Dynamic hedging strategies: an application to the crude oil market

Delphine Lautier, Alain Galli

DYNAMIC HEDGING STRATEGIES:

AN APPLICATION TO THE CRUDE OIL MARKET

Delphine LAUTIER[1], Professor at University Paris-Dauphine, DRM Finance, CNRS UMR-7088

Associate research fellow at MINES ParisTech (Cerna).

Postal address: DRM-Finance, Université Paris-Dauphine,

Place du Maréchal de Lattre de Tassigny,

75 775 Paris Cedex 16

France

E-mail address: delphine.lautier@dauphine.fr

Telephone: 33 1 44 05 46 73

Fax number: 33 1 44 05 40 23

Alain GALLI, Professor at MINES ParisTech (Cerna)

Postal address: Cerna, MINES ParisTech,

60 boulevard Saint-Michel,

75 272 Paris Cedex 06

France

E-mail address: galli@cerna.ensmp.fr

Telephone: 33 1 40 51 93 14

Fax number: 33 1 40 51 91 45

This draft: 03/2010

Acknowledgments: Financial support of the Conseil Français de l'Energie (CFE) is gratefully acknowledged.

ABSTRACT

This article analyses long-term dynamic hedging strategies relying on term structure models of commodity prices and proposes a new way to calibrate the models which takes into account the error associated with the hedge ratios. Different strategies, with maturities up to seven years, are tested on the American crude oil futures market. The study considers three recent and efficient models respectively with one, two, and three factors. The continuity between the models makes it possible to compare their performances which are judged on the basis of the errors associated with a delta hedge. The strategies are also tested for their sensitivity to the maturities of the positions and to the frequency of the portfolio rollover. We found that our method gives the best of two seemingly incompatible worlds: the higher liquidity of short-term futures contracts for the hedge portfolios, together with markedly improved performances. Moreover, even if it is more complex, the three-factor model is by far, the best.

JEL Classification: G13 – C13 – C52 – Q49

Key words: Dynamic Hedging – Commodities – Futures – Long-term commitment – Calibration

Empirical studies on long-term dynamic hedging strategies in commodity markets are rare. However, such strategies are important for almost every derivative market, especially when long-term transactions may be undertaken, like in interest rate, credit, or commodity markets. Moreover, little work has been done on comparing strategies relying on different term structure models. This article fills this gap and proposes a method that significantly improves the performance of dynamic hedging strategies. This method relies on a simple idea. In the context of dynamic hedging, the sensitivity of the present value of the hedged position must be the same as that of the hedge portfolio. This sensitivity is obtained by computing the first derivatives of the prices functions with respect to each of the model's state variables. This is why we propose that for hedging purposes, the models estimations rely on the derivatives themselves instead of the prices, as is usually done. Our method is tested empirically on the American crude oil futures market on a long term horizon: up to seven years.

The dynamic hedging strategies studied rely on three recent and efficient models of commodity futures prices: a one-, a two-, and a three-factor models. The one- and two-factor models are those proposed by Schwartz (1997). The single state variable of the first one is the spot price, which is mean reverting. The two factor model is an extension of the previous one. It is also mean reverting and includes an additional variable: the convenience yield[2]. Lastly, the three-factor model of Cortazar and Schwartz (2003) includes a third variable: the long-term price. So there is continuity between the different models. The calibration relies on rolling windows and non constant interest rates in order to make the study more realistic.

The dynamic hedging strategies considered in this article aim to hedge a long term commitment with short-term futures contracts. The advantages of such strategies are twofold. Firstly, it is important to be able to propose long-term commitments. Secondly, these strategies take advantage of the higher liquidity of the near-term contracts. Naturally, the use of short-term maturities supposes that the hedge portfolio is rebalanced regularly when the futures' expiry date approaches, in order to maintain the position on the futures market. Consequently, dynamic hedging strategies entail rollover basis risk.

Three reasons explain the choice of the crude oil market for empirical tests. First, at the beginning of the 1990s, Metallgesellschaft's[3] attempt to cover long-term forward commitments on the physical market of petroleum products with short-dated futures contracts ended in a resounding failure (USD 2.4 billions were lost), but it initiated research on whether this could be safely undertaken. Second, dynamic hedging strategies are especially questionable in commodity markets, as term structure models rely on non observable state variables. The hedge portfolio is a combination of futures contracts with different maturities and one may ask whether such a hedge is reliable. Third, the crude oil market is nowadays the best developed commodity futures market with maturities out as far as nine years, which is exceptional for commodities.

Relatively few studies of long-term dynamic hedging strategies have been carried out. Most of the time, the hedging problem was tackled by modelling the relationship between the futures prices – *ie* term structure models. The various strategies differ from each other mainly in the assumptions concerning the behaviour of futures prices. So studies of hedging strategies are above all associated with theoretical studies of the term structure, and empirical works have been much rarer[4].

Brennan and Crew (1997) compared the hedging strategy used by Metallgesellschaft on the crude oil market with strategies relying on several term structure models. Studying hedging strategies up to 24 months, they showed that those relying on the term structure models outperform by that of the German firm by far, all the more as the term structure model is able to correctly replicate the price curve empirically observed. In 1997, Schwartz proposed three new term structure models involving the spot price, the convenience yield and the interest rate. In order to take into account the storage behaviour of the operators in commodities markets, he introduced mean reverting processes. He also computed the hedge ratios associated with these models, but did not undertake empirical tests. Neuberger (1999) compared the performances of hedging strategies based on Schwartz's two factor model and on a new model. In the latter, no assumption is made about the number of variables, the process they follow, or the way risk is priced. The key assumption is that the expected price at which the long dated contract starts trading is a linear function of the price of existing contracts. While theoretically inconsistent with most models of the term structure of commodity prices, this new model still gives good results in practice, when considering hedging strategies up to 36 months. However, it requires balancing the portfolio more frequently than with Schwartz's one. In their model, Routledge, Seppi, and Spatt (2000) took into account the fact that in commodity markets, the basis has an asymmetrical behaviour: it is more volatile in backwardation than in contango. Once again, they computed the hedge ratios based on their term structure model but they did not test dynamic hedging strategies. Lastly, Veld-Merkoula and de Roon (2003) used a one factor term structure model based on convenience yield to construct hedge strategies that minimize both spot price risk and rollover risk by using futures of two different maturities. They showed that their strategy outperforms the naïve hedging strategy. However, they did not compare their results with previous work.

Compared to previous works, three of the major contributions of the paper are, first to have explicitly calculated the hedge portfolios of the term structure models employed in the study, second, to have found a methodology that leads better optimal hedge ratios than those traditionally employed and third, to have demonstrated its usefulness empirically. The article is organized as follows. The first part 1 describes the hedging problem: the forward commitment on the physical commodity market and the choice of short-term futures contracts to protect the commitment against prices fluctuations. Part 2 presents the term structure models which serve as a basis for the dynamic hedging strategies and the hedge ratios associated with these models. Part 3 is devoted to the analysis of the performance of the hedging strategies. Part 4 presents empirical results and Part 5 concludes.

I. THE HEDGING PROBLEM

This example which considers an operator on the physical market[5], is based on Metallgesellschaft's strategy: at a date t, a trader sells forward a barrel of crude oil for delivery at T. He does not have this barrel now and decides to wait until a few days before delivery to buy it on the spot market, thereby avoiding the storage costs associated with the commodity between t and T, but risking a rise in spot prices.

In order to protect himself against fluctuations in the spot price, the trader initiates a dynamic hedging strategy on the futures market: the maturity of the position hold on the paper market is shorter than the one of the commitment. This could be because of the lack of liquidity on the longer maturities in most derivative markets[6], and/or because the available futures contracts do not have a sufficient maturity to cover the position held on the physical market (as was the case for Metallgesellschaft). The gap between the maturities of the physical and paper positions leads naturally to a stack and roll strategy. Indeed, the use of shorter maturities supposes that the hedge portfolio is rebalanced regularly as the future's expiry date approaches, in order to constantly maintain the position on the futures market.

II. TERM STRUCTURE MODELS OF COMMODITY PRICES AND HEDGE RATIOS

In this article, the dynamic hedging strategies are based on three term structure models of commodity prices[7]. So, it relies on the relationship between the futures prices of different maturities and uses combinations of futures contracts having different maturities. Each of these term structure models has specific implications for hedging strategies, and leads to different hedge ratios.

A. Three term structure models

The three term structure models chosen for the study of dynamic hedging strategies are a one-, a two-, and a three-factor model. The first two were proposed by Schwartz (1997). They are currently well known and extensively used. The single state variable of the first one is naturally the spot price. It is mean reverting, in order to ensure the realism of the model. The two-factor model is an extension of the previous one. It is also mean reverting and includes an additional variable: the convenience yield. Lastly, the three-factor model of Cortazar and Schwartz' (2003) includes a third variable: the long-term price. So there is continuity between the different models, from the simplest to the most complicated.

1. One-factor model

A futures price is often defined as the expectation of the future spot price, conditional on the available information at date t and under the risk neutral probability. As the spot price is the main determinant of the futures price, most one-factor models rely on the spot price. There have been several one factor models in the literature on commodity prices. Most of them[8] retain a mean reverting process for their one-factor model. The storage theory (Brennan (1958)) and the Samuelson effect (Samuelson (1965)) show that this kind of dynamics, especially for commodity prices, is more relevant than a simple geometric Brownian motion. Among these models, that of Schwartz (1997), inspired by Ross (1995) is the best-known. In that case, the dynamic of the spot price under the historical probability is:

$$dS = \kappa(\mu - \ln S)Sdt + \sigma Sdz \qquad [1]$$

with : $\kappa, \sigma > 0$

where: - S is the spot price,

- μ the long-run mean of lnS,
- κ is the speed of adjustment of lnS,
- σ is the volatility of the spot price,
- dz is the increment of a standard Brownian motion associated with S.

Defining X = ln S and applying Ito's Lemma, this implies that the log price is an Ornstein-Uhlenbeck stochastic process:

$$dX = \kappa(\alpha - X)dt + \sigma dz$$

with: $\alpha = \mu - \frac{\sigma^2}{2\kappa}$

Solving this equation under the risk-neutral probability, gives the relationship between the futures price *F* at *t* for delivery in *T* (which is observable) and the variable *S*:

$$F(S,t,T) = \exp\left(\alpha + \left(\ln S - \hat{\alpha}\right)e^{-\kappa\tau} + \frac{\sigma^2}{4}H\left(\kappa,\tau\right)\right) \qquad [2]$$

with: $\tau = T - t$, $\widehat{\alpha} = \alpha - (\lambda / \kappa)$ and $H(x, \tau) = \dfrac{1 - e^{-2x\tau}}{x}$

The term structure of proportional futures volatility is:

$$\sigma_F^2 = \sigma^2 e^{-2\kappa\tau}$$

Two features of this model are that as the time to maturity of the futures contract tends to infinity, the volatility of its price converges to zero but that the futures price converges toward a fixed value, which is independent of the spot price:

$$F(S,\infty) = \exp\left(\widehat{\alpha} + \frac{\sigma^2}{4\kappa}\right)$$

In this model, when the spot price is above *F(S,∞)*, the term structure is in backwardation. Conversely, when the spot price is below, it is in contango. The main limitation of this model is that it does not allow for much flexibility in the prices curves.

