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OPTIMAL POPULATION AND EDUCATION

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Abstract. If raising and educating children is a private cost to households, while the availability of skilled labor supply resulting from the households' fertility and education choices is a public good, then competitive equilibria typically deliver a suboptimal mix of size and skills of the population. In particular, households would underinvest in their children education compared to the optimal level. This is the case even if households are aware of the increase in savings returns implied by a higher supply of skilled labor and manage to coordinate to try to exploit this effect. This paper shows that a tax-financed compulsory education is unlikely to implement the optimal steady state, even if the mandatory level of education is the optimal one (the system of equations is overdetermined). Nevertheless, a pensions scheme that makes payments contingent to the household fertility and investment in its children's education can implement the first-best steady state. The pension scheme is balanced period by period by financing pensions through a payroll tax on the increase in children's labor income resulting from their parents' human capital investment.

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1. Introduction

The most obvious economic decisions that households make routinely are how much to work and whether to save or borrow and how (i.e. in what assets) in order to smooth consumption over time. But households decide also whether to reproduce, to what extent, and how many resources (time and income) to invest in their children. These decisions have huge economic consequences in the aggregate. For instance, everything else remaining constant, changes in the overall fertility rate propagate across the population pyramid producing variations in the dependency ratio that may reduce the output per capita for generations as a result. On the other hand, the impact on output per capita of a fall in the fertility rate may be offset by a higher investment in children's education that increases their productivity, so that a quantity-quality trade-off is faced in the choice of population.

Nevertheless, households make typically reproductive and educational decisions (possibly only beyond some compulsory elementary schooling in the case of the latter) independently of each other and disregarding their impact on the aggregate, given the negligible size of each individual household compared to the entire economy. As a consequence, the resulting fertility and allocation of resources (including those devoted to educate children) will typically be suboptimal. In the case of reproductive and education decisions this will certainly be so if the cost of raising children is a private cost to the household, while its benefit is a public good (through an increase in the amount and skills of future labor supply, which raises both the return to savings and the possible pension transfers to the current generation), since under such conditions households will try to free-ride on other households fertility and education efforts. One may guess that the misalignment of incentives arising from the fact that households cannot appropriate the returns to their fertility and education investments is offset by households' altruism, by which they derive a direct utility from the quantity and quality of their children or from their well-being. As it will be shown below the inefficiency of the decentralized fertility and education choices actually persists even with altruistic parents, the reason being that the source of inefficiency is actually the inability of households to internalize the externality that their fertility and education choices have, in the aggregate, on the return to their own savings.

In this paper, I characterize in an overlapping generations setup devoid of altruism[1]

[1]This is without loss of generality, since the results do not change qualitatively if altruistic households are considered instead.

the optimal steady state fertility and human capital investment (along with the optimal savings and consumption profile), and show that they cannot be a laissez-faire competitive equilibrium outcome. This is a consequence of the inability of households to internalize the aggregate impact of their fertility and education decisions on the return to savings. It is important to note that this result is not just a consequence of the competitive behavior assumption since, as it will be shown below, the suboptimality of decentralized allocations still happens even if the agents were able to coordinate their fertility and educational efforts to try to exploit their impact on the factor prices, and on the return to savings in particular.

The most usual policy used historically to address the problem of households' underinvestment in education, namely a tax-financed compulsory education, is shown not to deliver the optimal steady state generically, since the system of equations characterizing this happens to be overdetermined. I show nonetheless that a pay-as-you-go social security that makes pensions contingent to the household fertility and investment in their children's human capital —and financed by a payroll tax *on the returns of the human capital investment*, and not on the entire labor income— implements the optimal steady state as a competitive equilibrium steady state.

Research addressing the issue of optimal population size goes back to at least Phelps (1967), followed by the characterization in Samuelson (1975) of the optimal (exogenous) growth rate of the population[2] and a subsequent extensive literature. Most of the literature addresses the issue of population size from the viewpoint of the sustainability of pay-as-you-go pension systems, and the need to tie pension payments to individual fertility in order to make social security sustainable and implement the optimal population size has been repeatedly been put forward (see Eckstein and Wolpin (1985) and, more recently, Abio, Mahieu and Patxot (2004), Michel and Wigniolle (2007)).

Although many papers on the optimal population size for the sustainability of PAYG pension schemes have addressed the issue separately from that of parental investments in their children's education, there are nonetheless papers in which the two decisions have been analyzed jointly. Galor and Weil (2002) consider for instance a household quantity-quality choice of children following the model of household fertility behavior in Becker (1960). Nevertheless households are supposed to derive utility from the total income earned by its children, again to offset the fact that children (both their quantity and quality) are supposed to be costly to parents (in

[2]Deardoff (1976) and Michel and Pestieau (1993) qualified the results showing the that a solution to first-order conditions used in Samuelson (1975) could be a minimum instead of a maximum.

terms of time here, and hence of lost labor income).[3]

Schoonbroodt and Tertilt (2010) consider parents deriving utility from both the number of children and their utility, but they address directly the problem of the misalignment of parents' incentives because of their inability appropriate the returns of the cost in making children. The authors explore the consequences of granting to parents property rights over some of theirs children income, but there is no human capital dimension in their analysis, so that the quantity-quality trade-off is overlooked.

In Cremer et al. (2006) individuals' utility depend, as in this paper, only on their own consumption. As a consequence, parents do not invest in increasing the probability of having children, which still they somehow arbitrarily have at the lowest of two exogenously given rates. Moreover, no quantity-quality trade-off is faced by the households and the only technology available is a storage technology allowing to transfer the endowment from young to old at an exogenously given fixed return. Thus any link between reproductive (and educational) choices and savings returns is again missing.

In this paper, I choose to make the households' utility depend only on their consumption and not on the number of children or their education. The reason is that the results are qualitatively the same otherwise, while the presentation gains in clarity.[4] I also disconnect the cost of child rearing from that of reproduction: what is costly is not reproduction per se, but training children to acquire skills beyond those they are naturally endowed with and are normalized to 1.[5] Making such modeling choices I want to make stand out, albeit in an admittedly oversimplified manner, relevant mechanisms to take into account in designing the optimal population and education policies.

The rest of the paper is organized as follows. Section 2 presents the economy. Sec-

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[3]As a matter of fact, the goal of Galor and Weil (2002) is rather to provide a framework with endogenous fertility and technological change able to account for the observed pattern of demographic and technological transition. As a consequence, the paper makes modeling choices leading to an economy where households do not face a savings problem, voiding of meaning any social security concern.

[4]I note nonetheless the differences in footnotes.

