unified way.

Keywords : arbitrage - investment opportunities - numéraire - market imperfections - transaction costs - Yan's theorem

1. Introduction

A classical problem in mathematical finance is the pricing of financial assets. This problem is usually solved through the use of the so-called Fundamental Theorem of Asset Pricing (see Harrison and Kreps (1979) or Harrison and Pliska (1981)). This theorem asserts that in a perfect financial model, the assumption of no-arbitrage (which amounts to saying that there is no plan yielding some profit without a countervailing threat of loss) is essentially equivalent to the existence of an equivalent martingale measure. The problem of fair pricing of financial assets is then reduced to taking their expected value with respect to equivalent martingale measures.

We want in this paper to find the analog of the Fundamental Theorem of Asset Pricing for very general models of investment opportunities. Such opportunities

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are described in a deterministic setting by Cantor-Lippman (1983, 1995), Dermody-Rockafellar (1991, 1995) and Adler-Gale (1997). More precisely, we adopt a model where all investment opportunities are described by their cash flows; for instance, in such a model, the investment opportunity which consists in buying, in a perfect financial model, at date s_1 one unit of a risky asset, whose price process is given by $(S_t)_{t\geq 0}$ and selling at date s_2 with $s_1 \leq s_2$ the unit bought, is described by a process $(\Phi_t)_{t\geq 0}$ which is null outside $\{s_1, s_2\}$ and which satisfies $\Phi_{s_1} = -S_{s_1}$ and $\Phi_{s_2} = S_{s_2}$. Our investment opportunities are assumed to be very general: they are not specifically related to a market model, like in the just mentioned example. The time horizon is not supposed to be finite. The framework is the one of continuous time. We don't assume that there exists a numéraire, enabling investors to transfer money from one date to another and even if such possibilities exist, we do not assume that the lending rate is equal to the borrowing rate or that we have possibilities to borrow.

We find in this very general model that the assumption of no-free lunch is essentially equivalent to the existence of a normalization process such that the "net present value" of any investment opportunity is nonpositive, i.e. there exists a process such that the "net present value" of the normalized cash flows is nonpositive. We emphasize that neither interest rate nor net present value are part of our model. As in Dermody-Rockafellar (1991, 1995) and Adler-Gale (1997), there is no externally given term structure, which would be needed if one were to apply the classical criterion of net present value; these notions arise however as a consequence of the assumption of no-arbitrage.

We then use this very general result for specific financial market models: perfect financial model, for which we prove the well-known characterization of the assumption of no-arbitrage given by the Fundamental Theorem of Asset Pricing in finite time, and a slightly different version in infinite time; but mainly financial models with imperfections like imperfections on the numéraire, proportional transaction costs, short sale constraints, convex cone constraints, etc., for which we generalize existing results.

Initial results on the Fundamental Theorem of Asset Pricing in the perfect case were achieved by Harrison-Kreps (1979), Harrison-Pliska (1979), Kreps (1981) and Duffie-Huang (1986). Various generalizations are now available in the literature: in Dalang, Morton and Willinger (1989), the problem was solved in the case of finite discrete time, by only using the assumption of no-arbitrage. For discrete infinite or continuous time, the notion of free lunch is needed; Schachermayer (1994) mainly deals with the case of discrete infinite time. Continuous time models are studied by (among others) Ansel-Stricker (1990), Delbaen (1992), Delbaen-Schachermayer (1994, 1998), Frittelli-Lakner (1994), Stricker (1990). In all these models, securities markets are assumed to be frictionless.

In the context of imperfect financial markets, Jouini-Kallal (1995a) characterize the assumption of no-free lunch in a model with proportional transaction costs and give fair pricing intervals for contingent claims in such a model. Cvitanic-Karatzas (1996) study the problem of hedging contingent claims, in continuous time, for a diffusion model (with one bond and one risky asset) with proportional transaction costs, and give a dual formula for the so-called super-replication price of a contingent claim (i.e. the minimum initial wealth needed to hedge the contingent claim or in other words, to obtain, through the investment opportunities available on the market, at least the contingent claim). Delbaen-Kabanov-Valkeila (1998) and Kabanov (1998) generalize this result to the multivariate case, in discrete as well as in continuous time,

and with a semimartingale price process.

As for other imperfections, Jouini-Kallal (1995b) study the case of short sale constraints or shortselling costs with possibly different borrowing and lending rates. For convex constraints (and also with possible higher interest rates for borrowing), the dual formulation for the superreplication price is obtained in a diffusion framework in Cvitanic-Karatzas (1993). In a more general framework, the result is obtained in Föllmer-Kramkov (1997). Pham-Touzi (1996) consider the Fundamental Theorem of Asset Pricing with cone constraints, in a discrete and finite time setting and by only using the assumption of no-arbitrage. Brannath (1997) studies the same problem in the more general setting of convex constraints.

This paper generalizes Carassus-Jouini (1997), that considers discrete models, i.e. with finite time horizon, discrete time and finite number of states of the world at each time.

We generalize existing results in the following ways: first, we don't assume that there exists a numéraire available to investors and allowing them to transfer money from one date to another; this enables us to consider any type of friction on the numéraire like no borrowing, different borrowing and lending rates, bonds with default risk, etc., which have been barely studied, or simply to take into account the fact that all investors are not equal with regard to borrowing and lending, namely some investors may enjoy special borrowing facilities while others may not; second, we are lead to introduce a new notion of no-free lunch, which is similar to the "usual one" (with deterministic times) in finite time but does not exclude a free lunch at infinity and is therefore maybe more economically meaningful; last, we characterize the assumption of no-arbitrage (or more precisely of no-free lunch) for very general investments, which enables us to consider investment opportunities that are not necessarily related to a market model and, more interestingly, to generalize the results obtained for imperfect markets and to obtain them all in a unified way.

The paper is organized as follows: in section 2, we obtain, under a "reasonable" assumption, an analog of Yan (1980)'s result in more complex spaces than the ones usually considered. Indeed, since we are not allowed to transfer money from one date to another, we cannot consider net gains anymore; in all papers dealing with the Fundamental Theorem of Asset Pricing (with simple integrands), the assumption of no-arbitrage or no-free lunch essentially amounts to saying that the set $Lin\left\{\theta_s \cdot \left(\bar{S}_t - \bar{S}_s\right)\right\}$, where θ describes all feasible strategies and $\bar{S}$ denotes the discounted underlying price process, contains no nonnegative nonnull random variable. Implicit in such an approach is the fact that there is an externally given term structure, enabling investors to borrow and lend money at the same rate. In our context with flows, we must use separation techniques in more complex spaces to obtain the Fundamental Theorem of Asset Pricing. We apply this characterization of the no-free lunch assumption to different market models with frictions in sections 3 and 4.

All the proofs are in the Appendix.

2. The Fundamental Theorem of Asset Pricing in a model with flows

As we have sen in the Introduction, since we are not necessarily allowed to transfer wealth through time, we must consider more general spaces. We start by introducing these spaces. Then we describe our general model with flows. Finally, we obtain,

through an analog of Yan's (1980) result, the characterization of the no free lunch assumption in such a model.

2.1. The space $L_P^1(\Omega, \mathcal{M}_b)$

For details about most notions introduced in this section, see Marle (1974) or Diestel-Uhl (1977).

We denote by $\mathcal{C}_0$ the set of continuous functions from R to R which converge to 0 at infinity; endowed with the uniform convergence topology, $\mathcal{C}_0$ is a Banach space. We denote by $\mathcal{M}_b$ the space of bounded Radon measures, i.e. the space of continuous linear functionals on $\mathcal{C}_0$; the space $\mathcal{M}_b$, endowed with the usual dual norm $\|\|_{\mathcal{M}_b}$ defined by $\|\mu\|_{\mathcal{M}_b} \equiv \sup\{|\mu(f)|; f \in \mathcal{C}_0; \|f\| \leq 1\}$ for all μ in $\mathcal{M}_b$, is a Banach space.

