Existence of competitive equilibrium in an optimal growth model with heterogeneous agents and endogenous leisure

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Abstract: This paper proves the existence of competitive equilibrium in a single-sector dynamic economy with heterogeneous agents, elastic labor supply and complete assets markets. The method of proof relies on some recent results concerning the existence of Lagrange multipliers in infinite dimensional spaces and their representation as a summable sequence and a direct application of the inward boundary fixed point theorem.
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1 Introduction

Since the seminal work of Ramsey (1928), optimal growth models have played a central role in modern macroeconomics. Classical growth theory relies on the assumption that labor is supplied in fixed amounts, although the original paper of Ramsey did include the disutility of labor as an argument in consumers' utility functions. Subsequent research in applied macroeconomics (theories of business

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cycle fluctuations) has reassessed the role of the labor-leisure choice in the process of growth. Nowadays, intertemporal models with elastic labor continue to be the standard setting used to model many issues in applied macroeconomics.

Our purpose is to prove existence of competitive equilibrium for the basic neoclassical model with elastic labor with less stringent assumptions than in the literature using some recent results (see Le Van and Saglam (2004)) concerning the existence of Lagrange multipliers in infinite dimensional spaces and their representation as a summable sequence.

Lagrange multiplier techniques have facilitated considerably the analysis of constrained optimization problems. The application of these techniques in the analysis of intertemporal models inherits most of the tractability found in a finite setting. However, the passage to an infinite dimensional setting raises additional questions. These questions concern both the extension of the Lagrangean in an infinite dimensional setting as well as the representation of the Lagrange multipliers as a summable sequence.

Previous work addressing existence of competitive equilibrium in intertemporal models attacks the problem of existence from an abstract point of view. Following the early work of Peleg and Yaari (1970), this approach is based on separation arguments applied to arbitrary vector spaces (see Bewley (1972), Bewley (1982), Aliprantis, et al. (1990), Dana and Le Van (1991)). The advantage of this approach is that it yields general results capable of application in a wide variety of models. However, it requires a high level of abstraction and some strong assumptions.

Le Van and Vailakis (2004) in order to prove the existence of competitive equilibrium in a model with a representative agent and elastic labor supply impose relatively strong assumptions.[1] In this paper, the existence of equilibrium cannot be established by using marginal utilities since we may have boundary solutions.

Recently, Le Van, et al. (2007) extended the canonical representative agent Ramsey model to include heterogeneous agents and elastic labor supply and supermodularity is used to establish the convergence of optimal paths. The novelty in their work is that relatively impatient consumers have their consumption and leisure converging to zero and any Pareto optimal capital path converges to a limit point as time tends towards infinity. However, if the limit points of the Pareto optimal capital paths are not bounded away from zero, then their convergence results do not ensure existence of equilibrium.

To obtain the convergence results, they impose strong assumptions which are not used in our paper.[2] Following the Negishi approach (1960), our strategy for tackling the question of existence relies on exploiting the link between

[1] They assumed $\frac{u(\epsilon,\epsilon)}{\epsilon} \to +\infty$ as $\epsilon \to 0$ for showing $c_t > 0, l_t > 0$ and $\frac{u_{cc}}{u_c} \leq \frac{u_{cl}}{u_l}$ for the proof of $k_t > 0$ for all t.

[2] Le Van, et al. (2007) assume that the cross-partial derivative u^i_{cl} has constant sign, $u^i_c(x,x)$ and $u^i_l(x,x)$ are non-increasing in x, the production function F is homogenous of degree $\alpha \leq 1$ and $F_{kL} \geq 0$ (Assumptions U4, F4, U5, F5).

Pareto-optima and competitive equilibria. We show that there exist Lagrange multipliers which can be used as a price system such that together with the Pareto-optimal solution they constitute an equilibrium with transfers. These transfers depend on the individual weights involved in the social welfare function. An equilibrium exists provided that there is a set of welfare weights such that the corresponding transfers equal zero. The model in which we establish existence is with complete contingent commodity Arrow-Debreu markets (as opposed to trading in sequential markets) and the prices and transfers are sufficient for decentralizing the optimal allocation. We also do not require, with additional assumptions, as in Le Van, et al. (2007) that the optimal capital stock converges in the long run to a strictly positive value in order to get prices in ℓ^1_+.

The organization of the paper is as follows. In section 2, we present the model and provide sufficient conditions on the objective function and the constraint functions so that Lagrange multipliers can be presented by an ℓ^1_+ sequence. We characterize some dynamic properties of the Pareto optimal paths of capital and of consumption-leisure. In particular, we prove that the optimal consumption and leisure paths of the more impatient agents will converge to zero in the long run (see Becker (1980) for a similar result in a sequential trading model) with a very elementary proof compared to the one in Le Van, et al. (2007) which uses supermodularity for lattice programming. In section 3, we prove the existence of competitive equilibrium by using the Negishi approach and the inward boundary fixed point theorem.

2 The model

We consider an intertemporal model with $m \geq 1$ consumers and one firm. There is a single produced good in each period that is either consumed or invested as capital. The preferences of each consumer, $i = 1, \ldots, m$, take the additive form: $\sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i)$ where $\beta_i \in (0, 1)$ is the discount factor. At date t, consumer i consumes c_t^i of the good, enjoys a quantity of leisure l_t^i and supplies a quantity of labor L_t^i which are normalized so that $l_t^i + L_t^i = 1$. Production possibilities are given by the gross production function F and a physical depreciation $\delta \in (0, 1)$. Denote $F(k_t, \sum_{i=1}^m L_t^i) + (1 - \delta)k_t = f(k_t, \sum_{i=1}^m L_t^i)$.

We next specify a set of restrictions on preferences and the production technology.[3]

U1: *u^i is continuous, concave, increasing on $\mathbb{R}_+ \times [0, 1]$ and strictly increasing, strictly concave on $\mathbb{R}_{++} \times (0, 1)$.*

U2: $u^i(0, 0) = 0$.

[3]We relax some important assumptions in the literature. For example, Bewley (1972) assumes that the production set is a convex cone (Theorem 3, page 525). Bewley (1982) assumes the strict positiveness of derivatives of utility functions on $\mathbb{R}^L_+$ (strict monotonicity assumption, page 240).

U3: *u^i is twice continuously differentiable on $\mathbb{R}_{++} \times (0,1)$ with partial derivatives satisfying the Inada conditions:* $\lim_{c\to 0} u^i_c(c,l) = +\infty$, $\forall l \in (0,1]$ and $\lim_{l\to 0} u^i_l(c,l) = +\infty$, $\forall c > 0$.

We extend the utility functions on $\mathbb{R}^2$ by imposing $u^i(c,l) = -\infty$ if $(c,l) \in \mathbb{R}^2 \setminus \{\mathbb{R}_+ \times [0,1]\}$.

The assumptions on the production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ are as follows:

F1: *F is continuous, concave, increasing on $\mathbb{R}^2_+$ and strictly increasing, strictly concave on $\mathbb{R}^2_{++}$.*

F2: $F(0,0) = 0$.

F3: *F is twice continuously differentiable on $\mathbb{R}^2_{++}$ with partial derivatives satisfying the Inada conditions:* $\lim_{k\to 0} F_k(k,L) = +\infty$, $\forall L > 0$, $\lim_{k\to +\infty} F_k(k,m) < \delta$ and $\lim_{L\to 0} F_L(k,L) = +\infty$, $\forall k > 0$.

We extend the function F over $\mathbb{R}^2$ by imposing $F(k,L) = -\infty$ if $(k,L) \notin \mathbb{R}^2_+$.

