Common knowledge: finite calculus with a syntactic cut-elimination procedure

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Abstract

In this paper we present a finitary sequent calculus for the **S5** multi-modal system with common knowledge. The sequent calculus is based on indexed hypersequents which are standard hypersequents refined with indices that serve to show the multi-agent feature of the system **S5**. The calculus has a non-analytic right introduction rule. We prove that the calculus is contraction- and weakening-free, that (almost all) its logical rules are invertible, and finally that it enjoys a syntactic cut-elimination procedure. Moreover, the use of the non-analytic rule can be restricted so that the calculus can be considered as suitable for proof search.

1 Introduction

Common knowledge is a key feature of multi-agent systems of knowledge which was first discussed by [11] and [4]. The books [7] and [12] provide an excellent introduction to logics of knowledge in general and of common knowledge in particular.

The common knowledge operator is standardly interpreted as the infinite conjunction "all agents know A, and all agents know that all agents know A and so on". From a syntactic point of view, the traditional way to capture common knowledge is by means of Hilbert-style systems comprising of a fixed point axiom, which states that common knowledge is a fixed point, and an induction rule that states that this fixed point is the greatest fixed point. From a semantic point of view, the common knowledge operator is formally defined as the modality of reachability that uses accessibility edges corresponding to any of the knowledge operators for the agents.

In this paper we consider common knowledge from the perspective of Gentzen-style sequent calculi. Whilst considerable progress has been made in developing other sorts of calculi for common knowledge, such as tableaux systems [1, 9], the situation regarding Gentzen-calculi is not entirely satisfactory. Two sorts of calculi have been explored: finitary calculi, for example in [3, 10] and infinitary calculi, for example [2, 17, 6]. None of the finitary systems presents a syntactic cut-elimination procedure; cut-elimination, if it is established, is proved indirectly by showing completeness of the cut-free system. Among the cited infinitary systems, only [6] propose a cut-elimination procedure.

The aim of this paper is to develop a Gentzen-style calculus for common knowledge that is composed of a finite set of finitary rules, but that nevertheless admits a syntactic cut-elimination procedure. The proposed calculus has other desirable structural properties, in particular the admissibility of all the structural rules and the invertibility of all but one of logical rules. However, in the light of the difficulties in finding Gentzen-style sequent calculi for common knowledge [2], these advantages come at a price: the rule that introduces the common knowledge operator on the right side of the sequent is non-analytic, i.e. there is a principal formula B in the premises of the rule that does not occur in the conclusion. In order to mitigate this shortcoming, we note that one can always identify a single appropriate principal formula B for any given application of the $\boxplus R$ rule; as such, the calculus retains much of the interest of a fully analytic calculus as regards proof search.

The calculus proposed in this paper is for the modal logic **S5** plus common knowledge. Since the system **S5** is used to formalise knowledge, this logic is the most appropriate for possible applications in the domain of common knowledge. However, we underline that the main results in this paper are not **S5**-dependent, i.e. they could be straightforwardly adapted to other normal modal systems, by exploiting the sequent calculi for these systems introduced in [14].

The calculus introduced in this paper is based on *indexed hypersequents*. Hypersequents were used in [15] in order to construct a cut-free sequent calculus for the system **S5**. Then hypersequents were refined by adding indices in order to build a cut-free sequent calculus for the multi-agent version of the system **S5** [16]. We exploit this last result as a base for building a sequent calculus for **S5** plus common knowledge. In the papers [15] and [16], the intuitive ideas that are behind hypersequents and indexed hypersequents are fully explained, and shall not be repeated here; instead, we focus on their formal interpretation.

The paper is organised as follows. In the next section we present the calculus for **S5** with common knowledge, while in Section 3. we show the admissibility of the structural rules and the invertibility of (almost all) logical rules. In Section 4. we prove that the calculus is sound and complete with respect to the Hilbert system for common knowledge. Finally, in Section 5., we present a syntactic cut-elimination procedure for our calculus, and we show that one can always identify a single principal formula B for any given application of the $\boxplus R$ rule.

2 The calculus HS5C

Definition 1. *We consider a language $\mathcal{L}_h^{\Box}$ with a set Φ of agents $\{a, b, c, ...\}$. Propositions S are atoms. The set of atoms is denoted by Ψ. Formulas are denoted by capital letters A, B, C, D. They are given by the following grammar:*

$$A ::= S \mid \neg A \mid (A \wedge A) \mid \Box_z A \mid \boxplus A$$

where $z \in \Phi$, the formula $\Box_z A$ is read as "agent z knows A" and the formula $\boxplus A$ is read as "A is common knowledge". The other propositional connectives, as well as the (dual) modal operators are defined as usual. We will use the formula $\Box A$ as an abbreviation for "everybody knows A":

$$\Box A = \Box_1 A \wedge ... \wedge \Box_h A$$

Definition 2. *In what follows we will use the following syntactic conventions:*

- *M, N,...: finite multisets of formulas,*
- *Γ, Δ, ...: classical sequents,*
- *G, H, ...: indexed hypersequents,*
- *α, β, ...: finite (perhaps empty) sets of indices of the form nz, where* $n \in \mathbb{N}$ *and* $z \in \Phi$*, and, for each set α and for each* $z \in \Phi$*, there exists at most one index* $nz \in \alpha$*. So, for instance, α could be the set* $\{1a, 1b, 2c\}$*, but* $\{1a, 2a\}$ *is not a legal set of indices.*

We use $\alpha\hat{\ }nz$ to denote the set of indices (understood to satisfy the property just mentioned) formed by adding the index nz to α. This notation serves to draw the reader's attention to the index nz. We use $||H||$ to denote the union of all the sets of indices contained in the hypersequent H. Classical sequents are defined in the standard way (i.e. they are objects of the form $M \Rightarrow N$); indexed hypersequents are defined as follows.

Definition 3. *An* indexed hypersequent *is a syntactic object of the form:*

$$\alpha_1 : M_1 \Rightarrow N_1 \mid \alpha_2 : M_2 \Rightarrow N_2 \mid \ \ldots \ \mid \alpha_n : M_n \Rightarrow N_n$$

where $M_i \Rightarrow N_i$ *(i = 1, ..., n) is a classical sequent,* α_i *is a finite set of indexes as defined above, and, for all m p,* $1 \leq m, p \leq n$ *and* $m \neq p$*,*

1. $\alpha_m \cap \alpha_p$ *contains at most one element;*
2. *there exists a sequence* $k_1, ..., k_q$ *with* $k_1 = m$ *and* $k_q = p$*, and for all r,* $1 \leq r < q$*,* $\alpha_{k_r} \cap \alpha_{k_{r+1}} \neq \emptyset$*.*
3. *there does not exist a sequence of indexed sequents* $\beta_1 : P_1 \Rightarrow Q_1 \mid \beta_2 : P_2 \Rightarrow Q_2 \mid \ ... \ \mid \beta_q : P_q \Rightarrow Q_q$ *such that:*
   - *for each pair of indexed sequents* $\beta_r : P_r \Rightarrow Q_r$*,* $\beta_{r+1} : P_{r+1} \Rightarrow Q_{r+1}$*, with* $1 \leq r < q$*,* $\beta_r \cap \beta_{r+1}$ *contains one element;*
   - $\beta_1 : P_1 \Rightarrow Q_1$ *is the same sequent as* $\beta_q : P_q \Rightarrow Q_q$*.*

Let us call disconnected indexed hypersequent*, for short* DIH*, an indexed hypersequent that satisfies 1 and 3, but not necessarily 2. We use the same syntactic notation for DIH as for indexed hypersequents, without risk of confusion.*

As a point of notation, empty sets of indices may be omitted (e.g. we write Γ rather than $\emptyset : \Gamma$). Moreover, with slight abuse of notation for a indexed sequent $\alpha : \Gamma$ and an indexed hypersequent H, we write $\alpha : \Gamma \in H$ to express the statement that $\alpha : \Gamma$ appears in H.