Fix a probability space (Ω, F, P). Let $(X, \|\|_X)$ be a Banach space. The set $St(\Omega, X)$ denotes the set of simple random variables, i.e., the set of random variables f of the form $f = \sum_{i=1}^m a_i 1_{A_i}$ for some m in N^*, A_i in F and a_i in X. The set $M_P(\Omega, X)$ denotes the set of P-measurable random variables, i.e. the set of random variables f such that there exists a sequence $(f_n)_{n\in N}$ in $St(\Omega, X)$ for which $f = \lim_n f_n$ a.s. P. Then $L_P^1(\Omega, X)$ denotes the set of P-measurable random variables f such that $\|f\|_X$ belongs to $L^1(\Omega, R)$, i.e. such that $\|f\|_{L_P^1(\Omega,X)} \equiv E^P[\|f\|_X] < \infty$.
Denote by $(X', \|\|_{X'})$ the dual space of X, endowed with the dual norm. We know that $(X', \|\|_{X'})$ is itself a Banach space. We define the space $\mathcal{L}_*^\infty(\Omega, P, X')$ to be the set of random variables $g : \Omega \to X'$, for which

$$\langle g, e\rangle : \omega \to \langle g(\omega), e\rangle \text{ is measurable for all } e \text{ in } X \text{ and}$$

$$\|g\|_{\mathcal{L}_*^\infty(\Omega,P,X')} \equiv \inf\{M \geq 0, \|g\|_{X'} \leq M \quad P \text{ a.s.}\} < \infty.$$

Then the set $L_*^\infty(\Omega, P, X')$ is defined to be the set of equivalence classes $g^\bullet$ of elements of $\mathcal{L}_*^\infty(\Omega, P, X')$, where $f \sim g \Leftrightarrow \langle f, e\rangle = \langle g, e\rangle$ P a.s. for all e in X. We consider the norm $\|\|_{L_*^\infty(\Omega,P,X')}$ given by $\|g\|_{L_*^\infty(\Omega,P,X')} = \inf_{g\in g^\bullet} \|g\|_{\mathcal{L}_*^\infty(\Omega,P,X')}$ for all $g \in L_*^\infty(\Omega, P, X')$.

It is shown in Schwartz (1974) that the dual space of $L_P^1(\Omega, X)$ is the space $L_*^\infty(\Omega, P, X')$. More precisely, there is an isomorphism between $L_*^\infty(\Omega, P, X')$ and $\left[L_P^1(\Omega, X)\right]'$ that associates with any random variable g in $L_*^\infty(\Omega, P, X')$ the continuous linear functional Ψ on $L_P^1(\Omega, X)$ given by

$$\Psi : f \to \Psi(f) = E\left[\langle f, g\rangle_{X,X'}\right] \quad \text{for all } f \text{ in } L_P^1(\Omega, X)$$

where $\langle f, g\rangle_{X,X'} : \omega \mapsto \langle f(\omega), g(\omega)\rangle_{X,X'}$. Notice then that $L_*^\infty(\Omega, P, X')$ is a Banach space.

Let Γ denote the set of random variables γ from Ω to $\mathcal{M}_b$ of the form $\gamma = \sum_{i=1}^m \gamma_{t_i}\delta_{t_i}$ i.e. $\gamma : \omega \mapsto \gamma_{t_1}(\omega)\delta_{t_1} + ... + \gamma_{t_m}(\omega)\delta_{t_m}$ with γ_{t_i} in $L^1(\Omega, R)$ for all $t_i \geq 0$. We should write $\gamma = \sum_{i=1}^{m_\gamma} \gamma_{t_i^\gamma}\delta_{t_i^\gamma}$ but we will omit the index γ for the simplicity of the notations.
We have the following

Lemma 2.1. *The set Γ is included in $L_P^1(\Omega, \mathcal{M}_b)$ and for all γ in Γ, $\gamma = \sum_{i=1}^m \gamma_{t_i}\delta_{t_i}$, $\|\gamma\|_{L_P^1(\Omega,\mathcal{M}_b)} = \sum_{i=1}^m \|\gamma_{t_i}\|_{L^1(\Omega,R)}$.*

Let Γ_+ (resp. Γ_-)$\equiv \left\{\gamma \in \Gamma; \gamma = \sum_{i=1}^m \gamma_{t_i}\delta_{t_i}; \gamma_{t_i} \geq 0 \text{ (resp. } \leq 0) \ P \text{ a.s.}\right\}$.

2.2. The model

We consider a model in which agents face investment opportunities described by their cash flows. A probability space (Ω, F, P) is specified and fixed. The set Ω represents all possible states of the world. An information structure, which describes how information is revealed to investors, is given by a filtration $(F_t)_{t\in\mathbf{T}}$ satisfying the "usual conditions" and such that $F_0 = \{\emptyset, \Omega\}$. In finite discrete time, we take $\mathbf{T} = \{1, ..., d\}$, in infinite discrete time $\mathbf{T} = N$ and in infinite continuous time $\mathbf{T} = R_+$. We model investment opportunities which are available to investors in the following way.

Definition 2.2. *An investment is an $(F_t)_{t\in\mathbf{T}}$-adapted process $\Phi = (\Phi_t)_{t\in\mathbf{T}}$, null outside a finite number of dates, i.e. there exists $\left(t_1^\Phi, ..., t_N^\Phi\right)$ such that $\Phi_t = 0$ for all $t \notin \left(t_i^\Phi\right)_{i=1}^N$, and such that Φ_t is in $L^1(\Omega, F_t, P)$ for all t in $\mathbf{T}$.*

We consider a convex cone J of investment opportunities: this amounts to saying that an investor has a right to subscribe to (a finite number of) different investment plans and that he can decide at the starting date of any investment opportunity which amount of this particular investment he wants to buy. We shall see in sections 3 and 4, with specific examples, that we are lead to consider convex cones in order to take into account the fact that investors are not necessarily able to sell an investment plan (see for instance the case of short sale constraints or transaction costs).

We don't specify the elements of J so far. We consider any investment $\Phi = (\Phi_t)_{t\in\mathbf{T}}$ as a random variable from Ω to the set of discrete bounded Radon measures of the form

$$\Phi = \sum_{t\geq 0} \Phi_t \delta_t$$

where $\Phi_t = 0$ for $t \notin \left(t_i^\Phi\right)_{i=1}^N$, i.e. as an element of Γ.
We introduce the following assumption.

Assumption A: there exists a sequence $d = (d_n)_{n\geq 0}$ such that for all $t^* \geq 0$, for all B_{t^*} in F_{t^*} of positive probability, there exists Φ in J of the form $\Phi_{t^*} = 0$ outside B_{t^*}, $\Phi_t = 0$ for all $t < t^*$, $\Phi_t \geq 0$ for all $t > t^*$, and there exists $d_n \in d$, $P\left[\Phi_{d_n} > 0\right] > 0$.

In words, if a convex cone J of available investments satisfies *Assumption A*, this means that there exists a sequence of trading dates such that, for all date and for all event at that date, there exists an investment plan in our admissible set of investment opportunities that starts at that date and in that event, that can take any value at that date and in that event but that is nonnegative after that date and nonnull at one date belonging to the above mentioned sequence of dates. Roughly, *Assumption A* corresponds to the possibility of transferring "some money" from any date and event to some particular date. This assumption is not too restrictive: it is satisfied if we can buy at every date and event a bond with a given maturity even if this bond is defaultable and even if there is no secondary market for that bond (i.e. we have to wait until the maturity in order to recover any money with a positive probability, which may be different from 1); this includes market models with frictions on the numéraire like no borrowing, different borrowing and lending rates, bonds with default risk, different borrowing facilities among investors. More generally, it is satified if there is at least one asset whose price cannot be negative (which is usually the case for stocks or for options, defaultable bonds, etc.). Notice that it is satisfied in all cases of market

models with frictions that we have mentioned in the introduction and that we shall consider more precisely in sections 3 and 4.

We now come to the notion of no-arbitrage.

Definition 2.3. *There is no arbitrage opportunity for J if and only if $J \cap \Gamma_+ = \{0\}$.*

Let us check that this definition corresponds to the usual notion of no-arbitrage, i.e. an impossibility to have access to an investment that yields a positive gain in some circumstances without a countervailing threat of loss in other circumstances. In our framework, an arbitrage opportunity would consist in a nonnegative nonnull investment. And $\Phi = (\Phi_t)_{t \in \mathbf{T}}$ is a nonnegative nonnull investment process if and only if $\Phi : \omega \mapsto \sum_{i=1}^{N} \Phi_{t_i^\Phi}(\omega)\, \delta_{t_i^\Phi}$, which is in $L_P^1(\Omega, \mathcal{M}_b)$, belongs to Γ_+ and is not null.

It is easy to see that the notion of no-arbitrage introduced in Definition 2.3 can be written in the form $(J - \Gamma_+) \cap \Gamma_+ = \{0\}$. A free lunch denoting the possibility of getting arbitrarily close to an arbitrage opportunity, we introduce the following

Definition 2.4. *There is no free lunch for J if and only if $\overline{(J - \Gamma_+)} \cap \Gamma_+ = \{0\}$, where the bar denotes the closure in $L_P^1(\Omega, \mathcal{M}_b)$.*

Notice that if there is a numéraire, the assumption $\overline{(J - \Gamma_+)} \cap \Gamma_+ = \{0\}$, where the bar denotes the closure in $L_P^1(\Omega, \mathcal{M}_b)$, is less restrictive than the "usual" assumption of no-free lunch (with deterministic times). The difference between the two notions lies in the fact that the nonnegative nonnull random variable that we can "almost reach" with admissible investments "is" at a finite date for the new notion whereas it can be "at infinity" for the usual notion. The notion of absence of free lunch adopted in this paper does not exclude free lunches "at infinity" and can therefore be considered as more economically meaningful.