2. Two-factor model

When a second state variable is introduced in a term structure model, it is usually the convenience yield. However, two-factor models with a long term price have also been developed. In all models, the introduction of a second state variable allows for more varied shapes of curves than one factor models and more varied volatility structures. Schwartz' model (1997) is probably the most famous term structure model of commodity prices. It was used as a reference to develop several models that are more sophisticated[9].

Based on the one proposed by Gibson and Schwartz in 1990, Schwartz' model (1997) is more tractable than the earlier version: it has an analytical solution. The two-factor model supposes that the spot price *S* and the convenience yield *C* can explain the behavior of the futures price *F*. The dynamics of these state variables is:

$$\begin{cases} dS = (\mu - C)Sdt + \sigma_S S dz_S \\ dC = \left[k(\alpha - C)\right]dt + \sigma_C dz_C \end{cases} \qquad [3]$$

with: κ, σ_S, $\sigma_C > 0$

where: - μ is the drift of the spot price,

- σ_S is the spot price volatility,

- dz_S is an increment to a standard Brownian motion associated with *S*,

- α is the long-run mean of the convenience yield,

- κ is the speed of adjustment of the convenience yield,

- σ_C is the volatility of the convenience yield,

- dz_C is an increment to a standard Brownian motion associated with *C*.

The increments to standard Brownian motions are correlated, with:

$$E[dz_S \times dz_C] = \rho dt$$

where ρ is the correlation coefficient.

Solving this equation gives the relationship at *t* between an observable futures price *F* for delivery in *T* and the state variables *S* and *C*:

$$F(S,C,t,T) = S(t) \times \exp\left[-C(t)\frac{1-e^{-\kappa\tau}}{\kappa} + B(\tau) \right] \quad [4]$$

with:

- $B(\tau) = \left(r - \hat{\alpha} + \frac{\sigma_C^2}{2\kappa^2} - \frac{\sigma_S \sigma_C \rho}{\kappa} \right) \times \tau + \left(\frac{\sigma_C^2}{4} \right) \times \left(\frac{1-e^{-2\kappa\tau}}{\kappa^3} \right) + \left(\hat{\alpha}\kappa + \sigma_S \sigma_C \rho - \frac{\sigma_C^2}{\kappa} \right) \times \left(\frac{1-e^{-\kappa\tau}}{\kappa^2} \right)$

- $\hat{\alpha} = \alpha - (\lambda / \kappa)$

where: - *r* is the risk free interest rate, assumed constant,

- λ is the market price of convenience yield risk,

- $\tau = T - t$ is the maturity of the futures contract.

Various shapes of prices curves can be obtained with two-factor models: sunken, humped, or flat. Furthermore, the volatility of the futures prices decreases with the maturity τ:

$$\sigma_F^2(\tau) = \sigma_S^2 + \sigma_C^2 \left(\frac{1-e^{-\kappa\tau}}{\kappa} \right)^2 - \left[2 \times \frac{1-e^{-\kappa\tau}}{\kappa} \times \rho \sigma_S \sigma_C \right]$$

When the contract reaches its expiry date, the futures price's volatility converges towards the spot price's volatility. Conversely, when the maturity tends towards infinity, the volatility of the futures price tends towards a fixed value:

$$\lim_{\tau \to \infty} \sigma_F^2 = \sigma_S^2 + \frac{\sigma_C^2}{\kappa^2} - \frac{2\rho\sigma_S\sigma_C}{\kappa}$$

The price to pay for the increased flexibility is increased complexity, because the model has six parameters, as opposed to two for the one factor model.

3. Three-factor model

In 2003, Cortazar and Schwartz proposed a three-factor model related to Schwartz (1997). In this model, the authors consider as a third risk factor the long term spot price return, allowing it to be stochastic and to return to a long term average. The two other stochastic variables are the spot price and the convenience yield. The latter models short-term variations in prices due to changes in inventory, whereas the long term return is due to changes in technologies, inflation, and demand pattern. The dynamic of these state variables is the following:

$$\begin{cases} dS = (v - y)Sdt + \sigma_1 Sdz_1 \\ dy = -\kappa y dt + \sigma_2 dz_2 \\ dv = a(\overline{v} - v)dt + \sigma_3 dz_3 \end{cases} \qquad [5]$$

with:

$$dz_1 dz_2 = \rho_{12} dt$$

$$dz_1 dz_3 = \rho_{13} dt$$

$$dz_2 dz_3 = \rho_{23} dt$$

where:
- *S* is the spot price
- *y* is the demeaned convenience yield, with $y = C - \alpha$, where α is the long run mean of the convenience yield *C*
- *v* is the expected long-term spot price return
- κ is the speed of adjustment of the demeaned convenience yield
- *a* is the speed of adjustment of *v*

- $\bar{v}$ is the long-run mean of the expected long term spot price return
- σ_i is the volatility of the variable i
- ρ_{ij} is the correlation between the variables i and j
- dz_i is the increment of a standard Brownian motion associated with the variable i

The expression for the futures price is:

$$F(S,y,v,t,T) = S(t) \times \exp\left(-y(t)H(\kappa,\tau) + v(t)H(a,\tau) + \varphi(\tau)\right) \quad [6]$$

with: $H(x,\tau) = \dfrac{1-e^{-x\tau}}{x}$

$$\begin{aligned}\varphi(\tau) = {} & \mu\tau + \frac{1}{2}\sigma_3^2\left[\frac{\tau - H(a,\tau)}{a^2} - \frac{H(a,\tau)^2}{2a}\right] + \frac{1}{2}\sigma_2^2\left[\frac{\tau - H(\kappa,\tau)}{\kappa^2} - \frac{H(\kappa,\tau)^2}{2\kappa}\right] \\ & + \frac{\sigma_1\sigma_3\rho_{13}}{a}\left(\tau - H(a,\tau)\right) - \frac{\sigma_1\sigma_2\rho_{12}}{\kappa}\left(\tau - H(\kappa,\tau)\right) \\ & - \frac{\sigma_3\sigma_2\rho_{23}}{a+\kappa}\left[\tau\left(\frac{1}{a} + \frac{1}{\kappa}\right) - \frac{1}{\kappa}H(\kappa,\tau) - \frac{1}{a}H(a,\tau) - H(a,\tau)H(\kappa,\tau)\right]\end{aligned}$$

$$\mu = \bar{v} - (\lambda_3 + \lambda_2 + \lambda_1)$$

The volatility of futures returns is:

$$\sigma_F^2(\tau) = \sigma_1^2 + \sigma_2^2\frac{\left(1-e^{-\kappa\tau}\right)^2}{\kappa^2} + \sigma_3^2\frac{\left(1-e^{-a\tau}\right)^2}{a^2} - 2\sigma_1\sigma_2\rho_{12}\frac{\left(1-e^{-\kappa\tau}\right)}{\kappa} + 2\sigma_1\sigma_3\rho_{13}\frac{\left(1-e^{-a\tau}\right)}{a} - 2\sigma_2\sigma_3\rho_{23}\frac{\left(1-e^{-\kappa\tau}\right)\left(1-e^{-a\tau}\right)}{a\kappa}$$

As τ goes to infinity, this converges to:

$$\sigma_F^2(\tau \to \infty) = \sigma_1^2 + \frac{\sigma_2^2}{\kappa^2} + \frac{\sigma_3^2}{a^2} - \frac{2\sigma_1\sigma_2\rho_{12}}{\kappa} + \frac{2\sigma_1\sigma_3\rho_{13}}{a} - \frac{2\sigma_2\sigma_3\rho_{23}}{a\kappa}$$

This volatility term structure decreases with maturity and converges to a positive constant. This is consistent with a mean reverting non stationary process.

Each model containing a different number of underlying factors, each of them has specific implications for hedging strategies. This leads naturally to different hedge ratios.

B. Hedge ratios

In this paragraph, we compute the hedge ratios associated to the term structure models[10], in a way that provides an easy way to compare the models.

To properly hedge the forward commitment, the sensitivity of the present value of the commitment with respect to each of the underlying factors must equal that of the portfolio of futures contracts used to hedge the commitment with respect to the same factors. So the number of positions w_i in futures contract with maturity τ_i required to hedge the forward commitment to deliver one unit of a commodity at time T is obtained by solving a system of equations. Lastly, since the three models assume constant interest rates, and therefore that futures prices are equal to forward prices, the present value of the forward commitment per unit of the commodity is obtained by discounting the future (forward) price.

The one factor model requires only one position in the futures market[11]:

$$w_1 F_S(S,\tau_1) = e^{-r\tau} F_S(S,\tau) \quad [7]$$

The solution of this equation is:

$$w_1 = e^{-r\tau} \frac{Y(\tau)}{Y(\tau_1)} \quad [8]$$

where: $Y(\tau_i) = F_S(S,\tau_i)$

This hedge position represents the uncertainty associated with the spot price. As transaction costs are high in physical commodity markets, it seems natural to choose a short-term futures contract as the proxy for this hedge position.

In the two factor model the second source of uncertainty is the convenience yield. As there is no traded asset representing the convenience yield, a second position in the futures market is taken as a hedge. So we require two positions for delivery at times t_1 and t_2:

$$\begin{cases} w_1 F_S(S,C,\tau_1) + w_2 F_S(S,C,\tau_2) = e^{-r\tau} F_S(S,C,\tau) \\ w_1 F_C(S,C,\tau_1) + w_2 F_C(S,C,\tau_2) = e^{-r\tau} F_C(S,C,\tau) \end{cases} \quad [9]$$

The solution of the system is:

$$w=e^{-r\tau}Y(\tau)\begin{bmatrix}\dfrac{H(\kappa,\tau_2)-H(\kappa,\tau)}{(H(\kappa,\tau_2)-H(\kappa,\tau_1))Y(\tau_1)}\\ \dfrac{H(\kappa,\tau)-H(\kappa,\tau_1)}{(H(\kappa,\tau_2)-H(\kappa,\tau_1))Y(\tau_2)}\end{bmatrix} \quad [10]$$

where: $Y(\tau_i)=F_S(S,C,\tau_i)$

Lastly, the three-factor model requires three positions, respectively representing the spot price, the convenience yield, and the long term price:

$$\begin{cases} w_1F_S(S,y,v,\tau_1)+w_2F_S(S,y,v,\tau_2)+w_3F_S(S,y,v,\tau_3)=e^{-r\tau}F_S(S,y,v,\tau)\\ w_1F_y(S,y,v,\tau_1)+w_2F_y(S,y,v,\tau_2)+w_3F_y(S,y,v,\tau_3)=e^{-r\tau}F_y(S,y,v,\tau)\\ w_1F_v(S,y,v,\tau_1)+w_2F_v(S,y,v,\tau_2)+w_3F_v(S,y,v,\tau_3)=e^{-r\tau}F_v(S,y,v,\tau)\end{cases} \quad [11]$$

As there is no traded asset associated with the long-term price, a third futures position is taken as a hedge. So, the hedge portfolio is a combination of futures contracts.