Definition 4. *For $\alpha_i : \Gamma_i$ an indexed sequent belonging to an indexed hypersequent H, define the set of all the indexed sequents belonging to H that have at least one common index with $\alpha_i : \Gamma_i$ as follows:*

$$\Sigma_{\alpha_i:\Gamma_i} = \{\alpha_j : \Gamma_j \in H \mid \alpha_i \cap \alpha_j \neq \emptyset\}$$

Definition 5. *Given an indexed hypersequent H containing a sequent $\alpha_i : \Gamma_i$, we define:*

$$H \setminus \alpha_i : \Gamma_i = \alpha'_1 : \Gamma_1 \mid \ldots \mid \alpha'_{i-1} : \Gamma_{i-1} \mid \alpha'_{i+1} : \Gamma_{i+1} \mid \ldots \mid \alpha'_n : \Gamma_n$$

where $\alpha'_j = \alpha_j \setminus \alpha_i$. That is $H \setminus \alpha_i : \Gamma_i$ is the result of dropping, from H, the sequent $\alpha_i : \Gamma_i$ and each of the indices belonging to α_i that occur in other indexed sequents of H. Note that $H \setminus \alpha_i : \Gamma_i$ is a DIH.

For any α_i, α_j with a single common element nz, we use $f(\alpha_i, \alpha_j)$ to denote the agent z.

Definition 6. *The* interpretation τ of a DIH H rooted at $\alpha_i : \Gamma_i$, $(H)^\tau_{\alpha_i:\Gamma_i}$ *is inductively defined as follows:*

- *if $H = \Gamma_i$ or $H = \Gamma_i \mid G$, and $\Gamma_i = M \Rightarrow N$, then $(H)^\tau_{\Gamma_i} = \bigwedge M \rightarrow \bigvee N$*
- *if $H = \alpha_1 : \Gamma_1 \mid \ldots \mid \alpha_i : \Gamma_i \mid \ldots \mid \alpha_n : \Gamma_n$, then $(H)^\tau_{\alpha_i:\Gamma_i} =$*

$$(\Gamma_i)^\tau_{\Gamma_i} \vee \bigvee_{\alpha_j:\Gamma_j \in \Sigma_{\alpha_i:\Gamma_i}} \Box_{f(\alpha_j,\alpha_i)} (H \setminus \alpha_i : \Gamma_i)^\tau_{\alpha_j:\Gamma_j}$$

Definition 7. *The* interpretation of an indexed hypersequent H *is defined in the following way:*

$$(H)^\tau = \bigwedge_{\alpha_i:\Gamma_i \in H} (H)^\tau_{\alpha_i:\Gamma_i}$$

We have thus introduced the notion of indexed hypersequent and its syntactic interpretation. In order to introduce the calculus **HS5C** which exploits indexed hypersequents, we require the following definitions.

Definition 8. *For any pair of sets of indices α, β,*

$$\overline{\beta}_\alpha = \{nz \in \beta \mid \exists m \in \mathbb{N}, mz \in \alpha\}$$

Moreover, for any $nz \in \overline{\beta}_\alpha$, call the corresponding element in α (if it exists), $n_\alpha z$. Finally,

$$\alpha + \beta = (\alpha \cup \beta)[n_{1\alpha} z_1 \ldots n_{l\alpha} z_l / n_1 z_1 \ldots n_l z_l]$$

where $\overline{\beta}_\alpha = \{n_1 z_1, \ldots, n_l z_l\}$.

Definition 9. *Let H be a DIH, and let α and β be sets of indices. We define $H_{\alpha/\beta}$ as follows:*

$$H_{\alpha/\beta} = H[m_{1\alpha}w_1 \dots m_{l\alpha}w_l / m_1 w_1 \dots m_l w_l]$$

where $\overline{\beta}_\alpha = \{m_1 w_1, \dots, m_l w_l\}$. For a set of indices γ, $\gamma_{\alpha/\beta}$ is defined similarly.

In the previous definitions, the substitution of indices for indices in an indexed hypersequent is defined in the standard way, and the standard notation is used.

The rules of the calculus **HS5C** are given in Figure 1. Note that, despite the restriction, the cut rule is indeed general as standard, due to the possibility of renaming indices which will be shown in Lemma 1 below.

As remarked in the Introduction, the rule $\boxast R$ is non-analytic: B does not appear in the conclusion. We shall discuss some consequences of this in Section 5. Note that a similar rule has been studied in the literature on temporal logics [13], using semantic techniques.

3 Admissibility of the Structural Rules

In this section we show which structural rules are admissible in the calculus **HS5C**. Moreover, we prove that the propositional rules, the modal rules and the rules $\boxast L_1$ and $\boxast L_2$ are invertible. The cut-elimination proof is given in the Section 5.

Definition 10. *For a formula A, we define its complexity, $dg(A)$, as follows:*

$dg(S) = 0$

$dg(\Box_z A) = dg(\neg A) = dg(A) + 1$

$dg(A \wedge B) = max(dg(A), dg(B)) + 1$

$dg(\boxast A) = \omega + dg(A)$

Definition 11. *We associate to each derivation d in* **HS5C** *three natural numbers $h(d)$ (the height of d), $crk(d)$ (the cut-rank of d), and $prk(d)$ (the pr-rank of d). The height corresponds to the length of the longest branch in a tree-derivation d, minus one. The cut-rank corresponds to the complexity of the cut-formulas in d. $crk(d)$ is the smallest $n \in \mathbb{N}$ such that each cut-formula A occurring in d is such that $dg(A) < n$. If $crk(d) = 0$, then d is a cut-free derivation. Finally the pr-rank corresponds to the maximal number of applications of the rule $\boxast R$ in any branch of a tree-derivation d. We omit the standard inductive definitions of height and cut-rank of a derivation [18].*

Definition 12. *$d \vdash^n_{p,q} G$ means that d is a derivation of G in* **HS5C***, with $h(d) \leq n$, $crk(d) \leq p$ and $prk(d) \leq q$. We write $\overset{\langle n \rangle}{\underset{\langle p,q \rangle}{}} G$, for: "there exists a derivation d in* **HS5C** *such that $d \vdash^n_{p,q} G$."*

Initial Indexed Hypersequents

$$G \mid \alpha : M, S \Rightarrow N, S \qquad\qquad G \mid \alpha : M, \boxplus A \Rightarrow N, \boxplus A$$

Propositional Rules

$$\frac{G \mid \alpha : M \Rightarrow N, A}{G \mid \alpha : \neg A, M \Rightarrow N}\ \neg L \qquad\qquad \frac{G \mid \alpha : A, M \Rightarrow N}{G \mid \alpha : M \Rightarrow N, \neg A}\ \neg R$$

$$\frac{G \mid \alpha : A, B, M \Rightarrow N}{G \mid \alpha : A \wedge B, M \Rightarrow N}\ \wedge L \qquad \frac{G \mid \alpha : M \Rightarrow N, A \qquad G \mid \alpha : M \Rightarrow N, B}{G \mid \alpha : M \Rightarrow N, A \wedge B}\ \wedge R$$