2.3. Characterization of the no-arbitrage assumption in a model with flows

By adapting the proof of Yan (1980), we obtain, under *Assumption A*, the following Fundamental Theorem of Asset Pricing in our general framework with flows.

Theorem 2.5. *Let J denote a convex cone of investments satisfying Assumption A. There is no free lunch for J if and only if there exists a process $g = (g_t)_{t \in \mathbf{T}}$ satisfying for all t in $\mathbf{T}$, $P(0 < g_t < M) = 1$ for some M in R_+^*, and such that*

$$E\left[\sum_{t \in \mathbf{T}} g_t \Phi_t\right] \leq 0 \quad \text{for all } \Phi = (\Phi_t)_{t \in \mathbf{T}} \in J. \tag{2.1}$$

Moreover, the process g can be taken $(F_t)_{t \in \mathbf{T}}$- adapted.

In the case where the set of investment opportunities is related to a countable set of dates, which is the case in finite (resp. infinite) discrete time where $\mathbf{T} = \{1, ..., d\}$ (resp. $\mathbf{T} = N$), then *Assumption A* is not needed to obtain Theorem 2.5.

Corollary 2.6. *Let J denote a convex cone of investments. In finite or infinite discrete time, there is no free lunch for J if and only if there exists a P-essentially bounded process $g = (g_t)_{t \in \mathbf{T}}$ such that inequation 2.1 holds.*

So, in continuous time, starting from the assumption of no free-lunch in a very general model with flows, without any assumption on the existence of a numéraire, but under *Assumption A* (for which we have given an interpretation and that will reveal to be suited to our market frictions), we have proved the existence of a "discount process" such that, using this process as a deflator, all available investments have non-positive present value; this means that there exists a term structure such that the market consisting of the primitive investment opportunities and of the additional borrowing and lending facilities is still "arbitrage-free". Besides, the existence of such a discount process prevents from any arbitrage opportunity. In other words, there is no free lunch for a convex cone of available investments satisfying *Assumption A* if and only if a given convex set of "admissible" discount processes is non-void. We recall that *Assumption A* is not needed if we deal with discrete, possibly infinite, time.

We have seen that the "discount process" can be taken adapted to our filtration. We have obtained so far no other regularity conditions (such as right-continuity); we shall see in the following section that under some regularity conditions on the investments, we obtain regularity properties on the "discount process" itself. Besides, the taking into account of frictions on the numéraire will enable us to better understand the nature of these admissible discount processes.

Since most market models with frictions can fit in the model with flows for a specific convex cone of available investments, our model provides a unified framework for the study of the characterization of the absence of free lunch in such imperfect market models. Notice however that economies with fixed transactions costs do not fall in our model with flows, since the set of all available investments is not a cone.

3. Application to market models with frictions on the numéraire

In this section, we still consider a general framework with flows, but we introduce a numéraire, possibly submitted to constraints. This enables us to give a better interpretation of the "admissible" discount processes found in the previous section. Moreover, we obtain a characterization of the no-free lunch assumption in general models with flows with possible frictions on the numéraire. Since market imperfections like convex cone constraints, proportional transaction costs, etc. can fit in our model with flows, our approach will enable us in the next section to characterize the absence of free lunch in these imperfect models, when there are, in addition, imperfections on the numéraire.

We introduce a few notations. Let $\mathcal{G}$ denote the set of all (equivalence classes) of processes $g = (g_t)_{t\geq 0}$, such that for all $t \in R_+$, $0 < g_t \leq M^g$ for some scalar M^g. For a convex cone K in Γ, let $\mathcal{G}_K$ denote the convex set of processes $g \in \mathcal{G}$ such that $E\left[\sum_{t\geq 0} g_t \Phi_t\right] \leq 0$ for all $\Phi \in K$.

3.1. With a "perfect" numéraire

We assume that there is a perfect numéraire, i.e. that there are possibilities to transfer money, without friction, from one date to another. Let $S^0 = \left(S^0_t\right)_{t\geq 0}$ denote a positive, adapted (numéraire) process such that for all $(t_1, t_2) \in R^2_+$, for all $\theta \in L^\infty(\Omega, F_{t_1\wedge t_2}, P)$, the process denoted by $\Phi^{(0;\theta,t_1,t_2)}$ and given by

$$\Phi^{(0;\theta,t_1,t_2)}_t = \theta\left[-S^0_{t_1} 1_{t=t_1} + S^0_{t_2} 1_{t=t_2}\right] \qquad \text{for all } t \geq 0$$

belongs to Γ. Notice that this is equivalent to $S_t^0 \in L^1(\Omega, F_t, P)$ for all $t \geq 0$. Let J^{num} denote the convex cone generated by a given convex cone J in Γ and all the investments $\Phi^{(0;\theta,t_1,t_2)}$. The set J^{num} corresponds to all available investments in a model with flows, where agents can invest in a perfect numéraire. Notice that J^{num} satisfies *Assumption A* so that, according to Theorem 2.5, there is no free lunch in a model with a perfect numéraire if and only if the set $\mathcal{G}_{J^{\text{num}}}$ is non-void. We obtain the following characterization of the set of admissible discount processes.

Lemma 3.1. *The set $\mathcal{G}_{J^{num}}$ consists of all processes $g \in \mathcal{G}_J$ such that gS^0 is a P-martingale for $(F_t)_{t\geq 0}$.*

This means that if we introduce a term structure given by a numéraire S^0, then in the absence of free lunch, the process g must be equal to $\frac{1}{S^0}$ multiplied by a positive martingale (which is the stochastic analog of a (positive) constant function). Up to a martingale, the process g gives us the possible term structures, i.e. which would be compatible with the assumption of no-free lunch. Note that the process gS^0 admits a right-continuous left-limited (RCLL) modification, so that if $1/S^0$ is RCLL, then g itself admits a RCLL modification.

In finite time, according to Lemma 3.1, the absence of free lunch for $\mathcal{G}_{J^{\text{num}}}$ implies that there exists an equivalent probability measure under which the net present value (using S^0 as a deflator) of any available investment is nonpositive, i.e. there exists a probability measure $Q \sim P$, such that $E^Q\left[\sum_{t\geq 0}\frac{\Phi_t}{S_t^0}\right] \leq 0$ for all $\Phi \in J$. In infinite time, the situation is somewhat different.

Corollary 3.2. *1. If S^0 is uniformly integrable, then there is no free lunch for J^{num} if and only if there exists an absolutely continuous probability measure $\bar{P}$ on (Ω, F, P), such that $\left(\frac{1}{S_t^0}E\left[\frac{d\bar{P}}{dP} \mid F_t\right]\right) \in \mathcal{G}$ and $E^{\bar{P}}\left[\sum_{t\geq 0}\frac{\Phi_t}{S_t^0}\right] \leq 0$ for all investment Φ in J.*

2. If..., there is no free lunch for J^{num} if and only if there exists a probability $\bar{P}$ on (Ω, F), such that $\left(\frac{1}{S_t^0}E\left[\frac{d\bar{P}}{dP} \mid F_t\right]\right) \in \mathcal{G}$ and $E^{\bar{P}}\left[\sum_{t\geq 0}\frac{\Phi_t}{S_t^0}\right] \leq 0$ for all investment Φ in J.

Compared with the "classical case" with a perfect numéraire constantly equal to 1 (see e.g. Stricker (1990)), in the infinite time framework, we only find an absolutely continuous probability measure, whose restricted density to all F_t is positive, instead of an equivalent probability measure and this is due to the fact that, as mentioned in section 2, our assumption of no-free lunch is less restrictive than the "usual" one, since it does not exclude free lunches at infinity.

We now turn to cases where the numéraire is subject to some constraints.

3.2. With lending and borrowing opportunities

Let $S^0 = \left(S_t^0\right)_{t\geq 0}$ and $S^1 = \left(S_t^1\right)_{t\geq 0}$ denote two positive, right-continuous and adapted processes with $S_0^0 = S_0'^0 = 1$, such that for all $(t_1, t_2) \in R_+^2$ with $t_1 \leq t_2$, for all $\theta \in L_+^\infty(\Omega, F_{t_1}, P)$, the processes denoted by $\Phi^{(0;\theta,t_1,t_2)}$ and $\Phi^{(1;\theta,t_1,t_2)}$ and given by

$$\begin{aligned}
\Phi_t^{(0;\theta,t_1,t_2)} &= \theta\left[-S_{t_1}^0 1_{t=t_1} + S_{t_2}^0 1_{t=t_2}\right] \qquad \text{for all } t \geq 0 \\
\Phi_t^{(1;\theta,t_1,t_2)} &= \theta\left[S_{t_1}^1 1_{t=t_1} - S_{t_2}^1 1_{t=t_2}\right] \qquad \text{for all } t \geq 0
\end{aligned}$$

belong to Γ. Let J^{lb} denote the convex cone generated by a given convex cone J in Γ and all the investments $\Phi^{(0;\theta,t_1,t_2)}$ and $\Phi^{(1;\theta,t_1,t_2)}$. The set J^{lb} corresponds to all available investments in a market model where agents have lending and borrowing opportunities, but not in the same conditions. Notice that J^{lb} satisfies *Assumption A* so that, according to Theorem 2.5, there is no free lunch in a model with such constraints on the lending and borrowing opportunities if and only if the set $\mathcal{G}_{J^{\text{lb}}}$ is non-void. We obtain the following characterization of the set $\mathcal{G}_{J^{\text{lb}}}$.