For any initial condition $k_0 \geq 0$, when a sequence $\mathbf{k} = (k_0, k_1, k_2, \ldots, k_t, \ldots)$ is such that $0 \leq k_{t+1} \leq f(k_t, m)$ for all t, we say it is feasible from k_0 and we denote the class of feasible capital paths by $\Pi(k_0)$. Let $(\mathbf{c}^1, \mathbf{c}^2, \ldots, \mathbf{c}^i, \ldots, \mathbf{c}^m)$ where $\mathbf{c}^i = (c^i_0, c^i_1, \ldots, c^i_t, \ldots)$ denotes the vector of consumption and $(\mathbf{l}^1, \mathbf{l}^2, \ldots, \mathbf{l}^i, \ldots, \mathbf{l}^m)$ where $\mathbf{l}^i = (l^i_0, l^i_1, \ldots, l^i_t, \ldots)$ the vector of leisure of all agents. A pair of consumption-leisure sequences $(\mathbf{c}^i, \mathbf{l}^i) = (c^i_t, l^i_t)_{t=0}^{\infty}$ is feasible from $k_0 \geq 0$ if there exists a sequence $\mathbf{k} \in \Pi(k_0)$ that satisfies $\forall t$,

$$\sum_{i=1}^{m} c^i_t + k_{t+1} \leq f\left(k_t, \sum_{i=1}^{m}(1 - l^i_t)\right) \quad \text{and } 0 \leq l^i_t \leq 1.$$

The set of feasible consumption-leisure sequences from k_0 is denoted by $\sum(k_0)$. Assumption **F3** implies that

$$\begin{aligned} f_k(+\infty, m) &= F_k(+\infty, m) + (1-\delta) < 1, \\ f_k(0, m) &= F_k(0, m) + (1-\delta) > 1. \end{aligned}$$

It follows that there exists $\overline{k} > 0$ such that: (i) $f(\overline{k}, m) = \overline{k}$, (ii) $k > \overline{k}$ implies $f(k,m) < k$, (iii) $k < \overline{k}$ implies $f(k,m) > k$. Therefore for any $\mathbf{k} \in \Pi(k_0)$, we have $0 \leq k_t \leq \max(k_0, \overline{k})$. Thus, a feasible sequence $\mathbf{k}$ is in ℓ^∞_+ which in turn implies that any feasible sequence $(\mathbf{c}, \mathbf{l})$ belongs to $\ell^\infty_+ \times [0,1]^\infty$.

We now give the characterization of the competitive equilibrium. For each consumer i, let $\alpha^i > 0$ denote the share of the profit of the firm which is owned by consumer i. We have $\sum_{i=1}^m \alpha^i = 1$. Let $\vartheta^i > 0$ be the share of the initial endowment owned by consumer i. Clearly, $\sum_{i=1}^m \vartheta^i = 1$, and $\vartheta^i\ k_0$ is the endowment of consumer i.

Definition 1 *Let $k_0 > 0$. A competitive equilibrium for this model consists of a sequence of prices $\mathbf{p}^* = (p^*_t)_{t=0}^{\infty}$ for the consumption good, a wage sequence*

$\mathbf{w}^* = (w_t^*)_{t=0}^{\infty}$ *for labor, a price* r *for the initial capital stock* k_0 *and an allocation* $\{\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i}, \mathbf{L}^{*i}\}$ *such that*

i)

$$\begin{aligned} \mathbf{c}^* &\in \ell_+^{\infty}, \mathbf{l}^{*i} \in \ell_+^{\infty}, \mathbf{L}^{*i} \in \ell_+^{\infty}, \mathbf{k}^* \in \ell_+^{\infty}, \\ \mathbf{p}^* &\in \ell_+^1 \backslash \{0\}, \mathbf{w}^* \in \ell_+^1 \backslash \{0\}, r > 0. \end{aligned}$$

ii) For every i, $(\mathbf{c}^{*i}, \mathbf{l}^{*i})$ *is a solution to the problem*

$$\begin{array}{rc} & \max \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i) \\ s.t & \sum_{t=0}^{\infty} p_t^* c_t^i + \sum_{t=0}^{\infty} w_t^* l_t^i \ \le \sum_{t=0}^{\infty} w_t^* + \vartheta^i r k_0 + \alpha^i \pi^* \end{array}$$

where π^* *is the maximum profit of the single firm.*

iii) $(\mathbf{k}^*, \mathbf{L}^*)$ *is a solution to the firm's problem*

$$\begin{aligned} \pi^* &= \max \sum_{t=0}^{\infty} p_t^* [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{\infty} w_t^* L_t - r k_0 \\ st \quad 0 &\le k_{t+1} \le f(k_t, L_t), 0 \le L_t, \forall t \end{aligned}$$

iv) Markets clear: $\forall t$,

$$\begin{aligned} & \sum_{t=1}^{m} c_t^{*i} + k_{t+1}^* = f\left(k_t^*, \sum_{i=1}^{m} L_t^{*i}\right), \\ & l_t^{*i} + L_t^{*i} = 1, L_t^* = \sum_{i=1}^{m} L_t^{i^*} \textit{ and } k_0^* = k_0. \end{aligned}$$

Observe that we have for any i

$$\sum_{t=0}^{\infty} \beta_i^t u^i(c_t^{*t}, l_t^{*i}) \ge \sum_{t=0}^{\infty} \beta_i^t u^i(0, 1) = \frac{u^i(0,1)}{1 - \beta_i}$$

In other words, in equilibrium, every agent is individually rational. We will therefore study the individually rational Pareto optimum problem (or Pareto problem, in short). We show that the Lagrange multipliers are in ℓ_+^1. Then these multipliers will be used to define a price and wage system for the equilibrium.

Let $\Delta = \{\eta_1, \eta_2, \ldots, \eta_m | \eta_i \ge 0 \text{ and } \sum_{i=1}^{m} \eta_i = 1\}$. Given a vector of welfare weights $\eta \in \Delta$, define the Pareto problem

$$\max \sum_{i=1}^{m} \eta_i \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i) \tag{Q}$$

$$\begin{aligned} s.t. \quad & \sum_{i=1}^{m} c_t^i + k_{t+1} \le f\left(k_t, \sum_{i=1}^{m} (1 - l_t^i)\right), \forall t \\ & c_t^i \ge 0, \ l_t^i \ge 0, \ l_t^i \le 1, \ \forall i, \forall t \\ & k_t \ge 0, \ \forall t, \ k_0 \text{ given,} \end{aligned}$$

and the individual rationality constraints:

$$\sum_{t=0}^{\infty} \beta_i^t u^i(c_t^t, l_t^i) \geq \sum_{t=0}^{\infty} \beta_i^t u^i(0,1) = \frac{u^i(0,1)}{1-\beta_i}, \quad \forall i = 1, \ldots, m.$$

Note that, for all $k_0 \geq 0$, $0 \leq k_t \leq \max(k_0, \overline{k})$, then $0 \leq c_t^i \leq f(\max(k_0, \overline{k}), m) \equiv A$, $\forall t$, $\forall i = 1, \ldots, m$. Therefore, the sequence $(u^i)_n = \sum_{i=1}^n \beta_i^t u^i(c_t^i, l_t^i)$ is increasing and bounded and will converge. Thus, we can write

$$\sum_{i=1}^{m} \eta_i \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i) = \sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u^i(c_t^i, l_t^i).$$

Let $\mathbf{x} = (\mathbf{c}, \mathbf{k}, \mathbf{l}) \in (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m$.

Define

$$\begin{aligned}
\mathcal{F}(\mathbf{x}) &= -\sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u^i(c_t^i, l_t^i) \\
-\Phi_{-1}^i(\mathbf{x}) &= \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^t, l_t^i) - \frac{u^i(0,1)}{1-\beta_i} \\
\Phi_t^1(\mathbf{x}) &= \sum_{i=1}^{m} c_t^i + k_{t+1} - f\left(k_t, \sum_{i=1}^{m} (1 - l_t^i)\right) \\
\Phi_t^{2i}(\mathbf{x}) &= -c_t^i \\
\Phi_t^3(\mathbf{x}) &= -k_t \\
\Phi_t^{4i}(\mathbf{x}) &= -l_t^i \\
\Phi_t^{5i}(\mathbf{x}) &= l_t^i - 1 \\
\Psi_t &= (\Phi_t^1, \Phi_t^{2i}, \Phi_{t+1}^3, \Phi_t^{4i}, \Phi_t^{5i}), \ \forall t, \forall i = 1, \ldots, m, \text{ and } \Phi = (\Phi_{-1}, \Psi)
\end{aligned}$$

The Pareto problem can be written as:

$$\left\{\min \ \mathcal{F}(\mathbf{x}) \mid \Phi(\mathbf{x}) \leq \mathbf{0}, \ \mathbf{x} \in (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m\right\} \qquad (P)$$

where:

$$\begin{aligned}
\mathcal{F} &: (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m \to \mathbb{R} \cup \{+\infty\} \\
\Psi &= (\Psi_t)_{t=0,\ldots,\infty} : (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m \to \mathbb{R} \cup \{+\infty\} \\
\Phi_{-1} &= (\Phi_{-1}^i)_{i=1,\ldots,m} : (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m \to \mathbb{R} \cup \{+\infty\} \\
\text{Let } C &= \operatorname{dom}(\mathcal{F}) = \{\mathbf{x} \in (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m | \mathcal{F}(\mathbf{x}) < +\infty\} \\
\Gamma &= \operatorname{dom}(\Phi) = \{\mathbf{x} \in (\ell_+^\infty)^m \times l_+^\infty \times (\ell_+^\infty)^m | \Psi_t(\mathbf{x}) < +\infty, \ \forall t, \ \text{ and } \Phi_{-1}^i(\mathbf{x}) < +\infty, \ \forall i\}.
\end{aligned}$$

The following theorem follows from Theorem 1 and Theorem 2 in Le Van and Saglam (2004) (see also Dechert (1982)).