[5]This endowment can encompass anything they learn effortlessly from the very process of socialization, which may include different skills depending on the cultural context. On the contrary, parental investments in human capital refers to those skills that require an effort (in terms of resources) to be acquired.

tion 3 characterizes the steady state that a planner maximizing the representative agent's utility would choose (Proposition 1). Section 4 addresses the problem from the viewpoint of a representative agent operating under competitive conditions. Section 5 characterizes the resulting competitive equilibria, in which the fertility rate remains undetermined and no effort is made by parents to endow their children with an extra amount of human capital beyond the costless one (Proposition 2). Since the planner instead does invest resources in educating children, it turns out that the planer's optimal steady state is never a competitive equilibrium steady state (Proposition 3). Section 6 shows that the conditions under which a compulsory education financed by taxes implements the optimal steady state generically not satisfied. Section 7 shows that the planner's steady state can nonetheless be implemented as a competitive equilibrium steady state by a pensions scheme contingent to both individual fertility *and parental education effort* —the scheme is financed by a tax *on the excess income resulting form education* (Proposition 3). At the competitive equilibrium decentralizing the planner's steady state, all the necessary intergenerational transfers are carried by the pension scheme and the demand for any other asset fulfilling that role is zero. Section 8 shows that nonetheless the presence of such an asset is essential, even if in zero demand at equilibrium, since removing it prevents the planner steady state to be implementable this way. Finally, Section 6 concludes.

2. The economy

Consider an economy of 2-period lived overlapping generations of agents (households) that, when young, can supply labor and reproduce at the rate of their choice. Consumption can be produced out of labor and capital (the amount previously produced but not consumed). Returns to scale are constant, both factors are needed for production, and without loss of generality capital is supposed to depreciate completely in one period for the sake of simplicity. Households derive utility only from consumption[6] so that they supply labor inelastically. Reproduction per se is not costly, but taking care of children (i.e. "educating" them) is. On the other hand, educating children increases the effective units of labor they will supply.

In principle, households have an interest in a high supply of effective labor when they will be old, in order to get the most of their capital savings. Indeed, the more

[6]While households can derive also utility from children and their education, the conclusions stay the same (differences are relegated to footnotes for the sake of clarity in the presentation).

households reproduce —costlessly as long as they do not make an extra effort increase their children's human capital—, the more labor (although unskilled) will be available for production next period, increasing output, but at the same time there will be more mouths to feed, putting pressure on resources tomorrow. Alternatively, effective labor can be increased tomorrow reproducing less today but increasing the investment in education instead. The problem is that while reproduction is costless, education is not, so that doing so puts pressure on resources today.

Clearly, as long as the education costs are born by households while its returns cannot be appropriated by them, households will underinvest in it, hoping to free-ride on the others. But if the returns to education exceed its cost this is inefficient. What is then the optimal mix of quantity and quality of labor for the society? Can that optimal combination be the result of decentralized choices of individuals in a competitive setup? If not, is there some some policy intervention that makes of the optimal quantity and quality of population a competitive outcome? These are the questions addressed in the following sections.

3. The optimal steady state

Consider a planner seeking to maximize the steady state utility of the representative household. The planner chooses a steady state profile of consumptions c_0, c_1, per worker capital savings k, a population growth factor n,[7] and an investment in its children education or human capital h, solution to[8]

$$\max_{0 \leq c_0, c_1, k, n, h} u_0(c_0) + u_1(c_1)$$
$$n\big[c_0 + k + e(nh)\big] + c_1 \leq F\big(k, n(1+h)\big)$$

On the right-hand side of the feasibility constraint, F being a neoclassical production function, no output is produced as soon as reproduction stops and therefore

[7]So that population grows between periods by a factor of n , meaning that a household has n households as descendants (the basic economic agent here is the household, and the mating process of individuals is thus overlooked). Therefore n does not correspond to the commonly used Total Fertility Rate defined as the average number of children born to a woman, but to roughly half of it. Specifically, the replacement rate, which in terms of TFR is (slightly over) 2, in this setup corresponds to $n = 1$ (in this simple setup abstraction is made of the slight excess of boys over girls in births and of factors like child mortality).
[8]Alternatively, the objective function is $u_0(c_0) + u_1(c_1) + \phi(nh)$ if the household derives utility from the number and education of its children.

labor supply evaporates. Also the term $e(nh)$ in the constraint is the cost of producing n children able to supply each $1+h$ (efficiency) units of labor (if $h=0$ they are supposed to be able to supply one unit of unskilled labor). This cost is supposed to satisfy $e(0)=0=e'(0)$ and $e'>0, e''>0$ for strictly positive levels of education investment.[9]

Since both labor and capital are assumed to be necessary for positive production, an assumption that $u_i'(0)=+\infty$, for $i=,1,2$, guarantees then that, at the solution, it must hold $k,n>0$. Also, for any given $k,n>0$, the output net of resources invested in education is maximized by an $h \geq 0$ such that

$$F_L\left(\frac{k}{n}, 1+h\right) - e'(nh)n \leq 0$$

and

$$h\left[F_L\left(\frac{k}{n}, 1+h\right) - e'(nh)n\right] = 0$$

so that $e'(0)=0$ implies $h>0$ as well. Therefore, the solution to the planner's problem is necessarily interior and, since the feasibility constraint can in that case be written (dividing both sides by $n>0$) as

$$c_0 + \frac{c_1}{n} + k + e(nh) \leq F\left(\frac{k}{n}, 1+h\right)$$

then the solution to the planner's problem is necessarily characterized by[10]

(1) the FOCs

$$\begin{pmatrix} u_0'(c_0) \\ u_1'(c_1) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \frac{1}{n} \\ 1 - F_K\left(\frac{k}{n}, 1+h\right)\frac{1}{n} \\ -\frac{c_1}{n^2} + e'(nh)h + F_K\left(\frac{k}{n}, 1+h\right)\frac{k}{n^2} \\ e'(nh)n - F_L\left(\frac{k}{n}, 1+h\right) \end{pmatrix}$$

for some $\lambda \geq 0$

[9]Note that the cost $e(nh)$ of producing any amount n of unskilled labor, i.e. with $h=0$, is zero, since $e(n0)=0$.

[10]The planner's problem is not convex (the constrained set is the upper contour set of a function that is not even quasi-concave) so that the FOCs are only necessary. Nonetheless, the sufficient conditions for a local maximum are satisfied if (i) the agent exhibits a relative risk aversion bigger than one in his second period of life, and (ii) the maximum output net of capital savings and fertility and education costs is concave in fertility and has a small enough elasticity (below 2 in absolute value, see Appendix).

(2) and the feasibility constraint

$$c_0 + \frac{c_1}{n} + k + e(nh) = F\Big(\frac{k}{n}, 1 + h\Big)$$

(since $\lambda > 0$ necessarily from any of the first two coordinates in the vectors above),

The conditions characterizing necessarily the steady state chosen by the planner are summarized in the following proposition.

Proposition 1. *In the overlapping generations economy considered —with a fertility choice and an education investment in the descendants labor productivity— the optimal steady state levels of consumption, savings, fertility, and education investment are all strictly positive and satisfy*[11]

$$\begin{aligned}
\frac{u_0'(c_0)}{u_1'(c_1)} = n = F_K\Big(\frac{k}{n}, 1 + h\Big) &= F_L\Big(\frac{k}{n}, 1 + h\Big)e'(nh)^{-1} \\
c_0 + \frac{c_1}{n} + k + e(nh) &= F\Big(\frac{k}{n}, 1 + h\Big) \\
\frac{c_1}{n} &= F_k\Big(\frac{k}{n}, 1 + h\Big)\frac{k}{n} + e'(nh)nh
\end{aligned}$$

In the next sections we'll characterize the steady state that would result from the decentralized choices of the agents.

