Growth vs level effect of population change on economic development: An inspection into human-capital-related mechanisms

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Abstract

We study the impact of demographic change on economic short and long-term dynamics in an enlarged Lucas-Uzawa model with intratemporal altruism. Demographics are summarized by population growth rate and initial size. In contrast to the existing literature, the long-run level effects of demographic changes, *i.e.* their impact on the levels of variables along the balanced growth paths, are deeply characterized in addition to the more standard growth effects. It is shown that the level effect of population growth is *a priori* ambiguous due to the interaction of three causation mechanisms, a standard one (dilution) and two non-standard, featuring in particular the transmission of demographic shocks into human capital accumulation. Overall, the sign of the level effect of population growth depends on preference and technology parameters, and on the initial conditions as well. In contrast, we prove that the long-run level effect of population size on per capita income is negative while its growth effect is zero. Finally, we show that the model is able to replicate complicated time relationships between economic and demographic changes. In particular, it entails a negative effect of population growth on per capita income, which dominates in the initial periods, and a positive effect which restores a positive correlation between population growth and economic performance in the long-run.

Keywords: Human Capital, Population Growth, Population Size, Endogenous Growth, Level Effect, Growth Effect.

JEL classification: C61, C62, E2, J10, O41.

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1 Introduction

The relationship between demographic change and economic development is an important topic which has suggested a huge empirical and theoretical literature in both demography and economics. While correlations between certain economic and demographic variables may sound as obvious at first glance, a general conclusion from most of the empirical studies performed is that such correlations are far from compelling, which has opened an ongoing intense *population debate*. For example, Kelley and Schmidt (2001) (see also Kelley, 1988, and Kelley and Schmidt, 1995) report "a general lack of correlation between the growth rates of population and per capita output", documented in more than two dozen studies. The same conclusion was reached by the demographer Ronald Lee (1983) two decades ago, who particularly pointed out the "inconclusivity" of cross-national studies. As mentioned by Kelley and Schmidt (2001), simple correlations between demographic and economic variables would be anyway difficult to interpret "...plagued as they are by failure to adequately account for reverse causation between economic and demographic change, complicated timing relationships associated with the Demographic Transition,..., complexity of economic-demographic linkages that are poorly modeled,...and data of dubious quality". Other problems come from the data limitations which has led to simplified specifications of the relationship between demographic and economic change. For example, while the levels of physical and human capital stocks are *a priori* key variables in the analysis of the latter relationship, they are quite difficult to construct, specially for developing countries. Usually, proxies of their respective growth rates are incorporated in modified relationships in terms of variables' growth rates. Even worse, a key variable like human capital, which sounds as the major variable connecting demographic and economic trends, is difficult to compile, be it in level or in growth rates.

This paper is a theoretical contribution to the population debate outlined above. Concretely, we study the impact of population change on human capital level and growth rates in a traditional setting where growth is endogenously generated by human capital accumulation in line with the Lucas-Uzawa two-sector model. In doing so, we abstract from the very well-known reverse causation highlighted by Kelley and Schmidt (2001). As in standard endogenous growth models with infinite-lived representative agents, we keep demographics exogenous, summarized in two parameters, population size (N) and population growth rate (n). There are some quite popular models studying the relationship between population, human capital, and growth under the assumption of endogenous fertility, mostly based on the well-known quality-quantity of children trade-off. An overwhelming part of the latter literature uses overlapping-generations models (see for example, Nerlove, Razin and Sadka, 1985). Here, we choose to investigate the demographic-economic link in a standard endogenous growth model with infinite-lived agents, and as most demographers, we do not incorporate any form of the traditional quality-quantity trade-off into the analysis, that's we keep fertility exogenous. As we shall see throughout the paper, our framework with exogenous demographics is already extremely complicated.