The solution of this system[12] is:

$$w=e^{-r\tau}Y(\tau)\begin{bmatrix}\dfrac{W_1}{Y(\tau_1)}\\ \dfrac{W_2}{Y(\tau_2)}\\ \dfrac{W_3}{Y(\tau_3)}\end{bmatrix} \quad [12]$$

where:

$$W=\begin{bmatrix}\dfrac{H(\kappa,\tau_2)H(a,\tau_3)-H(\kappa,\tau_3)H(a,\tau_2)+H(\kappa,\tau)(H(a,\tau_2)-H(a,\tau_3))+H(a,\tau)(H(\kappa,\tau_3)-H(\kappa,\tau_2))}{D}\\ \dfrac{H(\kappa,\tau_1)H(a,\tau_3)-H(\kappa,\tau_3)H(a,\tau_1)+H(\kappa,\tau)(H(a,\tau_1)-H(a,\tau_3))+H(a,\tau)(H(\kappa,\tau_3)-H(\kappa,\tau_1))}{D}\\ \dfrac{H(\kappa,\tau_2)H(a,\tau_1)-H(\kappa,\tau_1)H(a,\tau_2)+H(\kappa,\tau)(H(a,\tau_2)-H(a,\tau_1))+H(a,\tau)(H(\kappa,\tau_1)-H(\kappa,\tau_2))}{D}\end{bmatrix}$$

$$D=(H(\kappa,\tau_3)-H(\kappa,\tau_2))H(a,\tau_1)+(H(\kappa,\tau_1)-H(\kappa,\tau_3))H(a,\tau_2)+(H(\kappa,\tau_2)-H(\kappa,\tau_1))H(a,\tau_3)$$

One feature of hedge portfolios in commodity markets is quite specific, if we compare them with other derivative markets: there is no investment in the physical asset. So there is no need to finance this investment. In addition as all hedging positions are futures positions, they only require a deposit. In order to determine the weight w_i, it is necessary to choose the maturity τ_i. More traditionally, all the hedge ratios presented above are state dependent and decline with the maturity of the forward position.

Equations [7], [9] and [11] involve various first-order derivatives. For the models considered here, these derivatives are, up to a factor, the initial futures. This property proves to be very useful for the estimation procedure.

III. PERFORMANCES OF THE HEDGING STRATEGIES

The performances of the hedging strategies are judged by the ability of the hedge positions undertaken in the futures market to ensure a correct hedge for the commitment in the forward market. So the prices fluctuations recorded on the futures contracts must be as close as possible to the changes recorded on the forward contract. This kind of approach is comparable to a delta hedge.

A. Delta hedge of the forward commitment

Following Brennan and Crew (1997), the hedging period is decomposed into several sub periods, and new hedge ratios are computed each time the position is rolled on the futures market. So an intermediate hedging error corresponding to each sub period is computed. It compares – for example each month – the intermediate results on the paper and physical markets.

The intermediate result at *t* on the physical market is the variation of the present value of the forward commitment, between *(t-1)* and *t*:

$$\Delta L_{t,\tau} = L_{t,\tau} - L_{t-1,\tau+1} \qquad [13]$$

with: $$L_{t,\tau} = e^{-r_{t,\tau}\tau} F(t,\tau)$$

where:
- *L* is the present value of the forward commitment,
- τ is the maturity of the commitment,
- *r* is the interest rate observed at t for the maturity τ

- $F(t, \tau)$ is the futures price at t with maturity τ[13]

The intermediate result at t on the futures market depends on the hedge ratios and on the variations of the futures prices between *(t-1)* and *t*. For example, for the two-factor model, the hedge is made of two different positions w_1 and w_2, which maturities are respectively τ_1 and τ_2. In that case, the intermediate result recorded during one sub period is:

$$w_1 \Delta F_1 + w_2 \Delta F_2$$

where $\Delta_i F_i$ represents the variation of the futures price of maturity τ_i between *(t-1)* and *t*:

$$\Delta F_i = F(t, \tau_i - 1) - F(t-1, \tau_i)$$

Naturally, the pairs (w_i, τ_i) change with the model and maturities chosen for the hedge ratios.

Now, the intermediate error $e_{t,\tau}$ between *(t-1)* and *t* can be written:

$$e_{t,\tau} = w_1 \Delta F_1 + w_2 \Delta F_2 - \Delta L_{t,\tau} \qquad [14]$$

If the hedge was perfect, the error would be equal to zero. A positive error means that the futures positions overestimate the variations in the value of the forward commitment: there is an over-hedge. On the contrary, a negative error implies an underestimation of the fluctuations in the value of the commitment.

B. Estimation procedure

In order to test the dynamic hedging strategies, we need the parameter values of each model but these depend on non-observable state variables. The spot price, the convenience yield and the long-term price are regarded as non-observable because most of the time, there are no reliable time series for the spot price and because the convenience yield and long term price are not traded assets. A Kalman filter [14] is used to overcome this difficulty. However, it is used in a specific way, in order to take into account the errors associated with the futures prices or with the hedge ratios, depending on the objectives of the study.

1. The errors associated with futures prices and hedge ratios

When the Kalman filter is applied to term structure models of commodity prices, the measurement equation represents the relationship linking the futures prices and the state-variables (equations [2], [4] and [6]). Usually, the objective is to estimate the measurement equation's parameters, in order to obtain estimates of futures prices for different maturities $F^*(\tau_i)$, and to compare them with observed futures prices $F(\tau_i)$. The closer they are, the better is the model. The model's performance is usually judged by analyzing the estimation error for one specific maturity τ_i and for the whole estimation period.

The application of term structure models to dynamic hedging strategies however introduces another potential source of error: the one associated with hedge ratios. As shown by equations [7], [9] and [11], the hedge ratios are obtained by calculating the first derivatives of the function linking the futures prices and the state variables. Thus, in order to adjust the sensitivity of the present value of the commitment to that of the hedge portfolio, we suggest that for hedging purposes, the models estimations rely on the derivatives themselves instead of the prices, as is usually done.

In order to take into account the errors associated with hedge ratios as well as prices' errors, and to control all the dimensions of the first derivatives used to determine the hedge ratios, we minimized the weighted error E_t:

$$\|E_t\|^2 = \beta\|v_t\|^2 + (1-\beta)\|e_t\|^2 \qquad [15]$$

where:

- $v_t = y_t - y_t^*$ is the innovation in the Kalman filter. It represents the difference between the prices estimated with the model and the observed futures prices,

- e_t is the intermediate error of the hedging strategy that we define below,

- $\beta \in [0,1]$ is the weight associated with the price error,

- *(1- ß)* is the weight of the error associated with hedge ratios.

When the purpose of the estimation procedure is to determine which model is the best suited to replicate the price curve, the relevant criterion for the estimation is the innovation v_t. The errors associated with the hedge ratios are ignored, *ß* is set to 1, and E_t reflects only prices errors. Conversely, when the objective is to construct optimal dynamic hedging strategies, the prices innovations are not taken into account and *ß* is set to zero[15]. If we were interested in hedging and pricing, we would have to adjust empirically the parameter *ß.*

2. A focus on the errors associated with hedge ratios

The hedge ratios w_i are set to satisfy equations [8], [10] or [12]. However, as there is a discrepancy between the model and reality, if we replace the derivatives computed in the model by the actual ones, that is to say, the ones computed from futures prices, there will be an error $\tilde{e}_t$ which changes with the model. For example, for the one-factor model, $\tilde{e}_t$ is:

$$\tilde{e}_t = w_1 F_S^*(S,\tau_1) - e^{-r\tau} F_S^*(S,\tau) \qquad [16]$$

The star indicates the use of the true derivatives.

We would like to minimise the error of the hedging strategy but we cannot use the derivatives computed with the model because the error would by definition be zero. Furthermore this error assumes that the model is correct. So we would have to use the true derivatives of the futures which are unknown. One way would be to estimate them from the data but these types of estimates are often numerically unstable. At this point we use the fact that in all three term structure models considered in this study, the derivatives are proportional to the futures prices. So after multiplying the first equation in each model by S(t), we obtain a new error, involving the futures values themselves. For example, in the first model, this new error term e_t is*:*

$$e_t = e^{\kappa\tau_1} w_1 F(\tau_1) - e^{(\kappa - r)\tau} F(\tau)$$

For the other two models we do not have to change the second and third terms, because the derivatives are already expressed as a constant multiplied by the futures' value. So, for the two factor model, the term to minimize is:

$$\tilde{e}_t = \begin{bmatrix} \tilde{e}_t^1 \\ \tilde{e}_t^2 \end{bmatrix} = \begin{bmatrix} w_1 F_S^*(S,C,\tau_1) + w_2 F_S^*(S,C,\tau_2) - e^{-r\tau} F_S^*(S,C,\tau) \\ w_1 F_C^*(S,C,\tau_1) + w_2 F_C^*(S,C,\tau_2) - e^{-r\tau} F_C^*(S,C,\tau) \end{bmatrix}$$

From paragraph 2.2., it is possible to write:

$$F_C(S,C,\tau_i) = -H(\kappa,\tau_i)\times F(\tau_i)$$

Thus e_t becomes:

$$e_t = \begin{bmatrix} e_t^1 \\ e_t^2 \end{bmatrix} = \begin{bmatrix} w_1 F(\tau_1) + w_2 F(\tau_2) - e^{-r\tau} F(\tau) \\ w_1 H(\kappa,\tau_1) F(\tau_1) + w_2 H(\kappa,\tau_2) F(\tau_2) - e^{-r\tau} H(\kappa,\tau) F(\tau) \end{bmatrix}$$

Similarly, for the three factor model, e_t is:

$$e_t = \begin{bmatrix} e_t^1 \\ e_t^2 \\ e_t^3 \end{bmatrix} = \begin{bmatrix} w_1 F(\tau_1) + w_2 F(\tau_2) + w_3 F(\tau_3) - e^{-r\tau} F(\tau) \\ w_1 H(\kappa,\tau_1) F(\tau_1) + w_2 H(\kappa,\tau_2) F(\tau_2) + w_3 H(\kappa,\tau_3) F(\tau_3) - e^{-r\tau} H(\kappa,\tau) F(\tau) \\ w_1 H(a,\tau_1) F(\tau_1) + w_2 H(a,\tau_2) F(\tau_2) + w_3 H(a,\tau_3) F(\tau_3) - e^{-r\tau} H(a,\tau) F(\tau) \end{bmatrix}$$

These errors terms are directly related to our objective to hedge a long-term commitment with short-term futures contracts.

IV. EMPIRICAL TESTS

In this part of the article, we first present the data. Then we compare the ability of each model to replicate the prices curve. Lastly, we present the results obtained with the dynamic hedging strategies. The latter are tested in a way which is as close as possible to the situation of the trader. We suppose that the operator uses all available information and calibrates his model each time he initiates a new position in the futures market. In other words, we use out-of-sample procedures.