4. The market steady state

Consider the representative household born at period t. It chooses a profile of consumptions c_0^t, c_1^t, capital savings k^t, monetary savings M^t, fertility n^t, and children's

[11]When a utility $\phi(nh)$ is derived from the offspring, the third condition in the first line becomes

$$\frac{\phi'(nh)}{u'(c_0)} = e'(nh) - \frac{1}{n}F_L(\frac{k}{n}, 1 + h).$$

education h^t such that it solves

$$\max_{0 \leq c_0, c_1, k, M, n, h} u_0(c_0) + u_1(c_1)$$
$$c_0 + k + \frac{M}{p_t} + e(nh) \leq w_t(1 + h^{t-1})$$
$$c_1 \leq r_{t+1}k + \frac{M}{p_{t+1}}$$

given monetary prices for the consumption good p_t, p_{t+1}, the real wage w_t, the return to capital savings r_{t+1}, and the effective units of labor chosen by his parents h^{t-1}.

Since h^t enters only as a cost to the household, it will be zero at the solution. This leaves the fertility n^t undetermined at a level that can be assumed to be positive (and determined by sociological or cultural factors) since it entails no cost for the household given its choice $h^t = 0$.[12]

The solution to the representative agent's problem is necessarily characterized by

$$n^t > 0$$
$$h^t = 0$$

along with the first-order conditions

$$\begin{pmatrix} u_0'(c_0^t) \\ u_1'(c_1^t) \\ 0 \\ 0 \end{pmatrix} = \lambda^t \begin{pmatrix} 1 \\ 0 \\ 1 \\ \frac{1}{p_t} \end{pmatrix} + \mu^t \begin{pmatrix} 0 \\ 1 \\ -r_{t+1} \\ -\frac{1}{p_{t+1}} \end{pmatrix}$$

[12]Alternatively, given that there is a representative agent per generation, the second period budget constraint is actually

$$c_1 \leq \begin{cases} r_{t+1}k + \frac{M}{p_{t+1}} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

since capital and monetary savings become worthless should the representative agent (i.e. everyone) choose not to have any descendants. In case of heterogeneous agents, savings become worthless only if *all* agents within a generation choose not to reproduce. Anyway, given the assumptions made, there is no point in choosing that in order to avoid child-rearing costs since it is not reproduction but education that is costly. Thus a marginal utility of second period consumption $u_1'(c_1)$ going to infinity as c_1 vanishes will ensure that every agent chooses some positive fertility $n > 0$. In effect, doing so guarantees that his second period income is not zero without necessarily costing him anything since he has always the choice of setting $h^t = 0$.

for some $\lambda^t, \mu^t > 0$, and the budget constraints

$$c_0^t + k^t + \frac{M^t}{p_t} + e(n^t h^t) = w_t(1 + h^{t-1})$$
$$c_1^t = r_{t+1}k^t + \frac{M^t}{p_{t+1}}$$

That is to say, given $p_t, p_{t+1}, w_t, r_{t+1}$ and h^{t-1} generation t chooses $c_0^t, c_1^t, k^t, M^t, n^t$ and h^t such that

$$\frac{u_0'(c_0^t)}{u_1'(c_1^t)} = \frac{p_t}{p_{t+1}} = r_{t+1}$$
$$c_0^t + k^t + \frac{M^t}{p_t} + e(n^t h^t) = w_t(1 + h^{t-1})$$
$$c_1^t = r_{t+1}k^t + \frac{M^t}{p_{t+1}}$$
$$h^t = 0$$
$$n^t > 0$$

with fertility being guaranteed to be positive but at a level left undetermined, which allows for the possibility that the positive level fertility is pinned down by factors other than economic in a world in which children play no direct role in the old-age support of parents.

A solution to these conditions cannot be a minimum since the associated Lagrangian

$$\begin{aligned}\mathcal{L}(\lambda_0^t, \lambda_1^t, c_0^t, c_1^t, k^t, M^t) = u_0(c_0^t) + u_1(c_1^t) \\ &- \lambda_0^t[c_0^t + k^t + \frac{M^t}{p_t} + e(n^t h^t) - w_t h^{t-1}] \\ &- \lambda_1^t[c_1^t - r_{t+1}k^t - \frac{M^t}{p_{t+1}}]\end{aligned}$$

has a Hessian

$$\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & -\frac{1}{p_t} \\ 0 & 0 & 0 & -1 & r_{t+1} & \frac{1}{p_{t+1}} \\ -1 & 0 & u_0''(c_0^t) & 0 & 0 & 0 \\ 0 & -1 & 0 & u_1''(c_1^t) & 0 & 0 \\ -1 & r_{t+1} & 0 & 0 & 0 & 0 \\ -\frac{1}{p_t} & \frac{1}{p_{t+1}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

whose principal minors of order 5 and 6 satisfy respectively

$$\begin{aligned}
(-1)^5 H_5^{(6)} &= -(u_0'' + r_{t+1}^2 u_1'') > 0 \\
(-1)^5 H_5^{(5)} &= -(\frac{1}{p_t^2} u_0'' + \frac{1}{p_{t+1}^2} u_1'') > 0 \\
(-1)^5 H_5^{(4)} &= 0 \geq 0 \\
(-1)^5 H_5^{(3)} &= 0 \geq 0 \\
(-1)^6 H_6 &= 0 \geq 0
\end{aligned}$$

(where, for each i, $H_5^{(i)}$ stands for the principal minor of order 5 resulting from deleting the i-th row and column) which implies that a solution to the first-order conditions is not a minimum since the second order necessary conditions for a minimum are not satisfied. Moreover, the second order necessary conditions for a maximum are satisfied and, although the second order sufficient conditions are not, the existence itself of a maximum is guaranteed by the compactness of the budget set and the continuity of preferences.

The output per worker at t is given by

$$y_t = F\Big(\frac{k^{t-1}}{n^{t-1}}, 1 + h^{t-1}\Big)$$

where n^{t-1} is the rate of growth of the population chosen by generation $t-1$, so that

$$\begin{aligned}
w_t &= F_L\Big(\frac{k^{t-1}}{n^{t-1}}, 1 + h^{t-1}\Big) \\
r_{t+1} &= F_K\Big(\frac{k^t}{n^t}, 1 + h^t\Big)
\end{aligned}$$

A competitive equilibrium is then characterized by a sequence of prices p^t and profiles of consumptions, savings, fertility and education choices $c_0^t, c_1^t, k^t, M^t, h^t, n^t$

such that, for all t, they satisfy

$$\begin{aligned} \frac{u_0'(c_0^t)}{u_1'(c_1^t)} &= \frac{p_t}{p_{t+1}} = F_K\Big(\frac{k^t}{n^t}, 1\Big) \\ c_0^t + k^t + \frac{M^t}{p_t} &= F_L\Big(\frac{k^{t-1}}{n^{t-1}}, 1\Big) \\ c_1^t &= F_K\Big(\frac{k^t}{n^t}, 1\Big)k^t + \frac{M^t}{p_{t+1}} \\ \frac{M^t}{M^{t+1}} &= n^t \end{aligned}$$

—where the last condition is equivalent to the feasibility of the allocation of resources— along with

$$\begin{aligned} n^t &> 0 \\ h^t &= 0 \end{aligned}$$

Thus, at a steady state a competitive equilibrium is characterized by the conditions stated in the next proposition.