Theorem 1 *Let* $\mathbf{x}$, $\mathbf{y} \in (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m$, $T \in N$.

Define $x_t^T(\mathbf{x}, \mathbf{y}) = \begin{cases} x_t & if \ \ t \leq T \\ y_t & if \ \ t > T \end{cases}$

Suppose that two following assumptions are satisfied:

***T1**: If $\mathbf{x} \in C$, $\mathbf{y} \in (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m$ and $\forall T \geq T_0$, $\mathbf{x}^T(\mathbf{x},\mathbf{y}) \in C$ then $\mathcal{F}(\mathbf{x}^T(\mathbf{x},\mathbf{y})) \to \mathcal{F}(\mathbf{x})$ when $T \to \infty$.*

***T2**: If $\mathbf{x} \in \Gamma$, $\mathbf{y} \in \Gamma$ and $\mathbf{x}^T(\mathbf{x},\mathbf{y}) \in \Gamma$, $\forall T \geq T_0$, then*

$$a)\ \Phi_t(\mathbf{x}^T(\mathbf{x},\mathbf{y})) \to \Phi_t(\mathbf{x}) \textit{as } T \to \infty$$
$$b)\ \exists M\ s.t.\ \forall T \geq T_0, \|\Phi_t(\mathbf{x}^T(\mathbf{x},\mathbf{y}))\| \leq M$$
$$c)\ \forall N \geq T_0, \lim_{t\to\infty}[\Phi_t(\mathbf{x}^T(\mathbf{x},\mathbf{y})) - \Phi_t(\mathbf{y})] = 0$$

*Let $\mathbf{x}^*$ be a solution to (**P**) and $\bar{\mathbf{x}} \in C$ satisfy the Strong Slater condition:*

$$\sup_t \Phi_t(\bar{\mathbf{x}}) < 0.$$

Suppose $\mathbf{x}^T(\mathbf{x}^*,\bar{\mathbf{x}}) \in C \cap \Gamma$. Then, there exist $\mathbf{\Lambda} \in l_+^1 \backslash \{0\}$ such that

$$\mathcal{F}(\mathbf{x}) + \mathbf{\Lambda}\Phi(\mathbf{x}) \geq \mathcal{F}(\mathbf{x}^*) + \Lambda\Phi(\mathbf{x}^*),\ \forall \mathbf{x} \in (\ell^\infty)^m \times \ell^\infty \times (\ell^\infty)^m$$

and $\Lambda\Phi(\mathbf{x}^*) = 0$.

Obviously, for any $\eta \in \Delta$, an optimal path will depend on η. In what follows, if possible, we will suppress η and denote by $(\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{L}^{*i}, \mathbf{l}^{*i})$ any optimal path for each agent i. The following proposition characterizes the Lagrange multipliers of the Pareto problem.

Proposition 1 *If $\mathbf{x}^* = (\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i})$ is a solution to the Pareto problem (Q): then there exist $\forall i = 1,\ldots,m$, $\lambda = (\lambda_{-\mathbf{1}}^{\mathbf{i}}, \lambda^1, \lambda^{2i}, \lambda^3, \lambda^{4i}, \lambda^{5i}) \in \mathbf{R}_+^m \times \ell_+^1 \times (\ell_+^1)^m \times \ell_+^1 \times (\ell_+^1)^m \times (\ell_+^1)^m$, $\lambda \neq \mathbf{0}$ such that, for any $\big((\mathbf{c}^i, \mathbf{l}^i), \mathbf{k}, \mathbf{L}\big)$*

$$\begin{aligned}
&\sum_{t=0}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u^i(c_t^{*i}, l_t^{*i}) + \sum_{t=0}^{\infty}\sum_{i=1}^{m}\lambda_{-1}^i\beta_i^t u^i(c_t^{*i}, l_t^{*i}) - \sum_{t=0}^{\infty}\lambda_t^1\left(\sum_{i=1}^{m}c_t^{*i} + k_{t+1}^* - f(k_t^*, L_t^*)\right)\\
&+\sum_{t=0}^{\infty}\sum_{i=1}^{m}\lambda_t^{2i}c_t^{*i} + \sum_{t=0}^{\infty}\lambda_t^3 k_t^* + \sum_{t=0}^{\infty}\sum_{i=1}^{m}\lambda_t^{4i}l_t^{*i} + \sum_{t=0}^{\infty}\sum_{i=1}^{m}\lambda_t^{5i}(1 - l_t^{*i})
\end{aligned}$$

$$\begin{aligned}
\geq\ &\sum_{t=0}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u^i(c_t^i, l_t^i) + \sum_{t=0}^{\infty}\sum_{i=1}^{m}\lambda_{-1}^i\beta_i^t u^i(c_t^i, l_t^i) - \sum_{t=0}^{\infty}\lambda_t^1\left(\sum_{i=1}^{m}c_t^i + k_{t+1} - f(k_t, L_t)\right)\\
&+\sum_{t=0}^{\infty}\sum_{i=1}^{m}\lambda_t^{2i}c_t^i + \sum_{t=0}^{\infty}\lambda_t^3 k_t + \sum_{t=0}^{\infty}\sum_{i=1}^{m}\lambda_t^{4i}l_t^i + \sum_{t=0}^{\infty}\sum_{i=1}^{m}\lambda_t^{5i}(1 - l_t^i), \qquad (1)
\end{aligned}$$

$$\lambda_t^1\left[\sum_{i=1}^{m}c_t^{*i} + k_{t+1}^* - f(k_t^*, \sum_{i=1}^{m}L_t^{*i})\right] = 0 \qquad (2)$$

$$\lambda_t^{2i}c_t^{*i} = 0, \forall i = 1,\ldots,m \qquad (3)$$

$$\lambda_t^3 k_t^* = 0 \qquad (4)$$

$$\lambda_t^{4i} l_t^{*i} = 0, \forall i = 1, \ldots, m \tag{5}$$

$$\lambda_t^{5i}(1 - l_t^{*i}) = 0, \forall i = 1, \ldots, m \tag{6}$$

$$\lambda_{-1}^i \left[\sum_{t=0}^{\infty} \beta_i^t u^i(c_t^{*i}, l_t^{*i}) - \frac{u^i(0,1)}{1-\beta_i} \right] = 0, \ \forall i = 1, \ldots, m \tag{7}$$

$$0 \in (\eta_i + \lambda_{-1}^i)\beta_i^t \partial_1 u^i(c_t^{*i}, l_t^{*i}) - \{\lambda_t^1\} + \{\lambda_t^{2i}\}, \forall i = 1, \ldots, m \text{ with } \eta_i + \lambda_{-1}^i > 0 \tag{8}$$

$$0 \in (\eta_i + \lambda_{-1}^i)\beta_i^t \partial_2 u^i(c_t^{*i}, l_t^{*i}) - \lambda_t^1 \partial_2 f(k_t^*, L_t^*) + \{\lambda_t^{4i}\} - \{\lambda_t^{5i}\}, \forall i \text{ with } \eta_i + \lambda_{-1}^i > 0 \tag{9}$$

$$0 \in \lambda_t^1 \partial_1 f(k_t^*, L_t^*) + \{\lambda_t^3\} - \{\lambda_{t-1}^1\} \tag{10}$$

where, $L_t^* = \sum_{i=1}^m L_t^{*i} = \sum_{i=1}^m (1 - l_t^{*i})$, $\partial_j u(c_t^{*i}, l_t^{*i}), \partial_j f(k_t^*, L_t^*)$ *respectively denote the projection on the* j^{th} *component of the subdifferential of function* u *at* (c_t^{*i}, l_t^{*i}) *and the function* f *at* (k_t^*, L_t^*). [4]