A. Data

The crude oil data are daily settlement prices for the light, sweet, crude oil contract negotiated on the New York Mercantile Exchange (Nymex), from 17 March 1997 to 21 June 2005. When preparing the data, we reconstructed the Nymex calendar in order to determine precisely which contract has, for example, a one- or a two-month maturity and to determine when the contract changes from the two-month's maturity to one-month. Care is also required with the interest rates. We used the daily interest rates for US government bonds for the appropriate maturity. All the data (futures prices and interest rates) were extracted from Datastream.

Figure 1 gives an overview of the behavior of futures prices for the shortest and longest maturities, from 1997 to 2005. It shows that during this period, prices fluctuate markedly, ranging from a lower level of $11/b to the highest level of $60/b.

INSERT FIGURE 1 ABOUT HERE

B. The performance of the models in replicating the prices curve

We need the parameters of the models in order to measure their ability to reproduce the prices curve. Monthly observations on crude oil futures prices were used to estimate the optimal parameters associated with each term structure. Moreover, for the parameter estimations, and for the measure of the performances, four series of futures prices were retained and we used rolling windows. The length of these windows corresponds to the maturity of the commitment on the physical market. For the latter, we chose six different delivery dates: 7, 6, 5, 4, 3, and 2 years. The fourth price used for the calibration has a maturity that is identical to the one of the physical commitment. Thus, we took into account first, that the parameters change with time, second that they vary with maturity (Schwartz, 1997). Moreover, because the nearest futures price contains a lot of information but may also contain a lot of noise, we always retained the 2nd month maturity for the estimations.

Table 1 gives the performances obtained for the three models (M1 stands for the one-factor model, M2 for the two-factor model and M3 for the three-factor model), for different maturities on the physical market. On the basis of the RMSE computed on futures prices, the three factor model outperforms the two others whatever the delivery date of the commitment. Moreover, the RRMSE show that the differences between the performances increase with the maturity of the commitment: the longer, the horizon of analysis, the better is the three factor model. Lastly, the performances decrease with the length of the commitment on the physical market. These results suggest that dynamic hedging strategies relying on this model will be the best.

INSERT TABLE 1 ABOUT HERE

C. The hedging strategies

In this paragraph, after presenting the dynamic hedging strategies tested on the observation period and the choices we made in order to undertake the empirical tests, we give the results of the tests.

1. Presentation

The database allows us to test numerous strategies. For example, with a hedge portfolio rebalanced each month and a forward commitment having a seven years' maturity, it is possible to examine 16 strategies per term structure model: from March 1997 to March 2004, from April 1997 to April 2004, etc. Similarly, for a six-year commitment, it is possible to study 28 strategies; for a five year commitment, the number increases to 40 strategies, and so on. For each strategy, different combinations of maturities can be used for the hedge ratios. If we retain two different combinations of maturities for the hedge ratios, 32 strategies can be studied for a seven-year commitment, 56 for a six-year commitment, and so on. The number of strategies is even larger if we use different frequencies for the rebalancing of the portfolio.

Our choice of the maturities of the positions composing the hedge portfolio respects two different constraints: first, the necessity to keep a short-term contract in the hedge portfolio, in order to benefit from the higher liquidity on these contracts; second, the fact that each futures contract must represent a different source of uncertainty. If the maturities of the futures positions are too close to each other, the futures prices will evolve similarly and there is a risk that the hedge is not efficient. This is why we chose the maturities of 2, 6 and 18 months.

2. Performances of the dynamic hedging strategies

In order to measure the performance of the dynamic hedging strategies, we carried out and compared 3,912 strategies. Different tests were undertaken. On the basis of traditional statistics we show i) the influence of the choice of the model on the performances of the strategies; ii) the sensitivity of the performance to the frequency of the roll over; iii) the variations in the performances when the maturities chosen for the calibration change; iv) how the performances change when the errors associated with the hedge ratios are taken into account. We then confirm our findings by examining the frequency distributions of the errors for all these tests.

● The choice of the model

Table 2 presents the results obtained with the three term structure models on various delivery dates. For these tests, we rebalanced the hedge portfolios each month. For a 24 month strategy, 24 intermediate errors were computed. Between 1997 and 2005, it was possible to carry out 76 strategies for this maturity, which means 24 x 76 = 1,824 monthly errors. Likewise, for the 48 month delivery, there were 52 strategies and 48 x 52=2,496 errors; for the 60-month delivery, we obtained 2,400 errors; for the 72 months, 2016 and for the 84 months, 1,344. We computed the summary statistics for these errors: the mean, standard deviation, minimum and maximum. Statistics on the frequency distributions are left for further comments (see Table 7 below).

INSERT TABLE 2 ABOUT HERE

Table 2 shows that the relative performance of the different strategies depends on the model and maturity of the commitment on the physical market. For a delivery date situated between 24 and 36 months, the three-factor model outperforms the two factor model, which in turn outperforms the one factor model. These results are in line with the performances of the models in Table 1: the better, the model; the better, the performance of the strategies. However, this ordering changes as the maturity of the forward commitment increases: then, the best strategies are those relying on the one factor model, and the three factor model is the worst, whatever the criterion.

● The frequency of the roll over

When using dynamic hedging strategies, especially on long-term horizons, the frequency of the roll over is an important factor. The higher the frequency, the closer the hedging strategy becomes to a continuous time strategy with low errors. However, rolling the portfolio incurs transaction costs. Moreover, there is a need to optimize the procedure each time the portfolio is rebalanced.

The sensitivity of the performance to the frequency of the portfolio is illustrated by Tables 3 and 4. They show how performance changes with a portfolio rebalanced, respectively every two or three months. These results must be compared to those in Table 2, where the portfolio is rebalanced each month.

INSERT TABLE 3 ABOUT HERE

The results presented in Tables 3 and 4 are in line with those in Table 2, as far as the order of the different strategies / models is concerned: for short-term maturities, the three-factor model is the best, the one-factor the worst. However, for maturities exceeding 36 months, the hierarchy is reversed. The comparison between Tables 2, 3 and 4 also shows that the lower the frequency of the rollover, the worse the performance of the dynamic hedging strategies. The maximum and minimum hedging errors become really important, especially for the three-factor model. More interestingly, the sensitivity to the frequency of the rollover is higher for the one factor model, whereas the sensitivities of the two and three factor models are comparable.

INSERT TABLE 4 ABOUT HERE

● The maturities chosen for the calibration of the models and the hedge portfolio

Another important choice for dynamic hedging strategies is the maturity retained for calibrating the term structure models and the hedge portfolios. As mentioned before, different kinds of constraints must be taken into account. The first is the lack of liquidity for long-term futures contracts. The second constraint is the fact that the futures contracts' maturities must not be too close to each other. A third one, which is incompatible with the first constraint, is that the best way to reconstitute the term structure of futures prices is to take into account the two ends of the curve, because they have different informational contents (Lautier, (2005)).

In order to examine the sensitivity of the tests to the maturity chosen for the calibration, we used longer maturities: 2, 12 and 24 months, instead of 2, 6, and 18 months. Comparing Table 5 with

Table 2, the performances of all strategies are markedly improved. Moreover, the ordering of the different models is now more intuitive: the three-factor model outperforms the two others, whatever delivery date is concerned. Still, the one-factor model performs quite well. This shows that although using shorter maturities ensures better liquidity, the resulting hedging strategies do not perform as well.

INSERT TABLE 5 ABOUT HERE

- The errors associated with hedge ratios

The last test takes into account the errors associated with the hedge portfolios during the optimization procedure, as was described in paragraph 3.2.2. Table 6 presents the results obtained. These are significantly better than those in Tables 2 and 5 whatever maturity of commitment is considered, and whatever criterion is used. With this technique, the three-factor model clearly outperforms the others, whatever the delivery date taken into account. Moreover, the extreme values of the errors are substantially reduced.

INSERT TABLE 6 ABOUT HERE

So, taking account of the errors associated with hedge ratios gives us the best of both worlds: the higher liquidity of short term futures contracts for the hedge portfolio, together with high quality performance[16].

- Frequency distribution of errors

The better performance of the dynamic strategies when the error associated with the hedge ratios are taken into account, is confirmed by the frequency distributions of the monthly hedging errors. Table 7 summarizes the characteristics of the hedging errors corresponding to Tables 2, 5 and 6.

INSERT TABLE 7 ABOUT HERE

The frequency distributions of the monthly errors corresponding to the different tests share some general features. First, they all have very high kurtosis values (the Jarque-Bera tests always leads to rejecting the hypothesis of normality, at the 5% level). Second, the skewness is generally negative: the errors are spread more to the left of the mean, which means that the strategies are under-hedged. These results are consistent with those obtained by Harris and Chen (2006).

Table 7 summarizes three different situations. In the first one (β = 1 and F=2_6_18), the three models are compared on the basis of a “standard” calibration procedure. The maturities taken into account are rather short. The three-factor model not only performs better for short term commitments, as far as the standards statistics in Table 2 are concerned, but it also allows for an increase of low hedging errors: with the three-factor model, when the maturity of the commitment is set to 36 months, 87.78% of the errors are situated between -0.5 and +0.5 dollar per barrel, whereas the same figures for the one- and two-factor models are respectively 44.44% and 56.38%. However, when the maturity of the commitment increases, this ordering can change.

The second situation depicted by Table 7 corresponds to the standard calibration procedure, but using long-term maturities (see also Table 5). In that case, the information conveyed by the prices of the futures contracts retained for the optimization is higher. Compared with the previous tests, long-term maturities give better results. However, using long-term maturities amounts to giving up on liquidity.

The third situation corresponds to the new calibration procedure, when the errors associated with hedge ratios are taken into account and short-term maturities are used for the futures contracts (see also Table 6). The results in Table 7 and Figure 2 show that these hedging strategies outperform the others. While benefiting from the higher liquidity of short term contracts, the new calibration procedure leads to a reduction in the extreme values of the errors, a marked increase in the percentage of low hedging errors and a marked decrease in the percentage of high hedging errors. Moreover, the tendency to under hedge the physical commitment clearly diminishes.

INSERT FIGURE 2 ABOUT HERE

● Hedge ratios

Another difficulty that arises when carrying-out dynamic hedging strategies based on term structure models is that the number of the futures positions in the hedge portfolio is at least equal to the number of state variables in the model. With a three factor model, this might create some difficulties.

Figure 3 illustrates this point. It presents the average hedge ratios associated with the dynamic hedging strategies based on the three factor model, with a two-year maturity for the commitment on the physical market, with a standard calibration procedure, and with short-term maturities for the futures contracts. In the study period, we could carry out 76 hedging strategies having a two-year commitment.

Figure 3 shows that when the time to maturity is equal to 24 months (*i.e*. when the forward contract is initiated), in order to hedge one physical barrel, we need 2.67 paper barrels on average. The hedge portfolio is composed of a short position on the six month futures contract corresponding to 0.88 paper barrel (M3_B1_C2_2618_W2(M)) and two long positions; firstly 0.37 paper barrel on the two-month futures contract (M3_B1_C2_2618_W1(M)) and secondly 1.42 paper barrels on the eighteen-month futures (M3_B1_C2_2618_W3(M)).

