Some mathematical aspects of Newton's Principia

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Abstract
In this paper, I investigate some of the preliminary lemmas of *Principia* and deal with two important aspects of Newton's mathematics: the method of first and last ratios and the role of figures in the mathematical reasoning. In particular, I tackle the question of the relationship between the method of first and ultimate ratios and the modern theory of limits; then, I show that in Newton's mathematics, the figure continued to play one of the fundamental functions of the figure in Greek geometry: a part of the reasoning was unloaded on to it.

Keywords: Newton, limit, last and ultimate ratios, figures.

1. The method of first and ultimate ratios

In his *Correcting the Principia*, Rupert Hall wrote:

> The *Principia* was to remain a classic fossilized, on the wrong side of the frontier between past and future in the application of mathematics to physics [1, p. 301]

In effect, if Newton's physics was immediately winning, his mathematics was losing and his mathematical methodology was abandoned within some decades, after furious polemics with the continental mathematicians led by Leibniz and the Bernoullis[1].

In the present paper I will illustrate two important aspects of Newton's mathematics: in section 1, I will discuss the method of the first and ultimate ratios, whereas in section 2, I will deal with the use of geometric figures in the *Principia* and their role in mathematical demonstrations.

The most known exposition of the method of first and last ratios is found in Book I of the *Principia Mathematica Philosophia Naturalis*, which opens with Section I containing the famous preliminary mathematical lemmas and subtitled *De methodo rationum primarum et ultimarum, cujus ope sequentia demonstrantur.*

By the expression *ultima ratio* Newton attempted to give a meaning to the ratio $\frac{0}{0}$ which two variable quantities assume when become equal to zero, namely when they go to zero. In the final scholium of Section I, Book I, Newton justified the notion of "ultimate ratio" in this way*:*

(Q) Ultimæ rationes illæ quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum, sed limites ad quos quantitatum sine limite decrescentium rationes semper appropinquant, & quas propius assequi possunt quam pro data quavis differentia, nunquam vero transgredi, neque prius attingere quam quantitates diminuuntur in infinitum. Res clarius intelligetur in infinite magnis. Si quantitates duæ quarum data est differentia augeantur in infinitum, dabitur harum ultima ratio, nimirum ratio æqualitatis, nec tamen ideo dabuntur quantitates ultimæ seu maximæ quarum ista est ratio. Igitur in sequentibus, siquando facili rerum imaginationi consulens, dixero quantitates quam minimas, vel evanescentes vel ultimas, cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite[2].

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[1] On this question, I refer to *Reading the Principia* by Guicciardini [10].

[2] "Those ultimate ratios with which quantities vanish, are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities, decreasing without limit, do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished in infinitum. This matter will be understood more clearly in the case of quantities indefinitely great. If two quantities whose difference is given are increased indefinitely, their ultimate ratio will be given, namely the ratio of equality,

One might be tempted to consider this sentence as a definition of limit: in this case the expression *ultima ratio* would denote something very similar to the modern concept of limit. This is, for instance, the point of view of Bruce Pourciau. In his *Newton and the notion of limit*, he, first, stated:

> We find that Newton … was the first to present an epsilon argument, and that, in general, Newton's understanding of limits was clearer than commonly thought [21, p. 18].

and, then, he commented upon the above-mentioned Newton's words as follows:

> A surprise: this is not the confused Newton we were led to expect. It may be more an epsilon than an epsilon-delta definition, but the core intuition is clear and correct [21, p.19].

I view the matter differently. Indeed, I think that Newton's concept of first and ultimate ratio can be reduced to the modern concept of limits: it is true that Newton has a clear idea of what meaning "approaching a limit", but this is only an intuitive and non-mathematical idea that is entirely different from the modern, mathematical concept of limit[3].

To justify my opinion, I consider the first of the preliminary mathematical lemmas of Book 1 of *Principia*:

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and yet the ultimate or maximal quantities of which this is the ratio will not on this account be given" [14, p. 87].