Proof: We show that the Strong Slater condition holds. Since $f_k(0, m) > 1$,[5] for all $k_0 > 0$, there exists some $\widehat{k} \in (0, k_0)$ such that: $0 < \widehat{k} < f(\widehat{k}, m)$ and $0 < \widehat{k} < f(k_0, m)$. Thus, there exists a small positive number ε such that:

$$0 < \widehat{k} + \varepsilon < f(\widehat{k}, m - \varepsilon) \text{ and } 0 < \widehat{k} + \varepsilon < f(k_0, m - \varepsilon).$$

Denote $\bar{\mathbf{x}} = (\bar{\mathbf{c}}, \bar{\mathbf{k}}, \bar{\mathbf{l}})$ where $\bar{\mathbf{c}} = (\bar{\mathbf{c}}^i)_{i=1}^m$, and

$$\bar{\mathbf{c}}^i = (\bar{c}_t^i)_{t=0,\ldots,\infty} = (\frac{\varepsilon}{m}, \frac{\varepsilon}{m}, \frac{\varepsilon}{m}, \ldots)$$

$\bar{\mathbf{l}} = (\bar{\mathbf{l}}^i)_{i=1}^m$, where

$$\bar{\mathbf{l}}^i = (\bar{l}_t^i)_{t=0,\ldots,\infty} = (\frac{\varepsilon}{m}, \frac{\varepsilon}{m}, \frac{\varepsilon}{m}, \ldots).$$

and $\bar{\mathbf{k}} = (k_0, \widehat{k}, \widehat{k}, \ldots)$. We have

$$\Phi_0^1(\bar{\mathbf{x}}) = \sum_{i=0}^m c_0^i + k_1 - f\left(k_0, \sum_{i=1}^m (1 - l_0^i)\right) = \varepsilon + \widehat{k} - f(k_0, m - \varepsilon) < 0$$

$$\Phi_1^1(\bar{\mathbf{x}}) = \sum_{i=0}^m c_1^i + k_2 - f\left(k_1, \sum_{i=1}^m (1 - l_1^i)\right) = \varepsilon + \widehat{k} - f(\widehat{k}, m - \varepsilon) < 0$$

$$\begin{aligned} \Phi_t^1(\bar{\mathbf{x}}) &= \varepsilon + \widehat{k} - f(\widehat{k}, m - \varepsilon) < 0, \ \forall t \geq 2 \\ \Phi_t^{2i}(\bar{\mathbf{x}}) &= -\bar{c}_t^i = -\frac{\varepsilon}{m} < 0, \ \forall t \geq 0, \forall i = 1, \ldots, m \end{aligned}$$

$$\Phi_0^3(\bar{\mathbf{x}}) = -k_0 < 0, \Phi_t^3(\bar{\mathbf{x}}) = -\widehat{k} < 0 \quad \forall t \geq 1$$

[4] For a concave function f defined on $\mathbb{R}^n, \partial f(x)$ denotes the subdifferential of f at x.

[5] Assumption $f_k(0, 1) > 1$ is equivalent to the Adequacy Assumption in Bewley (1972), see Le Van and Dana (2003) Remark 6.1.1. This assumption is crucial to have equilibrium prices in ℓ_+^1 since it implies that the production set has an interior point. Subsequently, one can use a separation theorem in the infinite dimensional space to derive Lagrange multipliers.

$$\Phi_t^{4i}(\bar{\mathbf{x}}) = -\frac{\varepsilon}{m} < 0,\ \forall t \geq 0, \forall i = 1, \ldots, m$$
$$\Phi_t^{5i}(\bar{\mathbf{x}}) = \frac{\varepsilon}{m} - 1 < 0, \forall t \geq 0, \forall i = 1, \ldots, m.$$

To show that $\Phi_{-1}^i(\bar{\mathbf{x}}) < 0$, for any i, we just observe that the Inada condition $\lim_{c \to 0} u_c^i(c, l) = +\infty$ implies $u^i(\frac{\varepsilon}{m}, 1 - \frac{\varepsilon}{m}) > u^i(0, 1)$ if ε is small enough. Therefore, the Strong Slater condition is satisfied.

It is obvious that, $\forall T$, $\mathbf{x}^T(\mathbf{x}^*, \bar{\mathbf{x}})$ belongs to $(\ell_+^\infty)^m \times l_+^\infty \times (\ell_+^\infty)^m$. As in Le Van and Saglam (2004), Assumption **T2** is satisfied. We now check Assumption **T1**.

For any $\widetilde{\mathbf{x}} \in C$, $\widetilde{\widetilde{\mathbf{x}}} \in (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m$ such that for any T, $\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}}) \in C$ we have

$$\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) = -\sum_{t=0}^{T}\sum_{i=1}^{m}\eta_i\beta_i^t u^i(\widetilde{c}_t^i, \widetilde{l}_t^i) - \sum_{t=T+1}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u^i(\widetilde{\widetilde{c}}_t^i, \widetilde{\widetilde{l}}_t^i).$$

As $\widetilde{\widetilde{\mathbf{x}}} \in (\ell_+^\infty)^m \times \ell_+^\infty \times (\ell_+^\infty)^m$, $\sup_t |\widetilde{\widetilde{c}}_t| < +\infty$, there exists $A > 0, \forall t$, such that $|\widetilde{\widetilde{c}}_t| \leq A$. Since $\beta_i \in (0, 1)$, as $T \to \infty$ we have

$$0 \leq \sum_{t=T+1}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u^i(\widetilde{\widetilde{c}}_t^i, \widetilde{\widetilde{l}}_t^i) \leq u(A, 1)\sum_{t=T+1}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t = u(A, 1)\sum_{i=1}^{m}\sum_{t=T+1}^{\infty}\eta_i\beta_i^t \to 0$$

where $u(A, 1) = \max\{u_i(A, 1), i = 1, \ldots, m\}$. Hence, $\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) \to \mathcal{F}(\widetilde{\mathbf{x}})$ when $T \to \infty$. Taking account of the Theorem 1, we get (1)-(6).

Obviously, $\cap_{i=1}^m ri(dom(u^i)) \neq \emptyset$ where $ri(dom(u^i))$ is the relative interior of $dom(u^i)$. It follows from the Proposition 6.5.5 in Florenzano and Le Van (2001), we have

$$\partial\sum_{i=1}^{m}\eta_i\beta_i^t u^i(c_t^{*i}, l_t^{*i}) = \eta_i\beta_i^t\sum_{i=1}^{m}\partial u^i(c_t^{*i}, l_t^{*i}).$$

We then get (8)-(10) as the Kuhn-Tucker first-order conditions. ■

Remark 1 *1. We can prove that $\eta_i = 0 \Rightarrow c_t^{*i} = 0,\ l_t^{*i} = 1,\ \forall t$. Indeed, since the u^i are increasing, we have $c_t^{*i} = 0$, for any t. The individual rationality constraint implies $l_t^{*i} = 1$ for any t. The Inada condition on u^i, from (8), implies $\lambda_{-1}^i = 0$. Hence, $I = \{i : \eta_i > 0\} = \{i : \eta_i + \lambda_{-1}^i > 0\}$. It is easy to prove that $\sum_i \eta_i \sum_{t=0}^\infty \beta_i^t u^i(c_t^{*i}, l_t^{*i}) > \sum_i \eta_i \sum_{t=0}^\infty \beta_i^t u^i(0, 1)$. Therefore, there exists i with $\eta_i > 0$ and $\lambda_{-1}^i = 0$.*

2. For any optimal solution $(\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i})$, we have for any t, any $i \in I$, $\partial_1 u^i(c_t^{*i}, l_t^{*i}) \neq \emptyset$, $\partial_2 u^i(c_t^{*i}, l_t^{*i}) \neq \emptyset$, $\partial_1 f(k_t^*, L_t^*) \neq \emptyset$, $\partial_2 f(k_t^*, L_t^*) \neq \emptyset$, where $L_t^* = m - \sum_i l_t^{*i}$.