Proposition 2. *In the overlapping generations economy considered —with a fertility choice and an education investment in the descendants labor productivity— a competitive equilibrium steady state is characterized by* $c_0, c_1, k, m, n, h > 0$ *such that* $h = 0$ *and*[13]

$$\begin{aligned} \frac{u_0'(c_0)}{u_1'(c_1)} &= n = F_K\Big(\frac{k}{n}, 1+h\Big) \\ c_0 + k + m + e(nh) &= F_L\Big(\frac{k}{n}, 1+h\Big)(1+h) \\ \frac{c_1}{n} &= F_K\Big(\frac{k}{n}, 1+h\Big)\frac{k}{n} + m \end{aligned}$$

[13]When a utility $\phi(n^t h^t)$ is derived from the offspring, at the solution $h > 0$ holds, and moreover

$$\frac{\phi'(nh)}{u'(c_0)} = e'(nh).$$

Since, at a competitive equilibrium steady state $h = 0$ while the planner would choose a $h > 0$, the next property follows straightforwardly from Propositions 1 and 2.[14]

Proposition 3. *In the overlapping generations economy considered —with a fertility choice and an education investment in the descendants labor productivity— no competitive equilibrium implements the optimal steady state.*

Implementing the optimal steady state therefore requires, according to Proposition 3, intervening in order to try to steer the decentralized choices towards the optimal ones. Historically, a compulsory basic education financed through taxes has indeed been the most common policy to address the problem of a possible underinvestment in education by individual households. Alternatively, I consider also a policy that makes households pensions contingent to their choices on fertility and children's education. It turns out that the conditions under which such a policy implements the optimal steady state are much less stringent than those required for a tax-financed compulsory education to attain the same goal. In the following sections we will consider what does the introduction of tax-financed public education change, and then what fertility and education contingent pensions change.

Before that, next section addresses the question of whether the households could nonetheless attain the optimal steady state on their own in a decentralized way, without any need of intervention, by internalizing in their choices the impact they have on the factor prices (assuming they could coordinate on deviating from a competitive behavior). It will turn out that the answer is no because even though they will choose their fertility and education efforts only in order to maximize the returns to their own savings, taking the return to their labor services as given by the previous generation choices.

5. Impact of fertiltity and education on factor prices

In the absence of an altruistic motive by which the agents derive direct utility from having children, and possibly from educating them as well, it is clear that in the

[14]When a utility $\phi(n^t h^t)$ is derived from the offspring, Proposition 3 follows from the wedge that the planner drives between the marginal cost of educated children and the marginal rate of substitution between them and first-period consumption (see footnotes 11 and 13).

previous setup the agents fail to realize the indirect benefits from the skilled labor supplied by an educated offspring via its effect on factor prices in general, and on the returns to their savings when old in particular. Actually, even if they were aware of those effects, the lack of coordination on positive levels of n and h and the possibility of free-riding on everybody else's reproductive and educational efforts can only partially explain the suboptimality of the decentralized outcome. As a matter of fact, even if they were able to coordinate to exploit the impact of their fertility and education choices on the return to their savings, they would still miss the optimal steady state, as shown in what follows.

In effect, consider a representative agent born at t choosing a profile of consumptions c_0^t, c_1^t, capital savings k^t, monetary savings M^t, fertility n^t, and children's education h^t such that it solves

$$\max_{0\leq c_0,c_1,k,M,n,h} u_0(c_0)+u_1(c_1)$$
$$c_0+k+\frac{M}{p_t}+e(nh)\leq w_t(1+h^{t-1})$$
$$c_1\leq F_K\Big(\frac{k}{n},1+h\Big)k+\frac{M}{p_{t+1}}$$

given monetary prices for the consumption good p_t, p_{t+1}, the real wage w_t (which is determined by the choices $k^{t-1}, n^{t-1}, h^{t-1}$ made by the previous generation), and the effective units of labor chosen by his parents h^{t-1}. Note that in solving the problem above generation t is aware of the impact that its reproductive and educational choices (as well as its capital savings) have on the return to its own savings.

The solution to the representative agent's problem is now characterized by the first-order conditions[15]

$$\begin{pmatrix} u_0'(c_0^t) \\ u_1'(c_1^t) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda^t \begin{pmatrix} 1 \\ 0 \\ 1 \\ \frac{1}{p_t} \\ e'(n^t h^t)h^t \\ e'(n^t h^t)n^t \end{pmatrix} + \mu^t \begin{pmatrix} 0 \\ 1 \\ -F_K - F_{KK}\frac{k^t}{n^t} \\ -\frac{1}{p_{t+1}} \\ F_{KK}\frac{k^t}{(n^t)^2} \\ -F_{KL}k^t \end{pmatrix}$$

[15]with obvious notation for F_K, F_{KK} and F_{KL}.

for some $\lambda^t, \mu^t > 0$, and the budget constraints

$$c_0^t + k^t + \frac{M^t}{p_t} + e(n^t h^t) = w_t(1 + h^{t-1})$$

$$c_1^t = F_K\Big(\frac{k^t}{n^t}, 1 + h^t\Big)k^t + \frac{M^t}{p_{t+1}}$$

That is to say, an equilibrium is characterized by a sequence $\{c_0^t, c_1^t, k^t, M^t, h^t, p_t\}_{t \in \mathbb{Z}}$ such that, for all t,

$$\begin{aligned}
\frac{u_0'(c_0^t)}{u_1'(c_1^t)} = \frac{p_t}{p_{t+1}} &= F_K + F_{KK}\frac{k^t}{n^t} \\
&= F_{KL}\frac{k^t}{n^t}e'(n^t h^t)^{-1} \\
&= -F_{KK}\frac{k^t}{n^t}[e'(n^t h^t)n^t h^t]^{-1} \\
c_0^t + k^t + \frac{M^t}{p_t} + e(n^t h^t) &= F_L\Big(\frac{k^{t-1}}{n^{t-1}}, 1 + h^{t-1}\Big)(1 + h^{t-1}) \\
c_1^t &= F_K\Big(\frac{k^t}{n^t}, 1 + h^t\Big)k^t + \frac{M^t}{p_{t+1}} \\
\frac{M^t}{M^{t+1}} &= n^t
\end{aligned}$$

