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Coalition structures induced by the strength of a graph

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Abstract

We study cooperative games associated with a communication structure which takes into account a level of communication between players. Let us consider an undirected communication graph : each node represents a player and there is an edge between two nodes if the corresponding players can communicate directly. Moreover we suppose that a weight is associated with each edge. We compute the so-called strength of this graph and use the corresponding partition to determine a particular coalition structure. The strength of a graph is a measure introduced in graph theory to evaluate the resistance of networks under attacks. It corresponds to the minimum on all subsets of edges of the ratio between the sum of the weights of the edges and the number of connected components created when the set of edges is suppressed from the graph. The set of edges corresponding to the minimum ratio induces a partition of the graph. We can iterate the calculation of the strength on the subgraphs of the partition to obtain refined partitions which we use to define a hierarchy of coalition structures. For a given game on the graph, we build new games induced by these coalition structures and study the inheritance of convexity properties, and the Shapley value associated with them.

Keywords: communication networks, coalition structure, cooperative game.

1 Introduction

In this paper we consider communication games introduced by Myerson in 1977 [10]. These games are cooperative games (N, v) defined on the set of vertices N of an undirected graph $G = (N, E)$, where E is the set of

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edges. v is the characteristic function of the game, $v : 2^N \to \mathbb{R}$, $A \mapsto v(A)$ and verifies $v(\emptyset) = 0$. For every coalition $A \subseteq N$, we consider the induced restricted graph $G_A := (A, E(A))$. The set of edges $E(A)$ of G_A is the set of edges $e = \{i, j\} \in E$ such that i and j are in A. We denote by A/G the set of connected components of G_A. Myerson defined the restricted game $\overline{v}$ on N, also called a point game, by :

$$\text{(1)} \qquad \overline{v}(A) = \sum_{F \in A/G} v(F), \text{ for all } A \subseteq N.$$

Observe that it is sufficient to define v on A/G. The graph G describes how the players of N can communicate: $e = \{i, j\} \in E$ if and only if the players i and j can directly communicate. For every coalition $A \subseteq N$, a set $F \in A/G$ is defined as a maximal subcoalition of A such that all pairs of players i, j in F can communicate by a path in G_A starting from i and ending at j. Myerson provided an axiomatic characterization of the Shapley value of these games. The new game $\overline{v}$ takes into account how the players of N can communicate according to the graph G. Owen [11] proved that if v is superadditive then $\overline{v}$ is also superadditive without any assumption on G.

A graph $G = (N, E)$ is *cycle-complete* if every set of nodes corresponding to a cycle in G induces a complete subgraph of G. Van den Nouweland and Borm in 1991 [14] proved that if G is cycle-complete (particularly if G is cycle-free) and if v is convex, then $\overline{v}$ is also convex. Thus nice properties of the underlying game v are inherited by the restricted game $\overline{v}$.

In the present paper we associate weights or *strengths* with the edges of the graph. Each edge-strength $u(e)$ of $e = \{i, j\} \in E$ can be seen as a measure of the resistance or of the level of communication between players i and j. The model we propose takes more insight in the combinatorial structure of the graph than previous ones by using the concept of strength of a graph. This concept has been introduced by Gusfield [9] for graphs with edges of unit strengths and generalized to arbitrary edge-strengths by Cunningham [4]. The *strength* $\sigma(G, u)$ of G is defined by:

$$\text{(2)} \qquad \sigma(G, u) := \min_{A \subseteq E} \frac{u(A)}{k(A) - k(\emptyset)}$$

where $k(A)$ is the number of connected components of the graph $G = (N, E \setminus A)$ (and $k(\emptyset)$ the number of components of G). σ can be seen as a measure of the resistance of the network G under attack. Indeed, if we suppose that someone wants to destroy as much as possible the communication possibilities, and that the effort required to delete a link between two players is proportional to the strength of this link, then $\sigma(G, u)$ is the minimal average effort one has to make to augment the number of components as much as possible by deleting a subset A of edges of G. The partition of N in connected components corresponding to the graph $G = (N, E \setminus A)$ and to a minimizer A in the definition of σ provides a decomposition of N into connected components which are strongly coherent in the following sense: they take into account both the strength $u(e)$ of the links of communication and the combinatorial structure of the communication graph G. Besides, if B is such a component then the strength of the induced graph $G_B = (B, E(B))$ is larger than the strength of G.

We propose several new restricted games associated with (N, v) and the strength u. For the first one, we consider a chain of partitions of N. Partition $\mathcal{P}_1$ is the partition of N corresponding to the strength of G. Then, for each $A \in \mathcal{P}_1$, the strength of $G_A = (A, E(A))$ provides a new partition of A and therefore a refinement $\mathcal{P}_2$ of the partition $\mathcal{P}_1$ and so on. Now for a given $A \subseteq N$, we construct a partition $\mathcal{P}(A)$ of A by filling A as much as possible using at first subsets of $\mathcal{P}_1$, and then subsets of $\mathcal{P}_2$, and so on. The new restricted game $\tilde{v}$ is defined by :

$$\tilde{v}(A) := \sum_{F \in \mathcal{P}(A)} v(F). \tag{3}$$

One of the main interest of this definition is to provide natural examples where the results of Algaba, Bilbao and Lopez [1] and Faigle [5] about convex intersecting systems apply immediately to the family $\mathcal{F} := \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_n$. As a consequence we prove that convexity of the game (N, v) implies convexity of the new game $\tilde{v}$, and therefore the Shapley value of $\tilde{v}$ is still in the core of $\tilde{v}$. Thereafter, we construct another restricted game, called the main game associated with the strength, in the same spirit as the graph-restricted game of Myerson [10]. We do not use anymore the preceding chain of partitions but, for every $A \subseteq N$, we consider a subset $S \subseteq E(A)$ which is a minimizer of the strength of $G_A = (A, E(A))$ and the corresponding partition $\mathcal{P}(A)$ of A we obtain after deleting S. We define $\overline{v}$ by

$$\overline{v}(A) := \sum_{F \in \mathcal{P}(A)} v(F). \tag{4}$$

That is, in Myerson's definition we have replaced the partition of A into connected components by a partition of A into strongly coherent connected components associated with the strength of the graph $G_A = (A, E(A))$.

The article is organized as follows. We define in Section 2 the partitions associated with the strength of a graph. We study in Section 3 the inheritance of superadditivity and convexity for the game $\tilde{v}$. In Section 4 we give a simple counterexample to the inheritance of superadditivity (and therefore of convexity) for the main game $\overline{v}$ associated with the strength. Then we define a slightly weaker condition than convexity in Section 5 and establish necessary conditions on the edge-weights to have inheritance of this property. In Section 6 we prove that these conditions are also sufficient in the case of cycle-free graphs. Finally we compute the Shapley value of the game $\overline{v}$ in the case of cycle-free graphs in Section 7.

2 The partition associated with the strength of a graph

Let $G = (N, E)$ be a connected graph and let $u : E \to \mathbb{R}^+$ be a weight function on the set E of edges. The strength of G is defined by:

$$\sigma(G, u) := \min_{A \subseteq E} \frac{u(A)}{k(A) - 1} \tag{5}$$

where $k(A)$ is the number of connected components of the graph $G = (N, E \setminus A)$. When G and u are fixed and that there is no ambiguity, we simply denote by σ the strength $\sigma(G, u)$ of G. The computation of the strength is a polynomial problem [4].

Let r be the rank function associated with G, *i.e.*, for all $A \subseteq E$, $r(A)$ denotes the size of a maximal forest included in A. The rank function is *submodular*:

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B) \quad \forall A, B \subseteq E. \tag{6}$$

As G is a connected graph, we have $r(E) = |V| - 1$ and $r(E \setminus A) = |V| - k(A)$. Thus $k(A) - 1 = r(E) - r(E \setminus A)$ and:

$$\sigma(G, u) = \min_{A \subseteq E} \frac{u(A)}{r(E) - r(\overline{A})}. \tag{7}$$

We will define the strength of a not necessarily connected graph by this last formula because this last definition naturally extends to the case of a matroid or a polymatroid (cf [7, 6, 13]).

Let us define the auxiliary function f by, $\forall A \subseteq E$:

$$\begin{array}{rcl} f(A) & := & u(A) - \sigma(k(A) - 1) \\ & = & u(A) + \sigma r(\overline{A}) - \sigma r(E). \end{array} \tag{8}$$

As r is submodular and u is additive, f is submodular. By definition of σ, $\forall A \subseteq E$, $f(A) \geq 0$ and $f(A) = 0$ with $A \neq \emptyset$ is equivalent to $\frac{u(A)}{k(A)-1} = \sigma$, that is, A realizes the minimum of the strength. Thus $A \neq \emptyset$ realizes the minimum of the ratio in the definition of the strength if and only if A realizes the minimum of the submodular function f. It is a classical result that the family of sets which realizes the minimum of a given submodular function is closed under union and intersection. We give a short proof of this result for completeness.

Let us consider two subsets A and B of E satisfying $f(A) = f(B) = 0$. We have $f(A \cup B) \geq 0$ and $f(A \cap B) \geq 0$. Therefore, as f is submodular, we obtain $0 \leq f(A \cup B) + f(A \cap B) \leq f(A) + f(B) = 0$. Thus $f(A \cup B) = f(A \cap B) = 0$. Hence the family $\{A \subseteq E\ ; f(A) = 0\}$ is closed under union and intersection, and there exists a maximal element $A_{\max}$ in $\mathcal{F}$ and a minimal one $A_{\min}$ in $\mathcal{F}$. If $A_{\min} \neq \emptyset$, we have:

$$\sigma = \frac{u(A_{\max})}{k(A_{\max}) - 1} = \frac{u(A_{\min})}{k(A_{\min}) - 1} \tag{9}$$

and for all $A \subseteq E$ such that $\sigma = \frac{u(A)}{k(A)-1}$ we have:

$$A_{\min} \subseteq A \subseteq A_{\max}. \tag{10}$$

For example, if the graph G is a tree and if all weights are equal to 1, every subset $A \neq \emptyset$ is a minimizer of the strength:

$$\sigma = \frac{|A|}{k(A) - 1} = \frac{|A|}{|A|} = 1.$$

We have $A_{\max} = E$ and $A_{\min} = \emptyset$ and there are as many partitions of N associated with the strength as nonempty subsets A of E. Especially each edge e is a minimal minimizer but there is no smallest minimizer.