3. For $i \in I$, we have: $c_t^{*i} > 0$ iff $l_t^{*i} > 0$. In this case, $\partial_1 u^i(c_t^{*i}, l_t^{*i}) = \{u_c^i(c_t^{*i}, l_t^{*i})\}, \partial_2 u^i(c_t^{*i}, l_t^{*i}) = \{u_l^i(c_t^{*i}, l_t^{*i})\}$.

4. *For any $k_0 > 0$, there exists t with $\sum_i c_t^{*i} > 0$ and hence $\sum_i l_t^{*i} > 0$ (if not, $\sum_i \eta_i \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^{*i}, l_t^{*i}) = \sum_i \eta_i \sum_{t=0}^{\infty} \beta_i^t u^i(0,1)$: contradiction with the first statement.)*

In the following proposition, we will prove the positiveness of the optimal capital path.

Proposition 2 *If $k_0 > 0$, the optimal capital path satisfies $k_t^* > 0, \forall t$.*

Proof: Let $k_0 > 0$ but assume that $k_1^* = 0$. From (10), $L_1^* = 0$. This implies $\sum_i c_1^{*i} = 0$ and $l_1^{*i} = 1, \forall i$: a contradiction with (8). Hence $k_1^* > 0$. By induction, $k_t^* > 0, \forall t > 0$.
■

Remark 2 *From (10) and Proposition 2, if $k_0 > 0$, we have $L_t^* > 0$ for any $t \geq 0$. Hence, for any $t \geq 0$, $\partial_1 f(k_t^*, L_t^*) = \{f_k(k_t^*, L_t^*)\}$, $\partial_2 f(k_t^*, L_t^*) = \{f_L(k_t^*, L_t^*)\}$.*

Proposition 3 *Let $k_0 > 0$.*
(a) With any $\eta \in \Delta$, there exists a unique solution to the Pareto problem $\big((\mathbf{c}^{*i}), (\mathbf{l}^{*i}), \mathbf{k}^*\big)$. We have: For any $t \geq 0$,

$$\lambda_t^1(\eta) \in \cap_{i \in I} \ (\eta_i + \lambda_{-1}^i)\beta_i^t \partial_1 u^i(c_t^{*i}, l_t^{*i}) \tag{11}$$

$$\lambda_t^1(\eta) f_L(k_t^*, L_t^*) \in \cap_{i \in I} \ (\eta_i + \lambda_{-1}^i)\beta_i^t \partial_2 u^i(c_t^{*i}, l_t^{*i}) \tag{12}$$

$$\forall t \geq 1, \ 0 \in \lambda_t^1(\eta) \partial_1 f(k_t^*, L_t^*) - \lambda_{t-1}^1(\eta) \tag{13}$$

$$\text{and for any } i, \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^{*i}, l_t^{*i}) \geq \sum_{t=0}^{\infty} \beta_i^t u^i(0,1)$$

(b) Conversely, if the sequences $\mathbf{c}^{*i}, \mathbf{l}^{*i}, \mathbf{k}^*, \mathbf{L}^*$ satisfy

$$\begin{aligned} L_t^* &= \sum_i (1 - l_t^{*i}), \ \forall t \geq 0 \\ \sum_i c_t^{*i} &= f(k_t^*, L_t^*) - k_{t+1}^*, \ \forall t \geq 0 \\ k_0^* &= k_0 \\ \text{and for any } i, \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^{*i}, l_t^{*i}) &\geq \sum_{t=0}^{\infty} \beta_i^t u^i(0,1) \end{aligned}$$

and if there exist $\lambda^1 \in \ell_+^1$, $(\lambda_{-1}^i) \in \mathbf{R}_+^m$ which satisfy (11), (12) and (13), then $\mathbf{c}^{*i}, \mathbf{l}^{*i}, \mathbf{k}^*$ solve the Pareto problem with weights η and λ^1 is an associated multiplier.

Proof: It is easy. ■

Proposition 4 *Let $k_0 > 0$. Then there exists a unique multiplier $\lambda^{\mathbf{1}} \in \ell^1$.*

Proof: Existence has been proven. Let us prove uniqueness. First observe that, from Remark 2, we have $\partial_1 f(k_t^*, L_t^*) = \{f_k(k_t^*, L_t^*)\}$, $\partial_2 f(k_t^*, L_t^*) = \{f_L(k_t^*, L_t^*)\}$, for every t. We have three cases.

1. If for any t, $\sum_i c_t^{*i} > 0$, then $\lambda_t^1(\eta) = \eta_j \beta_j^t u^j(c_t^{*j}, l_t^{*j})$ with $c_t^{*j} > 0$ and $\lambda_{-1}^j = 0$ (see statement 1 of Remark 1).
2. Since $k_0 > 0$ there exists t with $\sum_i c_t^{*i} > 0$.
   (a) When $\sum_i c_0^{*i} > 0$, let T be the first date where $\sum_i c_T^{*i} = 0$ (and hence $\sum_i l_T^{*i} = 0$). From $t = 0$ to $t = T - 1$, $\lambda_t^1(\eta)$ is uniquely determined. We have, from (13), $\lambda_T^1(\eta) f_k(k_T^*, m) = \lambda_{T-1}^1(\eta)$ and $\lambda_T^1(\eta)$ is uniquely determined. But we also have $\lambda_{T+1}^1(\eta) f_k(k_{T+1}^*, L_{T+1}^*) = \lambda_T^1(\eta)$ and $\lambda_{T+1}^1(\eta)$ is uniquely determined. By induction, the result holds for every t.
   (b) When $\sum_i c_0^{*i} = 0$, let T be the first date where $\sum_i c_T^{*i} > 0$. In this case, $\lambda_T^1(\eta) = \eta_j \beta_j^t u_c^j(c_T^{*j}, l_T^{*j})$ with $c_T^{*j} > 0$. We have, from (13), $\lambda_T^1(\eta) f_k(k_T^*, L_T^*) = \lambda_{T-1}^1(\eta)$ and $\lambda_{T-1}^1(\eta)$ is uniquely determined. By backward induction $\lambda_t^1(\eta)$ is uniquely determined from 0 to $T-1$. We also have $\lambda_{T+1}^1(\eta) f_k(k_{T+1}^*, L_{T+1}^*) = \lambda_T^1(\eta)$ and $\lambda_{T+1}^1(\eta)$ is uniquely determined. By forward induction, the result holds for every $t \geq T + 1$.

■

Let us denote $I = \{i \mid \eta_i > 0\}$, $\beta = \max\{\beta_i | i \in I\}$, $I_1 = \{i \in I \mid \beta_i = \beta\}$ and $I_2 = \{i \in I \mid \beta_i < \beta\}$. We now show that the consumption and leisure paths of all agents with a discount factor less than the maximum one converge to zero. The proof is very simple compared to the one in Le Van, et al. (2007) which uses the supermodular structure inspired by lattice programming.

