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Abstract According to a minimalist version of Afriat's theorem, a consumer behaves as a utility maximizer if and only if a feasibility matrix associated with his choices is cyclically consistent. An "essential experiment" consists of observed consumption bundles $(x_1, \cdots, x_n)$ and a feasibility matrix $\boldsymbol{\alpha}$. Starting with a standard experiment, in which the economist has access to precise budget sets, we show that the necessary and sufficient condition for the existence of a utility function rationalizing the experiment, namely, the cyclical consistency of the associated feasibility matrix, is equivalent to the existence, for any budget sets compatible with the deduced essential experiment, of a utility function rationalizing them (and typically depending on them). In other words, the conclusion of the standard rationalizability test, in which the economist takes budget sets for granted, does not depend on the full specification of the underlying budget sets but only on the essential data that these budget sets generate. Starting with an essential experiment $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ only, we show that the cyclical consistency of $\boldsymbol{\alpha}$, together with a further consistency condition involving both $(x_1, \cdots, x_n)$ and $\boldsymbol{\alpha}$, guarantees the existence of a budget representation and that the essential experiment is rationalizable almost robustly, in the sense that there exists a single utility function which rationalizes at once almost all budget sets which are compatible with $(x_1, \cdots, x_n; \boldsymbol{\alpha})$. The conditions are also trivially necessary.

JEL classification numbers: D11,C81.

Keywords Afriat's theorem, budget sets, cyclical consistency, rational choice, revealed preference.

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1 Introduction

Afriat (1967)'s theorem has been revisited in a few recent papers, which propose new proofs (Fostel *et al.*, 2004; Chung-Piaw and Vohra, 2003), extensions (Forges and Minelli, 2009), new interpretations (Ekeland and Galichon, 2012) of the result. In all these papers, as already in the classical one (see, e.g., Varian, 1982), information on the choices of a given consumer at various dates $j = 1, \cdots, n$ is summarized by an $n \times n$ feasibility matrix. The (j, k) entry of this matrix takes the value -1, 0 or 1 and indicates to which extent the item (e.g., a consumption bundle) that has been chosen by the consumer at date k is affordable or not at date j. According to (a minimalist version of) Afriat's theorem, the consumer behaves as a utility maximizer if and only if the feasibility matrix satisfies a tractable property, referred to as "cyclical consistency". This version of Afriat's theorem is recalled in Section 2 as Proposition 1.[1]

In a standard framework, the observed choices of the consumer are bundles $x_1, \cdots, x_n \in \mathbb{R}^{\ell}_{+}$, which define, together with the associated feasibility matrix, what we call in this paper an "essential experiment". For the sake of exposition, we explicitly refer to the "economist" who analyzes the consumer's data. In order to test the consumer's rationality, the economist basically has to check whether the feasibility matrix is cyclically consistent. When performing this test, the economist typically has access to precise budget sets for every date. As shown by Forges and Minelli (2009), even if the budget sets are quite general (namely, just compact and monotonic), Afriat's original constructive approach applies: if the feasibility matrix is cyclically consistent, the economist can derive an explicit utility function rationalizing the data. This is another version of Afriat's theorem, which is stated in Section 3 as Lemma 1.

In the latter, classical, version of Afriat's theorem, the basic data take the form of an "experiment", namely, consumption bundles and budget sets. The utility function, if it exists, depends on these budget sets, which may be complex, e.g., involve tariffs or taxes. While the economist has access to these budget sets, he may not be sure that the consumer has made full use of the budget sets beyond what is revealed by the "essential experiment". For instance, let us assume that the unit price of, say, good 1, decreases with the quantity that is bought. If the consumer's chosen bundles at every date $j = 1, ..., n$, all contain small quantities of good 1, the economist has no reason to take for granted that the consumer knows the unit price of good 1 for large quantities. The "essential experiment" that we introduced above precisely captures the consumer's knowledge of the budget sets as it is revealed by his choices. The good news is that the essential experiment (as opposed to the standard one, which involves a precise description of the budget sets) is just what is needed to test the consumer's rationality. This result will be formally stated as Proposition 3. However, testing the consumer's rationality is not the only content of Afriat's contribution: it also provides a way to construct an explicit utility function when it exists.

We are thus led to the following question:

[1] Afriat's approach has also been applied to various economic environments, beyond classical consumer's theory (see, e.g. Brown and Calsamiglia, 2007; Carvajal, 2010; Chambers and Echenique, 2009) for recent references.

Given an essential experiment $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ in which the feasibility matrix $\boldsymbol{\alpha}$ is cyclically consistent, can we construct a utility function v which *robustly* rationalizes $(x_1, \cdots, x_n; \boldsymbol{\alpha})$, in the sense that $v(x_j)$ maximizes v over B_j, for *any* family (B_j) of budget sets compatible with $(x_1, \cdots, x_n; \boldsymbol{\alpha})$?

Proposition 4 gives an answer to this question. The motivation for such a utility function v is clear: v would not be sensitive to those specific aspects of the budget sets that the consumer might not perceive.

The previous informal discussion hides some difficulties. First of all, we will show that the previous question is not meaningful unless the essential experiment satisfies some basic consistency requirement (independent of cyclical consistency) guaranteeing that there indeed exists (compact, monotonic) budget sets that are compatible with it. We introduce the property that the essential experiment "contains no contradictory statement" and show that it captures such a requirement. This result is formally stated as Proposition 2. Equipped with this tool, we can give a formal statement of the "good news" announced above: an essential experiment $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ can be rationalized if and only if $\boldsymbol{\alpha}$ is cyclically consistent and $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ contains no contradictory statement.