Lemma 3.3. *The set $\mathcal{G}_{J^{\text{lb}}}$ consists of all processes $g \in \mathcal{G}_J$ such that gS^0 is a P-supermartingale and gS^1 is a P-submartingale for $(F_t)_{t\geq 0}$.*

We now introduce additional conditions on the processes S^0 and S^1.

(**C1**) For all (t, t_1, t_2) with $t \leq t_1 \leq t_2$, for all $A \in F_{t_1}$, the processes $\Phi^{\left(k;\frac{1_A}{S_t^k},t_1,t_2\right)} \in J^{\text{lb}}$ for $k = 0, 1$.

(**C2**) For all $s \leq t$, $S_t^0/S_s^0 \leq S_t^1/S_s^1$.

(**C3**) The processes S^0 and S^1 can be written in the form $S^0 = \exp\int_0^\cdot r_s^0 ds$ and $S^1 = \exp\int_0^\cdot r_s^1 ds$ for some bounded and progressively measurable processes r^0 and r^1.

Condition (C1) is an integrability condition; it is satisfied if for all $t \geq 0$, $1/S_t^0$ as well as $1/S_t^1$ are bounded. Condition (C2) essentially means that the "borrowing rate" is greater than or equal to the "lending rate". Condition (C3) means that the processes S^0 and S^1 are associated to interest rates, with a possible spread. Notice that contrarily to Jouini-Kallal (1995 b), we do not suppose $r^0 \leq r^1$, since we get it as a consequence of the absence of free lunch.

Lemma 3.4. *If there is no free lunch for J^{lb} and if Condition (C3) is satisfied, then for all $s \geq 0$, $r_s^0 \leq r_s^1$ and Conditions (C1) and (C2) are satisfied .*

We now obtain more specific characterizations of the set $\mathcal{G}_{J^{\text{lb}}}$ under the different conditions.

Lemma 3.5.

1. *Under Condition (C1), any $g \in \mathcal{G}_{J^{\text{lb}}}$ admits a right-continuous modification.*
2. *Under Conditions (C1) and (C2), the set $\mathcal{G}_{J^{\text{lb}}}$ consists of all processes $g \in \mathcal{G}_J$ for which for all $s \geq 0$, there exists a process $(Z_t^s)_{t\geq s}$ satisfying $S_t^0/S_s^0 \leq Z_t^s \leq S_t^1/S_s^1$ for all $t \geq s$ such that gZ^s is a (right-continuous) martingale for $(F_t)_{t\geq s}$.*
3. *Under Condition (C3), the set $\mathcal{G}_{J^{\text{lb}}}$ consists of all processes $g \in \mathcal{G}_J$ for which there exists a process $Z \equiv \exp\int_0^\cdot r_s ds$ for some bounded and progressively measurable processes r satisfying $r_s^0 \leq r_s \leq r_s^1$ for all $s \geq 0$ such that gZ is a (right-continuous) martingale.*

Part 1 says that the right continuity of the borrowing and lending processes induce the right-continuity of the admissible discount processes. Part 2 says up to a martingale, the returns of the admissible discount processes lie between the returns of the lending and borrowing processes; notice that if $1/S^0$ is LL, than any admissible discount process admits a LL modification. Part 3 says that if the lending and borrowing processes are associated to interest rates, then so are the admissible discount processes.

3.3. With lending opportunities only

With the notations adopted in the previous subsection, let J^{b} denote the convex cone generated by a given convex cone J in Γ and all the investments $\Phi^{(0;\theta,t_1,t_2)}$. The set J^{b} corresponds to all available investments in a market model where investors have lending opportunities, but not necessarily borrowing opportunities. Notice that J^{b} satisfies *Assumption A* so that there is no free lunch in a model with no borrowing if and only if the set $\mathcal{G}_{J^{\mathrm{b}}}$ is non-void.

Lemma 3.6. *The set $\mathcal{G}_{J^{\mathrm{b}}}$ consists of all processes $g \in \mathcal{G}_J$ such that gS^0 is a supermartingale.*

We can now turn to imperfections not only involving the numéraire.

4. Application to other market models with frictions

4.1. Models with dividends

We consider a model of financial market consisting of N financial assets, possibly paying dividends to their holders. We denote by $\left(S^k\right)_{1\leq k\leq N}$ the adapted price processes of the securities and by $\left(D^k\right)_{1\leq k\leq N}$ the associated (possibly null) adapted dividends processes. We assume that the dividends are discrete, i.e. that there is no dividend paid outside a countable set of dates $\left(u_n^k\right)_{n\geq 0}$: the random variable D_t^k corresponds to the dividends paid by the security k at time t and we have $D_t^k = 0$ for $t \notin \left(u_n^k\right)_{n\geq 0}$. We treat S_t^k by convention as the post-dividends market value of the security k at time t. In other words, if the dividend process jumps at time t, the market value S_t^k reflects the jump as already having been paid out, or S_t^k is ex-dividend.

For $1 \leq k \leq N$, for all $(t_1, t_2) \in R_+^2$ with $t_1 \leq t_2$, and for all $A \in F_{t_1}$, we assume that the process $\Phi^{(k;A,t_1,t_2)}$ given by

$$\Phi_t^{(k;A,t_1,t_2)} = 1_A \left[-S_{t_1}^k 1_{t=t_1} + D_t^k 1_{t_1<t\leq t_2} + S_{t_2}^k 1_{t=t_2}\right] \qquad \text{for all } t \ \geq 0$$

belongs to Γ and we let J_{Div} denote the linear space generated by all these investments. We suppose that *Assumption A* is satisfied. Using Theorem 2.5, we get that

Theorem 4.1. *There is no free lunch in the model with dividends (or equivalently for J_{Div}) if and only if there exists a process $g = (g_t)_{t\geq 0} \in \mathcal{G}$ such that for all $t_1 \leq t_2$*

$$E\left[g_{t_2}\left(S_{t_2} + \sum_{t_1<t\leq t_2} g_t D_t\right) \mid F_{t_1}\right] = g_{t_1} S_{t_1}.$$

Note that in the perfect case with no dividend, a finite time horizon, and a perfect numéraire (constantly equal to one), we obtain the "classical" Fundamental Theorem of Asset Pricing, which asserts that there is no free lunch if and only if there exists an equivalent probability measure with bounded density, which makes the (discounted) price process of the traded securities a martingale.

4.2. Convex cone constraints and applications to short sale constraints

We consider now a model of financial market where the quantities of the N different risky assets held by investors are constrained to lie in a convex cone $\mathcal{C}$ (in R^N). Notice that this situation includes the one with no constraint ($\mathcal{C} = R^N$).
We denote by $\left(S^k\right)_{1\leq k\leq N}$ the adapted price process of the risky assets. For $1 \leq k \leq N$, for all $(t_1, t_2) \in R_+^2$, and for all $A \in F_{t_1\wedge t_2}$, we assume that the process $\Phi^{(k;A,t_1,t_2)}$ given by

$$\Phi_t^{(k;A,t_1,t_2)} = 1_A \left[-S_{t_1}^k 1_{t=t_1} + S_{t_2}^k 1_{t=t_2}\right] \qquad \text{for all } t \geq 0$$

belongs to Γ. Let $\Phi^{(A,t_1,t_2)} = \left(\Phi^{(k;A,t_1,t_2)}\right)_{1\leq k\leq N}$. Let J_{Vex} denote the following set

$$\left\{x \cdot \Phi^{(A,t_1,t_2)}; x \in \mathcal{C}, (t_1, t_2) \in R_+^2, A \in F_{t_1\wedge t_2}\right\}.$$

The set J_{Vex} corresponds to all available investments in a model where portfolios are constrained to lie in a convex cone. Let $\mathcal{C}^0 = \left\{y \in R^N; y \cdot x \leq 0, \text{ for all } x \in \mathcal{C}\right\}$. In the unconstrained case, we have $\mathcal{C}^0 = \{0\}$. We get the following characterization of the set $\mathcal{G}_{J_{\mathrm{Vex}}}$.