The reference model is the Lucas (1988) two-sector model of endogenous growth with physical and human capital stocks, which distinguishes between the number of individuals (population) and the quality of individuals (human capital). Such a model endogenizes quality but leaves the number to follow an exogenous process. Human capital may be considered under different perspectives as knowledge, education, or experience and on the job training: knowledge and skills embodied in people are the cause of advances in technological and scientific knowledge, which in turn fosters economic development. We focus on the relationship between population, human capital, and growth by studying the impact of population growth and size on the long-run level (*level effect* [1]) and rate of growth (*growth effect* [2]) of human capital and income per capita. We also study the short-term dynamics in an attempt to distinguish between the *short-run* and *long-run* effects of population growth and size on economic performance and to uncover part of the "complicated timing relationships" pointed out by Kelley and Schmidt (2001). Moreover, the basic model has been enlarged to include the Benthamite principle of maximizing total utility (classical utilitarism), and the Millian principle of maximizing per capita utility (average utilitarism) as the two polar cases of social welfare criteria in line with Palivos and Yip (1993) and Razin and Yuen (1995).

Our contribution is therefore threefold. In first place, we do not restrict our analysis to the relationship between demographic and economic growth rates as it is the case in the related theoretical literature and in the vast majority of the empirical works. We also study the relationship between income and human capital levels and the demographic variables. Mankiw et al. (1992) do consider a two-sector growth model with physical and human capital and do estimate the shape of the relationship between the level of income per capita and the population growth rate. However, this was done in an exogenous growth setting with exogenous saving rates. When one turns to optimization-based endogenous growth theory with infinite-lived agents, our paper is the first which goes beyond the typical analysis of the link between demographic and economic growth rates. Strulik (2005) and Bucci (2008), among others, do study the latter link in proper endogenous growth frameworks but they do not account for any level effect. Investigating the impact of demographic change on the level of income per capita seems however a necessary task, especially if one is concerned with development policies in developing countries where level measures are generally much more meaningful than growth rate indicators as argued by Parente and Prescott (1993). The main reason why this task has not been undertaken so far in the class of endogenous growth models is technical: long-run levels are undetermined along balanced growth paths, only growth rates and ratios of variables are identifiable along these paths (see for example, chapter 5 of Barro and Sala-i-Martin's textbook, 1995, devoted to the Lucas-Uzawa model). Typically, these long-run levels depend on initial conditions, therefore implying that uncovering the long-run levels requires the characterization of transitional dynamics, a daunting task for non-AK endogenous growth models. In this paper, we rely on Boucekkine and Ruiz-Tamarit (2008) who produced analytical solutions to the Lucas-Uzawa model to construct closed-form solutions to the optimal paths of all variables in level in our enlarged Lucas model. The analytical solutions make use of a specific class of special functions, the so-called Gaussian hypergeometric functions, which naturally result from the resolution of the dynamic system formed by the first-order conditions. Because of the presence of the latter special functions, comparative statics with respect to the demographic parameters, while possible, are very complicated to handle analytically.

Second, we show that population size, that is the scale of the economy, is also an important determinant of economic performance. Precisely, we show that the size of population affects the levels of income and human capital but not the long-run growth of the economy, which in contrast depends on population growth rate. As outlined by Kelley and Schmidt (2001), "...curiously, even though studies in the economic-demographic tradition have long harkened the importance of population size and density, these influences have been strikingly missing in empirical growth in recent decades". The quite thin related literature points at a generally significant impact of population size and density on economic performance although it varies

[1] Changes in parameters that raise or lower balanced growth paths without affecting their slope.
[2] Changes in parameters that modify growth rates along balanced paths.

a lot across places and time (Kelley and Schmidt, 1999). Our theoretical analysis identifies a nonzero level effect of population size while its growth effect is shown to be nil, which is, to our knowledge, the first characterization in the related theoretical literature.