3 A first family of restricted games associated with the strength of a graph

Let (N, v) be a game on the set N of vertices of the graph $G = (N, E)$ and let $u : E \to \mathbb{R}^+$ be a weight function on the set of edges. For a family $\mathcal{F}$ of subsets of N and a subset A of N, we denote by $\mathcal{F}(A)$ the elements of $\mathcal{F}$ included in A.

$$\mathcal{F}(A) := \{F \in \mathcal{F}; F \subseteq A\}. \tag{11}$$

We consider on the set of players N a hierarchy of coalition structures, that is, a finite number of partitions $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_m$ of N such that $\mathcal{P}_0 = \{N\}$, $\mathcal{P}_m = \{\{1\}, \{2\}, \ldots, \{n\}\}$ is the singleton coalition structure, and:

$$\mathcal{P}_m \leq \mathcal{P}_{m-1} \leq \cdots \leq \mathcal{P}_{i+1} \leq \mathcal{P}_i \leq \cdots \leq \mathcal{P}_1 \leq \mathcal{P}_0 \tag{12}$$

where $\mathcal{P}_{i+1} \leq \mathcal{P}_i$ means that every block of $\mathcal{P}_{i+1}$ is a subset of a block of $\mathcal{P}_i$. $\mathcal{P}_1$ is one of the partitions of N given by the strength of the graph G. For every $A \in \mathcal{P}_i$, we consider the subgraph $G_A = (A, E(A))$. We select a minimizer of $\sigma(G_A)$ and consider the corresponding partition of A. This partition provides the blocks of $\mathcal{P}_{i+1}$ which are subsets of A. Let $\mathcal{F}$ be the family

$$\mathcal{F} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_m. \tag{13}$$

We define a new game $\tilde{v}$ on N by:

$$\tilde{v}(A) := \sum_{F \in \mathcal{F}(A),\, F \text{ maximal}} v(F) \quad \text{for all } A \in 2^N. \tag{14}$$

The family of sets $\mathcal{F}$ obviously satisfies the following property. For all $A, B \in \mathcal{F}$, one and only one of the following properties is verified: $A \cap B = \emptyset$, or $A \subset B$ or $B \subset A$ or $A = B$. We say that $\mathcal{F}$ is a *nested family*. Therefore $\mathcal{F}$ is also an *intersecting system*[1], $\mathcal{F}$ is also *weakly union-closed*[2] and therefore $\mathcal{F} \cup \emptyset$ is also a *partition system*[3] (cf. [1, 5, 8]).

We recall that a game (N, v) is *superadditive* if, for all $A, B \in 2^N$ such that $A \cap B = \emptyset$, $v(A \cup B) \geq v(A) + v(B)$. A game (N, v) is *convex* if the function v is *supermodular i.e.*, for all $A, B \in 2^N$, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$. We say v is $\mathcal{F}$*-superadditive* if for all $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, we have:

$$v(A \cup B) \geq v(A) + v(B). \tag{15}$$

Observe that for such games, since $\mathcal{F}$ is a nested family, we have for all $A, B \in \mathcal{F}$:

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B). \tag{16}$$

A game v defined on an intersecting system $(N, \mathcal{F})$ and satisfying condition (16) is called an *intersecting convex game* (cf. [5]). Algaba et al. [1] and Faigle [5] have proved:

Theorem 1. *If $(N, \mathcal{F}, v)$ is an intersecting convex game, then the restricted game $(N, 2^N, \tilde{v})$ defined by:*

$$\tilde{v}(A) = \sum_{F \in \mathcal{F}(A),\, F \text{ maximal}} v(F)$$

for all $A \subseteq N$, is a convex game.

This last theorem applies to any preceding nested family $\mathcal{F}$ we have constructed by (11), (12), (13) and (14) using the strength of a graph, and therefore the following theorem is a corollary of Theorem 1:

Theorem 2. *If $\mathcal{F}$ is a family of subsets of N associated with the strength of a graph $G(N, E)$ by means of the preceding hierarchy of coalitions structures, and if $(N, \mathcal{F}, v)$ is an $\mathcal{F}-$superadditive game on N, then the restricted game $(N, 2^N, \tilde{v})$ is a convex game.*

---

[1] If A and $B \in \mathcal{F}$ and if $A \cap B \neq \emptyset$ then $A \cap B$ and $A \cup B$ are in $\mathcal{F}$.

[2] If A and B are in $\mathcal{F}$ and if $A \cap B \neq \emptyset$ then $A \cup B \in \mathcal{F}$.

[3] For all $A \in \mathcal{F}$, the maximal subsets $F \in \mathcal{F}(A)$ form a partition of A, and the singletons are in $\mathcal{F}$.

We give a direct proof following the method of A. Van den Nouweland and P. Borm (1991) [14].

Proof. For all $A, B \in N$ such that $A \cap B = \emptyset$, we have $v(A \cup B) \geq v(A) + v(B)$. Let us now consider A, B and $i \in N$ such that $A \subset B \subseteq N \setminus \{i\}$. We have to prove $\tilde{v}(A \cup \{i\}) - \tilde{v}(A) \leq \tilde{v}(B \cup \{i\}) - \tilde{v}(B)$. By definition, we have:

$$\tilde{v}(A) = \sum_{C \in \mathcal{F}(A),\, C \text{ maximal}} v(C) \tag{17}$$

and

$$\tilde{v}(A \cup \{i\}) = \sum_{C \in \mathcal{F}(A \cup \{i\}),\, C \text{ maximal}} v(C). \tag{18}$$

Let us denote by $C(i)$ the unique maximal set $C \in \mathcal{F}(A \cup \{i\})$ such that $i \in C$. Let us denote by $\mathcal{C}$ the family:

$$\mathcal{C} := \{C \in \mathcal{F}(A),\ C \text{ maximal in } \mathcal{F}(A) \text{ and } C \subset C(i)\}. \tag{19}$$

Observe that as $i \notin A$, $C(i) = \{i\} \cup (\bigcup_{C \in \mathcal{C}} C)$ (If $C \in \mathcal{F}$, $C \subset A \cup \{i\}$, C is maximal in $\mathcal{F}(A \cup \{i\})$ and $i \notin C$, then $C \subset A$ and C is maximal in $\mathcal{F}(A)$.) Observe also that if $C \in \mathcal{F}(A \cup \{i\})$, C is maximal and $C \not\subset C(i)$ then $C \cap C(i) = \emptyset$ (partition) and $C \in \mathcal{F}(A)$ with C maximal in $\mathcal{F}(A)$. Hence:

$$\tilde{v}(A \cup \{i\}) - \tilde{v}(A) = v(C(i)) - \sum_{C \in \mathcal{C}} v(C). \tag{20}$$

Analogously, we define $D(i)$ as the maximal set D in $\mathcal{F}(B \cup \{i\})$ such that $i \in D$ and :

$$\mathcal{D} := \{D \in \mathcal{F}(B); D \text{ maximal in } \mathcal{F}(B), D \subset D(i)\}. \tag{21}$$

Then $D(i) = \{i\} \cup (\bigcup_{D \in \mathcal{D}} D)$ and :

$$\tilde{v}(B \cup \{i\}) - \tilde{v}(B) = v(D(i)) - \sum_{D \in \mathcal{D}} v(D). \tag{22}$$

Hence, it remains to prove that:

$$v(C(i)) - \sum_{C \in \mathcal{C}} v(C) \leq v(D(i)) - \sum_{D \in \mathcal{D}} v(D). \tag{23}$$

We want now to prove that for every $C \in \mathcal{C}$, there exists one and only one $D \in \mathcal{D}$ such that $C = D$. As $A \subset B$, $A \cup \{i\} \subset B \cup \{i\}$ and therefore $C(i) \subseteq D(i)$. Hence, for all $C \in \mathcal{C}$, there exists precisely one $D \in \mathcal{D}$ such that $C \subseteq D$. $D \cap C(i) \neq \emptyset$ because $D \cap C(i) \supset C \neq \emptyset$. $D \supseteq C(i)$ contradicts $i \notin D$. Therefore $D \subset C(i)$. But $i \notin D$ and $C(i) \subset A \cup \{i\}$, then $D \subset A$. But D is maximal in $\mathcal{F}(B)$, hence $D \subset A$ is maximal in $\mathcal{F}(A)$. As $C \subset D \subset A$ and C and D are maximal in $\mathcal{F}(A)$, we have $C = D$. We can now number the elements of $\mathcal{C}$ and $\mathcal{D}$ in such a way that $\mathcal{C} = \{C_1, C_2, \ldots, C_s\}$, $\mathcal{D} = \{D_1, D_2, \ldots, D_t\}$ with $s \le t$ and $C_r = D_r$ for all r, $1 \le r \le s$. Superadditivity of the game (N, v) implies :

$$v\left(\{i\} \cup \bigcup_{r=1}^{s} D_r \cup \bigcup_{r=s+1}^{t} D_r\right) \ge v\left(\{i\} \cup \bigcup_{r=1}^{s} D_r\right) + \left(\sum_{r=s+1}^{t} v(D_r)\right).$$

Then :

$$v\left(\{i\} \cup \bigcup_{D\in\mathcal{D}} D\right) - \sum_{D\in\mathcal{D}} v(D) \ge v\left(\{i\} \cup \bigcup_{r=1}^{s} D_r\right) - \left(\sum_{r=1}^{s} v(D_r)\right).$$

As $D_r = C_r$ for all r, $1 \le r \le s$, we obtain:

$$\begin{array}{rcl} v\left(\{i\} \cup \bigcup_{D\in\mathcal{D}} D\right) - \sum_{D\in\mathcal{D}} v(D) & \ge & v\left(\{i\} \cup \bigcup_{r=1}^{s} C_r\right) - \sum_{r=1}^{s} v(C_r) \\ & \ge & v\left(\{i\} \cup \bigcup_{C\in\mathcal{C}} C\right) - \sum_{C\in\mathcal{C}} v(C) \end{array}$$