Proposition 5 *If $(\mathbf{k}^*, \mathbf{c}^{*i}, \mathbf{l}^{*i})$ denotes the optimal path starting from k_0, then $\forall i \in I_2$, $c_t^{*i} \longrightarrow 0$ and $l_t^{*i} \longrightarrow 0$.*

Proof: First observe that any Individually Rational Pareto Optimum $((\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*))$ is also a Pareto Optimum without the Individually Rationality Constraint. That means there exists $\eta \in \Delta$ such that $((\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*))$ solve

$$\max \sum_{i=1}^{m} \eta_i \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i) \tag{Q}$$

$$
\begin{aligned}
s.t. \quad & \sum_{i=1}^{m} c_t^i + k_{t+1} \leq f\left(k_t, \sum_{i=1}^{m}(1-l_t^i)\right), \forall t \\
& c_t^i \geq 0,\ l_t^i \geq 0,\ l_t^i \leq 1,\ \forall i, \forall t \\
& k_t \geq 0,\ \forall t,\ k_0 \text{ given.}
\end{aligned}
$$

For this problem, one can show that $\eta^i = 0$ implies that the sequences of optimal consumptions and leisures equal to 0.
Consider problem $\mathcal{R}_t$

$$
\begin{aligned}
V_t(k_t, k_{t+1}) &= \max \sum_{i=1}^{m} \eta_i \beta_i^t u^i(c_t^i, l_t^i) \\
\text{s.t. } \sum_{i=1}^{m} c_t^i + k_{t+1} &\leq F\left(k_t, \sum_{i=1}^{m}(1-l_t^i)\right) + (1-\delta)k_t.
\end{aligned}
$$

It is easy to see that the Pareto problem is equivalent to

$$
\begin{aligned}
& \max \sum_{t=0}^{\infty} V_t(k_t, k_{t+1}) \\
\text{s.t. } 0 \leq \ & k_{t+1} \leq F(k_t, m) + (1-\delta)k_t,\ \forall t \geq 0 \\
& k_0 \text{ is given.}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
V_t(k_t, k_{t+1}) &= \beta^t \max \sum_{i=1}^{m} \eta_i \left(\frac{\beta_i}{\beta}\right)^t u^i(c_t^i, l_t^i) \\
\text{s.t. } \sum_{i=1}^{m} c_t^i + k_{t+1} &\leq F\left(k_t, \sum_{i=1}^{m}(1-l_t^i)\right) + (1-\delta)k_t.
\end{aligned}
$$

Denote $Z^t = \left(\eta_i (\frac{\beta_i}{\beta})^t\right)$. From the Berge Maximum Theorem (1959), the strict concavity and the increasingness of the utility functions, the optimal c^{*i}, l^{*i} are continuous with respect to (Z^t, k_t, k_{t+1}). Denote these functions by $\left(\Gamma^i(Z^t, k_t^*, k_{t+1}^*), \Lambda^i(Z^t, k_t^*, k_{t+1}^*)\right)_i$. Let κ^*, ξ^* denote the limit points of k_t^*, k_{t+1}^* when $t \to +\infty$. Then, for $i \in I_2$, $\Gamma^i(Z^t, k_t^*, k_{t+1}^*)$ converges to $\Gamma^i(0_{I_2}, (\eta_i)_{i\in I_2}, \kappa^*, \xi^*) = 0$, and $\Lambda^i(Z^t, k_t^*, k_{t+1}^*)$ converges to $\Lambda^i(0_{I_2}, (\eta_i)_{i\in I_2}, \kappa^*, \xi^*) = 0$. ■

3 Existence of competitive equilibrium

We have proved that there exist Lagrange multipliers $\left((\lambda_{-1}^i)_i, \lambda(\eta)\right)$, with $(\lambda_{-1}^i)_i \in \mathbb{R}_+^m$ and

$$
\begin{aligned}
\lambda(\eta) &= (\lambda^{\mathbf{1}}(\eta), \lambda^{\mathbf{2i}}(\eta), \lambda^{\mathbf{3}}(\eta), \lambda^{\mathbf{4i}}(\eta), \lambda^{\mathbf{5i}}(\eta)) \\
&\in \ell_+^1 \times (\ell_+^1)^m \times \ell_+^1 \times (\ell_+^1)^m \times (\ell_+^1)^m, i = 1...m,
\end{aligned}
$$

for the Pareto problem. In what follow, we will prove that, with given ($\mathbf{c}^*$, $\mathbf{k}^*, \mathbf{l}^*, \mathbf{L}^*$), one can associate a sequence of prices, $(p_t^*)_{t=0}^{\infty}$, and a sequence of wages, $(w_t^*)_{t=0}^{\infty}$, defined as

$$\begin{aligned} p_t^* &= \lambda_t^1, \ \forall t \\ w_t^* &= \lambda_t^1 f_L(k_t^*, L_t^*), \ \forall t \end{aligned}$$

where $f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*)$, and a price $r > 0$ for the initial capital stock k_0 such that $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r)$ is a *price equilibrium with transfers* (see Definition 2 below). The appropriate transfer to each consumer is the amount that just allows the consumer to afford the consumption stream allocated by the social optimization problem. Thus, for given weight $\eta \in \Delta$, the required transfers are:

$$\phi_i(\eta) = \sum_{t=0}^{\infty} p_t^*(\eta) c_t^{i*}(\eta) + \sum_{t=0}^{\infty} w_t^*(\eta) l_t^{i*}(\eta) - \sum_{t=0}^{\infty} w_t^*(\eta) - \vartheta^i r k_0 - \alpha^i \pi^*(\eta)$$

where

$$\pi^*(\eta) = \sum_{t=0}^{\infty} p_t^*(\eta)[f(k_t^*(\eta), L_t^*(\eta)) - k_{t+1}^*(\eta)] - \sum_{t=0}^{\infty} w_t^*(\eta) L_t^*(\eta) - r k_0.$$

According to the Negishi approach, a competitive equilibrium for this economy corresponds to a set of welfare weights $\eta \in \Delta$ such that these transfers equal to *zero*. Now we define an equilibrium with transfers.

Definition 2 *A given allocation $\{\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i}, \mathbf{L}^{*i}\}$, together with a price sequence $\mathbf{p}^*$ for consumption good, a wage sequence $\mathbf{w}^*$ for labor and a price r for the initial capital stock k_0 constitute an equilibrium with transfers if*

i)

$$\begin{aligned} \mathbf{c}^* &\in (\ell_+^\infty)^m, \mathbf{l}^* \in (\ell_+^\infty)^m, \mathbf{L}^* \in (\ell_+^\infty)^m, \mathbf{k}^* \in \ell_+^\infty, \\ \mathbf{p}^* &\in \ell_+^1 \backslash \{0\}, \mathbf{w}^* \in \ell_+^1 \backslash \{0\}, \ r > 0 \end{aligned}$$

ii) For every $i = 1, \ldots, m$, $(\mathbf{c}^{*i}, \mathbf{l}^{*i})$ is a solution to the problem

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i) \\ &st \quad \sum_{t=0}^{\infty} p_t^* c_t^i + \sum_{t=0}^{\infty} w_t^* l_t^i \leq \sum_{t=0}^{\infty} p_t^* c_t^{*i} + \sum_{t=0}^{\infty} w_t^* l_t^{*i} \end{aligned}$$

iii) $(\mathbf{k}^*, \mathbf{L}^*)$ is a solution to the firm's problem:

$$\begin{aligned} \pi^* &= \max \sum_{t=0}^{\infty} p_t^*[f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{\infty} w_t^* L_t - r k_0 \\ s.t. \quad 0 &\leq k_{t+1} \leq f(k_t, L_t), 0 \leq L_t, \forall t \end{aligned}$$

iv) Markets clear

$$\sum_{i=1}^{m} c_t^{*i} + k_{t+1}^* = f\left(k_t^*, \sum_{i=1}^{m} L_t^{*i}\right), \ \ \forall t,$$

$$L_t^* = \sum_{i=1}^{m} L_t^{*i}, l_t^{*i} = 1 - L_t^{*i} \textit{and } k_0^* = k_0.$$

The differences between two definitions - competitive equilibrium and price equilibrium with transfers - are the budget constraints of consumers. If the transfers $\phi_i(\eta) = 0$ for all i, a price equilibrium with transfers is a competitive equilibrium.

Before proving existence of an equilibrium, we will first prove that any *solution to the Pareto problem*, $\mathbf{x}^* = (\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i})$, associated with $k_0 > 0$ and $\eta \in \Delta$ is an equilibrium with transfers, with some appropriate prices $(p_t^*) \in \ell_+^1 \setminus \{0\}$ and wages $(w_t^*) \in \ell_+^1 \setminus \{0\}$.

The following result is required.