Next, we construct an essential experiment $(x_1, x_2; \boldsymbol{\alpha}) \in \mathbb{R}^2_+$ which contains no contradictory statement, where $\boldsymbol{\alpha}$ is cyclically consistent, and which cannot be rationalized robustly. This simple example is by no means pathological and shows that, formulated exactly as above, the question cannot be answered positively.

Nonetheless, we prove that every essential experiment $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ which contains no contradictory statement and where $\boldsymbol{\alpha}$ is cyclically consistent can be rationalized in an almost robust way, in the sense that for every sufficiently small ϵ, there exists an almost largest family (B_j^ϵ) of budget sets compatible with $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ and a utility function v^ϵ rationalizing $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ over (B_j^ϵ). It is not difficult to prove that, conversely, if $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ can be rationalized in an almost robust way, then $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ contains no contradictory statement and $\boldsymbol{\alpha}$ is cyclically consistent. This is the main content of Proposition 4 given in Section 4.

As suggested above, our results can be interpreted in the standard framework where the economist has access to precise budget sets. From these and the observed consumption bundles, he can deduce the corresponding essential experiment. A by-product of Proposition 3 (partially contained in Lemma 1) is that the conclusion of the standard rationalizability test, in which the economist takes budget sets for granted, does not depend on the full specification of the underlying budget sets but only on the essential data that these budget sets generate; the economist's conclusion automatically applies to a whole family of budget sets. If the essential experiment passes the rationalizability test, Proposition 4 gives a way to construct an almost robust utility function, which rationalizes the consumer's choices for basically all budget sets that are consistent with these choices.

In a much less classical interpretation of our results, the economist does not have access to a precise description of the budget sets, which remain private information of the consumer. The only available data on the consumer could then be the essential experiment, as the outcome of a survey. The economist would be interested in checking whether the data could have been generated by a rational consumer, making choices in monotonic budget sets.

The paper is organized as follows: notations are made precise in the next subsection; Section 2 deals with a consumer choosing over finitely many items and defines cyclical consistency; Section 3 deals with a consumer choosing over consumption bundles and defines budget sets; Section 4 turns to rationalization and states the results listed above; most of the proofs are given in the appendix.

1.1 Notations and terminology

$\mathbb{R}^\ell$ denote the Euclidean space of dimension ℓ;
For every $x, x' \in \mathbb{R}^\ell$, $x \geq x'$ iff $x_i \geq x'_i$, $\forall\, i = 1, \cdots, \ell$; $x > x'$ iff $x \geq x$ and $x \neq x'$; $x \gg x'$ iff $x_i > x'_i$, $\forall\, i = 1, \cdots, \ell$;
$\mathbb{R}^\ell_+ := \{x \in \mathbb{R}^\ell \mid x \geq 0\}$ and $\mathbb{R}^\ell_{++} := \{x \in \mathbb{R}^\ell \mid x \gg 0\}$ denote the non negative and positive orthant of $\mathbb{R}^\ell$ respectively;
Given a set $A \in \mathbb{R}^\ell$, $\mathrm{Fr}\, A = \{x \in A \mid \{x_k\} + \mathbb{R}^\ell_{++}\} \cap A = \emptyset\}$ and $[A]_+ = A \cap \mathbb{R}^\ell_+$ denote the frontier of A and the subset of the non negative elements of A respectively;
The vector $\mathbf{1}$ is the characteristic vector of $\mathbb{R}^\ell$ whose components are equal to 1;
A set $B \subseteq \mathbb{R}^\ell_+$ is monotonic if $[B - \mathbb{R}^\ell_+]_+ \subseteq B$; and if $x \in \mathrm{Fr}B$ then, for all $k \in [0, 1)$, $kx \in B \setminus \mathrm{Fr}B$;[2]
Let $N = \{1, \cdots, n\}$ be fixed once and for all.

2 Essential data

As announced in the introduction, we start with a feasibility matrix $\boldsymbol{\alpha} = (\alpha_{jk})_{j,k \in N}$, i.e. an $n \times n$ matrix which summarizes the affordability of n observed consumer choices: for every $j, k \in N$;

$\alpha_{jk} \in \{-1, 0, +1\}$ and $\alpha_{jj} = 0$,
$\alpha_{jk} = -1$ if item k is affordable at date j without exhausting the consumer's revenue;
$\alpha_{jk} = 0$ if item k is affordable at date j and exhausts the consumer's revenue;
$\alpha_{jk} = +1$ if item k is not affordable at date j.

The essential data determine a choice experiment in which the choice set reduces to the n items. A traditional question is to which extent the data are consistent with rational choice, namely whether there exists a rationalization of the data. This amounts to finding a number v_j for every item j, such that $v_j \geq v_k$ for every item k that is affordable at date j, with strict inequality if item k does not exhaust entirely the revenue of the agent.

Definition 1 Utils $(v_j)_{j \in N}$ *rationalize* the feasibility matrix $\boldsymbol{\alpha}$, if, for every $j \in N$:

$v_j \geq v_k$ for every $k \in N$ such that $\alpha_{jk} \leq 0$,
$v_j > v_k$ for every $k \in N$ such that $\alpha_{jk} < 0$.