That is precisely (23). □

4 The main restricted game associated with the strength of a graph

We define a second game $\overline{v}$ on 2^N by:

$$\overline{v}(A) = \sum_{F\in\mathcal{P}(A)} v(F) \text{ for all } A \subseteq N \tag{24}$$

where $\mathcal{P}(A)$ is a partition of A associated with the strength of the graph $G_A = (A, E(A))$. As we have already noticed in Section 2, there may be several minimizers for the strength of a given graph and therefore several possible partitions. We have, for each non empty subset A of N, to choose a partition $\mathcal{P}(A)$ among all possible partitions. Thereafter we will select the maximal subset $A_{\max}$ of $E(A)$ we can delete to achieve the minimum in the definition of the strength of G_A (as defined in Section 2), and denote by $\mathcal{P}_{\max}(A)$ the corresponding partition. We consider the *main restricted game* $\overline{v}$ on N defined by:

$$\overline{v}(A) = \sum_{F\in\mathcal{P}_{\max}(A)} v(F), \text{ for all } A \subseteq N.$$

For example, if G is a tree with all edge weights equal to 1, then for every subset $A \subseteq N$, $\mathcal{P}_{\max}(A)$ is the singletons partition of A and $\overline{v}$ is the trivial restricted game:

$$\overline{v}(A) = \sum_{i\in A} v(\{i\}).$$

We are going to study conditions on the edge-weights to have inheritance of the superadditivity and of the convexity. First of all we give a simple counterexample to the inheritance of superadditivity from a given game v to $\overline{v}$ in general. Let us consider the graph of Figure 1 where we specify two subsets S and T of nodes and two nodes a and b. In Figure 2 we represent

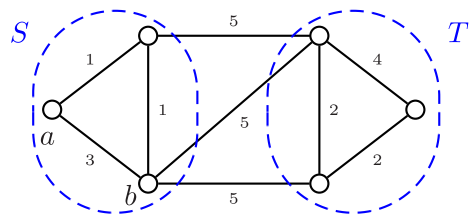

Figure 1:

at the top the graphs induced by $S \cup T$, S, and T and at the bottom the same graphs after the deletion of the maximal minimizer of the respective strengths. Let us consider the game v defined by $v(A) = \begin{cases} 1 \text{ if } A \supseteq \{a, b\} \\ 0 \text{ otherwise} \end{cases}$.

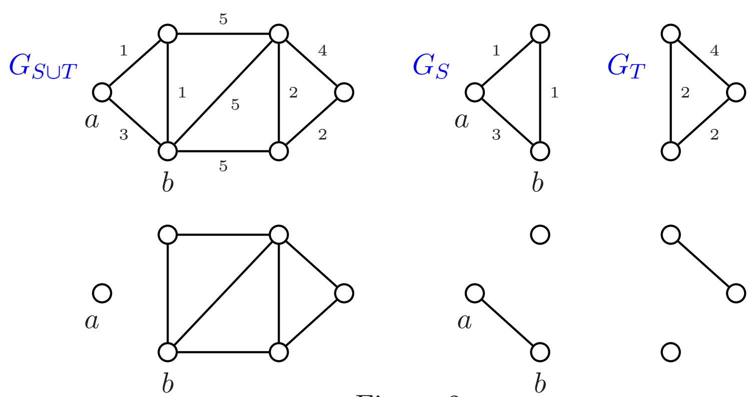

Figure 2:

Then v is superadditive but $\overline{v}(S \cup T) = 0 < 1 + 0 = \overline{v}(S) + \overline{v}(T)$. Therefore $\overline{v}$ is not superadditive.

Hence there is also no conservation of convexity in general (in the preceding example, the unanimity game v is convex but $\overline{v}$ is not superadditive and therefore not convex).

5 Necessary conditions

In this part we establish necessary conditions on the weight vector u for the inheritance of the convexity from the original communication game v to the new restricted game $\overline{v}$. For any given subset A of N, the unanimity game

$u_A : 2^N \to \mathbb{R}$ is defined by:

$$u_A(S) = \begin{cases} 1 & \text{if } S \supseteq A, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Actually, we are going to establish necessary conditions for a slightly weaker condition than convexity. We say that a subset $A \subseteq N$ is connected if the induced graph $G_A = (A, E(A))$ is connected. The family $\mathcal{F}$ of connected subsets of N is obviously weakly union-closed. A game v on 2^N is said to be *$\mathcal{F}$-convex* if for all $A, B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$, we have:

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B). \tag{26}$$

Of course convexity implies $\mathcal{F}$-convexity. The $\mathcal{F}$-convexity implies also the following condition. If a game v on 2^N is $\mathcal{F}$-convex then, $\forall A, B \subseteq N$ and $i \in N$ such that $A \subseteq B \subseteq N \setminus \{i\}$ and B and $A \cup \{i\}$ are connected, we have:

$$v(B \cup \{i\}) - v(B) \geq v(A \cup \{i\}) - v(A). \tag{27}$$

We first establish that, in the case of a graph without cycles, we have equivalence of these two conditions.

Theorem 3. *Let $G = (N, E)$ be a forest and let $\mathcal{F}$ be the family of connected subsets of N. Then the following conditions are equivalent:*

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B), \ \forall A, B \in \mathcal{F} \ s.t. \ A \cap B \neq \emptyset. \tag{28}$$

$$v(B \cup \{i\}) - v(B) \geq v(A \cup \{i\}) - v(A), \ \forall A, B \in \mathcal{F} \ s.t. \ A \subseteq B \subseteq N \setminus \{i\} \ and \ A \cup \{i\} \in \mathcal{F}. \tag{29}$$

The result is well known if $\mathcal{F} = 2^N$. The proof is the same as Schrijver's [12] (p. 767) with minor changes (as we are dealing with connected subsets of N). At first we have to prove the following lemma.

Lemma 4. *Let $G = (N, E)$ be a graph without cycles. Let S and T be two connected subsets of N such that $T \setminus S \neq \emptyset$. Then there exists a node $t \in T \setminus S$ which is a leaf node of T. Especially, $T \setminus \{t\}$ is still connected.*

Proof. Let $T' \subseteq T \setminus S$ be a connected component of $T \setminus S$. At first, let us assume $|T'| = 1$, $T' = \{t\}$ and let us assume t is not a leaf node of T. Then there exists two edges $e_1 = \{t, t_1\}$ and $e_2 = \{t, t_2\}$ in E with $t_1 \neq t_2$. As $|T'| = 1$ and $t \in T'$, t_1 and t_2 are in S. As S is connected, there exists a path γ from t_1 to t_2 in S. Therefore we obtain a cycle (e_1, γ, e_2) in G and we get a contradiction. Let us now assume $|T'| \geq 2$. As T' is a tree, T' has at least two leaf nodes t_1 and t_2. Let us now suppose neither t_1 nor t_2 are a leaf node of T. Then there exists two edges $e_1 = \{s_1, t_1\}$ and $e_2 = \{s_2, t_2\}$ with s_1 and $s_2 \in T \setminus T^{'}$. As $T^{'}$ is a connected component of $T \setminus S$, s_1 and s_2 belong to S. As S is connected, there exists a path γ in S from s_1 to s_2 and, as $T^{'}$ is connected, there exists a path $\gamma^{'}$ in $T^{'}$ from t_1 to t_2. But then $e_1 \cup \gamma \cup e_2 \cup \gamma^{'}$ forms a cycle and we get a contradiction. Therefore at least one of t_1 or t_2 is a leaf node of T. Of course if $t \in T \setminus S$ is a leaf node of T, then $T \setminus \{t\}$ is still connected. □

Proof of Theorem 3. We assume (29) is satisfied and we establish (28) by induction on $|A\Delta B|$. If $|A\Delta B| = 1$, then we have $A \subseteq B$ or $B \subseteq A$ and (28) is trivially satisfied. If $|A\Delta B| = 2$, we may suppose $|A \setminus B| = 1$ and $|B \setminus A| = 1$ (otherwise $A \subseteq B$ or $B \subseteq A$ and (28) is trivial). Setting $S := A \cap B$, $T := A$ and $B \setminus A = \{i\}$, (28) is equivalent to $v(T \cup \{i\}) + v(S) \geq v(T) + v(S \cup \{i\})$, which is equivalent to (29) applied to S and T and therefore (28) is satisfied. Let us observe that S is connected because A and B are connected and G has no cycle and that $S \cup \{i\} = B$ is connected. If now $|A\Delta B| \geq 3$, we may assume by symmetry of A and B that $|B \setminus A| \geq 2$. Applying Lemma 4, we can find $t \in B \setminus A$ such that $B \setminus \{t\}$ is still connected. By induction, we apply (28) to the pair $\{A, B \setminus \{t\}\}$:

$$v(A \cup (B \setminus \{t\})) - v(B \setminus \{t\}) \geq v(A) - v(A \cap B) \tag{30}$$

because $|A\Delta(B \setminus \{t\})| = |A\Delta B| - 1 < |A\Delta B|$. By induction we now apply (28) to the pair $\{A \cup (B \setminus \{t\}), B\}$:

$$v(A \cup (B \setminus \{t\}) \cup B) - v(B) \geq v(A \cup (B \setminus \{t\})) - v(B \setminus \{t\}) \tag{31}$$

because $|(A \cup (B \setminus \{t\}))\Delta B| = |A \setminus B| + 1 < |A \setminus B| + |B \setminus A| = |A\Delta B|$. Let us observe that $A \cup (B \setminus \{t\})$ is connected because A and $B \setminus \{t\}$ are connected and their intersection is $A \cap B \neq \emptyset$. As $A \cup (B \setminus \{t\}) \cup B = A \cup B$, (31) may be written as:

$$v(A \cup B) - v(B) \geq v(A \cup (B \setminus \{t\})) - v(B \setminus \{t\}). \tag{32}$$

(30) and (32) imply $v(A \cup B) - v(B) \geq v(A) - v(A \cap B)$. □

Let $\gamma = (e_1, e_2 \ldots, e_m)$ be an elementary path in G with $e_i = \{i, i+1\}$ for $1 \leq i \leq m$ and such that the subgraph induced by the set of vertices $\{1, 2, \ldots, m+1\}$ forms a tree. We denote by u_j the weight of e_j.

Proposition 5. *Let $G = (N, E)$ be an arbitrary graph. If there is inheritance of the $\mathcal{F}$-convexity from v to $\overline{v}$, then for all paths $\gamma = (e_1, e_2 \ldots, e_m)$ in G, such that its vertices define a tree in G, and for all i, j, k such that $1 \leq i < j < k \leq m$, the edge-weights u_j satisfy:*

$$u_j \leq \max(u_i, u_k). \tag{33}$$

Besides, if $u_1 = \min_{1 \leq i \leq m} u_i$ (resp. $u_m = \min_{1 \leq i \leq m} u_i$), then the sequence $(u_i)_{i=1}^m$ is non-decreasing (resp. non-increasing).