Modal Rules

$$\frac{G \mid \alpha : \Box_z A, A, M \Rightarrow N}{G \mid \alpha : \Box_z A, M \Rightarrow N}\ \Box_z L_1$$

$$\frac{G \mid \alpha\hat{}nz : \Box_z A, M \Rightarrow N \mid \beta\hat{}nz : A, P \Rightarrow Q}{G \mid \alpha\hat{}nz : \Box_z A, M \Rightarrow N \mid \beta\hat{}nz : P \Rightarrow Q}\ \Box_z L_2 \qquad \frac{G \mid \alpha\hat{}nz : M \Rightarrow N \mid nz : \Rightarrow A}{G \mid \beta : M \Rightarrow N, \Box_z A}\ \Box_z R$$

$$\text{where in the rule } \Box_z R, \beta = \begin{cases} \alpha\hat{}nz, & if \quad nz \in \|G\| \\ \alpha, & otherwise \end{cases}$$

Common Knowledge Rules

$$\frac{G \mid \alpha : \boxplus A, \Box A, M \Rightarrow N}{G \mid \alpha : \boxplus A, M \Rightarrow N}\ \boxplus L_1$$

$$\frac{G \mid \alpha : \boxplus A, M \Rightarrow N \mid \beta : \boxplus A, P \Rightarrow Q}{G \mid \alpha : \boxplus A, M \Rightarrow N \mid \beta : P \Rightarrow Q}\ \boxplus L_2 \qquad \frac{G \mid \alpha : M \Rightarrow N, B \quad B \Rightarrow \Box A \quad B \Rightarrow \Box B}{G \mid \alpha : M \Rightarrow N, \boxplus A}\ \boxplus R$$

Cut Rule

$$\frac{G \mid \alpha : M \Rightarrow N, A \quad H \mid \beta : A, P \Rightarrow Q}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\ cut_A$$

where $\|G\| \cup \alpha$ and $\|H\| \cup \beta$ are disjoint.

Figure 1: The calculus **HS5C**

Definition 13. *An inference rule $\mathcal{R}$ with premises $G_1, ..., G_n$ and conclusion H is height-, cut-rank- and pr-rank-preserving admissible in the calculus* **HS5C** *if, whenever* **HS5C** $\vdash^n_{p,q}$ *G_i, for each premise G_i, then* **HS5C** $\vdash^n_{p,q}$ *H. For each rule $\mathcal{R}$, we denote its inverse, which has the conclusion of $\mathcal{R}$ as its only premise and any premise of $\mathcal{R}$ as its conclu-*

$$\frac{G \mid \alpha : M \Rightarrow N}{G \mid \alpha : M, P \Rightarrow N, Q}\ IW \qquad \frac{G \mid \alpha : M \Rightarrow N \mid \beta : P \Rightarrow Q}{G \mid \alpha\hat{}nz : M \Rightarrow N \mid \beta\hat{}nz : P \Rightarrow Q}\ IndW$$

Figure 2: Internal Weakening and Indices Weakening

$$\frac{G \mid \alpha : M \Rightarrow N}{G \mid \beta : M \Rightarrow N \mid nz : P \Rightarrow Q}\ EW \qquad \frac{G \mid \alpha : M \Rightarrow N \mid \beta : P \Rightarrow Q}{G^{-} \mid \gamma : M, P \Rightarrow N, Q}\ me$$

$\beta = \alpha$, if $nz \in \|G\|$,

$\beta = \alpha\hat{}nz$, otherwise

$\alpha \cap \beta = nz,\ G^{-} = G[n_{1\alpha}z_1 \ldots n_{l\alpha}z_l / n_1z_1 \ldots n_lz_l]$ for $\overline{\beta}_\alpha = \{n_1z_1, \ldots, n_lz_l\}$, and

if $nz \in \|G\|$, $\gamma = \alpha + \beta$

if $nz \notin \|G\|$, $\gamma = \alpha + \beta \setminus \{nz\}$

Figure 3: External Weakening and Merge

sion, by $\overline{\mathcal{R}}$. An inference rule is height-, cut-rank- and pr-rank-preserving invertible in the calculus **HS5C** *if $\overline{\mathcal{R}}$ is height-, cut-rank- and pr-rank-preserving admissible in* **HS5C**.

Lemma 1. *For any indexed hypersequent G, if G is derivable in* **HS5C**, *then $G[n'_1z_1 \ldots n'_kz_k / n_1z_1 \ldots n_kz_k]$ is also derivable with the same height and the same cut- and pr-rank, provided that $G[n'_1z_1 \ldots n'_kz_k / n_1z_1 \ldots n_kz_k]$ is an indexed hypersequent (i.e. that it respects the conditions (i) and (ii) of Definition 3).*

Proof. By straightforward induction on the height of the derivation. □

Lemma 2. *In the calculus **HS5C** the following holds:*

1. *The rules of internal weakening and indices weakening (Figure 2) are height-, cut-rank- and pr-rank- admissible.*

2. *The rules of external weakening and merge (Figure 3) are height-, cut-rank- and pr-rank- admissible.*

3. *The propositional and modal rules, as well as the rules $\boxplus L_1$ and $\boxplus L_2$ are height-, cut-rank- and pr-rank- invertible.*

Proof. (i) and (ii) follow from a standard induction on the height of the proof. The same works for the propositional rules, and the rule $\Box_z R$ in (iii). The inverses of the rules $\Box_z L_1$, $\Box_z L_2$, $\boxplus L_1$ and $\boxplus L_2$ are just internal weakenings. □

Note that, for the rule of indices weakening, since the conclusion is an indexed hypersequent, there is an implicit restriction on the application of the rule to cases where the conditions 1.-3. in Definition 3 are respected.

Lemma 3. *The rule $\boxast R$ permutes down with respect to all the other rules of the calculus* ***HS5C****.*

Proof. The proof is straightforward. □

Lemma 4. *In the calculus* ***HS5C*** *the contraction rules*

$$\frac{G \mid \alpha : A, A, M \Rightarrow N}{G \mid \alpha : A, M \Rightarrow N}\ CL \qquad \frac{G \mid \alpha : M \Rightarrow N, A, A}{G \mid \alpha : M \Rightarrow N, A}\ CR$$

are cut- and pr-rank admissible.

Proof. The proof is by induction on the height of the derivation of the premise. The cases of the propositional rules and the rules $\Box_z L_1$, $\Box_z L_2$, $\boxast L_1$ and $\boxast L_2$ are straightforward. The case of the rule $\Box_z R$ is also straightforward, using the rule of merge. We analyse the following critical case:

$$\frac{\begin{array}{c} d_1 \\ \vdots \\ G \mid \alpha : M \Rightarrow N, B, \boxast A \end{array} \quad \begin{array}{c} d_2 \\ \vdots \\ B \Rightarrow \Box A \end{array} \quad \begin{array}{c} d_3 \\ \vdots \\ B \Rightarrow \Box B \end{array}}{G \mid \alpha : M \Rightarrow N, \boxast A, \boxast A}\ \boxast R$$

We go up the derivation d_1 to the point where the formula $\boxast A$ has been introduced. There we have several possibilities.

`Case 1` The formula $\boxast A$ comes from an initial indexed hypersequent. `Case 1a` The initial indexed hypersequent is of the form $G' \mid \alpha : S, M' \Rightarrow N', B', S, \boxast A$. We take the initial indexed hypersequent obtained by removing the occurrence of the formula $\boxast A$, and continue the derivation $d_1 + \boxast R$ as before. `Case 1b` The initial indexed hypersequent is of the form $G' \mid \alpha : \boxast A, M' \Rightarrow N', B', \boxast A$. Let us denote this initial indexed hypersequent by H. `Case 1b1` $B' = B$. We consider the initial indexed hypersequent H' obtained from H by removing the occurrence of the formula B, and continue the derivation d_1 as before, without applying the rule $\boxast R$ at the end. `Case 1b2` $B' \neq B$, so B has been constructed in the course of the derivation d_1. We consider the initial indexed hypersequent H'' obtained from H by removing all formulas, indices and indexed sequents that are used only to construct B, and develop the derivation d_1 as before omitting those inference rules that gave rise to the formula B. We no longer need to apply the rule $\boxast R$.