(where the last condition is equivalent to the feasibility of the allocation of resources) Note that fourth equality (of the second and the third lines) implies, given the 0-homogeneity of F_K, that

$$k^t h^t = 1 + h^t$$

meaning that necessarily $h^t > 0$, so that parents do invest in their children education. Nevertheless, such a steady state is characterized then by

$$\begin{aligned}
\frac{u_0'(c_0)}{u_1'(c_1)} = n = F_K + F_{KK}\frac{k}{n} &= F_{KL}\frac{k}{n}e'(nh)^{-1} \\
kh &= 1 + h \\
c_0 + k + m + e(nh) &= F_L\Big(\frac{k}{n}, 1 + h\Big)(1 + h) \\
\frac{c_1}{n} &= F_K\Big(\frac{k}{n}, 1 + h\Big)\frac{k}{n} + m
\end{aligned}$$

which cannot coincide with the optimal steady state either, since the latter requires rather

$$n = F_K.$$

Is this because only the impact on capital returns (and not on wages) is internalized? Consider a planner taking into account the impact on both returns to capital and wages but constrained to remunerate factors by their marginal productivities, so that it solves

$$\max_{0 \leq c_0, c_1, k, m, n, h} u_0(c_0) + u_1(c_1)$$

$$c_0 + k + m + e(nh) \leq F_L\Big(\frac{k}{n}, 1+h\Big)(1+h)$$

$$c_1 \leq F_K\Big(\frac{k}{n}, 1+h\Big)k + nm$$

The solution to the representative agent's problem is now characterized by the first-order conditions[16]

$$\begin{pmatrix} u_0'(c_0) \\ u_1'(c_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 1 - F_{LK}\frac{1+h}{n} \\ 1 \\ e'(nh)h \\ e'(nh)n - F_L - F_{LL}(1+h) \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -F_K - F_{KK}\frac{k}{n} \\ -n \\ F_{KK}\frac{k}{n^2} \\ -F_{KL}k \end{pmatrix}$$

for some $\lambda, \mu > 0$, and the budget constraints

$$c_0 + k + m + e(nh) = F_L\Big(\frac{k}{n}, 1+h\Big)(1+h)$$

$$c_1 = F_K\Big(\frac{k}{n}, 1+h\Big)k + nm$$

That is to say, an equilibrium is characterized by a sequence $\{c_0^t, c_1^t, k^t, M^t, h^t, p_t\}_{t\in\mathbb{Z}}$ such that, for all t,

$$\begin{aligned}
\frac{u_0'(c_0)}{u_1'(c_1)} = n &= \frac{F_K + F_{KK}\frac{k}{n}}{1 + F_{LK}\frac{1+h}{n}} \\
&= \frac{F_{KL}k}{e'(nh)n - F_L - F_{LL}(1+h)} \\
&= -F_{KK}\frac{k}{n}[e'(nh)nh]^{-1}
\end{aligned}$$

$$c_0 + k + m + e(nh) = F_L\Big(\frac{k}{n}, 1+h\Big)(1+h)$$

$$c_1 = F_K\Big(\frac{k}{n}, 1+h\Big)k + nm$$

[16] with obvious notation for F_K, F_{KK} and F_{KL}.

which again cannot coincide with the optimal steady state either, since the latter requires rather

$$n = F_K.$$

6. The market steady state with tax-funded compulsory education

In order to see whether in this setup tax-funded compulsory education is able to implement the optimal steady state, consider a representative agent born at t choosing a profile of consumptions c_0^t, c_1^t, capital savings k^t, monetary savings M^t, fertility n^t, and children's education h^t that solves[17]

$$\max_{0 \le c_0, c_1, k, M, n, h} u_0(c_0) + u_1(c_1)$$

$$c_0 + k + \frac{M}{p_t} + e(nh) \le w_t(1 + h^{t-1} + h_t^p) - T_t$$

$$c_1 \le r_{t+1}k + \frac{M}{p_{t+1}}$$

given monetary prices for the consumption good p_t, p_{t+1}, the real wage w_t, the return to capital savings r_{t+1}, the increase of his own endowment in effective units of labor chosen by his parents h^{t-1} and the increase imposed by the government h_t^p and financed through a lump-sum tax T_t.[18]

The solution to the representative agent's problem is characterized by

$$n^t > 0$$

$$h^t = 0$$

[17] If the household derives utility also from the number and education of its children, so that the objective function is $u_0(c_0^t) + u_1(c_1^t) + \phi(n^t(h^t + h_t^p))$, this policy leads to unbounded fertility, since the household will choose to set $h^t = 0$ and $n^t \to +\infty$. This is a consequence of the fact that costs induced by fertility other than education costs are left out of the picture for the sake of simplicity.

[18] Again in the second budget constraint, which is more precisely

$$c_1 \le \begin{cases} r_{t+1}k + \frac{M}{p_{t+1}} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

the representative agent internalizes the impact of the fertility choice on the real value of savings. A marginal utility of second period consumption $u_1'(c_1)$ going to infinity as c_1 vanishes ensures again that the representative agent chooses some $n > 0$.

(since $n = 0$ implies $c_1 = 0$ while $u_1'(0) = +\infty$) along with the first-order conditions

$$\begin{pmatrix} u_0'(c_0^t) \\ u_1'(c_1^t) \\ 0 \\ 0 \end{pmatrix} = \lambda^t \begin{pmatrix} 1 \\ 0 \\ 1 \\ \frac{1}{p_t} \end{pmatrix} + \mu^t \begin{pmatrix} 0 \\ 1 \\ -r_{t+1} \\ -\frac{1}{p_{t+1}} \end{pmatrix}$$

for some $\lambda^t, \mu^t > 0$, and

$$c_0^t + k^t + \frac{M^t}{p_t} + e(n^t h^t) = w_t(1 + h^{t-1} + h_t^p) - T_t$$
$$c_1^t = r_{t+1}k^t + \frac{M^t}{p_{t+1}}$$

That is to say, the households choice is characterized by

$$\frac{u_0'(c_0^t)}{u_1'(c_1^t)} = \frac{p_t}{p_{t+1}} = r_{t+1}$$
$$c_0^t + k^t + \frac{M^t}{p_t} = w_t(1 + h^{t-1} + h_t^p) - T_t$$
$$c_1^t = r_{t+1}k^t + \frac{M^t}{p_{t+1}}$$
$$h^t = 0$$
$$n^t > 0$$

The output per worker at t is hence given by

$$y_t = F\Big(\frac{k^{t-1}}{n^{t-1}}, 1 + h_t^p\Big)$$

where n^{t-1} is the rate of growth of the population chosen by generation $t-1$, so that

$$w_t = F_L\Big(\frac{k^{t-1}}{n^{t-1}}, 1 + h_t^p\Big)$$
$$r_{t+1} = F_K\Big(\frac{k^t}{n^t}, 1 + h_{t+1}^p\Big)$$

and the government budget will be balanced at t if, and only if,

$$T_t = e(n^t h_{t+1}^p)$$

Thus, given a compulsory education policy $\{h_t^p\}_t$ and the taxes allowing to finance it $\{T_t\}_t$, a competitive equilibrium is characterized by a sequence $\{c_0^t, c_1^t, k^t, M^t, h^t, p_t\}_t$ such that, for all t,

$$h^t = 0$$
$$n^t > 0$$

and

$$\frac{u_0'(c_0^t)}{u_1'(c_1^t)} = \frac{p_t}{p_{t+1}} = F_K\Big(\frac{k^t}{n^t}, 1 + h_{t+1}^p\Big)$$
$$c_0^t + k^t + \frac{M^t}{p_t} = F_L\Big(\frac{k^{t-1}}{n^{t-1}}, 1 + h_t^p\Big)(1 + h_t^p) - T_t$$
$$c_1^t = F_K\Big(\frac{k^t}{n^t}, 1 + h_t^p\Big)k^t + \frac{M^t}{p_{t+1}}$$
$$\frac{M^t}{M^{t+1}} = n^t$$

where the last condition is equivalent to the feasibility of the allocation of resources if

$$T_t = e(n^t h_{t+1}^p)$$

In particular, a competitive equilibrium steady state under an education policy h^p and the taxes $T = e(nh^p)$ paying for it, is then characterized by $h = 0$ and $\{c_0, c_1, k, m, n\}$ such that

$$\frac{u_0'(c_0)}{u_1'(c_1)} = n = F_K\Big(\frac{k}{n}, 1 + h^p\Big)$$
$$c_0 + k + m = F_L\Big(\frac{k}{n}, 1 + h^p\Big)(1 + h^p) - e(nh^p)$$
$$\frac{c_1}{n} = F_K\Big(\frac{k}{n}, 1 + h^p\Big)\frac{k}{n} + m$$

with

$$n > 0$$

but undetermined.