Thus we have a property of convexity on the edge-weights along every path in G which defines a tree.

Proof. Suppose there exists i, j, k such that $1 \leq i < j < k \leq m$ and $u_j > \max(u_i, u_k)$. We will construct a convex game v on N such that $\overline{v}$ is not convex. At first we make several reductions of this situation. We can fix e_i and e_k and select an edge e_j such that u_j is maximal for all e_j between e_i and e_k. We now fix such an e_j and we select a maximal index i such that $i < j$ and $u_i < u_j$. In the same way, we select k minimal such that $j < k$ and $u_k < u_j$. For all $l \in]i, k[$ we now have by construction $u_i < u_l = u_j$ and $u_k < u_l = u_j$ *i.e.* $\max(u_i, u_k) < u_l = u_j$, *i.e.* the weights are constant between u_i and u_k. We can now shrink the path γ to its restriction from i to $k+1$ and suppose that $i = 1$ and $k = m$. If necessary we can also exchange γ with the inverse path starting from $m+1$ and ending at 1 to have $u_1 \geq u_m$. Therefore we can suppose $u_1 \geq u_m$.

At first we consider the case $u_1 > u_m$. We are in the following situation:

$$\forall l, j \in]1, m[, u_l = u_j > u_1 > u_m. \tag{34}$$

We define the sets $A = \{2, 3, \ldots, m\}$, $B = \{1, 2, \ldots, m\}$ as represented in Figure 3 and we denote now by i the vertex $m+1$. We have $A \subset B \subset N \setminus \{i\}$ and A, B and $A \cup \{i\}$ are connected. As γ is a tree, the strength

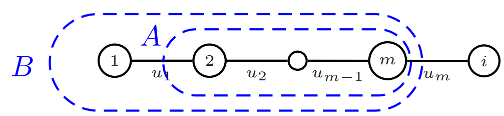

Figure 3:

of any subtree induced by a subset S of vertices of γ corresponds to the smallest edge-weight in $E(S)$. $\mathcal{P}_{\max}(S)$ is obtained by deleting all edges in $E(S)$ minimizing the weight in $E(S)$. Therefore, as a consequence of (34), we have $\mathcal{P}_{\max}(A \cup \{i\}) = \{A, \{i\}\}$ and $\mathcal{P}_{\max}(B \cup \{i\}) = \{B, \{i\}\}$. Thus $\overline{v}(A \cup \{i\}) = v(A) + v(i)$ and $\overline{v}(B \cup \{i\}) = v(B) + v(i)$. The $\mathcal{F}$-convexity of $\overline{v}$ would imply $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A)$ *i.e.*:

$$v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A). \tag{35}$$

Because of (34), u_1 is the smallest weight of edges in $E(B)$. Thus $\mathcal{P}_{\max}(B) = \{A, \{1\}\}$ and $\overline{v}(B) = v(A) + v(1)$. Following (34), all edges of $E(A)$ have the same weight, so we have to delete all edges of A to obtain $\mathcal{P}_{\max}(A)$. Thus $\mathcal{P}_{\max}(A) = \{\{2\}, \{3\}, \cdots, \{m\}\}$ and $\overline{v}(A) = \sum_{k \in A} v(k)$. (35) becomes:

$$v(B) - v(A) - v(1) \geq v(A) - \sum_{k \in A} v(k). \tag{36}$$

Now if we take $v = u_A$ then v is supermodular, but as $B \supset A$, (36) becomes equivalent to $1 - 1 - 0 \geq 1 - 0$, a contradiction.

We now consider the case $u_1 = u_m < u_l = u_j$ for all $l, j \in]1, m[$ and A, B, i as before. As $u_1 = u_m$ is the smallest weight of edges of $B \cup \{i\}$, we have to delete e_1 and e_m. Therefore $\mathcal{P}_{\max}(B \cup \{i\}) = \{A, \{1\}, \{i\}\}$ and $\overline{v}(B \cup \{i\}) = v(A) + v(1) + v(i)$. In the same way, we have also $\overline{v}(B) = v(A) + v(1)$, $\overline{v}(A \cup \{i\}) = v(A) + v(i)$, and $\overline{v}(A) = \sum_{k \in A} v(k)$. Therefore the inequality $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A)$ is equivalent to:

$$\sum_{k \in A} v(k) \geq v(A). \tag{37}$$

If we take $v = u_A$, then (37) is equivalent to $0 \geq 1$, a contradiction. □

Remark 1. If u_1 (resp. u_m) is the smallest weight of the edges of γ, then the condition of convexity of the u_i's means that the sequence $(u_i)_{i=1}^m$ is non-decreasing (resp. non-increasing) as $u_i \leq \max(u_1, u_{i+1}) = u_{i+1}$ (resp. $u_{i+1} \leq \max(u_i, u_m) = u_i$) for all $1 \leq i \leq m - 1$.

Remark 2. We cannot restrict the convexity condition to only every 3-uple of consecutive edges $u_i \leq \max(u_{i-1}, u_{i+1})$, $2 \leq i \leq m-1$, because of the obvious counter-example: $u_2 = u_3 > \max(u_1, u_4)$. Nevertheless $u_2 = \max(u_1, u_3)$ and $u_3 = \max(u_2, u_4)$.

Now we show there exists another necessary $\mathcal{F}$-convexity condition associated with every induced subgraph $(A, E(A))$ of G corresponding to a star. A star S_k corresponds to a tree with one internal node and k leaves. We first establish the result for stars with three leaves. The generalization to stars of greater size is immediate. We consider a star S_3 with vertices $\{1, 2, 3, 4\}$ and edges $e_1 = \{1, 2\}$, $e_2 = \{1, 3\}$ and $e_3 = \{1, 4\}$.

Proposition 6. *Let $G = (N, E)$ be an arbitrary graph. If for every $\mathcal{F}$-convex game v on N we have inheritance of the $\mathcal{F}$-convexity from v to the restricted game $\overline{v}$, then for every induced star of type S_3 of G, the weights u_1, u_2, u_3 of the three edges satisfy:*

$$u_1 \leq u_2 = u_3 \text{ after renumbering the weights if necessary.}$$

Proof. Suppose it is not true. At first we suppose $u_1 < u_2 < u_3$. We consider the situation of Figure 4 where $A = \{1, 4\}$, $B = \{1, 3, 4\}$ and $i = 2$. By deleting the edge of minimal weight we obtain successively: $\mathcal{P}_{\max}(B \cup \{i\}) = \{B, \{i\}\}$, $\mathcal{P}_{\max}(B) = \{A, \{3\}\}$, $\mathcal{P}_{\max}(A \cup \{i\}) = \{A, \{i\}\}$ and $\mathcal{P}_{\max}(A) = \{\{1\}, \{4\}\}$. Therefore $\overline{v}(B \cup \{i\}) = v(B) + v(i)$, $\overline{v}(B) = v(A) + v(3)$, $\overline{v}(A \cup \{i\}) = v(A) + v(i)$ and $\overline{v}(A) = v(1) + v(4)$. To get a contradiction, we have to construct a supermodular function v such that the following inequality is satisfied:

$$\overline{v}(A \cup \{i\}) - \overline{v}(A) > \overline{v}(B \cup \{i\}) - \overline{v}(B) \tag{38}$$

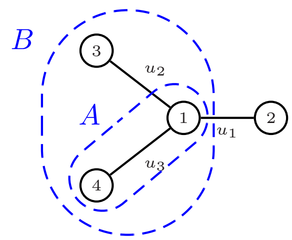

Figure 4:

which is equivalent to $2v(A) > v(B) + v(1) + v(4) - v(3)$. If we consider $v = u_A$ then v is supermodular, but (38) is satisfied and therefore $\overline{v}$ is not $\mathcal{F}$-convex. Therefore the three weights cannot be all distincts. Two of them, for instance u_2 and u_3, have to be equal. Thus we now suppose $u_1 > u_2 = u_3$. We consider now the situation of Figure 5 where $A = \{1, 2\}$,

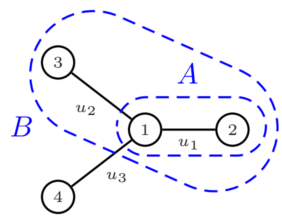

Figure 5:

$B = \{1, 2, 3\}$ and $i = 4$. Then we obtain $\overline{v}(B \cup \{i\}) = v(A) + v(3) + v(i)$, $\overline{v}(B) = v(A) + v(3)$, $\overline{v}(A \cup \{i\}) = v(A) + v(i)$ and $\overline{v}(A) = v(1) + v(2)$. (38) is now equivalent to

$$v(A) - v(1) - v(2) > 0. \tag{39}$$

Now if we take $v = u_A$, v is supermodular and (39) is satisfied. Therefore $\overline{v}$ is not $\mathcal{F}$-convex. $\square$

Remark 3. For an induced star with n edges $e_1, e_2, \ldots, e_n$ the weights verify $u_1 \leq u_2 = u_3 = \cdots = u_n$ after renumbering the edges if necessary.

We can easily obtain necessary conditions for inheritance of the convexity in the case of a chordless cycle in G induced by m vertices with $m \geq 3$. Denote by $1, 2, \ldots, m$ the nodes of a chordless cycle C and by $e_1, e_2, \ldots, e_m$ the edges with $e_i = \{i, i+1\}$ for $1 \leq i \leq m-1$ and $e_m = \{1, m\}$.

If $m = 3$, it is easy to see that for every choice of the weights u_1, u_2, u_3 we have conservation of the convexity. Let $N = \{1, 2, 3\}$, $i = 3$ and consider $\emptyset \neq A \subseteq B \subseteq N \setminus \{i\}$. If $A = B$ then $\overline{v}(B \cup \{i\}) - \overline{v}(B) = \overline{v}(A \cup \{i\}) - \overline{v}(A)$. If $A \subset B$, we can suppose $A = \{2\}$ and $B = \{1, 2\}$ as represented in Figure 6. Then $\overline{v}(A) = v(2)$, $\overline{v}(A \cup \{i\}) = v(2) + v(i)$ and $\overline{v}(B) = v(1) + v(2)$. If $\overline{v}(B \cup \{i\}) = v(B) + v(i)$ then $\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(\{1, 2\}) + v(i) - v(1) - v(2)$.