The following tractable condition of cyclical consistency is the usual test to check whether or not an experiment can be rationalized.

[2] The second part of the definition is technical and guarantees that $\mathrm{Fr}B$ is non-level on the boundary of $\mathbb{R}^\ell_+$.

Definition 2 An $n \times n$ real matrix $A = (a_{jk})_{j,k \in N}$ is *cyclically consistent* if for every chain $j, k, \ell, ..., r$, $a_{jk} \leq 0, a_{k\ell} \leq 0, ..., a_{rj} \leq 0$ implies all terms are 0.

The next proposition can be deduced from Afriat (1967)'s theorem, in which the feasibility matrix is actually implicit.

Proposition 1 *The following conditions are equivalent:*

1. *The feasibility matrix $\boldsymbol{\alpha}$ is cyclically consistent.*
2. *There exist utils $(v_i)_{i \in N}$ rationalizing the feasibility matrix $\boldsymbol{\alpha}$.*

Proof [1. $\Rightarrow$ 2.] is proved in Fostel *et al.* (2004, replacing A' by $\boldsymbol{\alpha}$ page 215). [2. $\Rightarrow$ 1.] is proved in Ekeland and Galichon (2012, replacing R_{ij} by α_{ij} in the proof of 3. $\Rightarrow$ 1., Theorem 0). $\square$

Remark 1 Ekeland and Galichon (2012) propose another, "dual", interpretation of the matrix $\boldsymbol{\alpha}$ in terms of a market with n traders and an indivisible good (house) to be traded (see also Shapley and Scarf (1974)). In the autarky allocation, each trader j owns house j. The matrix $\boldsymbol{\alpha}$ summarizes then the preferences of traders in the initial autarky allocation: $\alpha_{jk} = 1$ represents strict preference of his own house over house k; $\alpha_{jk} = -1$ represents strict preference of house k over his own house; $\alpha_{jk} = 0$ represents indifference of trader j between house k and his own house. In this dual interpretation, Proposition 1 actually amounts to: the autarky allocation is a *no trade equilibrium* allocation supported by prices $\pi_j = -v_j$ (condition 2. of Proposition 1) if and only if it is *Pareto optimal* (condition 1. of Proposition 1).

3 Budget sets

From now on we turn to the standard consumer problem, in which there are ℓ divisible consumption goods, and utility is thus defined by a function $v : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$. Hence, the data contain consumption bundles in addition to the feasibility matrix. This leads to the following notion of experiment, which becomes the basic data in our revealed preference analysis.

Definition 3 An *essential (consumer) experiment* $(\mathbf{x}, \boldsymbol{\alpha})$ consists of observed consumption bundles $(x_j)_{j \in N}$, $x_j \in \mathbb{R}^{\ell}_{++}$, and a feasibility matrix $\boldsymbol{\alpha}$.

We will distinguish such an essential experiment from a standard experiment which involves consumption bundles and budget sets. We follow Forges and Minelli (2009)'s model of budget sets, which is appropriate if finite bundles are consumed and free disposal is allowed. In particular, the formulation encompasses the following cases: classical linear budget sets; budget sets defined by the intersection of linear inequalities, as in Yatchew (1985); convex but non-linear budget sets, as in Matzkin (1991). More generally, the budget set of the consumer can result from quantity constraints, taxes and other sources of non convexities.

Besides compactness, the crucial requirement is monotonicity (see subsection 1.1).

Definition 4 A *budget set* is a compact and monotonic subset of $\mathbb{R}^{\ell}_{+}$.

The next definition is the natural extension of the classical notion of experiment with linear budget sets. In particular, the budget sets B_j are implicitly assumed to be observed by the economist, who will make inferences over the consumer's choices. Furthermore, consumption choices entirely exhaust the consumer's available revenue, at each given date. Note that the latter fact is also implicitly assumed in the classical theory with linear budget sets defined by prices.

Definition 5 An experiment $(\mathbf{x}, \mathbf{B})$ consists of observed consumption bundles $x_j \in \mathbb{R}^{\ell}_{++}$ and of budget sets B_j, such that $x_j \in \mathrm{Fr}B_j$ for every $j \in N$.[3]

In the standard approach of revealed preference analysis, an experiment $(\mathbf{x}, \mathbf{B})$ is given. This formulation implicitly assumes that a rational consumer perfectly knows his budget set B_j for every $j \in N$. The economist is interested in testing whether the consumer chooses every consumption bundle "rationally" given the budget sets at each date.

Definition 6 A utility function v is said to rationalize an experiment $(\mathbf{x}, \mathbf{B})$ if $v(x_j) = \max_{x \in B_j} v(x)$ for every $j \in N$.

Next we describe how to relate budget sets and the feasibility matrix $\boldsymbol{\alpha}$ to perform the consumer's rationalizability test in terms of essential data only.

Definition 7 Given an experiment $(\mathbf{x}, \mathbf{B})$, let $A^{x,B}$ denote the $n \times n$ matrix with entries $a^{x,B}_{jk} = -1$ if $x_k \in \mathrm{int}B_j$; $a^{x,B}_{jk} = 0$ if $x_k \in \mathrm{Fr}B_j$; $a^{x,B}_{jk} = 1$ if $x_k \notin B_j$.

An essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ admits *a budget representation* if there exists a family of budget sets $(B_j)_{j \in N}$ such that $(\mathbf{x}, \mathbf{B})$ is an experiment and $A^{x,B} = \boldsymbol{\alpha}$. A family $(B_j)_{j \in N}$ with this property is said *compatible with* $(\mathbf{x}, \boldsymbol{\alpha})$.

Given an experiment $(\mathbf{x}, \mathbf{B})$, the economist can deduce the corresponding essential experiment by setting $\boldsymbol{\alpha} = A^{x,B}$. Alternatively, let us imagine that the essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ is the only available one. As explained in the introduction, this can happen in at least two different situations. In the first one, the economist has access to the full experiment $(\mathbf{x}, \mathbf{B})$ but he suspects that the consumer only used the partial description in the associated essential experiment $(\mathbf{x}, A^{x,B})$. In the second situation, the economist has only access to survey data which take the form of an essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$.

If only the essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ is available, $(\mathbf{x}, \boldsymbol{\alpha})$ does not necessarily admit a budget representation. In the next section, we introduce a tractable necessary and sufficient condition, "no contradictory statement", for this property to hold (Proposition 2). For the time being, we just assume that $(\mathbf{x}, \boldsymbol{\alpha})$ admits a budget representation, as it is the case if the essential experiment is simply deduced from some standard experiment $(\mathbf{x}, \mathbf{B})$.

The next result can be deduced from Proposition 3 in Forges and Minelli (2009).

Lemma 1 *Let $(\mathbf{x}, \boldsymbol{\alpha})$ be an essential experiment which admits a budget representation. The following conditions are equivalent:*

1. *The matrix $\boldsymbol{\alpha}$ is cyclically consistent.*

[3] Note also that the definitions of budget set and experiment imply that every budget set considered hereafter has a nonempty interior.

2. *For any family of budget sets $(B_j)_{j\in N}$ compatible with $(\mathbf{x}, \boldsymbol{\alpha})$, there exists a locally non satiated, continuous utility function v^B rationalizing the experiment $(\mathbf{x}, \mathbf{B})$.*

Proof [1. ⇒ 2.] Since $(\mathbf{x}, \boldsymbol{\alpha})$ admits a budget representation, it holds that $\boldsymbol{\alpha}$ is cyclically consistent iff $(\mathbf{x}, \mathbf{B})$ satisfies GARP, for every family $(B_j)_{j\in N}$ compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ using straightforward arguments. Then apply Proposition 3 in Forges and Minelli (2009) to conclude the proof.[4] In particular, the construction of the utility functions relies on the following arguments: for every compatible family $(B_j)_{j\in N}$, construct continuous, monotone mappings $(g_j^B)_{j\in N}$ to describe the budget sets as $B_j = \left\{x \in \mathbb{R}_+^\ell : g_j^B(x) \leq 0\right\}$; use cyclical consistency of the matrix with entries $(g_j^B(x_k))_{j,k\in N}$ to derive inequalities *à la* Afriat; and finally, thanks to these inequalities, construct an explicit a utility function v^B depending on the mappings $(g_j^B)_{j\in N}$.[5]

[2. ⇒ 1.] Since $(\mathbf{x}, \boldsymbol{\alpha})$ admits a budget representation, there exist an experiment $(\mathbf{x}, \mathbf{B})$ and a locally non satiated, continuous utility function v^B rationalizing the experiment $(\mathbf{x}, \mathbf{B})$ where $A^{x,B} = \boldsymbol{\alpha}$. Hence, $v(x_j) \geq v(x_k)$ for every k such that $\alpha_{jk} \leq 0$; with strict inequalities if $\alpha_{jk} < 1$, using local non satiation. Then [2. ⇒ 1.] of Proposition 1 gives the result. □

Lemma 1 sheds further light on the standard rationalizability test, which is performed on the basis of the full experiment $(\mathbf{x}, \mathbf{B})$, but only uses the matrix $A^{x,B}$, equal here to $\boldsymbol{\alpha}$. The economist designs the test with specific budget sets $(B_j)_{j\in N}$ in mind but ends up checking the cyclical consistency (or rationalization) of the matrix $\boldsymbol{\alpha}$, which is equivalent to the rationalization of a whole class of budget sets. By proceeding in this way, we get a different utility function for every family of compatible budget sets. One can therefore question the predictiveness of such a utility function, defined up to a family of budget sets. This motivates the next section, together with the issue of the existence of a budget representation.

4 Rationalization

4.1 Existence of a budget representation and rationalization

Let us start with an essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$. Proposition 1 or Lemma 1 tells us which conclusion we can draw from the cyclical consistency of the matrix $\boldsymbol{\alpha}$ but takes for granted that there exists a family of budget sets $(B_j)_{j\in N}$ compatible with $(\mathbf{x}, \boldsymbol{\alpha})$. As already observed, an essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ cannot necessarily be generated by budget sets, which are monotonic, even if $\boldsymbol{\alpha}$ is cyclically consistent. For instance, $\boldsymbol{\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is cyclically consistent, but if the associated bundles are $x_1 = (1, 1)$ and $x_2 = (2, 2)$, there does not exist any (monotonic)

[4] The experiment $(\mathbf{x}, \mathbf{B})$ satisfies GARP if, for every $j, k \in N$, $x_k H x_j$ implies $x_k \notin \mathrm{int} B_j$, where H is the transitive closure of the direct revealed preference relation R: $x_k R x_j$ if $x_j \in B_k$. For easy constructive proofs of the equivalence between GARP and the existence of a rationalization, see, e.g., Varian (1982) in the linear case and Forges and Minelli (2009) in the general case.