`Case 2` The formula $\boxast A$ comes from the rule $\boxast R$, so we have:

$$\begin{array}{c} \dfrac{\begin{array}{c} d_1^2 \\ \vdots \\ G' \mid \alpha : M' \Rightarrow N', B', D \end{array} \quad \begin{array}{c} d_1^3 \\ \vdots \\ D \Rightarrow \Box A \end{array} \quad \begin{array}{c} d_1^4 \\ \vdots \\ D \Rightarrow \Box D \end{array}}{G' \mid \alpha : M' \Rightarrow N', B', \boxast A}\ \boxast R \\ d_1^1 \\ \vdots \\ G \mid \alpha : M \Rightarrow N, B, \boxast A \end{array}$$

Using Lemma 3, we permute down the application of the rule $\boxplus R$ to obtain a derivation of $G \mid \alpha : M \Rightarrow N, B, D$. Applying the rule $\vee R$[1] on this indexed hypersequent we obtain (i) $G \mid \alpha : M \Rightarrow N, B \vee D$. From $D \Rightarrow \Box A$ and $B \Rightarrow \Box A$, by application of the rule $\vee L$, we obtain (ii) $D \vee B \Rightarrow \Box A$. From $D \Rightarrow \Box D$ and $D \Rightarrow \Box B$, by weakening and $\vee L$, we get $D \vee B \Rightarrow \Box D, \Box B$. From $D \vee B \Rightarrow \Box D, \Box B$, we can derive (iii) $D \vee B \Rightarrow \Box(D \vee B)$. We use (i), (ii) and (iii) to obtain, by means of the rule $\boxplus R$, the conclusion $G \mid \alpha : M \Rightarrow N, \boxplus A$.

□

4 Adequateness Theorem

In this section we show that the calculus **HS5C** proves exactly the same formulas as its corresponding Hilbert-style system **S5C**. The Hilbert system **S5C** is fully described in [7, Ch 3].

Theorem 4.1. *For all indexed hypersequents G and for all formulas A,*

1. *if* $\vdash G$ *in* **HS5C**, *then* $\vdash (G)^\tau$ *in* **S5C**.
2. *if* $\vdash A$ *in* **S5C**, *then* $\vdash \Rightarrow A$ *in* **HS5C**.

Proof. The proof of (i) is relatively standard (it is similar to [14, Lemma 5.1]). In order to acquaint the reader with the calculus **HS5C**, we give as examples the proofs of the fixed point axiom and the induction rule; rest of (ii) is similar.

- fixed point axiom[2]

$$
\dfrac{\dfrac{\vdots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1z : \boxplus A, \cdots \Box_z A \cdots \Rightarrow \ \mid 1z : A \Rightarrow A}{1z : \boxplus A, \cdots \Box_z A \cdots \Rightarrow \ \mid 1z :\Rightarrow A}\,{\scriptstyle \Box_z L}}{1z : \boxplus A, \Box A \Rightarrow \ \mid 1z :\Rightarrow A}\,{\scriptstyle \wedge L^*}}{1z : \boxplus A \Rightarrow \ \mid 1z :\Rightarrow A}\,{\scriptstyle \boxplus L_1} \quad \dfrac{1z : \boxplus A \Rightarrow \ \mid 1z : \boxplus A \Rightarrow \boxplus A}{1z : \boxplus A \Rightarrow \ \mid 1z :\Rightarrow \boxplus A}\,{\scriptstyle \boxplus L_2}}{1z : \boxplus A \Rightarrow \ \mid 1z :\Rightarrow A \wedge \boxplus A}\,{\scriptstyle \wedge R}}{\boxplus A \Rightarrow \Box_z(A \wedge \boxplus A)}\,{\scriptstyle \Box_z R} \qquad \vdots}{\boxplus A \Rightarrow \Box(A \wedge \boxplus A)}\,{\scriptstyle \wedge R^*}}{\Rightarrow \boxplus A \rightarrow \Box(A \wedge \boxplus A)}\,{\scriptstyle \rightarrow R}
$$

- induction rule

$$
\dfrac{B \Rightarrow B \quad \dfrac{B \Rightarrow \Box A \wedge \Box B}{B \Rightarrow \Box A \quad B \Rightarrow \Box B}\,{\scriptstyle \overline{\wedge R}}}{B \Rightarrow \boxplus A}\,{\scriptstyle \boxplus R}
$$

□

[1] The rule $\vee R$, as well as the rule $\vee L$, can be straightforwardly formulated on the basis of the other propositional rules.

[2] We use the notation $\mathcal{R}_1^* + ... + \mathcal{R}_n^*$ to mean repeated applications of the rules $\mathcal{R}_1, ..., \mathcal{R}_n$. We take this notation for granted in what follows.

5 Cut-elimination

In this section we prove that the cut-rule is eliminable in the calculus **HS5C**. We end the section with a discussion of the non-analyticity of rule $\boxplus R$.

Lemma 5. *If*

$$\dfrac{\begin{matrix} d_1 \\ \vdots \\ G \mid \alpha : M \Rightarrow N, A \end{matrix} \quad \begin{matrix} d_2 \\ \vdots \\ H \mid \beta : A, P \Rightarrow Q \end{matrix}}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\ cut_A$$

and d_1 and d_2 do not contain any application of the cut-rule, then we can construct a derivation of $G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q$ with no application of the cut-rule.

Proof. The proof is developed by induction on the pr-rank of the derivation, with subinduction on the complexity of the cut-formula, and with a third subinduction on the sum of the heights of the derivations of the premises of the cut-rule. We distinguish cases according to the last rule applied on the left premise.

`Case 1.` $G \mid \alpha : M \Rightarrow N, A$ is an initial indexed hypersequent. Then either the conclusion is also an initial indexed tree-hypersequent, or the cut can be replaced by various applications of the rules IW, $IndW$ and EW on the right premise $H \mid \beta : A, P \Rightarrow Q$, and renaming of indices (Lemma 1).

`Case 2.` $G \mid \alpha : M \Rightarrow N, A$ is inferred by a rule $\mathcal{R}$ in which A is not principal. This case can be standardly solved by induction on the sum of the heights of the derivations d_1 and d_2.

`Case 3.` $G \mid \alpha : M \Rightarrow N, A$ is inferred by a rule $\mathcal{R}$ in which A is the principal formula. We distinguish three subcases: in the first subcase, 3.1., $\mathcal{R}$ is a propositional rule, in the second subcase, 3.2., $\mathcal{R}$ is a modal rule, in the third subcase, 3.3., $\mathcal{R}$ is a common knowledge rule.

`Case 3.1.` This case can be solved by applying Lemma 2 on the right premise, and by replacing the previous cut with one (or two, in case of the rule $\wedge R$) which is (are) eliminable by induction on the complexity.