Last but not least, using the closed-form solution paths, we are able along with Kelley and Schmidt (1995, 1996) to distinguish between *short* and *long term* effects of population growth and size on economic performance. This is a valuable exercise if one has in mind the ongoing population debate. Pessimistic theories of population growth would emphasize its short term adverse impacts given the apparent fixity of resources and diminishing returns. Optimistic theories would rather take a long term perspective where the short-run costs of population growth are counterbalanced by benefits. Therefore, having the possibility to compare short term and long term (level and growth) effects of demographic change is extremely worthwhile to deliver a global picture of the demographic-economic nexus. Again this contribution is quite original since most endogenous growth theories only focus on the long-run.

Empirical literature on the relationship between human capital and population, and the population scale effect

The interaction between population and human capital has been quantitatively studied at family and country levels. At family level, it has been shown that beyond a fixed family size, extra children are associated with lower average educational attainments, worse nutritional standards, and a lower spending on health services (King, 1985; Birdsall, 1977). Kelley (1996) reviews the available evidence from empirical studies and suggests that additional children reduce the years of schooling completed by other children in the household, although the size of this effect is usually small. In fact, the negative effect of larger families on the quantity of human capital is not always found, or may it not be statistically significant. For example, Mueller (1984) presents evidence from Botswana and Sierra Leone that children from larger families achieve higher average levels of schooling, controlling other pertinent variables. However, Birdsall (1977) points out that children from large families do less well in test intelligence, that mothers' health is negatively affected by pregnancies, especially among poor women, and that large families adjust to economic constraints transferring the burden on the children in the form of a declining quantity and quality of food and medical cares. At the aggregate level, the empirical evidence also shows an uncertain effect of demographic change on human capital accumulation measured by enrollment rates, years of school attainment of adults, school dropout rates, the student-teacher ratio, and scores on international examinations. For example, Schultz (1987) and Kelley (1996) find that rapid population growth is relatively unimportant in explaining the increased quantity of education (enrollment and attainment rates); however, it seems that it reduces the quality of the education provided, as it increases the student-teacher ratio and decreases the government expenditures per school-age child, mainly at the secondary level and during the sixties and the seventies.[3]

Concerning the population scale effect, one would conclude from the literature quoted above on the impact of family size on human capital level that it is not less disputed. As mentioned by Kelley and Schmidt (2001), population size has been traditionally viewed as a positive factor of long-run growth in countries with abundant resources, strong institutions and relatively low population densities. However, the latter conditions are seldom met, notably in developing countries. The empirical literature is rather thin on this question. Most of the existing related papers focus on the agricultural sector (see for example, Pingali and Binswanger, 1987) where

[3]For a deeper study of the relationship between quality of education, quantity of education, and the rate of growth of per capita income, see Castelló-Climent and Hidalgo (2010).

the impact of larger population densities on the efficiency of transportation and irrigation can be more directly apprehended. Still the available studies show a great variability in their conclusions (see again Kelley and Schmidt, 2001). More recently, some authors have studied whether "small states" have specific properties in terms of the development pattern. Among them, Easterly and Kraay (2000) have found that, controlling for location, smaller states are actually richer than other states in per capita GDP. That is there exists a negative correlation between population size and level of income per capita. However, they have also found that small states do not have different per capita growth rates, therefore concluding that population size looks uncorrelated with per capita growth rates. We shall show that our enlarged Lucas model displays a similar picture.