[5] The matrix with entries $(g_j^B(x_k))_{j,k\in N}$ is cyclically consistent iff the matrix $A^{x,B}$ is cyclically consistent.

budget sets that are compatible with $(\mathbf{x}, \boldsymbol{\alpha})$. This is not surprising as cyclical consistency characterizes the affordability of items without taking into account that they are consumption bundles. This motivates the following tractable condition, which expresses that the consumer understands that, at every date, free disposal is allowed.

Definition 8 An essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ admits *a contradictory statement* if there exist $j, k, k' \in N$ such that either $[\alpha_{jk} < \alpha_{jk'}$ and $x_k \geq x_{k'}]$ or $[\alpha_{jk} = \alpha_{jk'} = 0$ and $x_k \gg x_{k'}]$.

The previous property is completely independent of cyclical consistency (recall the example above). The next proposition states that, in the same way as cyclical consistency is necessary and sufficient for rationalization (Proposition 1), the absence of contradictory statement is necessary and sufficient for budget representation. The proof is given in the appendix.

Proposition 2 *The two following conditions are equivalent:*

1. *The essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ admits no contradictory statement.*
2. *The essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ admits a budget representation.*

We can now state our first main result: together, cyclical consistency and no contradictory statement are necessary and sufficient for an essential experiment to be consistent with the rational choices of a consumer facing budget sets.[6]

Proposition 3 *Let $(\mathbf{x}, \boldsymbol{\alpha})$ be an essential experiment. The following conditions are equivalent:*

1. *The essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ admits no contradictory statement and $\boldsymbol{\alpha}$ is cyclically consistent.*
2. *There exist a family of budget sets $(B_j)_{j \in N}$ compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ and a locally non satiated, continuous utility function v^B rationalizing the experiment $(\mathbf{x}, \mathbf{B})$.*

The proof is given in the appendix. Recalling Proposition 1, Proposition 3 enables us to disentangle the conditions which ensure that an essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ can be rationalized. The cyclical consistency of $\boldsymbol{\alpha}$ is crucial to assess the rationality of choices over items, independently of the fact that these items might be consumption bundles. The structure of consumption bundles matters to define the absence of contradictory statement, and this property, together with cyclical consistency, characterizes a stronger form of rationalization (namely, 2. in Proposition 3 instead of 2. in Proposition 1).

What does Proposition 3 teach us in the "dual" framework of Ekeland and Galichon (2012) (recall Remark 1)? The analogue of consumption bundle x_j could be a vector of "attributes"of house j, like its size, or other criteria that can be measured in real units. Proposition 3 would then say that if the n traders have monotonic preferences over the attribute vectors of the initial n houses, a continuous *equilibrium* price function can be constructed over the space of all attribute

[6] There is no hope to obtain testable restrictions in the consumer problem if one considers poorer information than the one contained in essential experiments.

vectors. Such a price function seems of little practical use.[7] However, the construction of such a utility function over the whole consumption space is the heart of Afriat's result (as recalled in Forges and Minelli, 2009, p. 139). But Proposition 3 is still not fully satisfactory in that respect, because, in condition 2., a different utility function v^B is associated with every family $(B_j)_{j\in N}$. This motivates our next section.

4.2 Robust rationalization

Taking again the essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ as basic data, the following definition of robust rationalization naturally emerges from the discussion at the end of Section 3: the utility function v *robustly rationalizes* the experiment $(\mathbf{x}, \boldsymbol{\alpha})$ if v rationalizes the experiment $(\mathbf{x}, \mathbf{B})$ for *every* family $(B_j)_{j\in N}$ compatible with $(\mathbf{x}, \boldsymbol{\alpha})$. The existence of a robust rationalization amounts therefore to the existence of a largest family of budget sets compatible with the essential experiment. Unfortunately, even if $(\mathbf{x}, \boldsymbol{\alpha})$ is well behaved (in particular, $\boldsymbol{\alpha}$ is cyclically consistent), such a family may not exist as the next simple example illustrates.

Example 1 Let $\big[(x_1, (\alpha_{11}, \alpha_{12})), (x_2, (\alpha_{21}, \alpha_{22}))\big]$ be an essential experiment such that $\alpha_{12} = 1$, $\alpha_{21} = -1$ and $x_1 \notin x_2 + \mathbb{R}^\ell_+$. First, it is an easy matter to verify that the experiment admits a budget representation (actually, $x_1 \notin x_2 + \mathbb{R}^\ell_+$ guarantees that there is no contradictory statement). For instance define a compatible family as follows: $B_1 = \big[\{x_1\} - \mathbb{R}^\ell_+\big]_+$ and $B_2 = \big[\{x_2\} - \mathbb{R}^\ell_+\big]_+ \cup \big[\{x_1 + \nu\mathbf{1}\} - \mathbb{R}^\ell_+\big]_+$ for some $\nu > 0$ sufficiently small.[8]