Theorem 4.2. *The set $\mathcal{G}_{J_{\mathrm{Vex}}}$ consists of all processes $g \in \mathcal{G}$ such that for all $t_1 \leq t_2$, $E\left[g_{t_2} S_{t_2} - g_{t_1} S_{t_1} \mid F_{t_1}\right] \in \mathcal{C}^0$.*

We can apply this result to market models with short sale constraints. We consider a model of financial markets where two sorts of securities can be traded. Short selling the first type of securities is not allowed, i.e. they can only be held in nonnegative amounts, whereas the second type of securities can only be held in nonpositive amounts. The model includes situations where holding negative amounts of a security is possible but costly as well as situations where some (or all) securities are not subject to any constraints, since we may include a security twice in the model, in the first and in the second set of securities. This model has been studied in Jouini-Kallal (1995b), in finite time, for price processes which are in $L^2(\Omega, F, P)$ and right-continuous.

This situation falls in our framework with convex cone constraints for

$$\mathcal{C}_s = \left\{x = (x_1, ..., x_N) \in R^N; x^k \geq 0 \text{ for } k \leq j; x^k \leq 0 \text{ for } k > j\right\}$$

in the case where the first j assets cannot be sold short. Then

$$\mathcal{C}_s^0 = \left\{y = (y_1, ..., y_N) \in R^N; y^k \leq 0 \text{ for } k \leq j \text{ and } y^k \geq 0 \text{ for } k \geq j\right\}.$$

We then easily obtain the following characterization of the set $\mathcal{G}_{J_{\mathrm{S}}}$, where J_{S} is given by J_{Vex} for $\mathcal{C} = \mathcal{C}_s$.

Corollary 4.3. *The set $\mathcal{G}_{J_{\mathrm{S}}}$ of admissible discount processes in the case with short sale constraints consists of all processes $g \in \mathcal{G}$, such that for all security $k \leq j$ that cannot be sold short, gS^k is a supermartingale, and for all security $k > j$ that can only be sold short, gS^k is a submartingale.*

4.3. Proportional transaction costs

We consider a model of financial market where the securities are subject to bid-ask spreads: at each date, there is not a unique price for a security but an ask price,

at which investors can buy the security and a bid price, at which they can sell the security. Notice that this model includes situations where there is a unique price process S and where the transaction cost remains constant overtime, i.e. situations where at each time $t \geq 0$, investors must pay $S_t(1+c)$ for some positive constant c to buy the security and receive $S_t(1-c)$ when selling it.

Such a model with transaction costs has been studied in Jouini-Kallal (1995a) using simple trading strategies like in Harrison-Kreps (1979), assuming that the time horizon is finite (equal to T), that there exists a numéraire process (identically equal to one) and that for all $t \geq 0$, the bid and ask prices at date t are in $L^2(\Omega, F_t, P)$. We intend here to obtain the result of Jouini-Kallal (1995a) by using our approach with general flows, and to extend it mainly to the case with imperfections on the numéraire and to infinite time.

More precisely, we consider $(N-1)$ securities and for $2 \leq k \leq N$, we let $\left(S_t^k\right)_{t\geq 0}$ and $\left(S_t'^k\right)_{t\geq 0}$ denote respectively the adapted ask and bid price process. We assume that, for each k, the processes S^k and S'^k are right-continuous and of class $\mathcal{D}_f$, i.e. that the families $\{S_\tau\}_{\tau\in\mathcal{S}_f}$ and $\{S'_\tau\}_{\tau\in\mathcal{S}_f}$ are uniformly integrable, where $\mathcal{S}_f$ denotes the collection of stopping times of $(F_t)_{t\geq 0}$, taking only a finite number of values in $[0,+\infty]$. Let $S^f_{t,\infty}$ denote the collection of stopping times τ in $\mathcal{S}_f$ such that $\tau \geq t$ a.s. P.

For each k, for any stopping times τ_1 and τ_2 in $\mathcal{S}_f$, we consider the process $\Phi^{(k;\tau_1,\tau_2)}$ given by

$$\Phi_t^{(k;\tau_1,\tau_2)} = -S^k_{\tau_1} 1_{t=\tau_1} + S'^k_{\tau_2} 1_{t=\tau_2} \qquad \text{for all } t \geq 0.$$

It is easy to see that for each k, for any stopping times τ_1 and τ_2 in S^f, the process $\Phi^{(k;\tau_1,\tau_2)}$ is an investment as defined in Definition 2.2 and we define the set J_{Costs} as the convex cone generated by all these investments.

With the notations adopted in the subsection 3.2, we make the additional assumption that there exist two processes S^0 and S^1 satisfying Condition (C1) and we denote by $J^{\text{lb}}_{\text{Costs}}$ the convex cone generated by J_{Costs} and all the investments $\Phi^{(0;\theta,t_1,t_2)}$ and $\Phi^{(1;\theta,t_1,t_2)}$. We denote by $\mathcal{G}_{\text{lb}}$ the set of processes $g \in \mathcal{G}$ such that gS^0 is a P-supermartingale and gS^1 is a P-submartingale for $(F_t)_{t\geq 0}$, according to Lemma 3.3, $\mathcal{G}_{\text{lb}}$ is the set of discout processes compatible with the absence of free lunch for the convex cone generated by all invesments related to the borrowing and lending processes.

Notice that the no-arbitrage assumption implies that the bid price process lies below the ask price process. We then have the following

Theorem 4.4. *There is no free lunch for J^{lb}_{Costs} if and only if there exist an adapted process $g = (g_t)_{t\geq 0} \in \mathcal{G}_{\text{lb}}$ and some adapted price process $\tilde{S}$ lying between the bid and ask price processes such that $g\tilde{S}$ is a (right-continuous) P-martingale for $(F_t)_{t\geq 0}$.*

If we now suppose that there is a perfect numéraire $S^0 \equiv 1$, we get the result obtained by Jouini-Kallal (1995a) in a finite time setting, i.e. there is no free lunch with proportional transaction costs if and only if there exists an equivalent probability measure, with bounded density (in $L^2(\Omega, F_T, P)$ in the setting of Jouini-Kallal (1995a)) that transforms some process between the bid and the ask price processes into a martingale. Theorem 4.4 enables to consider market models with proportional transaction costs and imperfections on the numéraire, in an infinite time setting.

5. Conclusion

We have characterized the assumption of no-arbitrage in financial markets, where any investment opportunity is described by the cash flows that it generates; the absence of free lunch is equivalent to the existence of a normalization process such that the "net present value" of any investment is nonpositive, i.e. there exists a process such that the "net present value" of the normalized cash flows is nonpositive. We have then applied this very general result to specific financial market models and mainly financial models with imperfections like imperfections on the numéraire, which had been barely studied, but also proportional transaction costs, short sale constraints, convex cone constraints, for which we have generalized, in a unified way, existing results.

It is easy to derive, from the characterization of the no-free lunch assumption in imperfect models, "free lunch bounds" on the price of contingent claims. For instance, in the model adopted in section 4.3 with transaction costs and a numéraire subject to constraints, we find that if the market consisting of the primitive securities and the contingent claim contains no free lunch, then the ask price for a contingent claim b at date T must be greater than the infimum of the expected values of $\frac{g_T}{g_0}b$ over all admissible discount processes g in $\mathcal{G}_{J^{\mathrm{lb}}_{\mathrm{Costs}}}$. In an analogous way, we find that the bid price for b must be smaller than the supremum of the expected values of $\frac{g_T}{g_0}b$ over all admissible discount processes. If the contingent claim b is not subject to any transaction cost then its fair price must lie in the interval $\left[\inf_{g\in\mathcal{G}_{J^{\mathrm{lb}}_{\mathrm{Costs}}}} E\left[\frac{g_T}{g_0}b\right], \sup_{g\in\mathcal{G}_{J^{\mathrm{lb}}_{\mathrm{Costs}}}} E\left[\frac{g_T}{g_0}b\right]\right]$. The same approach remains valid for the other market imperfections that we have considered.

Appendix

Proof of Lemma 2.1 1) We prove it for $\gamma = \gamma_t \delta_t$ in Γ. If γ_t is a simple real random variable, i.e., if it only takes a finite number of values $(a_i)_{i=1}^I$, then we can write γ in the form $\gamma = \sum_{i=1}^I 1_{A_i}(a_i \delta_t)$ for a partition $(A_i)_{1 \leq i \leq I}$ of Ω, and $\gamma \in St(\Omega, \mathcal{M}_b)$. For a general γ_t, γ_t can be written as the P a.s. limit of a sequence $(\gamma_t^n)_{n \geq 0}$ of simple real random variables; then $\gamma = \lim_n (\gamma_t^n \delta_t)$ a.s. P, and $\gamma \in M_P(\Omega, \mathcal{M}_b)$. Now, since $\|\gamma(\omega)\|_{\mathcal{M}_b} = \sup_{\|f\| \leq 1, f \in \mathcal{C}_0} |\gamma_t(\omega) f(t)|$, we get $E\left[\|\gamma\|_{\mathcal{M}_b}\right] = E\left[|\gamma_t(\omega)|\right] < \infty$.