Relation to the theoretical literature

We now briefly review the related theoretical literature. As mentioned above, an early contribution is due to Mankiw, Romer, and Weil (1992) who consider a two-sector exogenous growth model (so by definition there is no growth effect). According to this model, population size doesn't affect the long-run levels of both per capita human capital and income. On the contrary, the model predicts a negative level effect of population growth rate on the long-run level of per capita income due to the dilution effect experienced by both human and the physical capital. Since the corresponding investment rates are exogenous, both capital stocks cannot increase in proportion to population growth rate, resulting in decreasing stocks in per capita terms. In the case of the endogenous growth models, Jones (1999) has comprehensively evaluated the "demographic" properties of most of R & D based models of endogenous growth with no human capital accumulation, classified in the following categories: scale effect growth models, semi-endogenous growth models, and fully endogenous growth models.[4] Our paper is more closely related to Dalgaard and Kreiner (2001), Strulik (2005), and Bucci (2008) as these authors analyze the impact of population on economic growth in endogenous growth models with human capital accumulation, although they also consider endogenous R&D activities. They all focus on the relationship between population, human capital, and output per capita stressing the role played by the agents' degree of altruism. However, for the technical reasons mentioned above, none of these papers analyze separately the effect of population growth and size on the rate of growth of per capita income (*growth effect*) as well as on the long-run level of income per capita (*level effect*). As outlined above, our novel study of level effects is notably relevant for the design of theories primary concerned with policies which raise income levels and not growth rates (Parente and Prescott, 1993). Even more importantly, a substantial part of the empirical literature relies on direct level measures of human capital accumulation like enrollment rates or years of schooling: providing an explicit theory of how demographic change affects human capital in level is therefore not only theoretically challenging, it might be also illuminating from the empirical point of view.

Main findings

Four findings should be emphasized.

1. A first decisive outcome of our work is the separation of level Vs growth effects of demographic change on economic development. In particular, our paper is the first which does

[4] In particular, he noticed that *scale effect* growth models generate by construction a positive correlation between population size and the growth rate of per capita income, while *semi-endogenous* growth models do not only entail that the rate of growth of per capita income depends positively on the rate of population growth, they also deliver that the rate of growth of per capita income becomes zero in the absence of population growth.

the job in the class of endogenous growth models considered. While the study of growth effects of population change is standard (usually displaying the property that population growth has a non-negative impact on long-term economic growth), the mere inspection of its level effects is novel, and therefore, provides new insight into how demographic change influences economic development. Essentially, we have identified three causation mechanisms from population growth to the long-run income level. The first one is associated with the ratio of physical to human capital which originates in the standard effect of **physical capital dilution**: a larger population growth increases the size of the dilution effect, which is detrimental to the income per capita level. As outlined above, this sole effect explains the negative level effect obtained by Mankiw et al. (1992). A second more original mechanism is connected with the fraction of non-leisure time devoted to goods production (and consequently with preference parameters). This effect is nonzero if and only if economic agents are not selfish, and we therefore refer to it from now as the effect of **altruism utility**. As non-leisure time devoted to production of the final good is shown to be a decreasing function of the population growth rate, provided that economic agents are not selfish, this effect also generates a negative correlation between population growth and the level of per capita income.[5] Last but not least, a third causation line induced by the level of human capital arises, therefore representing the effect of **human capital**. Unfortunately, the third effect has a non-trivial sign. Consistently with the empirical literature on the link between the level of human capital and population growth, the relationship between the two latter variables is highly complex and depends nonlinearly on preference and production parameters, and on the initial conditions. Consequently, the total impact of population growth on the level of income per capita is ambiguous, which is again consistent with the empirical literature. This departs sharply from the simple comparative statics usually performed to study the impact of demographics on the long-run economic rate of growth: the level effects of population change are by far trickier.

2. Deeply inspecting the sources of ambiguity, we show that when the inverse of the intertemporal elasticity of substitution is equal to the value of the capital share in the final good sector, population growth rate has no effect on the long-run level of human capital, that is the effect of human capital mentioned above is nil. Consequently, the level effect on the long-run levels of per capita income and output is negative. However, when the inverse of the intertemporal elasticity of substitution is no longer equal to (say bigger than) the value of the capital share in the final good sector, things are substantially different. Considering the initial position of the economy with respect to its long-run equilibrium in terms of the ratio physical to human capital, we analytically show that if the economy starts from below or is exactly equal to the long-run value of the latter ratio, then population growth has a positive effect on the long-run level of human capital; that is, the effect of the human capital mechanism is positive. In such a case, the total level effect of population growth is ambiguous: the physical capital dilution and altruism have a negative level effect while the effect of human capital is positive. Resorting to numerical investigation, we find that for all the empirically relevant cases, population growth positively affects the (detrended) long-run level of human capital, and negatively affects the (detrended) long-run levels of per capita income and broad output. The sign of these effects are invariant to the configuration chosen for initial conditions and to the

[5] This correlation is nil in the absence of altruism.

assumed degree of altruism.