INSERT FIGURE 3 ABOUT HERE

The situation worsens when the maturity of the commitment on the physical market reaches seven years[17], as illustrated by Figure 4. In that case, when the time to maturity is 84 months, 20.28 paper barrels[18] are on average needed to hedge 1 physical barrel! This raises not only the problem of liquidity, but also that of transaction costs.

INSERT FIGURE 4 ABOUT HERE

Figures 5 and 6 illustrate what happens when the errors associated with hedge ratios are taken into account. Figure 4 shows that to hedge a 24 months commitment, there is now a need, on average, for 1.63 paper barrels per physical barrel[19]. And the improvement becomes spectacular when considering a seven-years commitment, as shown by Figure 5: then, 1.62 paper barrels are needed for 1 physical barrel, on average[20]. This figure must be compared with 20.28 given above.

INSERT FIGURE 5 ABOUT HERE

INSERT FIGURE 6 ABOUT HERE

One difficulty still remains: futures contracts are standardized, and it is not possible to buy or sell arbitrary quantities of paper barrels. More precisely, on the crude oil market, the futures contracts correspond to a volume of 1,000 barrels. This constraint probably also has an impact on the performances of the dynamic hedging strategies, but we leave this problem for future research.

CONCLUSION

This article presents a new and original way for calibrating term structure models for dynamic hedging strategies. These strategies are important in all derivative markets, especially when long-term transactions are considered. Term structure models constitute a natural theoretical framework to study them as they are based on arbitrage reasoning. Taking into account the errors associated with hedge ratios gives us the best of two seemingly incompatible worlds: the higher liquidity of short-term futures contracts for the hedge portfolios, together with markedly improved performances.

We studied the dynamic hedging strategies associated with three recent term structure models of commodity prices, with one, two, and three state-variables, respectively. The single factor of the first one is the spot price, which is mean reverting. The two factor model is an extension of the previous one. It is also mean reverting and includes an additional variable: the convenience yield. Lastly, the three factor model includes the long term price as the third variable. So, there is continuity between the different models, which enriches the comparison of the hedging performances.

We chose to perform our empirical tests on the crude oil market using the futures contract negotiated on the New York Mercantile Exchange as it is the most actively traded commodity futures contract. One reason for testing the performance of these strategies on commodities is that as the state variables of the term structure models are non observable, the hedge portfolios are combinations of futures positions having different maturities.

Our analysis of more than 3,900 strategies (including more than 45,000 hedging errors) showed that taking account of the errors associated with hedge ratios when constructing the portfolios improves the performances of the hedging strategies, while still allowing us to hedge with short-term instruments. It also reduces the mean and standard deviation of the errors, especially for long-term commitments; it drastically decreases the extreme values of the errors; the tendency to under hedge the forward commitment is lessened; fewer futures contracts are required in the hedge portfolio, which would otherwise be a cause for concern with the three-factor model. This is all the more important since the three-factor model is the best one. Finally, as the dynamic hedging strategies performed well during the study period from 1997 to 2005 when crude oil futures prices fluctuated strongly, we feel confident of their robustness and reliability.

References

Brennan, M.J.,1958, The Supply of Storage, *American Economic Review,* 47, 50-72.

Brennan, M.J.,1991, The Price of Convenience and the Valuation of Commodity Contingent Claims, In *Stochastic Models and Options Values*, edited by D., Land, Φksendal, B., 33-71 (Elsevier Science Publishers).

Brennan, M.J, Crew, N., 1997, Hedging Long Maturity Commodity Commitments with Short-Dated Futures Contracts. In *Mathematics of derivatives securities*, edited by Dempster, M., Pliska, S., 165-190, (Cambridge: Cambridge University Press).

Chen, S.S., Lee, C.F., Shrestha, K., 2004, An empirical analysis of the relationship between the hedge ratio and hedging horizon: a simultaneous estimation of the short- and long-run hedge ratios. *Journal of Futures Markets,* 24, 359-386.

Cortazar, G., Schwartz, E.S., 1997, Implementing a Real Option Model for Valuing an Undeveloped Oil Field, *International Transactions in Operational Research,* 4, 125-137.

Cortazar, G., Schwartz, E.S., 2003, Implementing a Stochastic Model for Oil Futures Prices, *Energy Economics,* 25, 215-238.

Culp, C.L., Miller, M.H., 1994, Hedging a Flow of Commodity Deliveries with Futures: Lessons from Metallgesellschaft, *Derivatives Quarterly,* 1, 7-15.

Culp, C.L., Miller, M.H., 1995, Metallgesellschaft and the Economics of Synthetic Storage, *Journal of Applied Corporate Finance,* 7, 62-76.

Edwards, F.R., Canter, M.S., 1995, The Collapse of Metallgesellschaft: Unhedgeable Risks, Poor Hedging Strategy, or Just Bad Luck? *Journal of Futures Markets,* 15, 221-264.

Gibson, R., Schwartz, E.S., 1990, Stochastic Convenience Yield and the Pricing of Oil Contingent Claims, *Journal of Finance,* 45, 959-975.

Harris, R.D.F., Shen, J., 2006, Hedging and Value At Risk, *Journal of Futures Markets,* 26(4), 369-390.

Harvey, A.C., 1989, *Forecasting, Structural Time Series Models and the Kalman Filter* (Cambridge University Press, Cambridge, UK).

Hilliard, J.E., Reis, J., 1998, Valuation of Commodity Futures and Options Under Stochastic Convenience Yield, Interest Rates, and Jump Diffusions in the Spot, *Journal of Financial and Quantitative Analysis,* 33, 61-86.

Javaheri, A., Lautier, D., Galli, A., 2003, Filtering in finance, *Wilmott magazine*, 5, 67-83.

Lautier, D., 2005, The Informational Value of Crude Oil Futures Prices, *Quarterly Journal of Finance,* 9(4), 67-83.

Lautier, D., 2005, Term structure models of commodity prices: a review, *The Journal of Alternative Investments,* 9(2), 42-64.

Neuberger, A., 1999, Hedging Long term Exposures with Multiple Short term Futures Contracts, *Review of Financial Studies,* 12, 429-459.

Richter, M.C., Sorensen, C., 2002, Stochastic Volatility and Seasonality in Commodity Futures and Options: the Case of Soybeans, *Working Paper, Copenhagen Business School.*

Ross, S.A., 1995, Hedging Long Run Commitments: Exercises in Incomplete Market Pricing, *Working Paper, Yale School of Management*.

Routledge, B.R., Seppi, D.J., Spatt, C.S., 2000, Equilibrium Forward Curves for Commodities, *Journal of Finance,* 55, 1297-1338.

Samuelson, P.A, 1965, Proof that Properly Anticipated Prices Fluctuate Randomly, *Industrial Management Review,* 6, 41-49.

Schwartz, E.S, 1997, The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging, *Journal of Finance*, 52, 923-973.

Schwartz, E.S, Smith, J.E., 2000, Short term Variations and Long term Dynamics in Commodity Prices, *Management Science,* 46, 893-911.

Veld-Merkoulova, Y.V., de Roon, F.A., 2003, Hedging Long term Commodity Risk, *Journal of Futures Markets,* 23, 109-133.

Yan X., 2002, Valuation of Commodity Derivatives in a New Multi-Factor Model, *Review of Derivatives Research,* 5, 251-271.

Appendix 1. The Kalman filter

The main principle of the Kalman filters is to use temporal series of observable variables in order to reconstitute the values of non-observable variables.

The Kalman filter is an iterative process. The model has to be expressed in a state-space form characterized by a transition equation and a measurement equation. This transition equation describes the dynamics of the state variables $\tilde{\alpha}$, for which there are no empirical data. During the first step of the iteration – the prediction phase – this equation is used to compute the values of the non-observable variables at time *t*, conditionally on the information available at time *(t-1)*. The predicted values $\tilde{\alpha}_{t/t-1}$ are then substituted into the measurement equation to determine the value of the measures $\tilde{y}_t$. The measurement equation represents the relationship linking the observable variables $\tilde{y}$ with the non-observable $\tilde{\alpha}$. In the second iteration step – or innovation phase – the innovation v_t, which is the difference, at *t*, between the measure $\tilde{y}_t$ and the empirical data y_t is calculated. The innovation is used, in the third iteration step – or updating phase – to obtain the value of $\tilde{\alpha}_t$ conditionally on the information available at *t*. Once this calculation has been made, $\tilde{\alpha}_t$ is used to begin a new iteration. Thus, the Kalman filter makes it possible to evaluate the non-observable variables $\tilde{\alpha}$, and it updates their value in each step using the new information.

The state-space form model, in the filter, is characterized by the following equations:

- Transition equation: $\alpha_{t/t-1} = T\alpha_{t-1} + c + R\eta_t$

where α_t is the m-dimensional vector of non-observable variables at t, also called state vector, *T* is a matrix (m × m), *c* is an m-dimensional vector, and R is (m × m)

- Measurement equation: $y_{t/t-1} = Z\alpha_{t/t-1} + d + \varepsilon_t$

where $y_{t/t-1}$ is an N-dimensional temporal series, Z is a (N×m) matrix, and *d* is an m-dimensional vector.

η_t and ε_t are white noises whose dimensions are respectively m and N. They are supposed to be normally distributed, with zero mean and with *Q* and *H* as covariance matrices:

$$E[\eta_t] = 0\ ,\ Var[\eta_t] = Q$$

$$E[\varepsilon_t] = 0\ ,\ Var[\varepsilon_t] = H$$

The initial value of the system is supposed to be normal, with mean and variance:

$$E[\alpha_0] = \tilde{\alpha}_0\ ,\ Var[\alpha_0] = P_0$$

If $\tilde{\alpha}_t$ is a non biased estimator of α_t, conditionally on the information available at t, then:

$$E_t[\alpha_t - \tilde{\alpha}_t] = 0$$

As a consequence, the following expression defines the covariance matrix P_t :

$$P_t = E_t[(\tilde{\alpha}_t - \alpha_t)(\tilde{\alpha}_t - \alpha_t)']$$

During one iteration, three steps are successively tackled: prediction, innovation and updating.

• Prediction :

$$\begin{cases} \tilde{\alpha}_{t/t-1} = T\tilde{\alpha}_{t-1} + c \\ P_{t/t-1} = T\ P_{t-1}\ T' + R\ Q\ R' \end{cases}$$

where $\tilde{\alpha}_{t/t-1}$ and $P_{t/t-1}$ are the best estimators of $\alpha_{t/t-1}$ and $P_{t/t-1}$, conditionally on the information available at (t-1).

• Innovation :

$$\begin{cases} \tilde{y}_{t/t-1} = Z\tilde{\alpha}_{t/t-1} + d \\ v_t = y_t - \tilde{y}_{t/t-1} \\ F_t = ZP_{t/t-1}Z' + H \end{cases}$$

where $\tilde{y}_{t/t-1}$ is the estimator of the observation y_t conditionally on the information available at (t-1), and v_t is the innovation process, with F_t as a covariance matrix.