Suppose now that $x_2 \notin x_1 + \mathbb{R}^\ell_+$, we can add a piece to the budget set B_1 without modifying the resulting matrix $A^{x,B}$. More precisely, there exists $\eta > 0$ such that, for all $\epsilon \in (0, \eta)$, $\frac{1}{1+\epsilon}x_2 \notin B_1$. Thus the family $(B_1^\epsilon, B_2^\epsilon)$, where $B_1^\epsilon = B_1 \cup \big[\{(\frac{1}{1+\epsilon}x_2\} - \mathbb{R}^\ell_+\big]_+$ and $B_2^\epsilon = B_2$ is compatible with the essential experiment. Suppose that there exists a well-behaved v rationalizing robustly the essential experiment, then v rationalizes the experiments $\big((x_1, B_1^\epsilon), (x_2, B_2^\epsilon)\big)$ for all $\epsilon \in (0, \eta)$. It follows that $v(x_1) \geq v(\frac{1}{1+\epsilon}x_2)$ since $\frac{1}{1+\epsilon}x_2 \in B_1^\epsilon$ and $v(x_2) \geq v(x_1)$ since $x_1 \in B_2^\epsilon$. From local non satiation, $v(x_2) > v(x_1)$ since $x_1 \in \mathrm{int}B_2^\epsilon$ but this contradicts the continuity of v as ϵ tends to 0.

The construction of the budget sets is given in Figure 1.

To obtain a contradiction in the above construction we assumed that $x_2 \notin \{x_1\} + \mathbb{R}^\ell_+$, which is by no means pathological. The previous essential experiment can be rationalized for any compatible family of budget sets, but we cannot hope for a robust rationalization. The previous example shows that, by enlarging gradually a family of budget sets which is compatible with a given essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$, we get at the limit budget sets which are well-behaved but are not compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ anymore. We will nevertheless achieve an almost robust rationalization, a concept that we define precisely below.

[7] This is not surprising as Ekeland and Galichon (2010)'s interpretation does not require the full power of Afriat's theorem, but only Proposition 1.

[8] Note that the essential experiment satisfies cyclical consistency and therefore $(B_1^\epsilon, B_2^\epsilon)$ satisfies GARP.

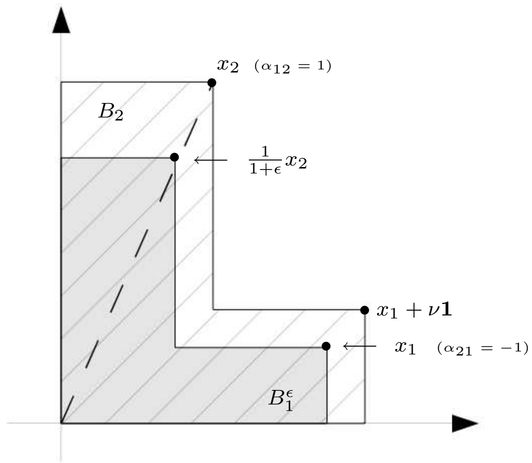

Fig. 1 Non-existence of a robust rationalization as ϵ tend to 0.

Definition 9 Let $(\mathbf{x}, \boldsymbol{\alpha})$ be an essential experiment. Let $\epsilon > 0$, the pair $((B_j^\epsilon)_{j\in N}, v^\epsilon)$ where $(B_j^\epsilon)_{j\in N}$ is a family of budget sets and v^ϵ is a utility function, is said to *ϵ-robustly rationalize* $(\mathbf{x}, \boldsymbol{\alpha})$ if:

(i). The family $(B_j^\epsilon)_{j\in N}$ is compatible with $(\mathbf{x}, \boldsymbol{\alpha})$,
(ii). The function v^ϵ rationalizes the experiment $(\mathbf{x}, \mathbf{B}^\epsilon)$,
(iii). For every family $(B_j)_{j\in N}$ compatible with $(\mathbf{x}, \boldsymbol{\alpha})$, $B_j \subseteq (1+\epsilon)B_j^\epsilon$ for every $j \in N$.

The justification for the terminology is that (ii) implies that v^ϵ rationalizes experiment $(\mathbf{x}, \mathbf{B})$, for every compatible family $(B_j)_{j\in N}$ included in $(B_j^\epsilon)_{j\in N}$ and, by (iii), every compatible family is almost included in $(B_j^\epsilon)_{j\in N}$. To show the former statement, note that x_j is such that $v^\epsilon(x_j) \geq v^\epsilon(x)$ for all $x \in B_j^\epsilon$ then a fortiori $v^\epsilon(x_j) \geq v^\epsilon(x)$ for all $x \in B_j$; and since $x_j \in B_j$, it follows that v^ϵ rationalizes the experiment $(\mathbf{x}, \mathbf{B})$.

We are now ready for our second main result which states that the conditions in Proposition 3 are also necessary and sufficient for the existence of an almost robust rationalization.

Proposition 4 *Let $(\mathbf{x}, \boldsymbol{\alpha})$ be an essential experiment. The following conditions are equivalent:*

1. *The essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ admits no contradictory statement and $\boldsymbol{\alpha}$ is cyclically consistent.*
2. *There exists $\eta > 0$ such that, for all $\epsilon \in (0, \eta)$, there exists a locally non satiated, continuous utility function v^ϵ rationalizing ϵ-robustly the experiment $(\mathbf{x}, \boldsymbol{\alpha})$.*

The proof is given in the appendix. It shows in particular that the utility function v^ϵ is easy to construct. As already suggested above, the construction of an almost robust rationalization (as in condition 2. of Proposition 4) is justified if we want to recover Afriat's conclusion (namely, a well-behaved explicit rationalization) in a situation where we suspect that the consumer only has partial knowledge of the experiment $(\mathbf{x}, \mathbf{B})$.