Since the planner's steady state is necessarily characterized by

$$\begin{aligned}
\frac{u_0'(c_0)}{u_1'(c_1)} &= n = F_K\Big(\frac{k}{n}, 1+h\Big) = F_L\Big(\frac{k}{n}, 1+h\Big) e'(nh)^{-1} \\
c_0 + \frac{c_1}{n} + k + e(nh) &= F\Big(\frac{k}{n}, 1+h\Big) \\
\frac{c_1}{n} &= F_k\Big(\frac{k}{n}, 1+h\Big)\frac{k}{n} + e'(nh)nh
\end{aligned} \tag{P}$$

then the planner's steady state can be decentralized as a competitive equilibrium steady state if under the policy $h^p = h$ the choice c_0, c_1, k, m, n is such that

$$F_K\Big(\frac{k}{n}, 1+h\Big) = F_L\Big(\frac{k}{n}, 1+h\Big) e'(nh)^{-1}$$

$$m = e'(nh)nh$$

Note that, while the characterization of a competitive equilibrium steady state under this policy has one degree of freedom (it leaves fertility undetermined), the conditions for it to implement the optimal steady state impose two additional equations, which makes the system overdeterminate, generically. Proposition 4 below summarizes this result.

Proposition 4. *In the overlapping generations economy considered —with a fertility choice and an education investment in the descendants labor productivity— a competitive equilibrium steady state c_0, c_1, k, m, n under a balanced-budget tax-financed compulsory education h^p does not implement the planner's steady state, generically.*[19]

Note that, while the characterization of a competitive equilibrium steady state has one degree of freedom (it leaves the fertility undetermined), the conditions for it to implement the optimal steady state impose on it two additional equations, with the risk of overdeterminacy.

The next section shows that an old-age transfer scheme that is contingent to households' fertility and education choices can implement the optimal steady instead.

[19] In the space of endowments, technology, and preferences of the economy.

7. Fertility-education contingent pensions

Consider instead an overlapping generations economy with a representative agent born at t choosing a solution $(c_0^t, c_1^t, k^t, M^t, n^t, h^t)$ to the problem[20]

$$\max_{0\leq c_0, c_1, k, M, n, h} u_0(c_0) + u_1(c_1)$$

$$c_0 + k + \frac{M}{p_t} + e(nh) \leq w_t(1 + (1-\tau)h^{t-1})$$
$$c_1 \leq r_{t+1}k + \frac{M}{p_{t+1}} + \tau w_{t+1} nh$$

given $p_t, p_{t+1}, w_t, r_{t+1}, h^{t-1}$ and τ.

The solution to the problem of agent t is interior under the assumptions made,[21] characterized by the first-order conditions

$$\begin{pmatrix} u_0'(c_0^t) \\ u_1'(c_1^t) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda^t \begin{pmatrix} 1 \\ 0 \\ 1 \\ \frac{1}{p_t} \\ h^t e'(n^t h^t) \\ n^t e'(n^t h^t) \end{pmatrix} + \mu^t \begin{pmatrix} 0 \\ 1 \\ -r_{t+1} \\ -\frac{1}{p_{t+1}} \\ -\tau w_{t+1} h^t \\ -\tau w_{t+1} n^t \end{pmatrix}$$

[20] Once more the second-period budget constraint is, more precisely,

$$c_1 \leq \begin{cases} r_{t+1}k + \frac{M}{p_{t+1}} + \tau w_{t+1} n(h-1) & \text{if } -1 < n \\ 0 & \text{if } -1 = n \end{cases}$$

[21] As with the planner, $e'(0) = 0$ prevents h^t from being 0, since the maximum present value of the pension net of education investment is characterized by

$$\frac{p_{t+1}}{p_t} \tau w_{t+1} n - e'(nh)n \leq 0$$

and

$$h\Big[\frac{p_{t+1}}{p_t} \tau w_{t+1} n - e'(nh)n\Big] = 0$$

for any $n > 0$ (the convexity of e guarantees that the present value of the pension net of education investment is not maximized by $n = 0$).

for some $\lambda^t, \mu^t > 0$, along with

$$c_0^t + k^t + \frac{M^t}{p_t} + e(n^t h^t)) = w_t(1 + (1-\tau)h^{t-1})$$
$$c_1^t = r_{t+1}k^t + \frac{M^t}{p_{t+1}} + \tau w_{t+1} n^t h^t$$

That is to say, the agent would choose $c_0^t, c_1^t, k^t, M^t, n^t, h^t$ such that[22]

$$\frac{u_0'(c_0^t)}{u_1'(c_1^t)} = \frac{p_t}{p_{t+1}} = r_{t+1} = \tau w_{t+1} e'(n^t h^t)^{-1}$$
$$c_0^t + k^t + \frac{M^t}{p_t} + e(n^t h^t) = w_t(1 + (1-\tau)h^{t-1})$$
$$c_1^t = r_{t+1}k^t + \frac{M^t}{p_{t+1}} + \tau w_{t+1} n^t h^t$$

while the output per worker at t is given by

$$y_t = F\Big(\frac{k^{t-1}}{n^{t-1}}, 1 + h^{t-1}\Big)$$

where n^{t-1} is the rate of growth of the population chosen by generation $t-1$, so that

$$w_t = F_L\Big(\frac{k^{t-1}}{n^{t-1}}, 1 + h^{t-1}\Big)$$
$$r_{t+1} = F_K\Big(\frac{k^t}{n^t}, 1 + h^t\Big)$$

Therefore, for a given τ, a competitive equilibrium is characterized by a sequence $\{c_0^t, c_1^t, k^t, M^t, n^t, h^t, p_t\}_t$ such that, for all t,

$$\frac{u_0'(c_0^t)}{u_1'(c_1^t)} = \frac{p_t}{p_{t+1}} = F_K\Big(\frac{k^t}{n^t}, 1 + h^t\Big) = \tau F_L\Big(\frac{k^t}{n^t}, 1 + h^t\Big) e'(n^t h^t)^{-1}$$
$$c_0^t + k^t + \frac{M^t}{p_t} + e(n^t h^t) = F_L\Big(\frac{k^{t-1}}{n^{t-1}}, 1 + h^{t-1}\Big)(1 + (1-\tau)h^{t-1})$$
$$c_1^t = F_K\Big(\frac{k^t}{n^t}, 1 + h^t\Big)k^t + \frac{M^t}{p_{t+1}} + \tau F_L\Big(\frac{k^t}{n^t}, 1 + h^t\Big) n^t h^t$$
$$\frac{M^t}{M^{t+1}} = n^t$$

[22]When utility $\phi(n^t h^t)$ is derived from the offspring, the third condition in the first line becomes
$$\phi'(n^t h^t) = u_0'(c_0^t)e'(n^t h^t) - u_1'(c_1^t)\tau w_{t+1}.$$

—where the last condition is equivalent to the feasibility of the allocation of resources.