• Updating :

$$\begin{cases} \tilde{\alpha}_t = \tilde{\alpha}_{t/t-1} + P_{t/t-1}Z'F_t^{-1}v_t \\ P_t = (I - P_{t/t-1}Z'F_t^{-1}Z)P_{t/t-1} \end{cases}$$

The matrices *T, c, R, Z, d, Q,* and *H* are the system matrices associated with the state-space model.

Figure 1. Crude oil futures prices, March 1997- June 2005

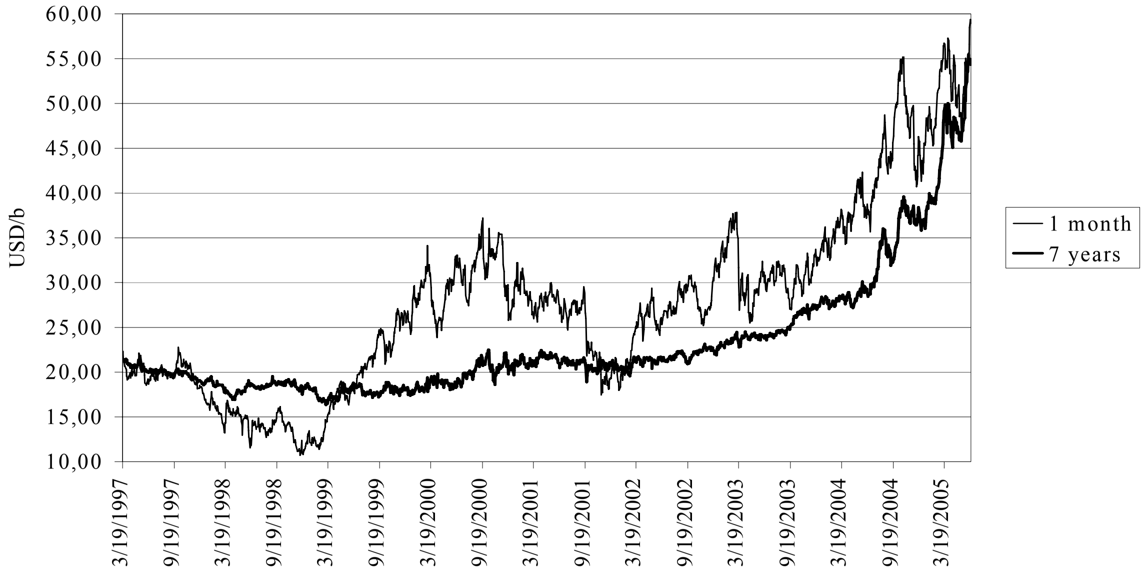

Figure 2. Frequency distribution of monthly hedging errors, for the three factor model (M3), when the hedging error is taken into account (β = 0) or not (β = 1), when the maturities used for the calibration correspond to 2, 6 and 18 months, or to 2, 12 and 24 months, and for a seven years commitment on the physical market.

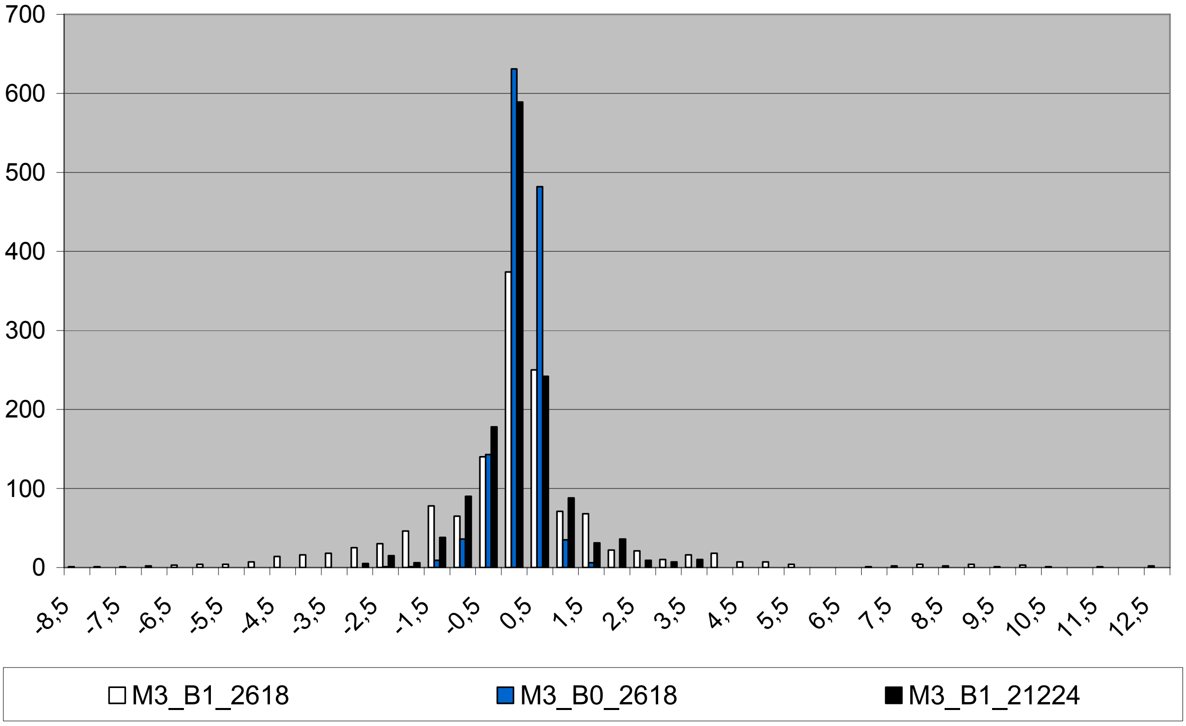

Figure 3. Average hedge ratios corresponding to the three factor model, when the maturity of the commitment is set to 2 years, and when the hedging error is not taken into account (β = 1).

(The maturities of the futures positions correspond to 2, 6 and 18 months)

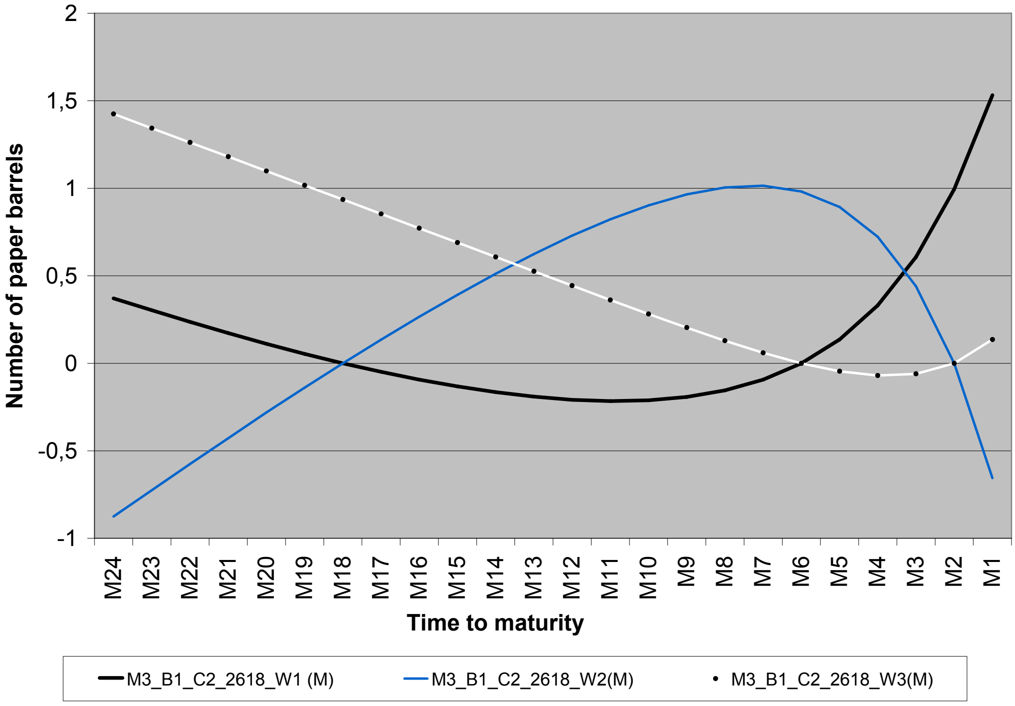

Figure 4. Average hedge ratios corresponding to the three factor model, when the maturity of the commitment is set to seven years and the hedging error is not taken into account (β = 1).

(The maturities of the futures positions correspond to 2, 6 and 18 months)

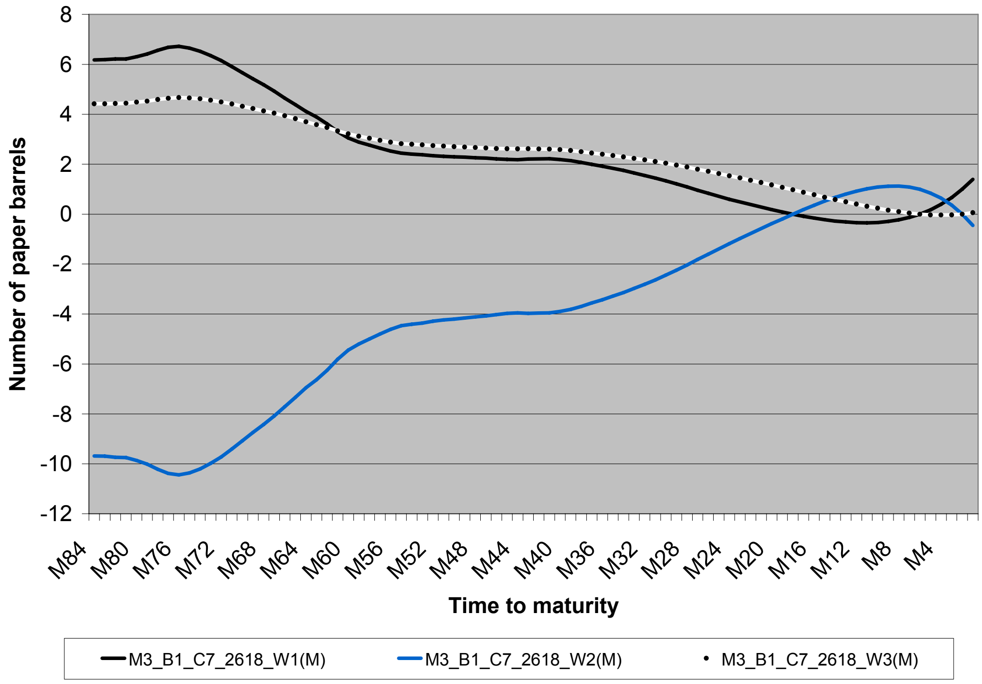

Figure 5. Average hedge ratios corresponding to the three factor model, when the maturity of the commitment is set to two years, and when the hedging error is taken into account (β = 0).