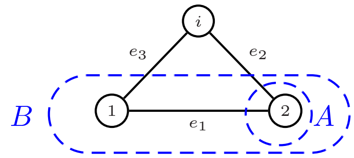

Figure 6:

If $\overline{v}(B \cup \{i\}) = v(\{1, i\}) + v(2)$ then $\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(\{1, i\}) - v(1)$. As v is supermodular, we have in these two cases $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(i)$. If $\overline{v}(B \cup \{i\}) = v(1) + v(2) + v(i)$ then $\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(i)$. Therefore in all cases we have $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A)$.

For $m \geq 4$ the inheritance of the convexity implies strong restrictions on the edge-weights.

Proposition 7. *If $C = (v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1)$ is an induced chordless cycle of $G = (N, E)$ with $m \geq 4$ and if the $\mathcal{F}$-convexity property is inherited from v to $\overline{v}$ for all $\mathcal{F}$-convex game v on N, then, after renumbering the edges if necessary, we have*

$$u_1 \leq u_2 \leq u_3 = \cdots = u_m.$$

Proof. After a permutation of the edges if necessary we can suppose $u_1 \leq u_i$ for all i, $1 \leq i \leq m$. Applying Proposition 5 to the path $(e_2, e_3, \ldots, e_m)$

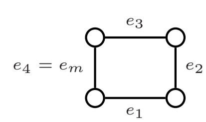

Figure 7:

of G we have $u_i \leq \max(u_2, u_m)$ for all i, $2 \leq i \leq m$. We may suppose $\max(u_2, u_m) = u_m$ after a permutation of $e_2, \ldots, e_m$ if necessary, that is:

$$u_1 \leq u_i \leq u_m, \forall i, 1 \leq i \leq m. \tag{40}$$

We now apply Proposition 5 to the path $(e_1, e_m, e_{m-1}, \ldots, e_4, e_3)$. As u_1 is the smallest weight, the sequence of weights is non-decreasing:

$$u_1 \leq u_m \leq u_{m-1} \leq \cdots \leq u_4 \leq u_3. \tag{41}$$

(40) and (41) imply $u_m = u_{m-1} = \cdots = u_4 = u_3$. □

Now we construct another example proving that the conditions of convexity of weights of the Propositions 5 and 6 are not sufficient to imply the inheritance of the convexity of games (even of the $\mathcal{F}$-convexity). We take all weights equal to 1. We add to a 3-cycle $\{e_1, e_2, e_3\}$ with

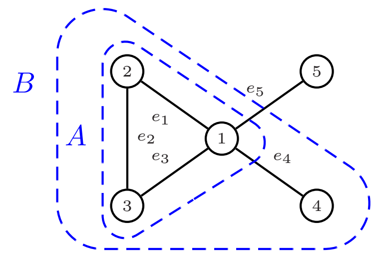

Figure 8:

$e_1 = \{1,2\}$, $e_2 = \{2,3\}$ and $e_3 = \{3,1\}$, two edges $e_4 = \{1,4\}$ and $e_5 = \{1,5\}$. We take $A = \{1,2,3\}$, $B = \{1,2,3,4\}$ and $i = 5$. We obtain $\mathcal{P}_{\max}(A) = \{\{1\},\{2\},\{3\}\}$, $\mathcal{P}_{\max}(A \cup \{i\}) = \{A,\{i\}\}$, $\mathcal{P}_{\max}(B) = \{A,\{4\}\}$ and $\mathcal{P}_{\max}(B \cup \{i\}) = \{A,\{4\},\{i\}\}$. Therefore $\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(i)$ and $\overline{v}(A \cup \{i\}) - \overline{v}(A) = v(A) + v(i) - v(1) - v(2) - v(3)$. $\overline{v}$ is not $\mathcal{F}$-convex if $\overline{v}(A \cup \{i\}) - \overline{v}(A) > \overline{v}(B \cup \{i\}) - \overline{v}(B)$, *i.e.* if:

$$v(A) > v(1) + v(2) + v(3). \tag{42}$$

If we take $v = u_A$, v is supermodular but (42) is satisfied therefore $\overline{v}$ is not $\mathcal{F}$-convex.

We now study the inheritance of the superadditivity.

Proposition 8. *If for every unanimity game u_S on N, $\overline{u_S}$ is superadditive then for all subset $A \subseteq N$ and all $i \in N \setminus A$, the partition $\mathcal{P}_{\max}(A)$ of A is a refinement of the restriction of $\mathcal{P}_{\max}(A \cup \{i\})$ to A.*

More precisely, the proposition means that if $\mathcal{P}_{\max}(A) = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{P}_{\max}(A \cup \{i\}) = \{A_1' \cup \{i\}, A_2', \ldots, A_p'\}$, then if $A_1' \neq \emptyset$, $\{A_1', A_2', \ldots, A_p'\}$ is another partition of A, and if $A_1' = \emptyset$, $\{A_2', \ldots, A_p'\}$ is another partition of A and for all j, $1 \leq j \leq m$, A_j is a subset of one and only one set A_k'.

Proof. For any given subset S of N the unanimity game u_S is supermodular and therefore $\overline{u_S}$ is superadditive, hence $\overline{u_S}(A \cup \{i\}) \geq \overline{u_S}(A) + \overline{u_S}(i)$. As $\overline{u_S}(i) \geq 0$, it implies

$$\overline{u_S}(A \cup \{i\}) \geq \overline{u_S}(A). \tag{43}$$

Let us consider $\mathcal{P}_{\max}(A) = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{P}_{\max}(A \cup \{i\}) = \{A_1' \cup \{i\}, A_2', \ldots, A_p'\}$. For every $1 \leq j, k \leq m$, we have $A_j \cap A_k = \emptyset$ for $j \neq k$ and therefore $u_{A_k}(A_j) = 0$, and for $j = k$, $u_{A_k}(A_k) = 1$. Thus $\overline{u_{A_k}}(A) = \sum_{j=1}^{m} u_{A_k}(A_j) = u_{A_k}(A_k) = 1$. Using (43), with $S = A_k$, we obtain $\overline{u_{A_k}}(A \cup \{i\}) \geq 1$. Therefore, we have $u_{A_k}(A_1' \cup \{i\}) + \sum_{j=2}^{p} u_{A_k}(A_j') \geq 1$. As the function u_{A_k} only takes values 0 or 1, we necessarily have for at least one A_j'

$(j \geq 2)$, $u_{A_k}(A_j^{'}) = 1$ or $u_{A_k}(A_1^{'} \cup \{i\}) = 1$. Therefore, for at least one $A_j^{'}$ we have $A_k \subseteq A_j^{'}$ (following the definition of u_{A_k}). As the $A_j^{'}$'s are disjoint, there exists for each A_k one and only one $A_j^{'}$ such that $A_k \subseteq A_j^{'}$. This proves that (A_k) is a refinement of the partition $(A_j^{'})$ of A. $\square$

For $A \subseteq B \subseteq N$, we can now repeatedly apply Proposition 8 adding successively to A every element $i \in B \setminus A$. We have proved:

Theorem 9. *If for every unanimity game u_S on N the main restricted game $\overline{u_S}$ is superadditive, then for all subsets $A \subseteq B \subseteq N$, $\mathcal{P}_{\max}(A)$ is a refinement of the restriction of $\mathcal{P}_{\max}(B)$ to A.*

The converse is true but we can establish a stronger result:

Theorem 10. *If for all subsets $A \subseteq B \subseteq N$, $\mathcal{P}_{\max}(A)$ is a refinement of the restriction of $\mathcal{P}_{\max}(B)$ to A, then for all superadditive set function v on 2^N, the restricted game $\overline{v}$ is still superadditive on 2^N.*

Proof. Suppose $A \cap B = \emptyset$ $(A, B \subseteq N)$. Then we have:

$$\overline{v}(A \cup B) = \sum_{C \in \mathcal{P}_{\max}(A \cup B)} v(C) = \sum_{C \in \mathcal{P}_{\max}(A \cup B)} v((C \cap A) \cup (C \cap B)). \tag{44}$$

As $C \cap A$ and $C \cap B$ are disjoint and v is superadditive, (44) implies:

$$\overline{v}(A \cup B) \geq \sum_{C \in \mathcal{P}_{\max}(A \cup B)} \left(v(C \cap A) + v(C \cap B) \right). \tag{45}$$

As $\mathcal{P}_{\max}(A)$ (resp. $\mathcal{P}_{\max}(B)$) is a refinement of $\mathcal{P}_{\max}(A \cup B)$, for every $C \in \mathcal{P}_{\max}(A \cup B)$ such that $C \cap A \neq \emptyset$ (resp. $C \cap B \neq \emptyset$), $C \cap A$ (resp. $C \cap B$) is a disjoint union of blocks of $\mathcal{P}_{\max}(A)$ (resp. $\mathcal{P}_{\max}(B)$). As v is superadditive, we obtain:

(46)

$$\overline{v}(A \cup B) \geq \sum_{C \in \mathcal{P}_{\max}(A \cup B)} \left[\sum_{F \subseteq C \cap A, F \in \mathcal{P}_{\max}(A)} v(F) + \sum_{F \subseteq C \cap B, F \in \mathcal{P}_{\max}(B)} v(F) \right]$$

which yields:

$$\overline{v}(A \cup B) \geq \sum_{F \in \mathcal{P}_{\max}(A)} v(F) + \sum_{F \in \mathcal{P}_{\max}(B)} v(F) = \overline{v}(A) + \overline{v}(B). \tag{47}$$

Therefore $\overline{v}$ is superadditive. $\square$

Corollary 11. *If $G = (N, E)$ is a forest then superadditivity is inherited from v to $\overline{v}$.*

Proof. For a given $B \subseteq N$, we denote by $\Sigma(B) \subseteq E(B)$ the maximal subset of edges which realizes the minimum in the definition of the strength of the subgraph G_B induced by B. As G_B is a forest, $e \in \Sigma(B)$ if and only if $u(e) = \min_{e' \in E(B)} u(e')$. Suppose now $A \subseteq B \subseteq N$. We have either $\Sigma(B) \cap E(A) = \Sigma(A)$ or $\Sigma(B) \cap E(A) = \emptyset$. $\mathcal{P}_{\max}(A)$ is the partition of A corresponding to the components of the subgraph $(A, E(A) \setminus \Sigma(A))$. If two elements of A are connected by a path γ in the subgraph $(A, E(A) \setminus \Sigma(A))$, then they are also connected by γ in the subgraph $(B, E(B) \setminus \Sigma(B))$. Therefore $\mathcal{P}_{\max}(A)$ is a subdivision of the restriction of $\mathcal{P}_{\max}(B)$ to A and the result is a consequence of Theorem 10. $\square$

Applying successively Theorems 9 and 10 we obtain:

Corollary 12. *If for all unanimity game u_S, $\overline{u}_S$ is superadditive, then for all superadditive function v, $\overline{v}$ is superadditive. In particular inheritance of the convexity for all unanimity games implies inheritance of the superadditivity.*

The graph represented in Figure 9 shows that we do not have inheritance of the superadditivity in general, even if all weights are equal to 1.