For a given τ, a competitive equilibrium steady state is characterized by $\{c_0, c_1, k, m, n, h\}$ such that

$$\frac{u_0'(c_0)}{u_1'(c_1)} = n = F_K\Big(\frac{k}{n}, 1+h\Big) = \tau F_L\Big(\frac{k}{n}, 1+h\Big)e'(nh)^{-1}$$
$$c_0 + k + m + e(nh) = F_L\Big(\frac{k}{n}, 1+h\Big)(1+(1-\tau)h)$$
$$\frac{c_1}{n} = F_K\Big(\frac{k}{n}, 1+h\Big)\frac{k}{n} + m + \tau F_L\Big(\frac{k}{n}, 1+h\Big)h$$

Note that there is one degree of freedom in the conditions characterizing the competitive equilibrium steady state under this policy. This means that the competitive equilibrium steady state depends on the value taken by one of the variables. Specifically, if $m = 0$ the system of equations becomes the system characterizing the optimal steady state with $\tau = 1$,[23] since then the system becomes

$$\frac{u_0'(c_0)}{u_1'(c_1)} = n = F_K\Big(\frac{k}{n}, 1+h\Big) = F_L\Big(\frac{k}{n}, 1+h\Big)e'(nh)^{-1}$$
$$c_0 + k + e(nh) = F_L\Big(\frac{k}{n}, 1+h\Big)$$
$$\frac{c_1}{n} = F_K\Big(\frac{k}{n}, 1+h\Big)\frac{k}{n} + F_L\Big(\frac{k}{n}, 1+h\Big)h$$

which is equivalent to the planner's steady steady state system

$$\frac{u_0'(c_0)}{u_1'(c_1)} = n = F_K\Big(\frac{k}{n}, 1+h\Big) = F_L\Big(\frac{k}{n}, 1+h\Big)e'(nh)^{-1}$$
$$c_0 + \frac{c_1}{n} + k + e(nh) = F\Big(\frac{k}{n}, 1+h\Big)$$
$$\frac{c_1}{n} = F_k\Big(\frac{k}{n}, 1+h\Big)\frac{k}{n} + e'(nh)nh$$

given that

$$F_L\Big(\frac{k}{n}, 1+h\Big) = e'(nh)n.$$

Proposition 5 below summarizes this result.

[23] In the general case since

$$\frac{\phi'(nh)}{u_0'(c_0)} = e'(nh) - \tau\frac{F_L(\frac{k}{n}, 1+h)}{F_K(\frac{k}{n}, 1+h)}.$$

Proposition 5. *In the overlapping generations economy considered —with a fertility choice and an education investment in the descendants labor productivity— the optimal steady state is the competitive equilibrium steady state c_0, c_1, k, m, n, h such that $m = 0$, under a fertility and education contingent old-age transfer scheme financed by taxing at a rate $\tau = 1$ the increase in labor income resulting from parental investment in education.*

Note that all the intergenerational transfers needed to implement the planner's steady state are realized transferring as fertility-education contingent pension to agent t at $t + 1$ the amount $\tau w_{t+1} n^t h^t$ raised at $t + 1$ by the payroll tax paid by agent $t + 1$ on the increase in labor income coming from the education investment made by parents t. As a consequence, there is no need to use another asset to complement such transfers, whence the condition $m = 0$. Nevertheless, the presence of that other asset is essential, even if not demanded at equilibrium, to guarantee that the rate at which agents can transfer wealth across periods —either saving in capital or through the fertility-education contingent pension scheme— coincides with the population growth factor implied by their fertility choice. This becomes apparent computing the competitive equilibrium steady state of the same economy without money (see appendix A2).

8. Concluding remarks

The model above shows that the decisions on fertility and education taken by households in a decentralized way typically lead to a suboptimal steady state. The reason for that is that producing future skilled labor is a private cost on the returns of which other households can free-ride.

While the problem has been recognized in the literature, two main innovations are introduced in the approach followed in this paper. Firstly, rather than wondering what is the optimal population size households want to produce, I draw the attention to the fact that it is not just the quantity but also the quality of the population that matters for the future returns to capital savings. Thus I let the agents choose both their fertility and how much they educate their children. Secondly, having the previous literature unnecessarily intertwined the (low) costs of producing kids with the (high) costs of producing skilled labor out of kids, I disentangle the two and as a consequence need not relay on altruism or children in the utility function to avoid the population collapsing.

The main results in the paper are, on the one hand, that the competitive equilibria steady state are typically suboptimal, and on the other hand, that the optimal steady state can nonetheless be implemented as a competitive equilibrium outcome if it is put in place a social security whose pension payments are made to depend on the households' choices on both fertility and education, and that is financed by a payroll tax on the increase in labor income of the children. Moreover it is shown that the common policy of a tax-financed compulsory education is unlikely to implement the optimal steady state, even if the mandatory education is set to be the optimal one.

The analysis can and should be extended in many directions, some of which have been mentioned throughout the paper, but the message stemming from the simple setup considered here should not change much as a result.

Appendix A1

The planner's problem can be written equivalently as

$$\max_{c_0,c_1,n>0} u_0(c_0) + u_1(c_1)$$
$$c_0 + \frac{c_1}{n} \leq \phi(n)$$

where

$$\phi(n) = \max_{k,h>0} F\left(\frac{k}{n}, 1+h\right) - k - e(nh)$$

which, for a neoclassical linear homogeneous production function, is well defined by the first-order conditions

$$F_K\left(\frac{k}{n}, 1+h\right)\frac{1}{n} - 1 = 0$$
$$F_L\left(\frac{k}{n}, 1+h\right) - e'(nh)n = 0$$

since the associated Hessian matrix

$$\begin{pmatrix} F_{KK}\frac{1}{n^2} & F_{KL}\frac{1}{n} \\ F_{KL}\frac{1}{n} & F_{LL} - e''n^2 \end{pmatrix}$$

is everywhere negative definite, since

$$F_{KK}\frac{1}{n^2} < 0$$

and

$$F_{KK}[F_{LL} - e''n^2] - F_{KL}^2 > 0$$

for a linearly homogeneous neoclassical production function

$$F(K, L) = K^a L^{1-a}$$

Sufficient conditions for the solution to the first-order conditions

$$\begin{aligned} \phi(n) - c_0 - \frac{c_1}{n} &= 0 \\ u_0'(c_0) - \lambda &= 0 \\ u_1'(c_1) - \frac{\lambda}{n} &= 0 \\ \lambda\big[\phi'(n) + \frac{c_1}{n^2}\big] &= 0 \end{aligned}$$

for some $\lambda > 0$, of the planner's problem to be a local maximum require the fundamental principal minors of order two and three of Hessian (with border)

$$\begin{pmatrix} 0 & -1 & -\frac{1}{n} & \phi'(n) + \frac{c_1}{n^2} \\ -1 & u_0''(c_0) & 0 & 0 \\ -\frac{1}{n} & 0 & u_1''(c_1) & \frac{\lambda}{n^2} \\ \phi'(n) + \frac{c_1}{n^2} & 0 & \frac{\lambda}{n^2} & \lambda\Big(\phi''(n) - 2\frac{c_1}{n^3}\Big) \end{pmatrix}$$

to have signs $(-1)^2$ and $(-1)^3$ respectively , i.e.