Proposition 6 *Let $k_0 > 0$.*
1. For any $\varepsilon > 0$, there exists T such that, for any $\eta \in \Delta$,

$$\sum_{T}^{+\infty} \lambda^1{}_t(\eta) \sum_i c_t^{*i} \le \varepsilon$$

$$\sum_{T}^{+\infty} \lambda^1{}_t(\eta) f_L(k_t^*, L_t^*) \sum_i l_t^{*i} \le \varepsilon$$

$$\sum_{T}^{+\infty} \lambda^1{}_t(\eta) f_L(k_t^*, L_t^*) \le \varepsilon.$$

2. There exists M such that, for any $\eta \in \Delta$,

$$\begin{aligned} \sum_{t=0}^{+\infty} \lambda^1{}_t(\eta) \sum_i c_t^{*i} &\le M \\ \sum_{t=0}^{+\infty} \lambda^1{}_t(\eta) f_L(k_t^*, L_t^*) \sum_i l_t^{*i} &\le M \\ \sum_{t=0}^{+\infty} \lambda^1{}_t(\eta) f_L(k_t^*, L_t^*) &\le M. \end{aligned}$$

Proof: 1. We know that there exists A such that $c_t^{*i}(\eta) \le A$, $\forall t$, $\forall i$, $\forall \eta \in \Delta$. Therefore

$$\begin{aligned} \frac{\beta^T}{1-\beta} \sum_i u^i(A, 1) &\ge \sum_{T}^{+\infty} \sum_i \eta_i \beta_i^t [u^i(c_t^{*i}, l_t^{*i}) - u^i(0,0)] \\ &\ge \sum_{T}^{+\infty} \lambda_t^1 \sum_i c_t^{*i} + \sum_{T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) \sum_i l_t^{*i}. \end{aligned}$$

Let $\varepsilon > 0$. There exists T such that $\frac{\beta^T}{1-\beta} \leq \varepsilon$. Hence, $\sum_T^{+\infty} \lambda^1{}_t(\eta) \sum_i c_t^{*i} \leq \varepsilon$, $\sum_T^{+\infty} \lambda^1{}_t(\eta) f_L(k_t^*, L_t^*) \sum_i l_t^{*i} \leq \varepsilon$, for any η.

We now prove that for T large enough, $\sum_T^{+\infty} \lambda^1{}_t(\eta) f_L(k_t^*, L_t^*) \leq \varepsilon$ for any η. We have

$$\sum_i c_t^{*i} = f(k_t^*, L_t^*) - k_{t+1}^*.$$

Since

$$f(k_t^*, L_t^*) = f(k_t^*, L_t^*) - f(0,0) \geq f_k(k_t^*, L_t^*)k_t^* + f_L(k_t^*, L_t^*)L_t^*,$$

we obtain by using (10):

$$\sum_{t=T}^{T+\tau} \lambda_t^1 \sum_i c_t^{*i} \geq \lambda_T^1 f_k(k_T^*, L_T^*)k_T^* - \lambda_{T+\tau}^1 k_{T+\tau+1}^* + \sum_{t=T}^{T+\tau} \lambda_t^1 f_L(k_t^*, L_t^*)L_t^*.$$

Let $\tau \to +\infty$. Since $\lambda^1 \in \ell^1$, and $k_t^* \leq \max\{k_0, \bar{k}\}, \forall t$, we have

$$\begin{aligned} \sum_{t=T}^{+\infty} \lambda_t^1 \sum_i c_t^{*i} &\geq \lambda_T^1 f_k(k_T^*, L_T^*)k_T^* + \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*)L_t^* \\ &\geq \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*)L_t^* = \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*)(m - \sum_i l_t^{*i}) \end{aligned} \quad (14)$$

Hence, for T large enough,

$$m \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) \leq \sum_{t=T}^{+\infty} \lambda_t^1 \sum_i c_t^{*i} + \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) \sum_i l_t^{*i} \leq \varepsilon$$

for any η.

2. Obviously:

$$\begin{aligned} \sum_0^{+\infty} \lambda_t^1 \sum_i c_t^{*i} + \sum_0^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) \sum_i l_t^{*i} &\leq M_1 = \frac{1}{1-\beta} \sum_i u^i(A,1) \quad (15) \\ \sum_{t=0}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) &\leq M_2 = \frac{2}{m} \times \frac{1}{1-\beta} \sum_i u^i(A,1). \end{aligned}$$

■

Proposition 7 *Let $k_0 > 0$. Let $(\mathbf{k}^*, \mathbf{c}^*, \mathbf{L}^*, \mathbf{l}^*)$ solve the Pareto problem associated with $\eta \in \Delta$. Take*

$$\begin{aligned} p_t^* &= \lambda_t^1,\ w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*) \text{ for any } t \\ \text{and } r &= \lambda_0^1[F_k(k_0, 0) + 1 - \delta]. \end{aligned}$$

Then $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$ is an equilibrium with transfers.

Proof:
i) We have

$$\mathbf{c}^* \in (\ell_+^\infty)^m, \mathbf{l}^* \in (\ell_+^\infty)^m, \mathbf{k}^* \in \ell_+^\infty, \mathbf{p}^* \in \ell_+^1, \mathbf{w}^* \in \ell_+^1.$$

From Remark 1 statement 4, $\mathbf{p}^* \neq \mathbf{0}$, and together with Remark 2, $\mathbf{w}^* \neq \mathbf{0}$.

ii) We now show that $(\mathbf{c}^{*i}, \mathbf{l}^{*i})$ solves the consumer's problem. Let $(\mathbf{c}^i, \mathbf{l}^i)$ satisfy

$$\sum_{t=0}^{\infty} p_t^* c_t^i + \sum_{t=0}^{\infty} w_t^* l_t^i \leq \sum_{t=0}^{\infty} p_t^* c_t^{*i} + \sum_{t=0}^{\infty} w_t^* l_t^{*i}.$$

Let

$$\Delta = \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^{*i}, l_t^{*i}) - \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i)$$

Since u^i is concave, from Proposition 3, statement (a), we have

$$\begin{aligned} \Delta &\geq \sum_{t=0}^{\infty} \frac{\lambda_t^1}{\eta_i + \lambda_{-1}^i}(c_t^{*i} - c_t^i) + \sum_{t=0}^{\infty} \frac{\lambda_t^1 f_L(k_t^*, L_t^*)}{\eta_i + \lambda_{-1}^i}(l_t^{*i} - l_t^i) \\ &= \sum_{t=0}^{\infty} \frac{p_t^*}{\eta_i + \lambda_{-1}^i}(c_t^{*i} - c_t^i) + \sum_{t=0}^{\infty} \frac{w_t^*}{\eta_i + \lambda_{-1}^i}(l_t^{*i} - l_t^i) \geq 0. \end{aligned}$$

This means $(\mathbf{c}^{*i}, \mathbf{l}^{*i})$ solves the consumer's problem.

iii) We now show that $(\mathbf{k}^*, \mathbf{L}^*)$ is solution to the firm's problem. Since $p_t^* = \lambda_t^1$, $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)$, we have

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*)\ L_t^* - rk_0.$$

Let :

$$\begin{aligned} \Delta_T &= \sum_{t=0}^{T} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*)\ L_t^* - rk_0 \\ &\quad - \left(\sum_{t=0}^{T} \lambda_t^1 [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) L_t - rk_0 \right) \end{aligned}$$

By the concavity of f, we get

$$\begin{aligned} \Delta_T &\geq \sum_{t=1}^{T} \lambda_t^1 f_k(k_t^*, L_t^*)(k_t^* - k_t) - \sum_{t=0}^{T} \lambda_t^1 (k_{t+1}^* - k_{t+1}) \\ &= [\lambda_1^1 f_k(k_1^*, L_1^*) - \lambda_0^1](k_1^* - k_1) + \ldots \\ &\quad + [\lambda_T^1 f_k(k_T^*, L_T^*) - \lambda_{T-1}^1](k_T^* - k_T) - \lambda_T^1(k_{T+1}^* - k_{T+1}). \end{aligned}$$

From Proposition 3, statement (b), we have:

$$\Delta_T \geq -\lambda_T^1(k_{T+1}^* - k_{T+1}) = -\lambda_T^1 k_{T+1}^* + \lambda_T^1 k_{T+1} \geq -\lambda_T^1 k_{T+1}^*.$$

Since $\lambda^1 \in \ell^1_+$, $\sup_T k^*_{T+1} < +\infty$, we have

$$\lim_{T\to+\infty} \Delta_T \geq \lim_{T\to+\infty} -\lambda^1_T k^*_{T+1} = 0.$$

We have proved that the sequences $(\mathbf{k}^*, \mathbf{L}^*)$ maximize the profit of the firm.