[3] In my *Differentials and differential coefficients in the Eulerian foundations of the calculus*, investing Euler's concept of infinitesimals, I defined this intuitive and non-mathematical idea of quantities approaching a limit or approaching each other as "protolimit": 'However, apart from these crucial differences, there is something in common between the Eulerian procedure and the modern one based upon the notion of limit: evanescent quantities and endlessly increasing quantities were based upon an intuitive and primordial idea of two quantities approaching each other. I refer to this idea as "protolimit" to avoid any possibility of a modern interpretation' (see [5, p. 46]).

Lemma 1. Quantitates, ut & quantitatum rationes, quæ ad æqualitatem dato tempore constanter tendunt & eo pacto propius ad invicem accedere possunt quam pro data quavis differentia; fiunt ultimo æquales[4].

This lemma contains the following explicit hypotheses:

(**H1**) two quantities, say $A(t)$ and $B(t)$, approach closer and closer to one other, when t varies over a finite interval I, whose endpoints are a and c, and approach c.

(**H2**) $A(t)$ and $B(t)$ approach so close to one other that their difference is less than any given quantity, namely it is

$$|A(t)-B(t)|<\varepsilon,$$

when $t < c$ but near enough to c.

Hypothesis (**H1**) implies that $A(t)$ and $B(t)$ approach each other, but this does not necessarily mean that the distance between $A(t)$ and $B(t)$ becomes smaller than any quantity. For instance, the quantities

$$A(t)=-t^2 \text{ and } B(t)=t^2+1 \text{ (when } t \text{ goes to 0)}$$

satisfy hypothesis (**H1**).

Hypothesis (**H2**) guarantees that the distance actually becomes smaller than any given quantity. The thesis is

(**T**) $A(t)$ and $B(t)$ are ultimately equal.

The thesis states that the two quantities effectively reach each other when $t = c$. If we use the term "limit" in the same way as Newton does in the scholium, the thesis states that the quantities $A(t)$ and $B(t)$ have the same limit or that the limit of their difference is zero.

The proof runs as follows:

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[4] "Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal" [14, p. 73].

Si negas, sit earum ultima differentia D. Ergo nequeunt propius ad æqualitatem accedere quam pro data differentia D: contra hypothesin[5].

Newton assumes that D > 0 is the ultimate difference, namely

$$|A(c)-B(c)| = \mathrm{D};$$

then

$$|A(t) - B(t)|$$

does not become less than D, contrary to hypothesis (**H2**).

It is clear that if the proof is to be taken seriously, (**H2**) and (**T**) are not the same thing, and this implies that Newton does not think of (**H2**) as the definition of limit or ultimately equal: (**H2**) is an essential property of limit but not the definition.

In effect, Newton does not define the terms "limit" and "ultimate ratio": these terms have a clear intuitive meaning to him.

In the final scholium of Section 1, Book 1, of *Principia*, Newton illustrates this intuitive meaning by referring to the "limit" as the last place or the last velocity of a motion:

Objectio est, quod quantitatum evanescentium nulla sit ultima proportio; quippe quæ, antequam evanuerunt, non est ultima, ubi evanuerunt, nulla est. Sed & eodem argumento æque contendi posset nullam esse corporis ad certum locum pergentis velocitatem ultimam. Hanc enim, antequam corpus attingit locum, non esse ultimam, ubi attigit, nullam esse. Et responsio facilis est. Per velocitatem ultimam intelligi eam, qua corpus movetur neque antequam attingit locum ultimum & motus cessat, neque postea, sed tunc cum attingit, id est illam ipsam velocitatem quacum corpus attingit locum ultimum & quacum motus cessat.

Et similiter per ultimam rationem quantitatum evanescentium intelligendam esse rationem quantitatum non antequam evanescunt, non postea, sed quacum evanescunt. Pariter & ratio prima nascentium est ratio quacum nascuntur. Et summa prima & ultima est quacum esse (vel augeri & minui) incipiunt & cessant. Extat limes quem velocitas in fine motus attingere potest, non autem transgredi. Hæc est velocitas ultima. Et par est ratio limitis quantitatum & proportionum omnium inci-

[5] "If you deny this, let them become ultimately unequal, and let their ultimate difference be D. Then they cannot approach so close to equality that their difference is less than the given difference D, contrary to the hypothesis." [14, p. 73].

pientium & cessantium. Cumque hic limes sit certus & definitus, Problema est vere Geometricum eundem determinare. Geometrica vero omnia in aliis Geometricis determinandis ac demonstrandis legitime usurpantur [6].