$$\begin{vmatrix} 0 & -1 & -\frac{1}{n} \\ -1 & u_0''(c_0) & 0 \\ -\frac{1}{n} & 0 & u_1''(c_1) \end{vmatrix} > 0$$

(which holds, given the strict concavity of u_0 and u_1) and

$$\begin{vmatrix} 0 & -1 & -\frac{1}{n} & \phi'(n) + \frac{c_1}{n^2} \\ -1 & u_0''(c_0) & 0 & 0 \\ -\frac{1}{n} & 0 & u_1''(c_1) & \frac{\lambda}{n^2} \\ \phi'(n) + \frac{c_1}{n^2} & 0 & \frac{\lambda}{n^2} & \lambda\Big(\phi''(n) - 2\frac{c_1}{n^3}\Big) \end{vmatrix} < 0$$

i.e. (expanding the determinant by the second row and rearranging)

$$-\lambda\Big[u_1''(c_1)\Big(\phi''(n) - 2\frac{c_1}{n^3}\Big) - \frac{\lambda}{n^4}\Big]$$

$$-u_0''(c_0)\frac{\lambda}{n^2}\left[\frac{2}{n}\phi'(n)+\phi''(n)\right]-u_0''(c_0)u_1''(c_1)\Big(\frac{c_1}{n^2}+\phi'(n)\Big)^2<0$$

which is guaranteed as long as

$$u_1''(c_1)\Big(\phi''(n)-2\frac{c_1}{n^3}\Big)-\frac{\lambda}{n^4}>0$$

and

$$\frac{2}{n}\phi'(n)+\phi''(n)<0$$

i.e., since $\lambda=u_1'(c_1)n$, $\phi'(n)<0$ (from applying the Envelope Theorem to $\phi(n)$), it suffices respectively that

$$-\frac{u_1''(c_1)}{u_1'(c_1)}c_1>1$$
$$-\frac{\phi''(n)}{\phi'(n)}n<2$$

i.e. that the second-period relative risk aversion is high enough and the elasticity of ϕ is low enough (this is sufficient but not necessary).

Appendix A2

Consider an overlapping generations economy like the previous one, with the only difference that agents can only save in terms of capital. The same policy of fertility-education contingent pensions financed by a payroll tax on the increase of labor income due to educations investments is in place.

An interior solution to the problem of agent t is characterized by the first-order conditions

$$\begin{pmatrix} u_0'(c_0^t) \\ u_1'(c_1^t) \\ 0 \\ 0 \\ 0 \end{pmatrix}=\lambda^t\begin{pmatrix} 1 \\ 0 \\ 1 \\ h^te'(n^th^t) \\ n^te'(n^th^t) \end{pmatrix}+\mu^t\begin{pmatrix} 0 \\ 1 \\ -r_{t+1} \\ -\tau w_{t+1}h^t \\ -\tau w_{t+1}n^t \end{pmatrix}$$

for some $\lambda^t,\mu^t>0$, and

$$c_0^t+k^t+e(n^th^t)=w_t(1+(1-\tau)h^{t-1})$$
$$c_1^t=r_{t+1}k^t+\tau w_{t+1}n^th^t$$

That is to say, the agent would choose $c_0^t, c_1^t, k^t, n^t, h^t$ such that

$$\begin{aligned}\frac{u_0'(c_0^t)}{u_1'(c_1^t)} &= r_{t+1} = \tau w_{t+1} e'(n^t h^t)^{-1}\\ c_0^t + k^t + e(n^t h^t) &= w_t(1+(1-\tau)h^{t-1})\\ c_1^t &= r_{t+1}k^t + \tau w_{t+1} n^t h^t\end{aligned}$$

The output per worker at t is given by

$$y_t = F\Big(\frac{k^{t-1}}{n^{t-1}}, 1+h^{t-1}\Big)$$

where n^{t-1} is the rate of growth of the population chosen by generation $t-1$, so that

$$\begin{aligned}w_t =& F_L\Big(\frac{k^{t-1}}{n^{t-1}}, 1+h^{t-1}\Big)\\ r_{t+1} =& F_K\Big(\frac{k^t}{n^t}, 1+h^t\Big)\end{aligned}$$

For a given τ, a competitive equilibrium is characterized by $\{c_0^t, c_1^t, k^t, n^t, h^t, p_t\}_{t\in\mathbb{Z}}$ such that, for all t,

$$\begin{aligned}\frac{u_0'(c_0^t)}{u_1'(c_1^t)} &= F_K\Big(\frac{k^t}{n^t}, 1+h^t\Big) = \tau F_L\Big(\frac{k^t}{n^t}, 1+h^t\Big)e'(n^t h^t)^{-1}\\ c_0^t + k^t + e(n^t h^t) &= F_L\Big(\frac{k^{t-1}}{n^{t-1}}, 1+h^{t-1}\Big)(1+(1-\tau)h^{t-1})\\ c_1^t &= F_K\Big(\frac{k^t}{n^t}, 1+h^t\Big)k^t + \tau F_L\Big(\frac{k^t}{n^t}, 1+h^t\Big)n^t h^t\end{aligned}$$

—the feasibility of the allocation of resources is guaranteed by the budget constraints.

For a given τ, a competitive equilibrium steady state is characterized by $\{c_0, c_1, k, n, h\}$ such that

$$\begin{aligned}\frac{u_0'(c_0)}{u_1'(c_1)} &= F_K\Big(\frac{k}{n}, 1+h\Big) = \tau F_L\Big(\frac{k}{n}, 1+h\Big)e'(nh)^{-1}\\ c_0 + k + e(nh) &= F_L\Big(\frac{k}{n}, 1+h\Big)(1+(1-\tau)h)\\ \frac{c_1}{n} &= F_K\Big(\frac{k}{n}, 1+h\Big)\frac{k}{n} + \tau F_L\Big(\frac{k}{n}, 1+h\Big)h\end{aligned}$$

which is *not* equivalent, even if $\tau = 1$, to the planner's steady state system

$$\frac{u_0'(c_0)}{u_1'(c_1)} = n = F_K\Big(\frac{k}{n}, 1+h\Big) = F_L\Big(\frac{k}{n}, 1+h\Big)e'(nh)^{-1}$$
$$c_0 + \frac{c_1}{n} + k + e(nh) = F\Big(\frac{k}{n}, 1+h\Big)$$
$$\frac{c_1}{n} = F_k\Big(\frac{k}{n}, 1+h\Big)\frac{k}{n} + e(nh)nh$$

since, in the absence of money, nothing guarantees that the productivity of capital is the growth factor of the population implied by the agents' fertility choice.

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