3. We also investigate the growth and level effects of population size (or scale effects). We find that there is no **growth effect** due to population size. The common long-run rate of growth of average human capital stock, per capita broad output, and per capita income does not depend on population size. In contrast, the **level effect** of the population size is nonzero. Here results also depend on the relationship between the inverse of the intertemporal elasticity of substitution in consumption and the physical capital share in goods production. For example, we find that in the normal case (that is when the former parameter is bigger than the latter), a larger initial population size leads to lower long-run detrended levels of per capita income, per capita broad output, and average human capital independently of the initial conditions and of the degree of altruism of economic agents. This roughly illustrates a negative level effect of population size, just like the level effect of population growth rate is generally found to be, although population growth does raise the level of human capital in the most relevant parametric cases of the model. The non-positive growth and level effects of population size obtained may seem opposite to the corresponding empirical literature. Notice however that the largest part of the latter literature has been more concerned with growth effects of the economy scale and even more concerned with the agricultural sector. We believe that our results on the level effect of population size in a human-capital-based growing economy are truly original. On the other hand, they are clearly consistent with the recent empirical work of Easterly and Kraay (2000) highlighted above, one of the very few papers separating growth and level effects.

4. Last but not least, we study the effects of population change over time by depicting the optimal transition paths. In doing so, we depict the optimal paths accounting for both the level and growth effects together. The results are highly interesting if one has in mind the population debate. In particular, we find that the effect of a higher population growth rate on per capita income is generally negative in the short-run, reflecting the negative level effect outlined above, while this effect is positive in the long-run through the positive growth effect also mentioned above. As such, our theory neatly explains why the relationship between population change and economic development depends on time. The distinction between level and growth effects of population change allows to give a simple and powerful explanation to this complicated time relationship.

The paper is organized as follows. Section 2 is devoted to briefly present an enlarged version of the Lucas-Uzawa model which includes an altruism parameter. Section 3 examines the balanced growth path and exposes the closed-form solution for the variables involved in the relationship between population, human capital, and growth. Section 4 analyzes the growth effect of population size and growth. Sections 5 and 6 analyze the level effect of population growth and population size, respectively. Section 7 studies the impact of different demographic shocks on the optimal transition paths of the more significant variables. Section 8 concludes.

2 The Uzawa-Lucas model

We will now consider the Uzawa-Lucas two-sector endogenous growth model. The economy is closed with competitive markets and populated with many identical, rational agents. They choose the controls $c(t)$, consumption per capita, and $u(t)$ $\forall t \geq t_0$, the fraction of non-leisure time devoted to goods production, which solve the dynamic optimization problem

$$\max \int_0^\infty \frac{c(t)^{1-\sigma} - 1}{1-\sigma} N(t)^\lambda e^{-\rho t} dt \tag{P}$$

subject to

$$\overset{\bullet}{K}(t) = AK(t)^\beta (u(t) N(t) h(t))^{1-\beta} - \pi K(t) - N(t) c(t),$$

$$\overset{\bullet}{h}(t) = \delta (1 - u(t)) h(t) - \theta h(t),$$

$$K(0) = K_0, \quad h(0) = h_0, \quad N(0) = N_0,$$

$$c(t) \geqslant 0, \quad u(t) \in [0, 1], \quad K(t) \geqslant 0, \quad h(t) \geqslant 0.$$