This quotation clearly shows that for Newton, the notion of limit or ultimate value is *an idea borrowed from Nature*: it is not a mathematical notion determined by its definition and a translation into modern terminology would strain Newton's concept.

It is very interesting to compare lemma 1 (where Newton does not use the word "limit") and quotation (Q), which is considered by Pourciau to be "Newton's best definition of limit" [21, p.19]. In quota-

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[6] "Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument it may be alleged, that a body arriving at a certain place, and there stopping, has no ultimate velocity; because the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases. And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities, not before they vanish, nor afterwards, but with which they vanish.

In like manner the first ratio of nascent quantities is that with which they begin to be. And the first or last sum is that with which they begin and cease to be (or to be augmented or diminished). There is a *limit* which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may be allowed to use in determining and demonstrating any other thing that is likewise geometrical" [14, p. 87].

Newton goes on stating: "Contendi etiam potest, quod si dentur ultimæ quantitatum evanescentium rationes, dabuntur et ultimæ magnatudines; et sic quantitas omnis constabit ex indivisibilibus, contra quam *Euclides* de incommensurabilibus, in libro decimo *Elementorum*, demonstravit. Verum hæc Objectio falsæ innititur hypothesi." (It can also be contended, that if the ultimate ratios of vanishing quantities [that is, the limits of such ratios] are given, their ultimate magnitudes will also be given; and thus every quantity will consist of indivisibles, contrary to what Euclid has proved in Book X of *Elements*. But this objection is based on a false hypothesis) [14, p. 87].

tion (Q), Newton states that the last ratios of two quantities, say *f(t)* and *g(t)*, are the limits

(A) towards which the ratios of quantities decreasing without limit, do always converge,
(B) to which they approach nearer than by any given difference,
(C) but never go beyond, nor in effect attain to, till the quantities are diminished in infinitum.

In effect, in quotation (Q) Newton repeats the conditions of lemma 1:

if (A) and (B) are verified, then the ratio *f(t)/g(t)* is ultimately equal to the limit.

Since, Newton uses the word 'limit', we can also state that the thesis (**T**) means that

(**T1**) the quantities *A(t)* and *B(t)* have the same limit [or that the limit of their differences is zero].

However, (**H2**) and (**T**) do not mean the same thing. Newton do not feel (**H2**) as the definition of "limit" or "ultimately equal". Rather, he states that if (**H1**) and (**H2**) are verified, then *A(t)* and *B(t)* become equal when the process finishes (in the sense that they assume the same value).

In conclusion, differently from what Pourciau stated, Newton does not define the word "limit" by referring to quantities that approach a certain value becoming less than any fixed quantity ε: Newton uses the term "limit" without defining it since he thinks this term had a clear, intuitive meaning. The limit is identified with the ultimate value and is conceived as something physical: the final position of a body in motion, the final speed; at most, the limit is a geometric idealization: the final position of a point that describes a curve.

In any case Newton does not distinguish between the limit process

$$\lim_{x \to c} A(t)$$

and the ultimate value of this process

$$\left| A(t) \right|_{x=c} .$$

In the modern conception, the limit notion is the result of a conventional definition and it is precisely this definition that creates the mathematical object "limit". For modern mathematicians, the notion of limit is entirely reduced to its definition. Today, by the symbols

$$l = \lim_{z \to c} f(z)$$

we mean:

(D) given any $\varepsilon>0$ there exists a $\delta>0$ such that if z belongs to the domain S and $|z|<\delta$, then

$$|f(z)- l |<\varepsilon;$$

This definition formalizes the notion of a function $f(x)$ approaching to a number l; however, there is a gap between the intuitive idea of "reaching the limit" and definition (D); a gap that only intuition is able to fill.