2) For all γ in Γ, $\gamma = \sum_{i=1}^m \gamma_{t_i} \delta_{t_i}$, we have $\|\gamma\|_{\mathcal{M}_b}(\omega) = \sum_{i=1}^m \left|\gamma_{t_i}(\omega)\right|$ because for all $(t_1, ...t_m)$ in $(R_+)^m$, we can find f in $\mathcal{C}_0$, satisfying $\|f\| \leq 1$ and $f(t_i) = 1$ (resp. $f(t_i) = -1$) if $\gamma_{t_i}(\omega) \geq 0$ (resp. $\gamma_{t_i}(\omega) \leq 0$). $\square$

Proof of Theorem 2.5 Let $C \equiv (J - \Gamma_+)$. Assume first that there exists a process g satisfying the conditions of the theorem. Let $x \in \Gamma$, $x = \lim_{L_P^1(\Omega, \mathcal{M}_b)} x^n$, $x^n \in C$. Since $x^n \in C \subseteq \Gamma$, $E\left[\left|\sum_{t \geq 0} g_t x_t^n - \sum_{t \geq 0} g_t x_t\right|\right] \leq M \|x - x_n\|_{L_P^1(\Omega, \mathcal{M}_b)}$ (Lemma 2.1), thus $E\left[\sum_{t \geq 0} g_t x_t^n\right] \to_n E\left[\sum_{t \geq 0} g_t x_t\right]$. Since for all n, $E\left[\sum_{t \geq 0} g_t x_t^n\right] \leq 0$, we get $E\left[\sum_{t \geq 0} g_t x_t\right] \leq 0$ and $x \notin \Gamma_+ \setminus \{0\}$.

Suppose now that $\overline{C} \cap \Gamma_+ = \{0\}$. Let μ denote any nonnull element of Γ_+. We apply the Hahn-Banach separation theorem in the normed vector space $L_P^1(\Omega, \mathcal{M}_b)$to find $\mathfrak{g} \neq 0$ in $\mathcal{L}_*^\infty\left(\Omega, (\mathcal{M}_b)'\right)$, i.e. satisfying

1. $\langle \mathfrak{g}, \nu \rangle_{(\mathcal{M}_b)', \mathcal{M}_b} : \omega \to \langle \mathfrak{g}(\omega), \nu \rangle_{(\mathcal{M}_b)', \mathcal{M}_b}$ is measurable for all ν in $\mathcal{M}_b$
2. $\|\mathfrak{g}\|_{\mathcal{L}_*^\infty(\Omega, (\mathcal{M}_b)')} \equiv \inf\left\{M \geq 0, \|\mathfrak{g}\|_{(\mathcal{M}_b)'} \leq M \quad P \text{ a.s.}\right\} < \infty,$

for which, for all c in C, $E\left[\langle \mathfrak{g}, c \rangle_{(\mathcal{M}_b)', \mathcal{M}_b}\right] \leq \alpha_\mu < \beta_\mu \leq E\left[\langle \mathfrak{g}, \mu \rangle_{(\mathcal{M}_b)', \mathcal{M}_b}\right]$, where $\langle \mathfrak{g}, c \rangle_{(\mathcal{M}_b)', \mathcal{M}_b} : \omega \mapsto \langle \mathfrak{g}(\omega), c(\omega) \rangle_{(\mathcal{M}_b)', \mathcal{M}_b}$. Since C is a convex cone, we can take $\alpha_\mu = 0$.

Let now for all $t \geq 0$, $g_t \equiv \langle \mathfrak{g}, \delta_t \rangle_{(\mathcal{M}_b)', \mathcal{M}_b}$ and $g \equiv (g_t)_{t \geq 0}$. Then, using 1), g_t is a measurable random variable. Moreover, using 2), since for all $t \geq 0$, $\delta_t \in \mathcal{M}_b$ and satisfies $\|\delta_t\|_{\mathcal{M}_b} \leq 1$, we get the existence of M in R_+^* such that for all $t \geq 0$, $g_t \leq M \quad P$ a.s.

We claim that for all $t \geq 0$, g_t is almost surely nonnegative. Suppose that there exists $t \geq 0$ such that $V_t \equiv \left\{\langle \mathfrak{g}, \delta_t \rangle_{(\mathcal{M}_b)', \mathcal{M}_b} < 0\right\}$ has positive P-probability. The random variable ϕ_t in $L_P^1(\Omega, \mathcal{M}_b)$ given by $\phi_t : \omega \to -1_{V_t}(\omega) \delta_t$ belongs to Γ_-, so according to the separation theorem $E\left[\langle \mathfrak{g}, \phi_t \rangle_{(\mathcal{M}_b)', \mathcal{M}_b}\right] \leq 0$. But $E\left[\langle \mathfrak{g}, \phi_t \rangle_{(\mathcal{M}_b)', \mathcal{M}_b}\right] = E\left[-1_{V_t} \langle \mathfrak{g}, \delta_t \rangle_{(\mathcal{M}_b)', \mathcal{M}_b}\right] > 0$, a contradiction.

We claim that for all $\Phi \equiv \sum_{t \geq 0} \Phi_t \delta_t$ in J, $E\left[\sum_{t \geq 0} g_t \Phi_t\right] \leq 0$. This is immediate using 2) and the equality $\left\langle \mathfrak{g}(\omega), \sum_{t \geq 0} \Phi_t \delta_t \right\rangle_{(\mathcal{M}_b)', \mathcal{M}_b} = \sum_{t \geq 0} \Phi_t(\omega) \langle \mathfrak{g}(\omega), \delta_t \rangle_{(\mathcal{M}_b)', \mathcal{M}_b}$.

Denote by G the set of all equivalence classes of processes $g = (g_t)_{t \geq 0}$ such that for all t, $0 \leq g_t \leq M^g$ a.s. P and $E\left[\sum_{t \geq 0} g_t \Phi_t\right] \leq 0$ for all $\Phi \in J$. We claim that for all $s \geq 0$, there exists a process g^s in G satisfying $g_s^s > 0$. Fix s in R_+ and let $\mathcal{S}$ be the

family of equivalence classes of subsets of Ω formed by the supports of the g_s for all g in G. We see that the family $\mathcal{S}$ is not reduced to the empty set by considering $\mu = \delta_s$ in the separation theorem. The family $\mathcal{S}$ is closed under countable unions: indeed, consider a sequence $(g^n)_{n\geq 0}$ in G, and let $(a_n)_{n\geq 0}$ denote a sequence of positive scalars such that $\sum_{n\geq 0} a_n M^{g^n} < \infty$; the process h given by $h_t \equiv \sum_{n\geq 0} a_n g_t^n$ a.s. P for all $t \geq 0$, belongs to G. Hence there is g^s in G such that for $S^* = \{g_s^s > 0\}$ we have $P(S^*) = \sup\{P(S); S \in \mathcal{S}\}$. We shall now prove that $P(S^*) = 1$. If $P(S^*) < 1$, then we can apply the separation theorem to $\mu = 1_{(\Omega - S^*)}\delta_t$ which belongs to $\Gamma_+ \setminus \{0\}$ and proceeding as above, we find $g'^s \in G$ such that $E\left[1_{(\Omega-S^*)} g_s'^s\right] > 0$. Then, $g^s + g'^s$ would be an element of G, whose support has P-probability strictly greater than $P(S^*)$: a contradiction.

In the same way, we get that there exists g in G such that $g_{d_n} > 0$ almost surely for all $d_n \in d$, where d is the sequence introduced in *Assumption A*. We consider the process g such that for all $t \geq 0, g_t \equiv \sum_{n\geq 0} b_n g_t^{d_n}$, where $(b_n)_{n\geq 0}$ is a sequence of positive scalars such that $\sum_{n\geq 0} b_n M^{g_{d_n}} < \infty$.