(The maturities of the futures positions correspond to 2, 6 and 18 months)

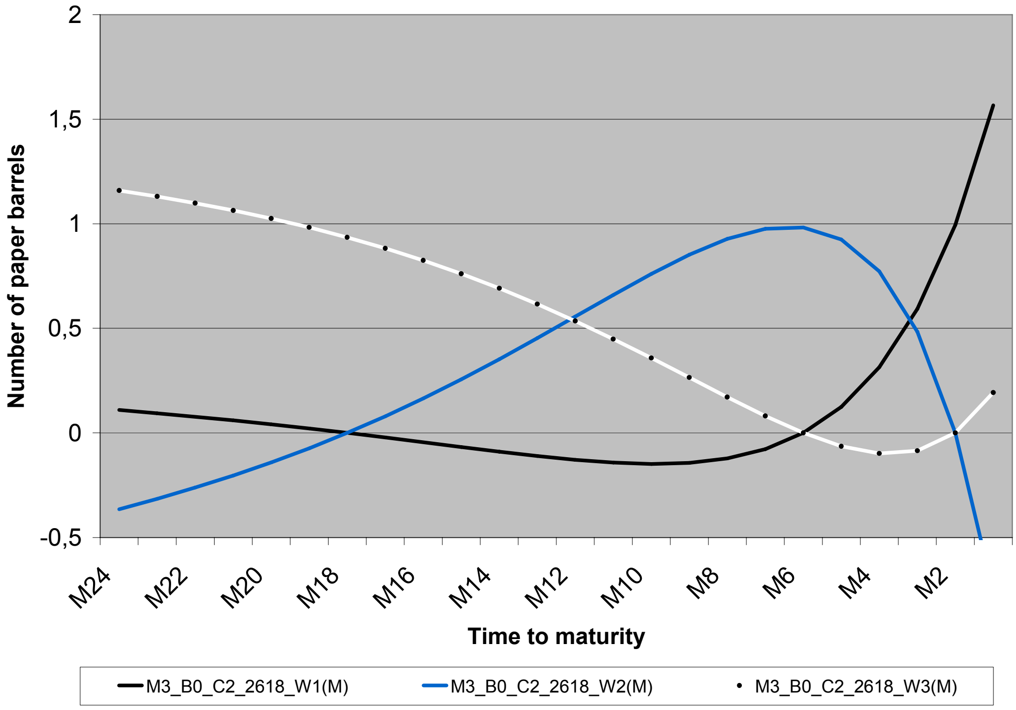

Figure 6. Average hedge ratios corresponding to the three-factor model, when the maturity of the commitment is set to seven years, and the hedging error is taken into account (β = 0).

(The maturities of the futures positions correspond to 2, 6 and 18 months)

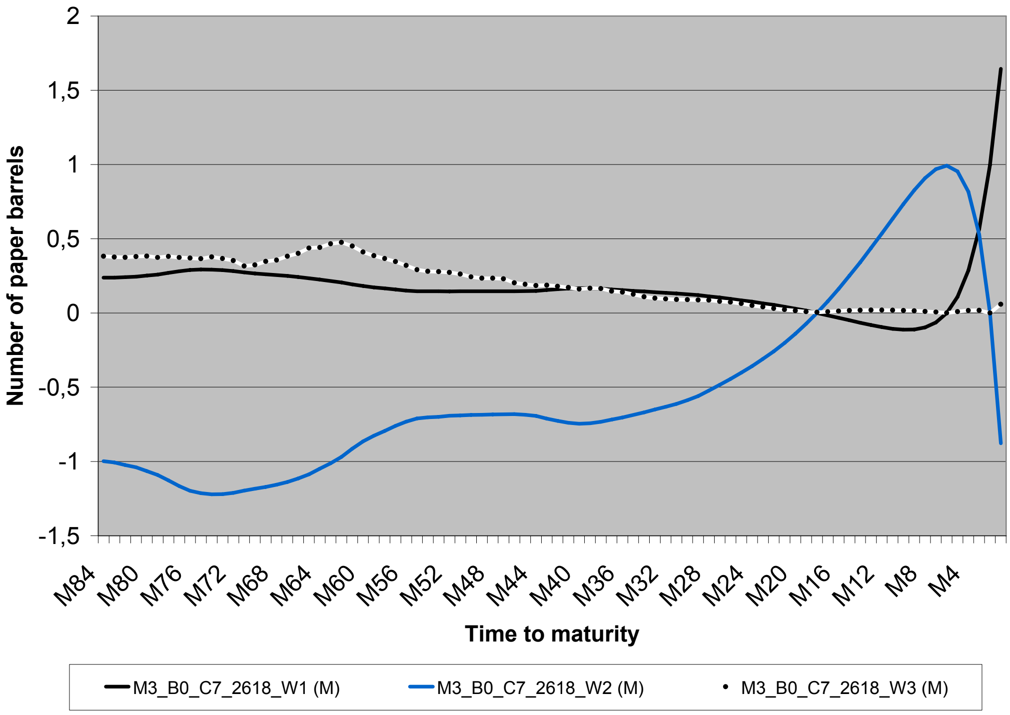

Table 1. Performances of the models: MPE, RMSE and RRMSE

Maturity of the commitment	Maturity retained for the optimization	M1		M2		M3	
		MPE	RMSE	MPE	RMSE	MPE	RMSE
24 months	2	0.1005		0,0823		0,2834	
	6	-0.1662	1.7875	0,3843	1.6189	0,0811	1.5400
	18	0.7941		0,5101	(0.9057)	0,4735	(0.8615)
	24	0.4180		0,2913		0,2811	
36 months	2	0.8512		0,0174		0,1295	
	6	0.2462	2.1948	0,2469	1.6983	0,0785	1.5491
	18	-1.0920		0,4544	(0.7738)	0,4915	(0.7058)
	36	0.3562		0,2071		0,1547	
48 months	2	1.0402		0,1345		0,1103	
	6	2.2487	3.3273	0,2330	1.7678	0,0879	1.5573
	18	-0.0958		0,3419	(0.5313)	0,4744	(0.4680)
	48	-0.5099		0,1865		0,1170	
60 months	2	1.2063		0,0787		0,1991	
	6	2.4586	3.5202	0,2442	1.7659	0,0523	1.5661
	18	-0.2367		0,4715	(0.5016)	0,5160	(0.4449)
	60	-0.4252		0,1787		0,0870	
72 months	2	1.4237		0,1157		0,2670	
	6	2.3788	3.8895	0,2353	1.8570	0,0588	1.5821
	18	-0.2124		0,4893	(0.4774)	0,5545	(0.4068)
	72	-0.1683		0,1688		0,0674	
84 months	2	2.1842		0,0796		0,2420	
	6	1.2975	3.2772	0,2626	2.0205	-0,0166	1.6714
	18	-1.7217		0,5185	(0.6165)	0,6017	(0.5100)
	84	0.3274		0,1990		0,0800	

The mean pricing error (MPE), expressed in USD, measures the estimation bias for one given maturity. When the estimation is good, the MPE is close to zero. The root mean squared error (RMSE) is also in USD. When there is no bias, it can be considered as an empirical variance. It measures the estimations stability. This second criterion is more representative because prices errors can offset themselves. In addition, in order to compare the relative performances of different models, we also computed the relative RMSE (RRMSE), which is the RMSE of the model being studied, divided by the RMSE of a "naïve" model, which is taken as a reference. In this study, the one factor model is taken as the reference. RRMSE are in parentheses.

Table 2. Dynamic strategies, summary statistics of the monthly errors, for the three models (M1, M2, M3) and for different forward commitments

	M1				**M2**				**M3**			
	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max
24 M	-0.0441	1.1975	-4.22	3.54	-0.0270	0.9733	-5.65	1.05	-0.0886	0.1950	-1.26	1.05
36 M	0.0263	1.3635	-4.44	3.72	0.0006	1.3285	-6.37	4.32	-0.1358	0.6039	-6.78	3.37
48 M	-0.3330	0.9688	-6.79	3.98	0.0218	1.6325	-6.52	4.23	-0.2161	1.6590	-7.80	15.70
60 M	-0.3134	0.9091	-6.80	2.67	0.0229	1.8339	-6.84	4.31	-0.2030	2.1812	-8.20	15.82
72 M	-0.3140	0.8900	-7.01	2.51	0.0286	1.9007	-6.79	4.24	-0.2886	2.5481	-10.94	14.40
84 M	0.0785	1.5603	-4.68	3.75	0.0043	1.9024	-6.56	4.17	-0.2423	2.0939	-8.76	12.33

The monthly errors are in dollar per barrel. The maturities chosen for the calibration of the parameters and hedge ratios are 2, 6, and 18 months (the fourth one is the maturity of the forward commitment). The errors associated with hedge ratios are not taken into account: β = 1.

Table 3. Dynamic strategies, summary statistics of the errors, for the three models (M1, M2, M3) and for different forward commitments, with a portfolio rolled every 2 months

	M1				M2				M3			
	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max
24 M	-0.3212	1.3898	-4.09	5.32	-0.3053	1.1521	-4.22	3.62	-0.3661	0.3297	-1.69	0.79
36 M	-0.1841	1.6513	-4.82	5.92	-0.2776	1.4966	-5.32	4.25	-0.3830	0.7616	-4.39	4.21
48 M	-0.7068	1.4650	-7.15	5.42	-0.2364	1.8012	-5.44	4.74	-0.4300	1.8514	-10.34	9.43
60 M	-0.6634	1.3788	-7.33	5.49	-0.2113	1.9934	-5.70	5.00	-0.3934	2.3763	-12.12	18.10
72 M	-0.6647	1.3248	-7.34	4.51	-0.2277	2.0783	-5.88	4.79	-0.4277	2.7903	-10.91	15.75
84 M	-0.0525	2.0377	-5.91	6.40	-0.2503	2.1223	-5.81	4.84	-0.3660	2.3222	-9.68	10.97

The errors are in dollar per barrel. The maturities chosen for the calibration of the parameters and hedge ratios are: 2, 6, and 18 months (the fourth is the maturity of the forward commitment). The errors associated with hedge ratios are not taken into account: $\beta = 1$.

Table 4. Dynamic strategies, summary statistics of the errors, for the three models (M1, M2, M3) and for different forward commitments, with a portfolio rolled every 3 months

	M1				M2				M3			
	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max
24 M	-0.5905	1.6308	-4.73	6.26	-0.5654	1.3281	-6.22	5.22	-0.6375	0.5037	-2.58	0.95
36 M	-0.3839	1.9066	-5.03	7.19	-0.5436	1.6820	-6.69	6.38	-0.6313	0.8556	-5.76	3.54
48 M	-1.0660	1.8902	-12.65	5.18	-0.5019	2.0577	-7.37	6.98	-0.6253	2.0376	-13.72	11.19
60 M	-0.9950	1.7800	-12.92	5.03	-0.4563	2.2602	-7.82	7.09	-0.5668	2.5370	-15.77	15.52
72 M	-0.9956	1.7240	-12.95	2.92	-0.4829	2.3551	-7.60	6.67	-0.5368	3.1168	-15.40	15.81
84 M	-0.1841	2.3243	-6.02	7.77	-0.4957	2.3439	-7.96	6.96	-0.4780	2.5393	-10.10	14.60

The errors are in dollar per barrel. The maturities chosen for the calibration of the parameters and hedge ratios are 2, 6, and 18 months (the fourth is the maturity of the forward commitment). The errors associated with hedge ratios are not taken into account: $\beta = 1$.