In the modern theory of limits no theorem is demonstrated by referring to the intuitive idea of approaching a number: only the formal definition is used. For Newton, the notion of reaching the limit is a natural notion (namely, a notion derived from the observation of natural phenomena), not a mathematical notion created by its definition. According to Newton, if anything has property (**H2**), then it reaches its limit; however, Newton's idea of limit cannot be reduced to such a property and entirely maintains its intuitiveness and also its ambiguity and so the method of the first and ultimate ratios cannot be reduced to the method of limits.

2. The role of diagrams

In this section I will examine another crucial aspect of Newton's mathematics closely[7]: the use of physical and geometric evidences in mathematical reasoning, an use that was mediated by geometric figures.

In the previous section, we already saw that, in *Principia*, there is no definition of the notion of first and ultimate ratios and that Newton refers to a physical and geometric evidence to explain such a notion. In effect, in the *Principia*, there are several other mathematical notions that Newton uses without definition, only basing on their geometric evidence. For instance, let us consider the second of the preliminary mathematical lemmas of the *Principia*:

> Lemma II. Si in figura quavis *Aa cE* rectis *Aa*, *AE*, et curva *AcE* comprehensa, inscribentur parallelogramma quotcunque *Ab, Bc, Cd*, &c. sub basibus *AB, BC, CD*, etc. æqualibus, & lateribus *Bb, Cc, Dd*, &c. figuræ lateri Aa parallelis comenta; et compleantur parallelogramma *aKbl*, *bLcm*, *cMdn*, etc, Dein horum parallelogrammorum latitudo minuatur, et numerus augeatur in infinitum: dico quod ultimæ rationes, quas habent ad se invicem figura inscripta *AKbLcMdD*, circumscripta *AalbmcndoE*, et curvilinea *AabcdE*, sunt rationes æqualitatis[8].

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[7] This is an aspect that Newton shared with all mathematicians of his time (on this matter, see [4]).

[8] LEMMA II. If in any figure *AacE* comprehended by the straight lines *Aa* and *AE* and the curve *acE* any number of parallelograms *Ab, Bc, Cd* ... are inscribed upon equal bases *AB*, *BC*, *CD* ... and sides, *Bb*, *Cc*, *Dd* ... parallel to the side *Aa* of the figure; and if the parallelograms *aKbl*, *bLcm*, *cMdn* ... are completed; if then the width of these parallelograms is diminished and their number increased indefinitely, I say that the ultimate ratios which the inscribed figure *AKbLcMdD* the circumscribed figure *AalbmcndoE* , and the curvilinear figure *AabcdE*, have to one another are ratios of equality [19, p. 433; 18, p. 29].

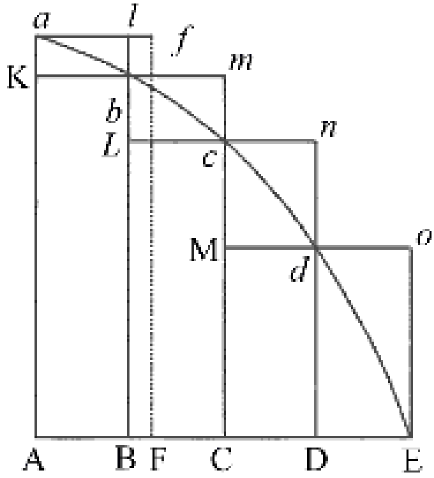

Figure 1

Newton gives the following proof:

> Nam figuræ inscriptæ et circumscriptæ differentia est summa parallelogrammorum *Kl+Lm+Mn+Do*, hoc est (ob æquales omnium bases) rectangulum sub unius basi *Kb* & altitudinum summa *Aa*, id est rectangulum *ABla*. Sed hoc rectangulum, eo quod latitudo ejus *AB* in infinitum minuitur, sit minus quovis dato. Ergo, per Lemma I, figura inscripta et circumscripta et multo magis figura curvilinea intermedia fiunt ultimo æquales. Q.E.D.[9]