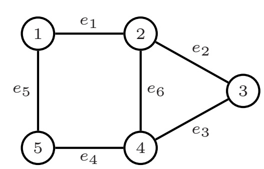

Figure 9:

We set $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$. We then have $\mathcal{P}_{\max}(B) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ and $\mathcal{P}_{\max}(A) = \{\{1\}, \{2, 3, 4\}\}$. Therefore $\mathcal{P}_{\max}(A)$ is not a refinement of the restriction of $\mathcal{P}_{\max}(B)$ to A and Theorem 9 proves that there is no inheritance of the superadditivity.

Now we present a general counterexample to the inheritance of convexity. Let us consider $i \in N$ and $A \subseteq B \subseteq N \setminus \{i\}$. We suppose G_B has two connected components G_{B_1} and G_{B_2}. Furthermore we suppose $A = A_1 \cup A_2$ with $A_1 \subseteq B_1$ and $A_2 \subset B_2$. We consider the graph formed by G_B plus an edge $e = \{i, j\}$ with $j \in A_1$, represented in Figure 10. We suppose that the

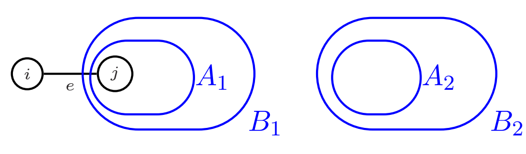

Figure 10:

edge-weights are so that the following inequalities are satisfied:

$$u(e) < \sigma(G_{B_2}) < \sigma(G_{B_1}) \leq \sigma(G_{A_1}) < \sigma(G_{A_2}) \tag{48}$$

As $u(e) < \sigma(G_{B_2}) = \sigma(G_B)$ and $u(e) < \sigma(G_{A_1}) = \sigma(G_A)$ and $j \in A_1$, we have $\overline{v}(B \cup \{i\}) = v(B_1)+v(B_2)+v(i)$ and $\overline{v}(A \cup \{i\}) = v(A_1)+v(A_2)+v(i)$. As $\sigma(G_{B_2}) < \sigma(G_{B_1})$, we have $\mathcal{P}_{\max}(B) = \mathcal{P}_{\max}(B_2) \cup \{B_1\}$. We denote by $\{B_{2,1}, B_{2,2}, \ldots, B_{2,N_2}\}$ the partition $\mathcal{P}_{\max}(B_2)$. Therefore we have $\overline{v}(B) = v(B_1) + \sum_{q=1}^{N_2} v(B_{2,q})$. In the same way, we denote by $\{A_{1,1}, A_{1,2}, \ldots, A_{1,N_1}\}$ the partition $\mathcal{P}_{\max}(A_1)$ and obtain $\overline{v}(A) = \sum_{p=1}^{N_1} v(A_{1,p}) + v(A_2)$. If $\overline{v}$ is convex, we have $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A)$, which is equivalent to:

$$v(B_2) - \sum_{q=1}^{N_2} v(B_{2,q}) \geq v(A_1) - \sum_{p=1}^{N_1} v(A_{1,p}). \tag{49}$$

Taking for v the unanimity game u_{A_1}, (49) reduces to $0 \geq 1$, which shows that convexity is not inherited.

We illustrate this general counterexample with the situation represented in Figure 11. We suppose:

$$u_2 = u_3 < u_4 < u_1. \tag{50}$$

We define $A_1 = B_1 = \{v_1, v_2\}$, $A_2 = \{v_5\}$, $B_2 = \{v_4, v_5\}$, $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. According to (50), we have $u_2 = u_3 < \sigma(B_2) =$

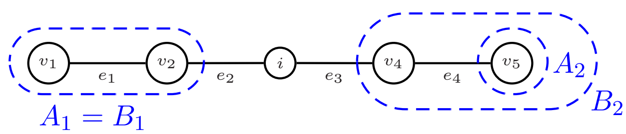

Figure 11:

$u_4 < \sigma(A_1) = \sigma(B_1) = u_1$. Therefore $\overline{v}(B \cup \{i\}) = v(B_1) + v(B_2) + v(i)$, $\overline{v}(B) = v(B_1) + v(A_2) + v(v_4)$, $\overline{v}(A \cup \{i\}) = v(A_1) + v(A_2) + v(i)$ and $\overline{v}(A) = v(A_2) + v(v_1) + v(v_2)$. Then $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A)$ is equivalent to:

$$v(B_2) - v(A_2) - v(v_4) \geq v(A_1) - v(v_1) - v(v_2). \tag{51}$$

Taking $v = u_{A_1}$, (51) reduces to $0 \geq 1$ which is a contradiction. Therefore $\overline{v}$ is not convex.

6 Sufficient conditions

Let $\mathcal{F}$ be the family of connected subsets of N. Henceforth G will be a cycle-free graph. We will now prove that the preceding necessary conditions are also sufficient in this case.

Theorem 13. *Let $G = (N, E)$ be a cycle-free graph and let $u : E \to I\!R^+$ be a weight function on the set E of edges. For every $\mathcal{F}$-convex game v on N, the restricted game $\overline{v}$ associated with v is $\mathcal{F}$-convex if and only if the following conditions are satisfied:*

1. *(Convexity condition) For all paths $\gamma = \{e_1, e_2, \ldots, e_m\}$ in G and for all i, j, k such that $1 \leq i < j < k \leq m$, we have $u_j \leq \max(u_i, u_k)$.*

2. *(Branching condition) For all stars S_n, $n \geq 3$, with edges $e_1, e_2, \ldots, e_n$, the weights satisfy $u_1 \leq u_2 = u_3 = \ldots = u_n$ after renumbering the edges if necessary.*

Proof. We have already seen in Section 5 that conditions 1 and 2 are necessary. We now prove they are sufficient. Let v be a given $\mathcal{F}$-convex game on N. We first prove that, for all $A \subseteq B \subseteq N \setminus \{i\}$ and $A, B \in \mathcal{F}$, we have:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A). \tag{52}$$

We will consider $\mathcal{P}_{\max}(A) = \{A_1, A_2, \ldots, A_p\}$ and $P_{\max}(B) = \{B_1, B_2, \ldots, B_q\}$. We begin with the special case $A = \emptyset$. We have to prove:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(i) \tag{53}$$

for all $B \subseteq N \setminus \{i\}$.

Case 1 We suppose there is no edge linking i with a node in B, *i.e.*, $E(B \cup \{i\}) = E(B)$. Thus the edges of minimal weight in $E(B \cup \{i\})$ are in $E(B)$ and $\mathcal{P}_{\max}(B \cup \{i\}) = \mathcal{P}_{\max}(B) \cup \{i\}$. Therefore $\overline{v}(B \cup \{i\}) = \overline{v}(B) + v(i)$ and (53) is satisfied.

Case 2 There is an edge between i and B. As B is connected, there exists only one edge $e = \{i, j\}$ with $j \in B$ otherwise there would be a cycle in G. We have to consider different subcases according to the weight $u(e)$ of e.

Case 2.1 $u(e) > \sigma(B) = \min_{e' \in E(B)} u(e')$. If we suppose $j \in B_1$, then $\mathcal{P}_{\max}(B \cup \{i\}) = \{B_1 \cup \{i\}, B_2, \ldots, B_q\}$. Therefore we have $\overline{v}(B \cup \{i\}) = v(B_1 \cup \{i\}) + \sum_{l=2}^{q} v(B_l)$. As v is superadditive, we obtain $\overline{v}(B \cup \{i\}) \geq v(B_1) + v(i) + \sum_{l=2}^{q} v(B_l) = \overline{v}(B) + v(i)$.

Case 2.2. $u(e) = \sigma(B) = \min_{e' \in E(B)} u(e')$. Then $\mathcal{P}_{\max}(B \cup \{i\}) = \{B_1, B_2, \ldots, B_q, \{i\}\}$ and $\overline{v}(B \cup \{i\}) = \sum_{l=1}^{q} v(B_l) + v(i) = \overline{v}(B) + v(i)$.

Case 2.3 $u(e) < \sigma(B) = \min_{e' \in E(B)} u(e')$. Then $u(e) = \sigma(B \cup \{i\})$, $\mathcal{P}_{\max}(B \cup \{i\}) = \{B, \{i\}\}$ and $\overline{v}(B \cup \{i\}) = v(B) + v(i)$. As v is superadditive, we have $v(B) = v(B_1 \cup B_2 \cup \ldots \cup B_m) \geq \sum_{l=1}^{m} v(B_l) = \overline{v}(B)$. Thus $\overline{v}(B \cup \{i\}) \geq \overline{v}(B) + v(i)$.

Therefore in these three cases (53) is satisfied.

We consider now the case $A \neq \emptyset$.

Case 3.1 We suppose there is no edge linking i with a node in B. As $A \subseteq B$

there is also no edge between i and A. We are for A and B in the situation of Case 1. Therefore $\overline{v}(B \cup \{i\}) = \overline{v}(B) + v(i)$, $\overline{v}(A \cup \{i\}) = \overline{v}(A) + v(i)$ and $\overline{v}(B \cup \{i\}) - \overline{v}(B) = \overline{v}(A \cup \{i\}) - \overline{v}(A)$.