`Case 3.2.` $\mathcal{R}$ is $\Box_z R$ and $A = \Box_z B$. Consider the last rule $\mathcal{R}'$ of d_2. If no rule $\mathcal{R}'$ introduces $H \mid \beta : \Box_z B, P \Rightarrow Q$ because $H \mid \beta : \Box_z B, P \Rightarrow Q$ is an initial indexed hypersequent, then we can solve the case as in the case 1. If $\Box_z B$ is not principal in the rule $\mathcal{R}'$, then we can solve the case as in the case 2. If $\Box_z B$ is the principal formula of the rule $\mathcal{R}'$, then there are two cases: 3.2.1. $\mathcal{R}'$ is $\Box_z L_1$, and 3.2.2. $\mathcal{R}'$ is $\Box_z L_2$. We consider first 3.2.1. We have[3]

$$\dfrac{\dfrac{G \mid \alpha\hat{\ } nz : M \Rightarrow N \mid nz :\Rightarrow B}{G \mid \alpha : M \Rightarrow N, \Box_z B}\ \Box_z R \quad \dfrac{H \mid \beta : \Box_z B, B, P \Rightarrow Q}{H \mid \beta : \Box_z B, P \Rightarrow Q}\ \Box_z L_1}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\ cut_{\Box_z B}$$

which we reduce to

[3]Note that we analyse the case where the index nz only appears in the displayed sequents $M \Rightarrow N$ and $\Rightarrow B$ in the premise of $\mathcal{R}$. The case where $nz \in \|G\|$ is dealt with analogously.

$$
\dfrac{\dfrac{G \mid \alpha\hat{}nz : M \Rightarrow N \mid nz :\Rightarrow B}{G \mid \alpha : M \Rightarrow N, B}\,{\scriptstyle me} \qquad \dfrac{G^* \mid \alpha^* : M \Rightarrow N, \Box_z B \quad H \mid \beta : \Box_z B, B, P \Rightarrow Q}{G^* \mid H_{\alpha^*/\beta} \mid \alpha^* + \beta : B, M, P \Rightarrow N, Q}\,{\scriptstyle cut_{\Box_z B}}}{\dfrac{G \mid G^*_{\alpha/(\alpha^*+\beta)} \mid (H_{\alpha^*/\beta})_{\alpha/(\alpha^*+\beta)} \mid \alpha + (\alpha^* + \beta) : M, M, P \Rightarrow N, N, Q}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\,{\scriptstyle C^*+merge^*}}\,{\scriptstyle cut_B}
$$

where $G^* \mid \alpha^* : M \Rightarrow N, \Box_z B$ is the result of renaming the indexed hypersequent $G \mid \alpha : M \Rightarrow N, \Box_z B$ so that $\|G\| \cup \alpha$, $\|G^*\| \cup \alpha^*$ and $\|H\| \cup \beta$ are mutually disjoint. We assume this notation in all the cases below.

The first cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule, while the second cut is eliminable by induction on the complexity of the cut-formula. Moreover, since $\alpha + (\alpha^* + \beta) = \alpha + \beta$ and $(H_{\alpha^*/\beta})_{\alpha/(\alpha^*+\beta)} = H_{\alpha/\beta}$, only repeated applications of merge and contraction to G and $G^*_{\alpha/(\alpha^*+\beta)}$ are required to obtain the conclusion.

As concerns case 3.2.2 ($\mathcal{R}'$ is $\Box_z L_2$), we have:

$$
\dfrac{\dfrac{G \mid \alpha\hat{}nz : M \Rightarrow N \mid nz :\Rightarrow B}{G \mid \alpha : M \Rightarrow N, \Box_z B}\,{\scriptstyle \Box_z R} \qquad \dfrac{H' \mid \beta\hat{}mz : \Box_z B, P \Rightarrow Q \mid \gamma\hat{}mz : B, Z \Rightarrow W}{H' \mid \beta\hat{}mz : \Box_z B, P \Rightarrow Q \mid \gamma\hat{}mz : Z \Rightarrow W}\,{\scriptstyle \Box_z L_2}}{G \mid H'_{\alpha/\beta\hat{}mz} \mid \alpha + (\beta\hat{}mz) : M, P \Rightarrow N, Q \mid \gamma\hat{}mz : Z \Rightarrow W}\,{\scriptstyle cut_{\Box_z B}}
$$

which we reduce to

$$
\dfrac{G \mid \alpha\hat{}nz : M \Rightarrow N \mid nz :\Rightarrow B \qquad \dfrac{G^* \mid \alpha^* : M \Rightarrow N, \Box_z B \quad H' \mid \beta\hat{}mz : \Box_z B, P \Rightarrow Q \mid \gamma\hat{}mz : Z \Rightarrow W}{G^* \mid H'_{\alpha^*/\beta\hat{}mz} \mid \alpha^* + (\beta\hat{}mz) : M, P \Rightarrow N, Q \mid \gamma\hat{}mz : B, Z \Rightarrow W}\,{\scriptstyle cut_{\Box_z B}}}{G \mid G^*_{nz/mz} \mid (H'_{\alpha^*/\beta\hat{}mz})_{nz/mz} \mid \alpha\hat{}nz : M \Rightarrow N \mid \alpha^* + (\beta\hat{}nz) : M, P \Rightarrow N, Q \mid \gamma\hat{}nz : Z \Rightarrow W}\,{\scriptstyle cut_B}
$$

By repeated applications of merge and contraction, an observation similar to that in the previous case, and an application of Lemma 1, we obtain the desired conclusion.

The first cut is eliminable by induction of the sum of the heights of the derivations of the premises of the cut-rule, while the second cut is eliminable by induction on the complexity of the cut-formula.

`Case 3.3.` $\mathcal{R}$ is $\boxast R$ and $A = \boxast B$. Let us suppose that $\boxast B$ is the principal formula of the rule $\mathcal{R}'$; the other cases are treated as in 3.2. There are two subcases: 3.3.1. $\mathcal{R}'$ is $\boxast L_1$, and 3.3.2. $\mathcal{R}'$ is $\boxast L_2$. In the former case, we have:

$$
\dfrac{\dfrac{G \mid \alpha : M \Rightarrow N, C \quad C \Rightarrow \Box B \quad C \Rightarrow \Box C}{G \mid \alpha : M \Rightarrow N, \boxast B}\,{\scriptstyle \boxast R} \qquad \dfrac{H \mid \beta : \boxast B, \Box B, P \Rightarrow Q}{H \mid \beta : \boxast B, P \Rightarrow Q}\,{\scriptstyle \boxast L_1}}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\,{\scriptstyle cut_{\boxast B}}
$$

which we reduce to

$$
\dfrac{\dfrac{G \mid \alpha : M \Rightarrow N, C \quad C \Rightarrow \Box B}{G \mid \alpha : M \Rightarrow N, \Box B}\,{\scriptstyle cut_C} \qquad \dfrac{G^* \mid \alpha^* : M \Rightarrow N, \boxast B \quad H \mid \beta : \boxast B, \Box B, P \Rightarrow Q}{G^* \mid H_{\alpha^*/\beta} \mid \alpha^* + \beta : \Box B, M, P \Rightarrow N, Q}\,{\scriptstyle cut_{\boxast B}}}{\dfrac{G \mid G^*_{\alpha/\alpha^*+\beta} \mid (H_{\alpha^*/\beta})_{\alpha/\alpha^*+\beta} \mid \alpha + (\alpha^* + \beta) : M, M, P \Rightarrow N, N, Q}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\,{\scriptstyle C^*+merge^*}}\,{\scriptstyle cut_{\Box B}}
$$

where the conclusion is obtained in a similar way to case 3.2 above. The cut_C is eliminable by induction on the pr-rank, the $cut_{\boxplus B}$ is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule, and the $cut_{\Box B}$ is eliminable by induction on the complexity of the cut-formula.