According to Pourciau, "in his geometric style, Newton has stated and proved a basic theorem of calculus … Every monotonic function on a closed and bounded interval must be integrable." [21, p. 24]. In effect, Pourciau recasts Newton's lemma in modern terms and interprets the figure *AacE* as the graph of a function *f* defined on the seg-

[9] For the difference of the inscribed and circumscribed figures is the sum of the parallelograms *Kl*, *Lm*, *Mn* and *Do* that is (because they all have equal bases), the rectangle having as base *Kb* (the base of one of them) and as altitude *Aa* (the sum of the altitudes) that is the rectangle *ABla* . But this rectangle, because its width *AB* is diminished indefinitely, becomes less than any given rectangle. Therefore (by lem. I) the inscribed figure and the circumscribed figure and all the more the intermediate curvilinear figure become ultimately equal. Q.E.D. [19, p. 433; 18, p. 29]

ment *AE*. In his opinion, "Newton clearly (but without saying so) takes f to be monotone decreasing with $f(E)=0$. Of course the areas of the inscribed and circumscribed figures, *AKbLcMdD* and *Aalbmcn-doE*, correspond to lower and upper sums,

$$L_n \equiv f(t_1)\Delta t + \ldots + f(t_n)\Delta t \qquad \text{and} \qquad U_n \equiv f(t_0)\Delta t + \ldots + f(t_{n-1})\Delta t$$

that arise from a partition

$$A = t_0 < t_1 < \ldots < t_{n-1} < t_n = E$$

of the segment *AE* into n subintervals of equal length $\Delta t = AE/n$". [21, p. 23].

I view the matter differently. First, I observe that Newton does not use the abstract concept of functions representing them geometrically by a graph, but uses geometric quantities represented by curves (sometimes, these curves are analytically expressed by letters, but this does not usually occur in the *Principia*). For this reason the expression "graph of f" is misleading.

Second, it is true that Newton conceived the area under the curve as the last ratio of the inscribed and circumscribed figures, it is however clear that he define neither the concept of integral, nor the area. In fact, for him, the area is only an entity that has a geometric and physical evidence. In lemma 2, Newton does not prove the existence of the area; rather he proves that the ultimate ratios which the inscribed figures, circumscribed figures and area under the curve have to one another are ratios of equality, namely the ratio of the area and the ultimate value of the inscribed and circumscribed figures is 1.

In a similar way, in lemma VI where Newton stated that the angle between the secant and tangent is evanescent, he simply reasons on the geometric evidence and does not give the definition of tangent to a curve.

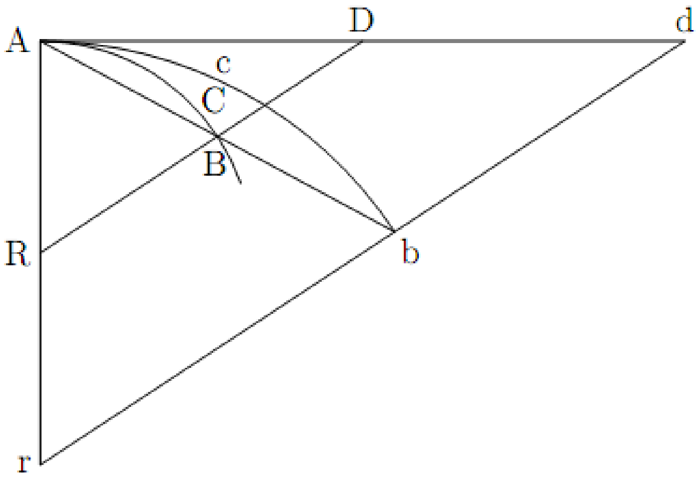

Figure 2

Lemma VI. Si arcus quilibet positione datus *ACB* subtendatur chorda *AB*, et in puncto aliquo *A*, in medio curvaturae continuae,tangatur a recta utrinque producta *AD*; dein puncta *A*, *B* ad invicem accedant et coeant; dico quod angulus *BAD*, sub chorda et tangente contentus, minuetur in infinitum et ultimo evanescet[10].