Case 3.2 There is an edge $e = \{i, j\}$ with $j \in B \setminus A$. Case 2 applies to B, $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(i)$. As $j \notin A$, Case 1 applies to A, $\overline{v}(A \cup \{i\}) - \overline{v}(A) = v(i)$ and therefore $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A)$.

Case 3.3 There is an edge $e = \{i, j\}$ with $j \in A$. Then we have several subcases to consider.

Case 3.3.1 We suppose $j \in A$ and $u(e) > \min_{e' \in E(A)} u(e') = \sigma(A) \geq \sigma(B) = \min_{e' \in E(B)} u(e')$. We are for A and B in a situation similar to case 2.1. If we suppose $j \in A_1 \cap B_1$ then $\mathcal{P}_{\max}(A \cup \{i\}) = \{A_1 \cup \{i\}, A_2, \ldots, A_p\}$ and $\mathcal{P}_{\max}(B \cup \{i\}) = \{B_1 \cup \{i\}, B_2, \ldots, B_q\}$. As in case 2.1, we have:

$$\overline{v}(A \cup \{i\}) - \overline{v}(A) = v(A_1 \cup \{i\}) - v(A_1) \tag{54}$$

and

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(B_1 \cup \{i\}) - v(B_1). \tag{55}$$

As $A \subseteq B$ and as G is a forest, we have $\Sigma(B) \cap E(A) = \emptyset$ or $\Sigma(B) \cap E(A) = \Sigma(A)$. Therefore $A_1 \subseteq B_1$ and by the convexity of v we have:

$$v(B_1 \cup \{i\}) - v(B_1) \geq v(A_1 \cup \{i\}) - v(A_1). \tag{56}$$

Using (54) and (55), (56) is equivalent to $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A)$. Let us observe that we have used the supermodularity of v and not only the superadditivity as in the preceding cases.

Case 3.3.2 We now suppose $j \in A$ and $u(e) = \sigma(A) \geq \sigma(B)$. Then, applying cases 2.1 and 2.2 to B, we have $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(i)$. We can now apply the case 2.2 to A and we have $\overline{v}(A \cup \{i\}) - \overline{v}(A) = v(i)$. Therefore we obtain $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A)$.

Case 3.3.3 We now suppose $j \in A$ and:

$$\sigma(A) > u(e) > \sigma(B). \tag{57}$$

We will see this case cannot occur in fact. Let γ_1 be a path in G_A, $\gamma_1 = (e_1, e_2, \ldots, e_m)$ such that $u(e_1) = \sigma(A)$ and j is an end-vertex of e_m. Similarly let γ_2 be a path in G_B, $\gamma_2 = (e_1', e_2', \ldots, e_r')$ such that $u(e_1') = \sigma(B)$ and j is an end-vertex of e_r'. The convexity condition applied to the path

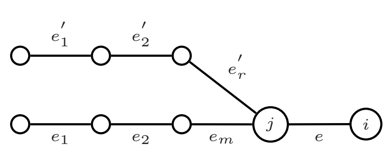

Figure 12:

$\gamma_1 \cup \{e\}$ and (57) imply $u(e_m) \leq \max(u(e_1), u(e)) = u(e_1) = \sigma(A)$. As $e_m \in E(A)$, $u(e_m) = \sigma(A)$ and using again (57), we obtain:

$$u(e_m) > u(e). \tag{58}$$

The convexity condition applied to the path $\gamma_2 \cup \{e\}$ and (57) imply:

$$u(e_r^{'}) \leq \max(u(e_1^{'}), u(e)) = u(e). \tag{59}$$

If $e_m = e_r^{'}$ then, from (58) and (59), we get a contradiction. If now $e_m \neq e_r^{'}$, the branching condition for the three edges $e_m, e_r^{'}, e$ and (58) imply $u(e_m) = u(e_r^{'}) > u(e)$ contradicting (59). Therefore (57) does not occur and we have proved the following lemma.

Lemma 14. *Let us consider a cycle-free graph $G = (N, E)$ and an edge-weight function u satisfying the convexity conditions 1) and 2) of Theorem 13. If A and B are connected, $A \subseteq B \subseteq N \setminus \{i\}$, $j \in A \cap B$ and $e := \{i, j\} \in E$, then either $u(e) \geq \sigma(A) \geq \sigma(B)$ or $\sigma(A) \geq \sigma(B) \geq u(e)$.*

We have now to consider the two last cases corresponding to $\sigma(A) \geq u(e) = \sigma(B)$ and $\sigma(A) \geq \sigma(B) > u(e)$. To solve these remaining cases, we need the following result.

Lemma 15. *Let us consider $A \subseteq B \subseteq N$ and $\{B_1, B_2, \ldots, B_p\}$ a partition of B. Then, for every supermodular function v on N we have:*

$$v(B) - \sum_{i=1}^{p} v(B_i) \geq v(A) - \sum_{i=1}^{p} v(B_i \cap A). \tag{60}$$

Proof of Lemma 15. By the definition of supermodularity, we have for every $i \in [1, p]$, $v(A \cup B_i) + v(A \cap B_i) \geq v(A) + v(B_i)$. Adding these inequalities, we obtain:

$$\sum_{i=1}^{p} v(A \cup B_i) + \sum_{i=1}^{p} v(A \cap B_i) \geq p\, v(A) + \sum_{i=1}^{p} v(B_i). \tag{61}$$

But $(A \cup B_1) \cap (A \cup B_2) = A \cup (B_1 \cap B_2) = A \cup \emptyset = A$, therefore supermodularity still implies $v(A \cup B_1 \cup B_2) + v(A) \geq v(A \cup B_1) + v(A \cup B_2)$. We can iterate. Suppose by induction:

$$v(A \cup (\cup_{i=1}^{k} B_i)) + (k-1)v(A) \geq \sum_{i=1}^{k} v(A \cup B_i). \tag{62}$$

By supermodularity, we have $v(A \cup (\cup_{i=1}^{k} B_i) \cup B_{k+1}) + v(A) \geq v(A \cup (\cup_{i=1}^{k} B_i)) + v(A \cup B_{k+1})$. Adding this last inequality to (62) we obtain $v(A \cup (\cup_{i=1}^{k+1} B_i)) + k\, v(A) \geq \sum_{i=1}^{k+1} v(A \cup B_i)$. For $k = p$, (62) becomes:

$$v(B) + (p-1)v(A) \geq \sum_{i=1}^{p} v(A \cup B_i). \tag{63}$$

(61) and (63) imply (60). $\square$

The following lemma gives a simple condition ensuring $\mathcal{P}_{\max}(A)$ is induced by $\mathcal{P}_{\max}(B)$.

Lemma 16. *Let us consider a cycle-free graph $G = (N, E)$, and $A \subseteq B \subseteq N$ such that $\sigma(A) = \sigma(B)$. Then $\mathcal{P}_{\max}(A)$ is the restriction of $\mathcal{P}_{\max}(B)$ to A and therefore, for all supermodular function v:*

$$v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A). \tag{64}$$

Proof. We have to prove for every component B_k of $\mathcal{P}_{\max}(B)$ with $B_k \cap A \neq \emptyset$, that $B_k \cap A$ is a component A_k of $\mathcal{P}_{\max}(A)$. Let α_0 be a fixed vertex of $A \cap B_k$ and A_k be the component of $\mathcal{P}_{\max}(A)$ which contains α_0. We will prove $A \cap B_k = A_k$. As $\sigma(A) = \sigma(B)$, $\Sigma(A) = E(A) \cap \Sigma(B)$. Let now α_1 be another vertex of A_k and γ be a path in A_k from α_0 to α_1. Each edge e of γ is in $E(A) \setminus \Sigma(A)$ and therefore satisfies $u(e) > \sigma(A)$. As $\sigma(A) = \sigma(B)$ and $A \subseteq B$, each edge e of γ is in $E(B)$ and satisfies $u(e) > \sigma(B)$. Therefore γ is a path from α_0 to α_1 in B and therefore $\alpha_1 \in B_k$. That is :

$$A_k \subseteq A \cap B_k. \tag{65}$$

Let now α_1 be a vertex in $A \cap B_k$. As A is connected, there exists a path γ from α_0 to α_1 in A and possibly another one $\gamma^{'}$ from α_0 to α_1 in B_k. But as G has no cycle $\gamma = \gamma^{'}$ and γ is a path in $A \cap B_k$. For each edge e of γ, e is in $E(A)$ and in $E(B) \setminus \Sigma(B)$, that is $u(e) > \sigma(B)$. As $\sigma(A) = \sigma(B)$, we have also $u(e) > \sigma(A)$ and therefore $e \in E(A) \setminus \Sigma(A)$. Thus γ is a path of a component of $\mathcal{P}_{\max}(A)$. As $\alpha_0 \in A_k$, γ is a path in A_k and then $\alpha_1 \in A_k$. That is:

$$A \cap B_k \subseteq A_k. \tag{66}$$

Following (65) and (66), we have shown $A \cap B_k = A_k$. Lemma 15 implies (64).