We now consider case 3.3.2 ($\mathcal{R}'$ is $\boxplus L_2$), where we have:

$$\dfrac{\dfrac{G \mid \alpha : M \Rightarrow N, C \quad C \Rightarrow \Box B \quad C \Rightarrow \Box C}{G \mid \alpha : M \Rightarrow N, \boxplus B}\,\boxplus R \qquad \dfrac{H' \mid \beta : \boxplus B, P \Rightarrow Q \mid \gamma : \boxplus B, Z \Rightarrow W}{H' \mid \beta : \boxplus B, P \Rightarrow Q \mid \gamma : Z \Rightarrow W}\,\boxplus L_2}{G \mid H'_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q \mid \gamma_{\alpha/\beta} : Z \Rightarrow W}\,cut_{\boxplus B}$$

We go up the derivation d_2 to the first rule $\mathcal{R}''$ that is not a $\boxplus L_2$ rule applied to some of the $\boxplus B$'s. We distinguish three cases.

- The premise of $\mathcal{R}''$ is an initial indexed hypersequent, call it I. If the formula $\boxplus B$ is not the principal formula in I, then even the conclusion of the cut is an initial indexed hypersequent and the case is solved. If the formula $\boxplus B$ is the principal formula, then I contains an indexed sequent $\delta : Z', \boxplus B \Rightarrow W', \boxplus B$.[4] So the conclusion of the cut has the following form $G \mid H''_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q \mid \gamma_{\alpha/\beta} : Z \Rightarrow W \mid \delta_{\alpha/\beta} : Z' \Rightarrow W', \boxplus B$.

  By the condition (ii) of Definition 3 we know that the set of indices β and δ in I are linked by a chain of indices $n_1 i_1, ..., n_m i_m$. We now build the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \Rightarrow \Box C}{C \Rightarrow \Box_{i_1} C}\,\overline{\wedge R}}{n_1 i_1 : C \Rightarrow \ \mid n_1 i_1 :\Rightarrow C}\,\overline{\Box_z R} \qquad C \Rightarrow \Box C}{n_1 i_1 : C \Rightarrow \ \mid n_1 i_1 :\Rightarrow \Box C}\,cut_C}{n_1 i_1 : C \Rightarrow \ \mid n_1 i_1 :\Rightarrow \Box_{i_2} C}\,\overline{\wedge R}}{n_1 i_1 : C \Rightarrow \ \mid n_1 i_1, n_2 i_2 :\Rightarrow \ \mid n_2 i_2 :\Rightarrow C}\,\overline{\Box_z R}$$

$$\vdots$$

  where the derivation is continued with the same succession of inferences to obtain as conclusion the indexed hypersequent $n_1 i_1 : C \Rightarrow \ \mid ... \mid n_m i_m : \Rightarrow C$, where $n_1 i_1, ..., n_m i_m$ are exactly those indices that link the sets β and δ in I. The cuts in this derivation are eliminable by induction on the pr-rank. We finish solving the case with the following derivation; the cut is also eliminable by induction on the pr-rank.

$$\dfrac{\dfrac{\dfrac{G \mid \alpha : M \Rightarrow N, C \quad n_1 i_1 : C \Rightarrow \ \mid ... \mid n_m i_m : \Rightarrow C}{G \mid \alpha + n_1 i_1 : M \Rightarrow N \mid ... \mid n_m i_m : \Rightarrow C}\,cut_C \qquad C \Rightarrow \Box B \quad C \Rightarrow \Box C}{G \mid \alpha + n_1 i_1 : M \Rightarrow N \mid ... \mid n_m i_m : \Rightarrow \boxplus B}\,\boxplus R}{G \mid H''_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q \mid \gamma_{\alpha/\beta} : Z \Rightarrow W \mid \delta_{\alpha/\beta} : Z' \Rightarrow W', \boxplus B}\,{}^{IW^*,\ EW^*}_{IndW^*,\ Lem\ 1}$$

[4] We consider the case where this sequent is in H', the case where it is $\gamma : \boxplus B, Z \Rightarrow W$ is treated similarly.

- None of the $\boxast B$ are principal formulas of $\mathcal{R}''$. This case is treated similarly to case 2.

- $\mathcal{R}'' = \boxast L_1$ and has (any of the) $\boxast B$ as principal formula. If the principal formula $\boxast B$ of the rule belongs to the indexed sequent $\beta : \boxast B, P \Rightarrow Q$, then we apply the rule $\boxast L_2$ n times on the premise of the $\boxast L_1$ and then operate as in case 3.3.1. Now consider the case where the principal formula does not belong to this indexed sequent. First, in a way analogous to the previous item, we construct a derivation of the indexed hypersequent $n_1 i_1 : C \Rightarrow \ | \ldots | \ n_m i_m :\Rightarrow \Box B$. Then we apply the rule $\boxast L_2$ n times on the premise of the rule $\boxast L_1$ to obtain the indexed hypersequent $H'' \mid \beta : \boxast B, P \Rightarrow Q \mid \gamma : Z \Rightarrow W \mid \delta : \Box B, Z' \Rightarrow W'$.[5] We proceed with the following cuts:

$$\frac{G \mid \alpha : M \Rightarrow N, C \quad n_1 i_1 : C \Rightarrow \ \mid \ldots \mid n_m i_m : \Rightarrow \Box B}{G \mid \alpha + n_1 i_1 : M \Rightarrow N \mid \ldots \mid n_m i_m : \Rightarrow \Box B} \ cut_C$$

$$\frac{G \mid \alpha : M \Rightarrow N, \boxast B \quad H'' \mid \beta : \boxast B, P \Rightarrow Q \mid \gamma : Z \Rightarrow W \mid \delta : \Box B, Z' \Rightarrow W'}{G \mid H''_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q \mid \gamma_{\alpha/\beta} : Z \Rightarrow W \mid \delta_{\alpha/\beta} : \Box B, Z' \Rightarrow W'} \ cut_{\boxast B}$$

where the former cut is eliminable by induction on the pr-rank, and the latter by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Renaming and applying the cut-rule on the conclusions of these cuts, with principal formula $\Box B$, we obtain the indexed hypersequent:

$$G^* \mid G_{(n_m i_m)^*/\delta_{\alpha/\beta}} \mid (H''_{\alpha/\beta})_{(n_m i_m)^*/\delta_{\alpha/\beta}} \mid (\alpha+\beta)_{(n_m i_m)^*/\delta_{\alpha/\beta}} : M, P \Rightarrow N, Q \mid \alpha+n_1 i_1 : M \Rightarrow N \mid \ldots \mid (\gamma_{\alpha^*/\beta})_{(n_m i_m)^*/\delta_{\alpha/\beta}} : Z \Rightarrow W \mid (n_m i_m)^* + \delta_{\alpha/\beta} : Z' \Rightarrow W'$$

This cut is eliminable by induction on the complexity of the cut-formula. We obtain the desired conclusion by renaming indices and several applications of the rules of merge and contraction.

□

The following theorem follows immediately from Lemma 5 by induction on the number of cuts.

Theorem 5.1. *Every derivation d in* **HS5C** *can be effectively transformed into a derivation d′ where there is no application of the cut-rule.*

[5]The $\Box B$ could belong to the indexed sequent $\gamma^* : Z \Rightarrow W$, instead of to the sequent $\delta^* : Z' \Rightarrow W', \boxast B$. The case is solved in the same way.

6 Discussion and refinements

The calculus thus admits a syntactic procedure for eliminating cuts. However, given the non-analyticity of the $\boxplus R$ rule, cut-elimination does not imply the subformula property. Moreover, one might consider this to be a partial cut-elimination result,[6] insofar as some "cut-like" elements are "built into" the $\boxplus R$ rule.