The proof goes as follows:

Nam si angulus ille non evanescit, continebit arcus ACB cum tangente AD angulum rectilineo aequalem, et propterea curvatura ad punctum A non erit continua, contra hypothesin[11].

Lemma I, II, and VI show a peculiar use of figures that makes Newton's mathematics deeply different from the modern one. This

[10] "If any arc *ACB*, given in position is subtended by its chord *AB*, and in any point *A*, in the middle of the continued curvature, is touched by a right line *AD*, produced both ways; then if the points *A* and *B* approach one another and meet, I say, the angle *BAD*, contained between the chord and the tangent, will be diminished in infinitum, and ultimately will vanish. [19, p. 443]

[11] "For if that angle does not vanish, the arc *ACB* will contain with the tangent *AD* an angle equal to a rectilinear angle; and therefore the curvature at the point A will not be continued, which is against the supposition." [19, p. 443]

way of using figures was shared by most mathematicians of that time and rooted in ancient Greek conception.

Indeed, Greeks did not manipulate algebraic symbols in their mathematical reasonings; rather, they reasoned upon figures. A figure is a symbolic representation as well; however, it has a different nature with respect to algebraic symbolism. It is iconic and imitative and reproduces the features of various real bodies by analogy. When a modern mathematicians uses figures, he considers them as dispensable tools for facilitating the comprehension. Their role is essentially pedagogical or illustrative. Indeed, a modern mathematical theory is a conceptual system, composed of explicit axioms and rules of inference, definitions and theorems derived by means of a merely linguistic deduction. For instance, consider the proposition

> Two equal circles of radius *r* intersect each other if the separation of their centers is less than 2*r*.

In modern geometry one can state this proposition if an appropriate axiom (or a theorem based upon appropriate axioms) guarantees their intersection. Modern verbal formulation of geometry implies that terms such as circle, radius, and center, only have the properties that derive from their definitions and the axioms of the theory.

Instead, Greek geometry used figures as parts of reasoning (and not as a merely pedagogical or illustrative tool). Thus, in order to derive the existence of the intersection between two circles, say C and C', Greek geometers could instead refer to the evidence of Fig. 3 and simply say: "Look!" This is precisely what Euclid did in the proof of his very first proposition, where he constructed an equilateral triangle. There was no necessity to clarify precisely all the relationships between the objects of a theory, to make all axioms explicit and to define all terms. The mere inspection of figures provided information that we would now consider missing.

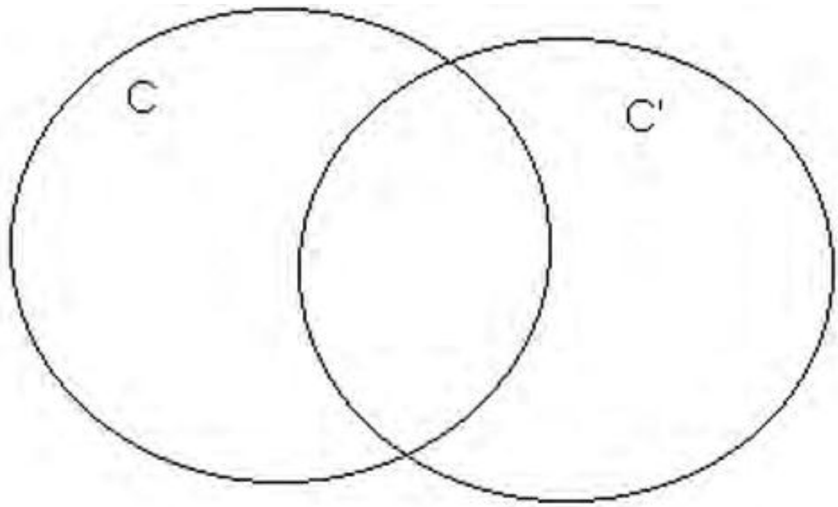

Figure 3

In conclusion, the previous analysis shows the permanence in Newton's mathematics of various traditional aspects, aspects that are to be investigated further to make the real nature of Newton's mathematics clearer.

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