We have found so far a process g such that for all $t \geq 0$, g_t is in $L_+^\infty(\Omega, F, P; R)$, $g_t \leq M$ for all t, $g_{d_n} > 0$ almost surely for all $d_n \in d$ and $E\left[\sum_{t\geq 0} g_t \Phi_t\right] \leq 0$ for all $\Phi = \sum_{t\geq 0} \Phi_t \delta_t$ in J. Then it is easy to check that the same holds for the $(F_t)_{t\geq 0}$-adapted process $\tilde{g} = (\tilde{g}_t)_{t\geq 0}$ defined by $\tilde{g}_t = E[g_t \mid F_t] \quad P$ a.s. for all t. To finish the proof, we only need to show that for all $t \geq 0$, $\tilde{g}_t > 0 \quad P$ a.s.. Assume that for some T outside the set of dates $\{d_n; n \in N\}$ we have just considered, the event $B_T \equiv \{\tilde{g}_T = 0\}$ has positive P-probability; we know that there exists $\Phi = \sum_{t\geq 0} \Phi_t \delta_t$ in C such that $\Phi_T = 0$ outside B_T, $\Phi_t = 0$ for all $t < T$, $\Phi_t \geq 0$ for all $t > T$, $\exists d_n \in d$, $P[\Phi_{d_n} > 0] > 0$. For this particular Φ, we would have $E\left[\sum_{t\geq 0} g_t \Phi_t\right] \geq E[g_{d_n}\Phi_{d_n}] > 0$, which is impossible and completes our proof.□

Proof of Corollary 2.6 The proof remains the same as for Theorem 2.5, replacing the sequence $(d_n)_{n\geq 0}$ by N. □

Proof of Lemma 3.1 The set $\mathcal{G}_{J^{\text{num}}}$ consists of all adapted process $g \in \mathcal{G}$ such that for all Φ in J^{num}, $E\left[\sum_{t\geq 0} \Phi_t g_t\right] \leq 0$. Applying this inequality to $\Phi^{(0;1_A,t_1,t_2)}$ and $\Phi^{(0;-1_A,t_1,t_2)}$ for all $t_1 \leq t_2$, and $A \in F_{t_1}$, we get that for all $t_1 \leq t_2$, $E\left[1_A S_{t_2}^0 g_{t_2}\right] \leq E\left[1_A g_{t_1} S_{t_1}^0\right]$ and $E\left[1_A S_{t_2}^0 g_{t_2}\right] \geq E\left[1_A g_{t_1} S_{t_1}^0\right]$, thus the process $\left(g_t S_t^0\right)_{t\geq 0}$ is a P-martingale for $(F_t)_{t\geq 0}$. The converse is immediate. □

Proof of Corollary 3.2 For 1) and 2), one implication is immediate taking $g_t \equiv 1/S_t^0 E\left[d\bar{P}/dP \mid F_t\right]$ and applying Lemma 3.1 and Theorem 2.5. For the converse implication, we know by Lemma 3.1 that for all $g \in \mathcal{G}_{J^{\text{num}}}$, $gS^0$is a martingale. The process gS^0 admits a right-continuous modification, that can be chosen so as to be a martingale with respect to $(F_t)_{t\geq 0}$ (see Karatzas-Shreve (1991)). For 1), the process $\left(g_t S_t^0\right)_{t\geq 0}$ is uniformly integrable, so it converges P a.s. to an integrable random variable that we shall denote by $g_\infty S_\infty^0$, such that $\left\{g_t S_t^0; F_t; 0 \leq t \leq \infty\right\}$ is a martingale (see Karatzas-Shreve (1991)). Since for all $t \geq 0$, $g_t S_t^0$ is P a.s. positive, we define an absolutely continuous probability measure $\bar{P}$ on (Ω, F) by setting $\frac{d\bar{P}}{dP} \equiv \frac{g_\infty S_\infty^0}{E[g_\infty S_\infty^0]}$. Its density restricted to any F_t, which is equal to $\frac{g_t S_t^0}{E[g_\infty S_\infty^0]}$, is positive. Then for all investment Φ in J, $\frac{\Phi_t}{S_t^0}$ belongs to $L^1\left(\Omega, F_t, \bar{P}\right)$ for all $t \geq 0$ and $E^{\bar{P}}\left[\sum_{t\geq 0} \frac{\Phi_t}{S_t^0}\right] = \frac{1}{E[g_\infty S_\infty^0]} E\left[\sum_{t\geq 0} \Phi_t g_t\right] \leq 0$.

For 2), □

Proof of Lemma 3.3 Any $g \in \mathcal{G}_{J^{\text{lb}}}$ satisfies $E\left[\sum_{t\geq 0} g_t \Phi_t^{(0;1_A,t_1,t_2)}\right] \leq 0$ for all $t_1 \leq t_2$, for all $A \in F_{t_1}$. Then for all $A \in F_{t_1}$, $E\left[1_A g_{t_2} S_{t_2}^0\right] \leq E\left[1_A g_{t_1} S_{t_1}^0\right]$ and the process gS^0 is a supermartingale. In an analogous way, we find that the process gS^1 is a submartingale. □

Proof of Lemma 3.4 Since $(C3)$ implies $(C1)$, using lemma 3.5 1), there is a right-continuous process $g \in \mathcal{G}$ such that for all $t_1 \leq t_2$, $E\left[g_{t_2}\left(\frac{S_{t_2}^0}{S_{t_1}^0} - \frac{S_{t_2}^1}{S_{t_1}^1}\right) \mid F_{t_1}\right] \leq 0$. Then,

$$\lim_{t_2 \to t_1} E\left[g_{t_2} \frac{\left(\frac{S_{t_2}^0}{S_{t_1}^0} - \frac{S_{t_2}^1}{S_{t_1}^1}\right)}{t_2 - t_1} \mid F_{t_1}\right] = E\left[g_{t_1}\left(r_{t_1}^0 - r_{t_1}^1\right) \mid F_{t_1}\right]$$

$= g_{t_1}\left(r_{t_1}^0 - r_{t_1}^1\right) \leq 0$, so that for all $s \geq 0$, $r_s^0 \leq r_s^1$ a.s. P. Now, let A be given by $A \equiv \left\{(s,\omega), r_s^0(\omega) > r_s^1(\omega)\right\}$. Since r^0 and r^1 are both progressively measurable, we know that $A \in \mathcal{B}([0,t]) \otimes F_t$ for all $t \geq 0$. By Fubini's theorem, letting λ denote the Lebesgue measure, we get that $\lambda \otimes P(A) = 0$, so that by Fubini's Theorem again, $\lambda(A_\omega) = 0$ a.s. P (where as usual, $A_\omega \equiv \{s, (s,\omega) \in A\}$) and for all $t_1 \leq t_2$, $\int_{t_1}^{t_2} r_s^0 ds \leq \int_{t_1}^{t_2} r_s^1 ds$ P a.s.□

Proof of Lemma 3.5 1) Let $s \geq 0$ be fixed. Using Condition $(C1)$ and proceding like in Lemma 3.3, we get that $X_t \equiv \left(g_t \frac{S_t^0}{S_s^0}\right)_{t\geq s}$ is an $(F_t)_{t\geq s}$-supermartingale and $Y_t \equiv \left(g_t \frac{S_t^1}{S_s^1}\right)_{t\geq s}$ is an $(F_t)_{t\geq s}$-submartingale. We know that P a.s., the limits X_{s+} and Y_{s+} exist. By considering the investments $\Phi^{\left(k;\frac{1_A}{S_s^k},s,t\right)} \in J^{\text{lb}}$, we get for all $t \geq s$, $E[X_t \mid F_s] \leq g_s \leq E[Y_t \mid F_s]$. Let $\{s_n\}$ be a sequence of rational numbers decreasing to s. Then $E[Y_{s_n}]$ is decreasing and bounded below by $E[Y_s]$. Then $\{Y_{s_n}\}$ is a u.i. sequence and a.s. convergence implies L^1 convergence. Proceding in the same way for X, we get

$$E\left[\frac{\left(gS^0\right)(s_+)}{S_s^0} \mid F_s\right] \leq g_s \leq E\left[\frac{\left(gS^1\right)(s_+)}{S_{s_+}^1} \mid F_s\right]$$

and, using the right-continuity of the filtration, g_{s_+} exists and $g_s = g_{s_+}$ a.s. P. thus g admits a right-continuous modification.

2) We prove it for $s = 0$. We know by Lemma 3.3 and 1) that for $g \in \mathcal{G}_{J^{\text{lb}}}$, gS^0 (resp. gS^1) is a right-continuous super-(resp. sub-)martingale . Moreover, by Condition $(C2)$, we have $gS^0 \leq gS^1$. Then we use the approach adopted in Jouini-Kallal (1995a, Lemma 3) or Choulli-Stricker (1997), adapting it to the case of infinite time to get the existence of a right-continuous martingale process $(m_t)_{t\geq 0}$ which satisfies $\left(gS^0\right)(t) \leq m_t \leq \left(gS^1\right)(t)$ for all $t \geq 0$ (see Lemma 1 in Appendix B for a detailed proof). Since for all $t \geq 0$, g_t is almost surely positive, we may define the process $(Z_t)_{t\geq 0}$ by $Z_t \equiv \frac{m_t}{g_t}$ for all $t \geq 0$.