5 Appendix

The proofs of Propositions 2, 3 and 4 are deduced from the next Theorem and its proof:

- The statements of Propositions 3 and 4 coincide with the equivalence given below.
- The proof of Proposition 2 is jointly obtained in the proof below (in [1. $\Rightarrow$ 2.] and in [2. $\Rightarrow$ 3.]).

Theorem 1 *Let $(\mathbf{x}, \boldsymbol{\alpha})$ be an essential experiment. The following conditions are equivalent:*

1. *There exist $(B_j)_{j\in N}$ compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ and a locally non satiated, continuous utility function v^B rationalizing the experiment $(\mathbf{x}, \mathbf{B})$.*
2. *The essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ admits no contradictory statement and $\boldsymbol{\alpha}$ is cyclically consistent.*
3. *There exists $\eta > 0$ such that, for all $\epsilon \in (0, \eta)$, there exists a locally non satiated, continuous utility function v^ϵ rationalizing ϵ-robustly the experiment $(\mathbf{x}, \boldsymbol{\alpha})$.*

Proof [1. $\Rightarrow$ 2.] [9] To show the cyclical consistency of $\boldsymbol{\alpha}$ we can proceed as in the proof of Lemma 1 (2. $\Rightarrow$ 1.). To show the property of no contradictory statement, suppose, first, on the contrary that there exist $j, k, k' \in N$ such that $[\alpha_{jk} < \alpha_{jk'}$ and $x_k \geq x_{k'}]$. Since $(B_j)_{j\in N}$ is compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ we have either $[x_k \in \mathrm{int}B_j$ and $x_{k'} \notin \mathrm{int}B_j]$ or $[x_k \in \mathrm{Fr}B_j$ and $x_{k'} \notin B_j]$ together with $x_k \geq x_{k'}$, but this contradicts the monotonicity property required in the definition of a budget set (see Definition 4 and also subsection 1.1). Second, suppose on the contrary that there exist $j, k, k' \in N$ such that $[\alpha_{jk} = \alpha_{jk'} = 0$ and $x_k \gg x_{k'}]$. Since $(B_j)_{j\in N}$ is compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ we obtain that $x_k, x_{k'} \in \mathrm{Fr}B_j$ but this contradicts $x_k \gg x_{k'}$.

[2. $\Rightarrow$ 3.] Let $m > 0$ be such that $x_j \leq m\mathbf{1}$ for every $j \in N$ and define the following family $(B_j^\epsilon)_{j\in N}$ (see also Figure 2):

$$B_j^\epsilon = \left[\widetilde{B}_j^\epsilon \cap \left(\{m\mathbf{1}\} - \mathbb{R}_+^\ell\right)\right]_+$$

where $\widetilde{B}_j^\epsilon = \left(\mathrm{int}\Big((\cup_{i\in N,\alpha_{ji}=0}(\{x_i\} + \mathbb{R}_+^\ell)) \bigcup (\cup_{i\in N,\alpha_{ji}=1}(\{\frac{1}{1+\epsilon}x_i\} + \mathbb{R}_+^\ell))\Big)\right)^c$

Let us check first the condition (iii) of Definition 9.

Claim For every compatible family $(B_j)_{j\in N}$ with $(\mathbf{x}, \boldsymbol{\alpha})$, it holds that $B_j \subseteq (1+\epsilon)B_j^\epsilon$ for any $j \in N$ and any $\epsilon > 0$.

[9] A by-product is the proof of [2. $\Rightarrow$ 1.] of Proposition 2.

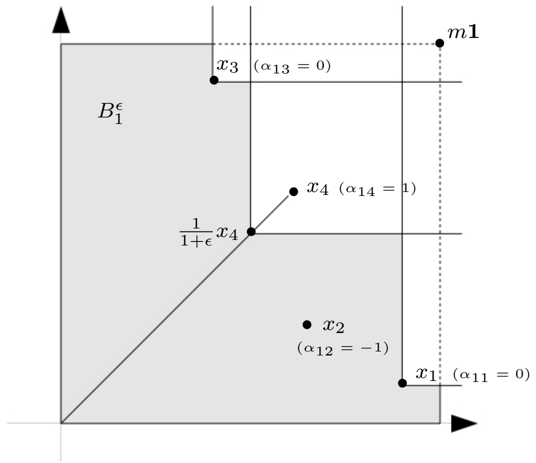

Fig. 2 Construction of the family $(B_j^\epsilon)_{j\in N}$ (here B_1^ϵ)

Proof Let $x \in B_j$ and assume that $\frac{1}{1+\epsilon}x \notin B_j^\epsilon$. Then it must be true that, for some $k \in N$, either $\alpha_{jk} = 0$ together with $\frac{1}{1+\epsilon}x \gg x_k$ or $\alpha_{jk} = 1$ together with $\frac{1}{1+\epsilon}x \gg \frac{1}{1+\epsilon}x_k$. In both cases, the fact that $(B_j)_{j\in N}$ is compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ and the monotonicity property imply that $x \notin B_j$, which is a contradiction. □

The next two claims establish that, for any $\epsilon > 0$ sufficiently small, the family of budget sets $(B_j^\epsilon)_{j\in N}$ is compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ (i.e. condition (i) of Definition 9).[10]

Claim $(\mathbf{x}, \mathbf{B}^\epsilon)$ is an experiment for any $\epsilon > 0$ sufficiently small.