Table 5. Dynamic strategies, summary statistics of the monthly errors, for the three models (M1, M2, M3) and for different forward commitments, with long term maturities

	M1				M2				M3			
	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max
36 M	0.0275	1.4191	-4.48	4	-0.0680	0.5608	-2.68	2.34	-0.1037	0.2233	-1.37	1.09
48 M	0.0313	1.5259	-4.95	4.49	-0.0840	0.7478	-3.70	2.32	-0.1320	0.3854	-3.16	4.45
60 M	-0.3057	0.8740	-6.79	2.36	-0.0846	0.8718	-3.57	2.42	-0.1344	0.4839	-2.40	2.61
72 M	-0.3059	0.8563	-6.85	2.38	-0.0798	0.9091	-3.49	2.34	-0.1413	0.6932	-4.44	4.32
84 M	-0.3364	0.9128	-7.01	2.51	-0.0979	0.9173	-3.21	2.32	-0.1588	0.8410	-3.27	3.45

The monthly errors are in dollar per barrel. The maturities chosen for the calibration of the parameters and hedge ratios are: 2, 12, and 24 months (the fourth is the maturity of the forward commitment). The errors associated with hedge ratios are not taken into account: $\beta = 1$.

Table 6. Dynamic strategies, summary statistics of the monthly errors, for the three models (M1, M2, M3) and for different forward commitments, when the errors of the hedge ratios are taken into account (*β* = 0)

	M1				M2				M3			
	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max	Mean	Std dev.	Min	Max
24 M	-0.011	1.3862	-4.17	3.53	-0.0147	0.6246	-2.16	2.85	-0.0881	0.1456	-0.86	0.62
36 M	0.0506	1.5376	-4.36	4.18	0.0171	0.8684	-4.49	3.44	-0.0957	0.2171	-1.21	0.71
48 M	0.0566	1.6250	-4.86	4.13	0.0198	1.0270	-4.19	3.90	-0.1027	0.3128	-2.19	1.36
60 M	0.1127	1.6128	-4.80	4.36	0.0562	1.0438	-4.42	3.84	-0.1011	0.3682	-3.44	4.17
72 M	0.1114	1.6119	-5.24	4.68	0.1004	1.0312	-5.59	4.03	-0.0404	0.4731	-4.15	3.87
84 M	0.0879	1.613	-4.86	3.8	0.0865	1.1045	-5.68	4.00	-0.1118	0.4097	-2.96	1.46

The monthly errors are in dollar per barrel. The maturities chosen for the calibration of the parameters and hedge ratios are 2, 6, and 18 months (the fourth is the maturity of the forward commitment). The frequency of the roll-over is one month.

Table 7. Frequency distributions of the monthly errors in different situations: when the error associated with the hedge ratios is taken into account (β = 0) or not (β = 1), and when different set of maturities are retained for the calibration (F = 2_6_18) or (F = 2_12_24)*

		β = 1 F=2_6_18			β = 1 F=2_12_24			β = 0 F=2_6_18		
		M1	M2	M3	M1	M2	M3	M1	M2	M3
24M	Mode	0	0	0				0	0	0
	Kurtosis	4.39	8.57	7.07				3.36	5.92	5.37
	Skewness	-0.12	-0.55	0.06				-0.18	0.67	0.14
	Jarque-Bera	152.3	2449	1262				19.51	784.25	433.11
	-0.5 ≤ p(e) ≤ 0.5	54.50	69.08	99.19				41.30	73.79	92.11
	P(e) >|1|	24.19	15.79	0.05				30.87	5.47	0.00
36M	Mode	0	0	0	0	0	0	0	0	0
	Kurtosis	3.52	6.32	17.11	3.43	7.69	6.95	3.03	5.60	6.12
	Skewness	-0.15	-0.58	-0.54	-0.14	-0.57	0.18	-0.14	0.13	-0.38
	Jarque-Bera	34.35	1184	19217	25.82	2230	1513	7.68	653.02	989.37
	-0.5 ≤ p(e) ≤ 0.5	44.44	56.38	87.85	43.01	85.29	98.13	38.72	67.62	98.96
	P(e) >|1|	33.51	25.04	4.73	35.50	4.60	0.04	41.45	12.98	0.00
48M	Mode	0	0	0	0	0	0	0	0	0
	Kurtosis	9.12	4.91	16.40	3.13	5.34	19.83	2.99	4.63	7.45
	Skewness	-0.57	-0.64	1.11	-0.20	-0.55	0.12	-0.16	-0.16	-0.22
	Jarque-Bera	4029	552.5	19192	18.63	694.9	29449	10.50	288.05	2075
	-0.5 ≤ p(e) ≤ 0.5	67.63	45.59	70.19	39.10	74.92	94.67	36.62	57.73	96.51
	P(e) >|1|	12.22	36.10	21.55	39.98	10.58	2.12	43.15	18.91	0.68
60M	Mode	0	0	1	0	0	0	0	0	0
	Kurtosis	10.73	4.26	17.50	10.69	4.50	10.29	2.98	4.49	18.21
	Skewness	-1.13	-0.63	1.87	-1.15	-0.56	0.59	-0.21	0.05	-0.64
	Jarque-Bera	6492	319.4	22417	6449	349.5	5456	17.71	222.95	23283
	-0.5 ≤ p(e) ≤ 0.5	71.29	38.21	64.17	72.25	67.13	91.08	38.17	56.46	96.25
	P(e) >|1|	10.13	43.71	25.75	9.54	14.71	3.71	43.50	19.42	0.67
72M	Mode	0	0	3	0	0	0	0	0	0
	Kurtosis	13.70	3.99	9.57	13.20	4.12	12.25	3.15	497.74	10.47
	Skewness	-1.81	-0.54	0.79	-1.76	-0.42	0.24	-0.29	-17.01	0.02
	Jarque-Bera	10721	181.4	3833	9784	163.6	7205	31.01	206576	4691
	-0.5 ≤ p(e) ≤ 0.5	74.06	35.91	55.01	74.70	63.64	85.17	35.86	54.32	86.51
	P(e) >|1|	8.28	46.38	34.52	7.79	15.53	6.94	44.20	21.24	2.18
84M	Mode	0	0	1	0	0	0	0	0	0
	Kurtosis	3.51	3.79	9.93	16.01	3.99	6.65	3.39	4.84	6.53
	Skewness	-0.40	-0.44	1.02	-2.44	-0.30	0.47	-0.35	0.09	-0.95
	Jarque-Bera	49.87	79.03	2924	10812	75.01	795.33	36.27	191.94	901.75
	-0.5 ≤ p(e) ≤ 0.5	38.17	34.08	56.85	77.38	62.87	75.07	37.05	55.06	93.45
	P(e) >|1|	40.03	47.54	33.04	6.70	14.96	11.68	41.44	22.54	1.26

(*) F = 2_6_18 (F=2_12_24) means that the maturities retained for the calibration are 2, 6 and 18 months (2, 12 and 24 months). The fourth maturity is to the maturity of the forward commitment. [-0.5 ≤ p(e) ≤ 0.5] represents the percentage of hedging errors that are superior to 0.5 dollar per barrel, and inferior to this value. [P(e) >|1]| represents the percentage of hedging errors that are superior to 1 dollar per barrel, in absolute value.

[1] Corresponding author.

[2] The two factor models of Schwartz (1997) and of Schwartz and Smith (2000) can be considered as two different versions of the same model.

[3] For more information on the Metallgesellschaft case, see for example Culp and Miller (1994, 1995) or Edwards and Canter (1995).

[4] Another kind of literature is focused on optimal hedge ratios, and do not explicitly consider the term structure of futures prices. For references on this type of literature, see for example Chen *et al* (2004).

[5] The commercial profit of the trader is set to zero for simplicity.

[6] In the American crude oil market, in 2005, more than 84% of the trading volume was concentrated on the first three months!

[7] For a review on the literature on term structure models of commodity prices, see for example Lautier (2005).

[8] Schwartz (1997), Cortazar and Schwartz (1997), Routledge, Seppi and Spatt (2000), among others.

[9] See for example Hilliard and Reis (1998), Schwartz (1998), Neuberger (1999), Schwartz and Smith (2000), Yan (2002), Richter and Sorensen (2002), Veld-Merkoulova and de Roon (2003), Cortazar and Schwartz (2003).

[10] To the best of our knowledge, the hedge ratios have never been computed before for the three-factor model.

[11] F_x stands for $\frac{\delta F}{\delta x}$

[12] Cortazar and Schwartz (2003) did not compute the hedge ratios associated to their term structure model. The details of these computations are available from the authors upon request.

[13] As we need to test the efficiency of the hedging strategies, we did not use the term structure models in order to compute the value of the forward delivery. Indeed, in our case, $F(t, \tau)$ is the futures price empirically observed.

[14] For more details on Kalman filters, see Appendix 1, Harvey (1989), or Javaheri *et al* (2003).

[15] In order to test our procedure, we tried to use intermediate values of $ß$. We found that the performance of all models in the replication of the prices curves increases linearly with this parameter; they reach their maximum level with $ß = 1$. Conversely, the performance of the dynamic hedging strategies depends linearly on $ß$ and reaches their maximum level with $ß = 0$. This result is in line with our assumptions.

[16] We did not take into account transactions costs in the results provided in the paper for the following reasons. First, our method necessitates far lower hedging ratios – and thus far lower transaction costs – than the traditional one. Second we aim to compare two situations, when the traditional or the new estimation method is used. If we add transaction costs in these two situations, it will reinforce the interest of our methodology. Lastly, transaction costs are extremely low in futures markets. For example, the Nymex reported that during February 2008, the direct transaction costs on the futures markets amounted to USD 0.25 per contract. One contract represents a volume of 1,000 barrels and during February 2008, the level of the nearest futures price averaged USD 94 per barrel, that is, transaction costs represented 0.00027% of the value of the contract (even less in our case, if we consider that for rollover transactions, the exchange gives the possibility to make two transactions in one through spread trading).

[17] As we were able to compute 16 dynamic strategies for the seven years maturity – compared to 76 for the two years maturity – the average hedge ratios evolve less regularly for the long term commitment.

[18] 9.69 paper barrels must be sold on the 6 months maturity, 4.41 must be bought on the 2 months maturity, and 6.18 on the 18 months maturity.

[19] 0.36 paper barrel must be sold on the 6 months maturity, 0.11 must be bought on the 2 months maturity, and 1.16 on the 18 months maturity.

[20] 1 paper barrel must be sold on the 6 months maturity, 0.24 must be bought on the 2 months maturity, and 0.38 on the 18 months maturity.