In reply to this worry, we show that the application of the $\boxplus R$ rule may be restricted. To this end, we first define the set of (disjunctive) normal forms, DNF, for our language, as follows:

$$\begin{array}{lcl} Lit & ::= & S \mid \neg A \mid \neg\Box_z\neg Term \\ Term & ::= & Lit \mid Term \wedge Term \\ Clause & ::= & Term \mid \neg \boxplus \neg Term \mid \boxplus \neg Term \mid Clause \wedge Clause \\ DNF & ::= & Clause \mid Clause \vee Clause \end{array}$$

It is straightforward to show that, for any formula A, there is an equivalent disjunctive normal form; we call it A^{DNF}.

Proposition 1. *Any formula A is equivalent to a formula* $A^{DNF} \in DNF$.

Proof. First define the modal depth of a formula A, $d(A)$ as standard: $d(A) = d(\neg A) = 0$; $d(A \wedge B) = \max(d(A), d(B))$, $d(\Box_z A) = d(\boxplus A) = d(A) + 1$. We reason by induction on $d(A)$. For $d(A) = 0$, the result is a standard application of the disjunctive normal form theorem for propositional logic. For $d(A) = n > 0$, A is equivalent to a Boolean combination of propositional atoms and formulas of the form $\neg\Box_z\neg B$ and $\neg \boxplus \neg C$, where $d(B), d(C) < n$; so, by the inductive hypothesis, $B = \bigvee B_i$ where the B_i are clauses of modal depth $n - 1$ (and likewise for C). We thus have $\neg\Box_z\neg B \equiv \bigvee \neg\Box_z\neg B_i$, by standard modal logic. Moreover, whenever B_i is of the form $B'_i \wedge \neg \boxplus \neg B''_i$ for some clauses B'_i, B''_i with $d(B''_i) < n-1$, it follows from the logic of common knowledge that $\neg\Box_z\neg B_i \equiv \neg\Box_z\neg B'_i \wedge \neg\boxplus\neg B''_i$. Similarly, if B_i is of the form $B'_i \wedge \boxplus\neg B''_i$ for some clauses B'_i and B''_i with $d(B''_i) < n-1$, $\neg\Box_z\neg B_i \equiv \neg\Box_z\neg B'_i \wedge \boxplus\neg B''$. Hence $\neg\Box_z\neg B$ is equivalent to a disjunction of clauses of the required form. Similarly, $\neg \boxplus \neg C \equiv \bigvee \neg \boxplus \neg C_i$; and whenever C_i is of the form $C'_i \wedge \neg \boxplus \neg C''_i$ (respectively $C'_i \wedge \boxplus\neg C''_i$), then $\neg \boxplus \neg C_i \equiv \neg \boxplus \neg C'_i \wedge \neg \boxplus \neg C''_i$ (respectively $\neg \boxplus \neg C_i \equiv \neg \boxplus \neg C'_i \wedge \boxplus\neg C''_i$). By the propositional disjunctive normal form theorem, there exists a formula in DNF equivalent to A, as required. □

Note that the definition of disjunctive normal forms, as well as this result, is similar to the (standard) notions and results proposed in [8] for modal logics without fixed point operators.

For a formula A, A^{DNF} is the disjunction of clauses D of the form $D_{prop\Box} \wedge D_{-\boxplus} \wedge D_{+\boxplus}$, where $D_{prop\Box}$ is a term (a conjunction of propositional atoms, formulas preceded by $\Box_z$ and their negations) and $D_{-\boxplus}$ and $D_{+\boxplus}$ are conjunctions of formulas of the form $\neg \boxplus \neg C$ and $\boxplus\neg C$ respectively. For each clause D, we define the *common knowledge reduction of D*, $D^{CK} = \neg \boxplus \neg D_{prop\Box} \wedge D_{-\boxplus} \wedge D_{+\boxplus}$. For any clause D and any set of propositional atoms $\mathcal{P}$, we define the *common knowledge reduction of D restricted*

[6] Partial cut-elimination results (eg. [2]) show that, as concerns the derivation of a given formula, all except a certain class of cuts can be eliminated.

to $\mathcal{P}$, $D^{CK}_{\mathcal{P}}$ to be result of removing from D^{CK} any propositional atom, not belonging to $\mathcal{P}$, that occurs in a formula of the form $\neg \boxplus \neg C$.[7] Similarly, for any formula A with disjunctive normal form $A^{DNF} = \bigvee D_i$, the *common knowledge reduction of A* is defined to be $A^{CK} = \bigvee D^{CK}_i$, and the *common knowledge reduction of A restricted to* $\mathcal{P}$ is defined to be $A^{CK}_{\mathcal{P}} = \bigvee_{D^{CK}_{i\mathcal{P}} \not\equiv \top} D^{CK}_{i\mathcal{P}}$.

Finally, for a formula A, define $\mathcal{P}_{-A}$ to be the set of propositional atoms occurring in the scope of a negative occurrence of the $\boxplus$ operator in A.[8] We have the following result concerning the principal formula in the application of the $\boxplus R$ rule.

Proposition 2. *If*

$$\dfrac{\begin{array}{c}\vdots d_1 \\ G \mid \alpha : M \Rightarrow N, B\end{array} \quad \begin{array}{c}\vdots d_2 \\ B \Rightarrow \Box A\end{array} \quad \begin{array}{c}\vdots d_3 \\ B \Rightarrow \Box B\end{array}}{G \mid \alpha : M \Rightarrow N, \boxplus A}\ \boxplus R$$

then there is a derivation of $G \mid \alpha : M \Rightarrow N, \boxplus A$*, concluding with an instance of the* $\boxplus R$ *rule whose principal formula is* $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}}$.[9]

Proof. The proof, which is quite long, uses a semantic argument, relying on Theorem 4.1 and the soundness and completeness of the Hilbert system for **S5C**. Further details are omitted for lack of space.

By Theorem 4.1 and Definition 7, since $G \mid \alpha : M \Rightarrow N, \boxplus A$ is derivable, $\vdash (G \mid \alpha : M \Rightarrow N, \boxplus A)^{\tau}_{\alpha:M\Rightarrow N,\boxplus A}$ in the system **S5C**. It follows from Definition 6 and propositional logic that $\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N} \vdash \boxplus A$. For brevity, let $C = \neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N}$. We now show that $C^{CK}_{\mathcal{P}_{-A}} \vdash \boxplus A$, reasoning semantically, and using the soundness and completeness of the standard Kripke semantics with respect to the system **S5C**.[10] We thus have $C \vDash \boxplus A$, and we wish to show that $C^{CK}_{\mathcal{P}_{-A}} \vDash \boxplus A$.

First note that, if C is true for some state in a CK-cell, then C^{CK} holds for all states in the cell; conversely, if C^{CK} holds for some state in a CK-cell, then there must be a state in the cell which satisifies C. Hence the set of CK-cells for which C is true for some state in the cell coincides with the set of CK-cells for which C^{CK} is true for some state in the cell. Since, by the form of $\boxplus A$, the truth of $\boxplus A$ in a state depends entirely on the CK-cell to which the state belongs, and since $C \vDash \boxplus A$, we have that $C^{CK} \vDash \boxplus A$.