The considered instantaneous utility function is standard, with $\sigma^{-1} > 0$ representing the constant elasticity of intertemporal substitution. Population size at time t is $N(t)$, which is assumed to grow at a constant exogenously given rate n starting from a given initial size N_0. Parameter ρ is the rate of time preference or discount rate. We assume $\rho > n$. Parameter $\lambda \in [0, 1]$ contributes to determine agents preferences, which are represented using a Millian, an intermediate, or a Benthamite intertemporal utility function. In one extreme, when $\lambda = 0$ (average utilitarism), agents maximize the per capita utility (average utility of consumption per capita). In the other, when $\lambda = 1$ (classical utilitarism), agents maximize total utility (the addition across total population of utilities of per capita consumption).[6]

In this model $h(t)$ is the human capital level, or the skill level, of a representative worker while $u(t)$ is the fraction of non-leisure time devoted to goods production. The output, $Y(t)$, which may be allocated to consumption or to physical capital accumulation depends on the capital stock, $K(t)$, and the effective workforce, $u(t) N(t) h(t)$. Parameter β is the elasticity of output with respect to physical capital. The efficiency parameter A represents the constant technological level in the goods sector of this economy. It is assumed that the growth of human capital do not depend on the physical capital stock. It depends on the effort devoted to the accumulation of human capital, $1-u(t)$, as well as on the already attained human capital stock. The efficiency parameter δ represents the constant technological level in the educational sector. It also represents the maximal rate of growth for $h(t)$ attainable when all effort is devoted to human capital accumulation. Technology in goods sector shows constant return to scale over private internal factors. Technology in educational sector is linear. Both physical and human capital depreciate at constant rates, which are $\pi \geqslant 0$ and $\theta \geqslant 0$, respectively. We shall also assume that $\delta + \lambda n > \theta + \rho$ for positive (long-run) growth to arise, as it will be transparent later. Note that this assumption also implies that $\delta + n + \pi - \theta > 0$.

As it is explained in Lucas (1988), the per capita human capital accumulation equation implies that there is no human capital dilution effect. Consequently, population growth *per se*

[6]The literature differentiates between two types of altruism depending on the two parameters ρ and λ. The first one is *intertemporal* altruism and depends on the discount rate applied to future population utility. The second one is *intratemporal* altruism and depends on the number of individuals which is taken into account each period. In particular, for representative and infinitely lived agent models, parameter λ controls for the degree of altruism towards future generations. When agents are (partially) selfish, $\lambda = 0$, they care only about per capita utility (current and future), and the size of population has no direct effect on the intertemporal utility. Instead, when agents are (almost perfectly) altruistic, $\lambda = 1$, they care not only about their own utility but also about that of their dynasties. In this case, the intertemporal utility function includes total population as a determinant, regardless of its value in the future. When $0 < \lambda < 1$ agents show an intermediate degree of intratemporal altruism.

do not reduce the current average knowledge of the representative worker. In other words, newborns enter the workforce endowed with a skill level proportional to the level already attained by older. Lucas' assumption is based on the social nature of human capital accumulation, which has no counterpart in the accumulation of physical capital.

The current value Hamiltonian associated with the previous intertemporal optimization problem is

$$H^c(K, h, \vartheta_1, \vartheta_2, c, u; A, \sigma, \lambda, \beta, \delta, \pi, \theta, \{N(t) : t \geq 0\}) =$$

$$= \frac{c^{1-\sigma} - 1}{1-\sigma} N^{\lambda} + \vartheta_1 \left[AK^{\beta}(uNh)^{1-\beta} - \pi K - Nc \right] + \vartheta_2 \left[\delta (1-u) h - \theta h \right] \tag{1}$$

where ϑ_1 and ϑ_2 are the co-state variables for K and h, respectively.