3) Consider a set of dates $(t_i)_{i\leq N}$ in $[0,T]$ such that $0 = t_0 < t_1 < ... < t_N = T$, where T is a given positive real number. For all $i \geq 0$, we know that there exists a process denoted by Y^i defined for $t \geq t_i$, which is a P-martingale for $(F_t)_{t\geq t_i}$ and such

that $g_t \frac{S_t^0}{S_{t_i}^0} \leq Y_t^i \leq g_t \frac{S_t^1}{S_{t_i}^1}$ for all $t \geq t_i$. Now let $\tilde{Y}$ denote the $(F_t)_{t \in (t_i)_{i \leq N}}$-adapted process defined for $t \in (t_i)_{i \leq N}$ by $\tilde{Y}_t = \prod_{j \leq i} \frac{Y_{t_j}^{j-1}}{g_{t_j}} Y_t^i$ for all $t_i \leq t < t_{i+1}$. It is easy to check that $\tilde{Y}$ is a P-martingale for $(F_t)_{t \in (t_i)_{i \leq N}}$. Moreover, if we let $X^0 \triangleq \frac{\tilde{Y}}{g}$, then for all $i \leq N-1$, $\frac{X_{t_{i+1}}^0}{X_{t_i}^0} = \frac{Y_{t_{i+1}}^i}{g_{t_{i+1}}}$, so that for all $i \leq N-1$,

$$\frac{S_{t_{i+1}}^0}{S_{t_i}^0} \leq \frac{X_{t_{i+1}}^0}{X_{t_i}^0} \leq \frac{S_{t_{i+1}}'^0}{S_{t_i}'^0}.$$

We extend the process found this way for each time grid $t_0 < t_1 < ... < t_N$ into a continuous time process, defined for $t \leq T$, by letting $X^0 = \exp \int_0^{\cdot} y_s ds$ with $y(s) = y_i(s)$ for every $t_i \leq t < t_{i+1}$ where $\frac{X_{t_{i+1}}^0}{X_{t_i}^0} = \exp \int_{t_i}^{t_{i+1}} y_i(s) ds$. Note that X^0 is absolutely continuous and can be written $X_t^0 = 1 + \int_0^t u_s ds$ where $u_s = y_s \exp \int_0^s y_v dv$. We do this for finer and finer grids of union equal to some dense subset of $[0, T]$. The sequence of slopes $u(s, \omega)$ constructed this way is bounded in $L^\infty([0,T] \times \Omega)$, so it admits a convergent (for the weak-star topology $\sigma(L^\infty, L^1)$) subsequence. Let us denote by $\tilde{u}$ its limit, and consider the process $Z = 1 + \int_0^{\cdot} \tilde{u}_s ds$, defined for $t \leq T$, which can be written in the form $Z_t = \exp \int_0^t r_s ds$, for some bounded process r.
The "associated" subsequence gX^0 is also convergent for the weak-star topology $\sigma(L^\infty, L^1)$ with limit equal to gZ. Using the fact that for all time grid $t_0 < t_1 < ... < t_N$, the associated process gX^0 is a P-martingale for $(F_t)_{t \in \{t_0, ..., t_N\}}$, it is easy to see that gZ is a P-martingale for $(F_t)_{t \leq T}$. We adopt the same approach between nT and $(n+1)T$ for $n \geq 1$, to find a process Z, defined for $t \geq 0$, which can be written in the form $Z_t = \exp \int_0^t r_s ds$, for some bounded process r and such that gZ is a P-martingale for $(F_t)_{t \geq 0}$. Proceding like in the proof of Lemma 3.4, we get that $r_s^0 \leq r_s \leq r_s^1$. □

Proof of Lemma 3.6 Immediate. □

Proof of Theorem 4.2 Using Theorem 2.5, for all $A \in F_{t_1}$ and $x \in \mathcal{C}$, we have $E[1_A x \cdot (g_{t_2} S_{t_2} - g_{t_1} S_{t_1})] \leq 0$ so that for all $x \in \mathcal{C}$, $x \cdot E[(g_{t_2} S_{t_2} - g_{t_1} S_{t_1}) \mid F_{t_1}] \leq 0$ and $E[g_{t_2} S_{t_2} - g_{t_1} S_{t_1} \mid F_{t_1}] \in \mathcal{C}^0$. Conversely, if $E[g_{t_2} S_{t_2} - g_{t_1} S_{t_1} \mid F_{t_1}] \in \mathcal{C}^0$, then for all $A \in F_{t_1}$ and $x \in \mathcal{C}$, $1_A x \cdot E[g_{t_2} S_{t_2} - g_{t_1} S_{t_1} \mid F_{t_1}] \leq 0$ a.s. P, thus $E\left[\sum_{t \geq 0} \left(x \cdot \Phi^{(A, t_1, t_2)}\right)_t g_t\right] \leq 0$ and Theorem 2.5 concludes. □

Proof of Corollary 4.3 Immediate using Theorem 4.2. □

Proof of Theorem 4.4 One implication is easy noticing that for all stopping times τ_1 and τ_2, we have, using the fact that $\tilde{S}$ lies between the bid and ask price processes, the law of iterated expectations and the optional sampling theorem,

$$E\left[\sum_{t \geq 0} \Phi_t^{(k; \tau_1, \tau_2)} g_t\right] \leq E\left[S_{\tau_2}'^k g_{\tau_2} - S_{\tau_1}^k g_{\tau_1}\right] \leq E\left[\tilde{S}_{\tau_2}^k g_{\tau_2} - \tilde{S}_{\tau_1}^k g_{\tau_1}\right] \leq 0.$$

The result then follows from Theorem 2.5.

For the converse implication, there exists a process $g = (g_t)_{t \geq 0} \in \mathcal{G}$ such that for all k, for all τ_1 and τ_2 in $S_{t,\infty}^f$, $E\left[-S_{\tau_1}^k g_{\tau_1} + S_{\tau_2}'^k g_{\tau_2} \mid F_t\right] \leq 0$ or

$$E\left[S_{\tau_2}'^k g_{\tau_2} \mid F_t\right] \leq E\left[S_{\tau_1}^k g_{\tau_1} \mid F_t\right]. \tag{5.1}$$

We consider the two n-dimensional processes Z and Z' given by $Z'_t = \underset{\tau \in S^f_{t,\infty}}{ess\sup} E\left[S'_\tau g_\tau \mid F_t\right]$ and $Z_t = \underset{\tau \in S^f_{t,\infty}}{ess\inf} E\left[S_\tau g_\tau \mid F_t\right]$ for all $t \geq 0$. In words, Z'^k is the supremum of the conditional expected value of the proceeds from the strategy that consists in going short in one security k (and investing the proceeds in security 0) after time t (but not necessarily at the same time in different events). The process Z is defined symmetrically. We have for all $t \geq 0$, $S'_t g_t \leq Z'_t$ as well as $Z_t \leq S_t g_t$ because for all $t \geq 0$, the stopping time $\tau = t$ belongs to $S^f_{t,\infty}$.
It is a standard result in optimal stopping that Z' is a P-supermartingale for $(F_t)_{t\geq 0}$ and that Z is a P-submartingale for $(F_t)_{t\geq 0}$. Using inequality (5.1), we have $Z' \leq Z$. The process g can be taken right-continuous (Lemma 3.5, 1.) Adapting Lemma 3 in Jouini-Kallal (1995a), we get that there is a process $\tilde{Z}$ lying between $(S'g)$ and (Sg), which is a P-martingale for $(F_t)_{t\geq 0}$ (Lemma 1 in Appendix B). Now, letting $\tilde{S} \equiv \frac{\tilde{Z}}{g}$, we get that $g\tilde{S}$ is a right-continuous P-martingale and for all $t \geq 0$, $S'_t \leq \tilde{S}_t \leq S_t$ P a.s.□

Appendix B

Lemma 1 *If Z (resp. Z') is an $\left\{(F_t)_{t\geq 0}, P\right\}$-supermartingale (resp. submartingale) such that $Z'_t \leq Z_t$ for all $t \geq 0$, then there exists an $\left\{(F_t)_{t\geq 0}, P\right\}$-martingale process $\tilde{Z}$ satisfying $Z'_{t+} \leq \tilde{Z}_t \leq Z_{t+}$ for all $t \geq 0$.*

Proof We adapt here the approach of Jouini-Kallal (1995a) to the case of infinite and possibly not right continuous processes. If the processes are discrete then the result is an application of the intermediate value Theorem. Otherwise, consider an increasing sequence of time grids $D_n = \left\{\frac{k}{2^n}; k \geq 0\right\}$ which converges to a dense subset D of R_+. Then proceed like in Jouini-Kallal (1995a) to find a process Y defined on D, which is a P-martingale for $(F_t)_{t\in D}$ and which satisfies $Z'_t \leq Y_t \leq Z_t$ for all $t \in D$. Then extend this process to R_+ by defining for all $t \notin D$, $\tilde{Z}_t = \lim_{\substack{s\to t, s>t \\ s\in D}} Y_s$; we recall that for any submartingale $X = \{X_t, F_t, t \geq 0\}$, P almost surely the limit $X_{t+} \triangleq \lim_{\substack{s\to t, s>t \\ s\in Q}}$ exists for all $t \geq 0$. The process $\tilde{Z}$ lies between Z'_+ and Z_+. Using the right-continuity of the filtration and the fact that Y itself is a martingale, we obtain that $\tilde{Z}$ is a martingale. □

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