Now, for any set P of CK-cells, let the $\mathcal{P}_{-A}$-closure of P be the largest set of CK-cells containing P such that the (states in the) cells all give the same valuation to all formulas of the form $\boxplus C$, and to all formulas of the form $\neg \boxplus C$ containing only propositional atoms in $\mathcal{P}_{-A}$. It is clear that the set of CK-cells satisfying $C^{CK}_{\mathcal{P}_{-A}}$ is the $\mathcal{P}_{-A}$-closure of the set satisfying C^{CK}. Moreover, since the only propositional atoms occurring in the scope of negative occurrences of $\boxplus$ in $\boxplus A$ belong to $\mathcal{P}_{-A}$, the set of CK-cells satisfying $\boxplus A$ is the $\mathcal{P}_{-A}$-closure of itself. Since the operation of $\mathcal{P}_{-A}$-closure

[7] Formally, removing corresponds to replacing a positive occurrence of p by $\top$ and any negative occurrence of p by $\bot$.

[8] Positive and negative occurrences of formulas are defined as standard, see eg. [5].

[9] Recall the notation from Definition 6.

[10] We assume standard Kripke semantics terminology (eg. [5]); moreover, we use the term CK-cell for the set of states accessible from a given state by the accessibility relation for the common knowledge operator.

is evidently monotonic (ie. if $P \subseteq Q$, then the $\mathcal{P}_{-A}$-closure of P is contained in the $\mathcal{P}_{-A}$-closure of Q), it follows that $C^{CK}_{\mathcal{P}_{-\mathcal{A}}} \models ⊞A$.

Since $C^{CK}_{\mathcal{P}_{-\mathcal{A}}} \models ⊞A$, it follows that $C^{CK}_{\mathcal{P}_{-\mathcal{A}}} \models \Box A$. By the completeness of the standard Hilbert calculus and Theorem 4.1, it follows that $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}} \Rightarrow \Box A$ is derivable. Moreover, by the semantic reasoning above and the definition of the common knowledge reduction of a formula, it can be shown that $G \mid \alpha : M \Rightarrow N, (\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}}$ is derivable. Furthermore, there is a simple derivation of $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK} \Rightarrow \Box(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}}$. Hence all the premises for the application of the $⊞R$ rule with $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}}$ as principal formula are derivable. There is thus a derivation of $G \mid \alpha : M \Rightarrow N, ⊞A$ whose last rule is $⊞R$ with principal formula $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}}$, as required.

□

By restricting the form of the principal formula of the $⊞R$ rule, this result limits the non-analyticity of the calculus. On the one hand, it indicates that, to search for a proof of a formula, it suffices to consider one sole possible application of the $⊞R$ rule (with the formula $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}}$); a major inconvenience of the lack of subformula property, namely the fact that it renders proof search impossible, because one would have to search for 'disappearing' principal formulas, is thus overcome. On the other hand as concerns the "partialness" of our cut-elimination, they strengthen the cut-elimination result, insofar as they greatly restrict the application of the $⊞R$ rule: to a single principal formula for each conclusion. Indeed, Proposition 2 could be thought of as a thought of elimination result for all applications of the $⊞R$ rule except one.

To give an idea of the strength of the restrictions Proposition 2 places on the application of the $⊞R$ rule, to give a comparison with partial cut-elimination results for finitary calculi elsewhere in the literature, as well as to give an example of an application of the calculus, suppose that there are only two agents a and b, and consider the following (derivable) sequent, taken from [2]: $\Box_a(P \wedge ⊞Q), \Box_b(Q \wedge ⊞P) \Rightarrow ⊞(P \vee Q)$. This sequent is not derivable in the finitary calculus proposed by [2] without the cut rule, and the partial cut-elimination result they have limits the set of cuts that can be used to derive the formula to (at least) an order of 2^{18}.[11]

By contrast, straightforward calculation shows that the formula proposed in Proposition 2 for this case is just $⊞P \wedge ⊞Q$. To search for a proof involving a final application of the $⊞R$ rule, it suffices to search for one where the principal formula is $⊞P \wedge ⊞Q$. And indeed, it is easy to see how to construct such a proof. The derivation of the leftmost premise of the rule is:

[11] [2] proposes a partial cut-elimination result according to which any derivable sequent can be derived using only cuts on formula in the disjunctive-conjunctive closure of the Fisher-Ladner closure of the sequent to be proven, though they cite stronger results involving only the conjunctive closure of the Fisher-Ladner closure. They state that the size of the Fischer-Ladner closure is of the order of the length of the formula (which in this case is 18), so the set of conjunctions of elements of the Fischer-Ladner closure is of order 2^{18}.

$$\dfrac{\dfrac{\dfrac{\Box_a(P \wedge \boxplus Q), P, \boxplus Q, \Box_b(Q \wedge \boxplus P), \Rightarrow \boxplus P}{\Box_a(P \wedge \boxplus Q), P \wedge \boxplus Q, \Box_b(Q \wedge \boxplus P), \Rightarrow \boxplus P}\,\wedge L}{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P) \Rightarrow \boxplus P}\,\Box L_1 \quad \dfrac{\dfrac{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P), Q, \boxplus P \Rightarrow \boxplus P}{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P), Q \wedge \boxplus P \Rightarrow \boxplus P}\,\wedge L}{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P) \Rightarrow \boxplus P}\,\Box L_1}{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P) \Rightarrow \boxplus P \wedge \boxplus Q}\,\wedge R$$

The derivation of the middle premise is:[12]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1a : \boxplus P, \Box_a P, \boxplus Q \Rightarrow \ \mid 1a : P \Rightarrow P, Q}{1a : \boxplus P, \Box_a P, \boxplus Q \Rightarrow \ \mid 1a : P \Rightarrow P \vee Q}\,\vee R}{1a : \boxplus P, \Box_a P, \boxplus Q \Rightarrow \ \mid 1a : \Rightarrow P \vee Q}\,\Box_a L_2}{\boxplus P, \Box_a P, \boxplus Q \Rightarrow \Box_a(P \vee Q)}\,\Box_a R}{\boxplus P, \boxplus Q \Rightarrow \Box_a(P \vee Q)}\,\boxplus_a L_1}{\boxplus P \wedge \boxplus Q \Rightarrow \Box_a(P \vee Q)}\,\wedge L$$

and similarly for $\boxplus P \wedge \boxplus Q \Rightarrow \Box_b(P \vee Q)$, with a final application of the $\wedge R$ rule. Finally, the derivation of the right premise is:

$$\dfrac{\dfrac{\dfrac{\dfrac{1a : \boxplus P, \boxplus Q \Rightarrow \ \mid 1a : \boxplus P, \boxplus Q \Rightarrow \boxplus P \quad 1a : \boxplus P, \boxplus Q \Rightarrow \ \mid 1a : \boxplus P, \boxplus Q \Rightarrow \boxplus Q}{1a : \boxplus P, \boxplus Q \Rightarrow \ \mid 1a : \boxplus P, \boxplus Q \Rightarrow \boxplus P \wedge \boxplus Q}\,\wedge R}{1a : \boxplus P, \boxplus Q \Rightarrow \ \mid 1a : \Rightarrow \boxplus P \wedge \boxplus Q}\,\boxplus L_2^*}{1a : \boxplus P \wedge \boxplus Q \Rightarrow \ \mid 1a : \Rightarrow \boxplus P \wedge \boxplus Q}\,\wedge L}{\boxplus P \wedge \boxplus Q \Rightarrow \Box_a(\boxplus P \wedge \boxplus Q)}\,\Box_a R$$

and similarly for $\boxplus P \wedge \boxplus Q \Rightarrow \Box_b(\boxplus P \wedge \boxplus Q)$.

We conclude that, though the proposed calculus is not strictly speaking analytic, it is remarkably easy to construct proofs using it, given the difficulty in finding finitary calculi for common knowledge, and in comparison to other proposals.

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[12] See footnote 1 concerning the $\vee R$ rule.

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