Proof By construction, each B_j^ϵ is a budget set since it is the intersection of two monotonic subsets of $\mathbb{R}_+^\ell$, one of those being compact. Suppose next that there exists $j \in N$ such that $x_j \notin \mathrm{Fr}B_j^\epsilon$ for all $\epsilon > 0$. Thus by construction there exists necessarily k such that either $\alpha_{jk} = 0$ and $x_k \ll x_j$ or $\alpha_{jk} = 1$ and $\frac{1}{1+\epsilon}x_k \ll x_j$, for all $\epsilon > 0$. This contradicts the fact that $(\mathbf{x}, \boldsymbol{\alpha})$ admits no contradictory statement, by using that ϵ tends to 0 if necessary in the latter case. □

Claim $A^{x,B^\epsilon} = \boldsymbol{\alpha}$ for any $\epsilon > 0$ sufficiently small.

Proof Let $j, k \in N$ be such that $\alpha_{jk} = -1$. Suppose that there exists k' such that $x_k \geq x_{k'}$ with $\alpha_{jk'} = 0$. Then it is a contradictory statement, which cannot be the case. Suppose then that there exists k' such that $x_k \geq \frac{1}{1+\epsilon}x_{k'}$ with $\alpha_{jk'} = 1$ for all $\epsilon > 0$. As ϵ tends to 0, this contradicts again the fact that $(\mathbf{x}, \boldsymbol{\alpha})$ admits no contradictory statement. Therefore for any $\epsilon > 0$ sufficiently small

[10] A by-product is the proof of [1. ⇒ 2.] of Proposition 2.

$x_k \notin (\cup_{i\in N,\alpha_{ji}=0}(\{x_i\}+\mathbb{R}^\ell_+))\bigcup(\cup_{i\in N,\alpha_{ji}=1}(\{\frac{1}{1+\epsilon}x_i\}+\mathbb{R}^\ell_+))$. Thus $x_k \in \mathrm{int}B^\epsilon_j$, that is $a^{x,B^\epsilon}_{jk} = -1$.

Let $j,k \in N$ be such that $\alpha_{jk} = 0$. Suppose that there exists k' such that $x_k \gg x_{k'}$ with $\alpha_{jk'} = 0$. Then it is then a contradictory statement, which cannot be the case. Suppose then that there exists k' such that $x_k \geq \frac{1}{1+\epsilon}x_{k'}$ with $\alpha_{jk'} = 1$ for all $\epsilon > 0$. As ϵ tends to 0, this contradicts again the fact that $(\mathbf{x},\boldsymbol{\alpha})$ admits no contradictory statement. Therefore for any $\epsilon > 0$ sufficiently small $x_k \notin \mathrm{int}\Big((\cup_{i\in N,\alpha_{ji}=0}(\{x_i\}+\mathbb{R}^\ell_+))\bigcup(\cup_{i\in N,\alpha_{ji}=1}(\{\frac{1}{1+\epsilon}x_i\}+\mathbb{R}^\ell_+))\Big)$ but since $x_k \in \cup_{i\in N,\alpha_{ji}=0}(\{x_i\}+\mathbb{R}^\ell_+)$, it follows that $x_k \in \mathrm{Fr}B^\epsilon_j$, that is $a^{x,B^\epsilon}_{jk} = 0$.

Let $j,k \in N$ be such that $\alpha_{jk} = 1$. Then clearly, for all $\epsilon > 0$, $x_k \in \mathrm{int}\Big((\cup_{i,\alpha_{ji}=0}(\{x_i\}+\mathbb{R}^\ell_+))\bigcup(\cup_{i\in N,\alpha_{ji}=1}(\{\frac{1}{1+\epsilon}x_i\}+\mathbb{R}^\ell_+))\Big)$ since $x_k \gg \frac{1}{1+\epsilon}x_k$. That is to say $x_k \notin B^\epsilon_j$, i.e. $a^{x,B^\epsilon}_{jk} = 1$. □

It remains to prove that one can construct a well behaved utility function v^ϵ with the desired properties (i.e. condition (*ii*) of Definition 9). Using condition 2. and the fact that $(B^\epsilon_j)_{j\in N}$ is compatible with $(\mathbf{x},\boldsymbol{\alpha})$ (for any $\epsilon > 0$ sufficiently small), Lemma 1 establishes the existence of a locally non satiated, continuous utility function v^ϵ rationalizing $(\mathbf{x},\mathbf{B}^\epsilon)$.

[3. ⇒ 1.] Consider the pair $(v^{\frac{\eta}{2}}, B^{\frac{\eta}{2}})$ which rationalizes the experiment $(\mathbf{x},\boldsymbol{\alpha})$ $\frac{\eta}{2}$-robustly, as given by condition 3. Then a fortiori the well-behaved function $v^{\frac{\eta}{2}}$ rationalizes the experiment $(\mathbf{x},\mathbf{B}^{\frac{\eta}{2}})$ as required by condition 1. □

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