Finally, the market is cleared as the utility function is strictly increasing. ■

Let $k_0 > 0$. From Proposition 4, we define the following mapping

$$\phi_i(\eta) = \sum_{t=0}^{\infty} p^*_t(\eta) c^{*i}_t(\eta) + \sum_{t=0}^{\infty} w^*_t(\eta) l^{*i}_t(\eta) - \sum_{t=0}^{\infty} w^*_t(\eta) - \vartheta^i r k_0 - \alpha^i \pi^*(\eta)$$

where

$$\begin{aligned} p^*_t &= \lambda^1_t, w^*_t = \lambda^1_t f_L(k^*_t, L^*_t), \forall t \\ \pi^*(\eta) &= \sum_{t=0}^{\infty} p^*_t(\eta)[f(k^*_t(\eta), L^*_t(\eta)) - k^*_{t+1}(\eta)] - \sum_{t=0}^{\infty} w^*_t(\eta) L^*_t(\eta) - r k_0. \end{aligned}$$

This mapping ϕ_i is uniformly bounded (see Proposition 6, statement 2).

We can now state our main result.

Theorem 2 *Assume* ***U1, U2, U3, F1, F2, F3****. Let $k_0 > 0$. Then there exists $\bar{\eta} \in \Delta, \bar{\eta} >> 0$, such that $\phi_i(\bar{\eta}) = 0, \forall i$. This means there exists a competitive equilibrium.*

Proof: We first prove that ϕ_i is continuous for any i. Let $(\eta^n) \to \eta$. Since,

$$c^{*i}_t(\eta^n) \to c^{*i}_t(\eta), l^{*i}_t(\eta^n) \to l^{*i}_t(\eta), k^*_t(\eta^n) \to k^*_t(\eta),$$

and if $\sum_j c^{*j}_t(\eta) > 0$ then $p^*_t(\eta^n) \to p^*_t(\eta), w^*_t(\eta^n) \to w^*_t(\eta)$. It remains to be proven that $p^*_t(\eta^n) \to p^*_t(\eta), w^*_t(\eta^n) \to w^*_t(\eta)$ even when $\sum_j c^{*j}_t(\eta) = 0$. Let $\mathcal{T} = \{t : \sum_j c^{*j}_t(\eta) = 0\}$. From the proof in Proposition 6, there exists M such that for any $\eta \in \Delta$,

$$\sum_{t=0}^{+\infty} w^*_t(\eta) = \sum_{t=0}^{+\infty} \lambda^1{}_t(\eta) f_L(k^*_t, L^*_t) \leq M$$

and for any $\varepsilon > 0$, there exists T_0 such that, for any $\eta \in \Delta$, for any $T \geq T_0$,

$$\sum_{T}^{+\infty} w^*_t(\eta) = \sum_{T}^{+\infty} \lambda^1{}_t(\eta) f_L(k^*_t, L^*_t) \leq \varepsilon$$

These inequalities show that $\{w^*(\eta^n)\}$ is in a relatively compact set of ℓ^1. We can assume that it converges to $(\bar{w}_t) \in \ell^1$. From (13), for $t \in \mathcal{T}, \lambda^1_t(\eta^n) \to \bar{\lambda}^1_t = \dfrac{\bar{w}_t}{f_L(k^*_t, m)}$.

When $\sum_j c^{*j}_0(\eta) > 0$, consider T, the first date where $\sum_j c^{*j}_T(\eta) = 0$. For $t = 0, \ldots, T-1$, we have $\lambda^1_t(\eta^n) \to \lambda^1_t(\eta)$. Since $\lambda^1_T(\eta^n) f_L(k^*_T(\eta^n), L^*_t(\eta^n)) =$

$\lambda^1_{T-1}(\eta^n)$, we have $\bar{\lambda}^1_t f_L(k^*_T(\eta), m) = \lambda^1_{T-1}(\eta)$. From Proposition 4, and relation (13), we have $\bar{\lambda}^1_T = \lambda^1_T(\eta)$. In other words, $\lambda^1_T(\eta^n) \to \lambda^1_T(\eta)$. By induction, $\lambda^1_t(\eta^n) \to \lambda^1_t(\eta)$ for any $t \geq T$.

Use the same arguments to prove that $\lambda^1_t(\eta^n) \to \lambda^1_t(\eta)$ for any t, when $\sum_j c_0^{*j}(\eta) = 0$.

From these results we get $\bar{w}_t = w^*_t(\eta)$ for any t.

It follows from (14) and (15) in Proposition 6 that for any $\eta \in \Delta$, any T

$$\frac{\beta^T}{1-\beta}\sum_i u^i(A,1) \geq \sum_{t=T}^{+\infty} \lambda^1_t \sum_i c^{*i}_t \geq \sum_T^{+\infty} \lambda^1_t f_L(k^*_t, L^*_t) L^*_t$$

or

$$\frac{2\beta^T}{1-\beta}\sum_i u^i(A,1) \geq \sum_{t=T}^{+\infty} \lambda^1_t \sum_i \left(c^{*i}_t + f_L(k^*_t, L^*_t) l^{*i}_t\right) \geq m \sum_T^{+\infty} \lambda^1_t f_L(k^*_t, L^*_t) \quad (16)$$

Let $\varepsilon > 0$. From inequality (16), there exists T such that for any n we have:

$$\begin{aligned}
&|\sum_{t\geq T} p^*_t(\eta^n) c^{*i}_t(\eta^n) + \sum_{t\geq T} w^*_t(\eta^n) l^{*i}_t(\eta^n) \\
&- \sum_{t\geq T} w^*_t(\eta^n) - \vartheta^i r k_0 - \alpha^i \sum_{t\geq T} p^*_t(\eta^n) \sum_i c^{*i}_t(\eta^n) \\
&- \sum_{t\geq T} w^*_t(\eta^n)(m - \sum_i l^{*i}_t(\eta^n)) - r k_0| \leq \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&|\sum_{t\geq T} p^*_t(\eta) c^{*i}_t(\eta) + \sum_{t\geq T} w^*_t(\eta) l^{*i}_t(\eta) \\
&- \sum_{t\geq T} {w^*}_t(\eta) - \vartheta^i r^*(\eta) k_0 - \alpha^i \sum_{t\geq T} {p^*}_t(\eta) \sum_i c^{*i}_t(\eta) \\
&- \sum_{t\geq T} {w^*}_t(\eta)(m - \sum_i l^{*i}_t(\eta)) - r^*(\eta) k_0| \leq \varepsilon
\end{aligned}$$

Consider $t \in \{0, \ldots, T-1\}$. One has: $p^*_t(\eta^n) \to {p^*}_t(\eta)$, $w^*_t(\eta^n) \to w^*_t(\eta)$, $c^{*i}_t(\eta^n) \to c^{*i}_t(\eta)$, $l^{*i}_t(\eta^n) \to l^{*i}_t(\eta)$, $k^*_t(\eta^n) \to k^*_t(\eta)$. Thus, for n large enough, we have $|\phi_i(\eta^n) - \phi_i(\eta)| \leq 3\varepsilon$. The proof that ϕ_i is continuous is complete.

Observe that $\sum_i \phi_i(\eta) = 0$ by Walras law. It follows from remark 1.1 that $\eta_i = 0 \Rightarrow \phi_i(\eta) < 0$. Let $\Psi_i(\eta) = \eta_i - \phi_i(\eta)$. We then have $\sum_i \Psi_i(\eta) = 1$ and $\eta_i = 0 \Rightarrow \Psi_i(\eta) > 0$. The mapping $\Psi = (\Psi_1, ..., \Psi_m)$ satisfies the inward boundary fixed-point theorem. There exists $\bar{\eta} \in \Delta$, $\bar{\eta} >> 0$ such that $\Psi(\bar{\eta}) = \bar{\eta}$, or equivalently $\phi_i(\bar{\eta}) = 0, \forall i$. ■

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