□

We can now achieve the proof of the convexity of $\overline{v}$. In the two remaining subcases we have:

$$u(e) \leq \sigma(B) \leq \sigma(A). \tag{67}$$

We can first prove that in fact we necessarily have $\sigma(A) = \sigma(B)$. Let e_1 (resp. e_2) be an edge of $E(A)$ (resp. $E(B)$) such that $u(e_1) = \sigma(A)$ (resp. $u(e_2) = \sigma(B)$). As A and B are connected and as $j \in A \cap B$, there exists a path γ_1 (resp. γ_2) in G_A (resp. in G_B) such that its first edge is e_1 (resp. e_2) and its last edge denoted by $e_1^{'}$ (resp. $e_2^{'}$) ends at j. Adding the edge e to γ_1 (resp. γ_2) we obtain the paths $\gamma_1 \cup \{e\}$ and $\gamma_2 \cup \{e\}$ ending at i as represented in Figure 13. (67) implies e is an edge of smallest weight in both paths $\gamma_1 \cup \{e\}$ and $\gamma_2 \cup \{e\}$. Hence the convexity property of the edge-weights

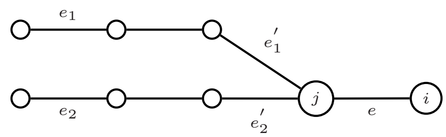

Figure 13:

of a path in G implies the edge-weights are non-increasing along the paths $\gamma_1 \cup \{e\}$ and $\gamma_2 \cup \{e\}$. Therefore we have $\sigma(A) = u(e_1) \geq u(e_1^{'}) \geq \sigma(A)$. Thus equality holds everywhere and we obtain:

$$u(e_1) = u(e_1^{'}) = \sigma(A). \tag{68}$$

In the same way, we have:

$$u(e_2) = u(e_2^{'}) = \sigma(B). \tag{69}$$

Using (68) and (69), (67) is equivalent to:

$$u(e_1^{'}) \geq u(e_2^{'}) \geq u(e). \tag{70}$$

If $e_1^{'} = e_2^{'}$ then it results from (68) and (69) that $\sigma(A) = \sigma(B)$. If $e_1^{'} \neq e_2^{'}$, the branching condition applied to the edges $e_1^{'}, e_2^{'}, e$ and (70) imply:

$$u(e_1^{'}) = u(e_2^{'}) \geq u(e). \tag{71}$$

(68), (69) and (71) imply $\sigma(A) = \sigma(B)$. We have proved the following lemma:

Lemma 17. *Let us consider a cycle-free graph $G = (N, E)$ and an edge-weight function u satisfying the convexity conditions 1) and 2) of Theorem 13. If A and B are connected, $A \subseteq B \subseteq N \setminus \{i\}$, $j \in A$ and $e := \{i, j\} \in E$, and $\sigma(A) \geq \sigma(B) \geq u(e)$, then $\sigma(A) = \sigma(B)$.*

Case 3.3.4 Let us hereafter assume that $u(e) < \sigma(B) = \sigma(A)$. Then $\mathcal{P}_{\max}(A \cup \{i\}) = \{A, \{i\}\}$, $\mathcal{P}_{\max}(B \cup \{i\}) = \{B, \{i\}\}$, and therefore $\overline{v}(A \cup \{i\}) = v(A) + v(i)$ and $\overline{v}(B \cup \{i\}) = v(B) + v(i)$. Then the inequality $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A)$ becomes equivalent to $v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A)$. As $\sigma(A) = \sigma(B)$, Lemma 16 implies that this last inequality is satisfied.

Case 3.3.5 It only remains to consider the case where $j \in A$, and $u(e) = \sigma(B) = \sigma(A)$. Then $\mathcal{P}_{\max}(A \cup \{i\}) = \mathcal{P}_{\max}(A) \cup \{i\}$ and $\mathcal{P}_{\max}(B \cup \{i\}) = \mathcal{P}_{\max}(B) \cup \{i\}$. Therefore $\overline{v}(A \cup \{i\}) = \overline{v}(A) + v(i)$ and $\overline{v}(B \cup \{i\}) = \overline{v}(B) + v(i)$ and the supermodularity property $\overline{v}(B \cup \{i\}) - \overline{v}(B) = \overline{v}(A \cup \{i\}) - \overline{v}(A) = v(i)$ is trivially satisfied.

$\square$

Let us observe that, as G has no cycle, if v is superadditive, $\overline{v}$ is superadditive using Corollary 11. Hence if A and B are connected and $A\cap B=\emptyset$ we still have $\overline{v}(A\cup B)\geq\overline{v}(A)+\overline{v}(B)$. For all connected subsets A and B of N, we have $\overline{v}(A\cup B)+\overline{v}(A\cap B)\geq\overline{v}(A)+\overline{v}(B)$ (assuming v is $\mathcal{F}$-convex and superadditive).

7 Shapley value

We now investigate the computation of the Shapley value of the game $\overline{v}$ in the case of cycle-free graphs. We assume that $G=(N,E)$ is a tree (results easily extend to forests). To compute the Shapley value, we have to compute $\overline{v}(S\cup i)-\overline{v}(S)$ for every $S\subseteq N\setminus i$ and every $i\in N$. Let us fix $i\in N$. Several cases occur.

1. $S, S\cup i$ are connected. Since G is a tree, it follows that there is a unique edge $e=\{i,j\}$ linking i to S. Put $\mathcal{P}_{\max}(S)=\{S_1,\dots,S_k\}$ with $S_1\ni j$ and u_0 the minimal weight in S. Then:

$$\overline{v}(S\cup i)-\overline{v}(S)=\begin{cases}v(i)+v(S)-v(S_1)-\cdots-v(S_k), & \text{if } u(e)<u_0\\ v(i), & \text{if } u(e)=u_0\\ v(S_1\cup i)-v(S_1), & \text{if } u(e)>u_0.\end{cases}$$

2. S is connected but not $S\cup i$. Then there is no edge between i and a node of S. Therefore:

$$\overline{v}(S\cup i)-\overline{v}(S)=v(i).$$

3. S is not connected but $S\cup i$ is. Let $S_1,\dots,S_k$ be the connected components of S. Then there are edges $\{i,j_1\},\dots,\{i,j_k\}$ from i to each $S_1,\dots,S_k$, with weights $u_1,\dots,u_k$, assuming w.l.o.g. $u_1\leq u_2\leq\cdots\leq u_k$, and u_0 is the minimal weight on S. Put $\mathcal{P}_{\max}(S_l)=\{S_{l,1},\dots,S_{l,p_l}\}$, $l=1,\dots,k$ with $S_{l,1}\ni j_l$. Then

$$\overline{v}(S\cup i)-\overline{v}(S)$$
$$=\begin{cases}v(S_{1,1}\cup\cdots\cup S_{k,1}\cup i)-v(S_{1,1})-\cdots-v(S_{k,1})\\ \qquad\text{if } u_0<u_1\\ \sum_{r=1}^{m}v(S_r)+v(S_{m+1}\cup\cdots\cup S_k\cup i)-\sum_{r=1}^{k}\sum_{s=1}^{p_r}v(S_{r,s})\\ \qquad\text{if } u_1=\cdots=u_m<\min(u_0,u_{m+1})\\ v(S_{m+1,1}\cup\cdots\cup S_{k,1}\cup i)-v(S_{m+1,1})-\cdots-v(S_{k,1})\\ \qquad\text{if } u_1=\cdots=u_m=u_0<u_{m+1}.\end{cases}$$

4. $S, S\cup i$ are not connected. Let $S_1, \ldots, S_k$ be the connected components of S. If i is not linked to any of the $S_1, \ldots, S_k$, then

$$\overline{v}(S \cup i) - \overline{v}(S) = v(i).$$

Assume now w.l.o.g. that i is linked to $S_1, \ldots, S_l$, with $u_1 \leq \cdots \leq u_l$ and same notation as above:

$$\begin{aligned}
&\overline{v}(S \cup i) - \overline{v}(S) \\
&\quad = \begin{cases}
v(S_{1,1} \cup \cdots \cup S_{l,1} \cup i) - v(S_{1,1}) - \cdots - v(S_{l,1}) \\
\quad \text{if } u_0 < u_1 \\
\sum_{r=1}^{m} v(S_r) + v(S_{m+1} \cup \cdots \cup S_l \cup i) - \sum_{r=1}^{l} \sum_{s=1}^{p_r} v(S_{r,s}) \\
\quad \text{if } u_1 = \cdots = u_m < \min(u_0, u_{m+1}) \\
v(S_{m+1,1} \cup \cdots \cup S_{l,1} \cup i) - v(S_{m+1,1}) - \cdots - v(S_{l,1}) \\
\quad \text{if } u_1 = \cdots = u_m = u_0 < u_{m+1}.
\end{cases}
\end{aligned}$$

It turns out that Case 4 includes all the others, provided we allow for $l = 0$ and $m = l$. Therefore, we have an explicit expression of the Shapley value for trees.

A player i is called a *dummy player* in a game (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every $S \subseteq N \setminus i$.

Proposition 18. *If there exists a dummy player i for $\overline{v}$, then either $\{i\}$ has degree at most 1, or all players are dummy for $\overline{v}$.*

Proof. Suppose i is dummy for $\overline{v}$. Then for any $S \subseteq N \setminus i$,

$$\overline{v}(S \cup \{i\}) = \overline{v}(S) + \overline{v}(\{i\}) = \overline{v}(S) + v(\{i\}). \tag{72}$$

According to case 2 above, this happens if there is no edge from i to S. Therefore, one possibility is that i has no edge, i.e., it has degree 0. Suppose then that this is not the case. Then there exist, say m edges adjacent to i, denoted by $\{i, j_1\}, \ldots, \{i, j_m\}$ with weight $u_1, \ldots, u_m$. According to case 4, only the 3d equation with $m = l$ permits to get $\overline{v}(S \cup i) - \overline{v}(S) = v(i)$, which means that $u_1 = u_2 = \cdots = u_m = \min_{e \in E(S)} u(e)$.

Claim: any edge in the graph has same weight u_1 if $m > 1$.

Indeed, take any edge $e = \{k, l\}$ in E. If k or $l = i$, then we know already that $u(e) = u_1$. Suppose now there exists $p \in \{1, \ldots, m\}$ such that k or $l = j_p$ and consider $S = \{k, l\}$. Then imposing (72) for S shows that $u(e) = u_1$ is the only possibility. Suppose finally that both k, l differ from $i, j_1, \ldots, j_m$. If there is no edge linking j_1 to k or l, let us take $S = \{j_1, k, l\}$. The partition of S is the partition in singletons since $\{k, l\}$ is not connected with j_1. Therefore to satisfy (72), the partition of $S \cup \{i\}$ must also be in singletons, which implies that $u(e) = u_1$. If there is an edge between j_1 and

k or l, then, as the graph is cycle free, there is no edge linking k or l to j_p for $p \in \{2, \ldots, m\}$. Taking now $S = \{j_2, k, l\}$ we still get $u(e) = u_1$.

Now if all weights are equal, $\overline{v}$ is additive, therefore all players are dummy. □

8 Conclusion

All preceding games are point games on the set N of vertices of G. Borm, Owen and Tijs in 1990 [3] have introduced arc games on the set E of edges of G and the position value. We could also consider arc games in the same spirit, by substituting to the partition into connected components the partition associated with the strength of the graph. Aziz et al. [2] have investigated some properties of the wiretapping game associated with a given graph. The value of this game is precisely equal to the reciprocal of the strength of the graph. Using the strength of the subgraphs they construct a prime partition of the set of edges which is of main interest to analyse the wiretapping game. By means of this prime partition we could also construct for a given arc game v on E a new restricted game. It would be interesting to study inheritance of superadditivity and convexity for this type of games.

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