The first order necessary conditions are

$$N^{\lambda-1} c^{-\sigma} = \vartheta_1, \tag{2}$$

$$\vartheta_1 (1-\beta) AK^{\beta} (uNh)^{-\beta} N = \vartheta_2 \delta, \tag{3}$$

the Euler equations

$$\overset{\bullet}{\vartheta}_1 = (\rho + \pi) \vartheta_1 - \vartheta_1 \beta AK^{\beta-1} (uNh)^{1-\beta}, \tag{4}$$

$$\overset{\bullet}{\vartheta}_2 = (\rho + \theta) \vartheta_2 - \vartheta_1 (1-\beta) AK^{\beta} (uN)^{1-\beta} h^{-\beta} - \vartheta_2 \delta (1-u), \tag{5}$$

the dynamic constraints

$$\overset{\bullet}{K} = AK^{\beta}(uNh)^{1-\beta} - \pi K - Nc, \tag{6}$$

$$\overset{\bullet}{h} = \delta (1-u) h - \theta h, \tag{7}$$

the boundary conditions K_0, h_0, and the transversality conditions

$$\lim_{t\to\infty} \vartheta_1 K \exp\{-\rho t\} = 0, \tag{8}$$

$$\lim_{t\to\infty} \vartheta_2 h \exp\{-\rho t\} = 0. \tag{9}$$

Notice that by (2), $\vartheta_1(t)$ cannot be equal to 0 at any finite date t because this would require that consumption is infinite at a finite date, which violates the resource constraint of the economy. Then, according to (3), $\vartheta_2(t) \neq 0$ at a finite t, provided the economy starts with finite and strictly positive endowments of physical and human capital, implying also finite and strictly positive output levels at any finite date.

From (2) and (3) we get the control functions

$$c = \vartheta_1^{-\frac{1}{\sigma}} N^{\frac{\lambda-1}{\sigma}}, \tag{10}$$

$$u = \left(\frac{(1-\beta) A}{\delta} \right)^{\frac{1}{\beta}} \left(\frac{\vartheta_1}{\vartheta_2} \right)^{\frac{1}{\beta}} \frac{K}{h} N^{\frac{1-\beta}{\beta}}. \tag{11}$$

After substituting the above expressions into equations (4)-(7), we obtain

$$\overset{\bullet}{\vartheta}_2 = -(\delta - \rho - \theta) \vartheta_2 \tag{12}$$

$$\overset{\bullet}{\vartheta}_1 = (\rho + \pi) \vartheta_1 - \psi_1(t) \vartheta_1^{\frac{1}{\beta}} \tag{13}$$

$$\overset{\bullet}{K} = \psi_2 (t) K - \psi_3 (t) \tag{14}$$

$$\overset{\bullet}{h} = (\delta - \theta) h - \psi_4 (t) \tag{15}$$

where

$$\psi_1(t) = \beta A \left(\frac{(1-\beta) A}{\delta} \right)^{\frac{1-\beta}{\beta}} N^{\frac{1-\beta}{\beta}} \vartheta_2^{-\frac{1-\beta}{\beta}}, \tag{16}$$

$$\psi_2(t) = A \left(\frac{(1-\beta) A}{\delta} \right)^{\frac{1-\beta}{\beta}} N^{\frac{1-\beta}{\beta}} \left(\frac{\vartheta_1}{\vartheta_2} \right)^{\frac{1-\beta}{\beta}} - \pi, \tag{17}$$

$$\psi_3(t) = N^{\frac{\sigma+\lambda-1}{\sigma}} \vartheta_1^{-\frac{1}{\sigma}}, \tag{18}$$

$$\psi_4(t) = \delta \left(\frac{(1-\beta) A}{\delta} \right)^{\frac{1}{\beta}} N^{\frac{1-\beta}{\beta}} \left(\frac{\vartheta_1}{\vartheta_2} \right)^{\frac{1}{\beta}} K. \tag{19}$$

These equations, together with the initial conditions, K_0 and h_0, and the transversality conditions (8) and (9) constitute the dynamic system which drives the economy over time. This dynamic system can be recursively solved in closed form. Boucekkine and Ruiz-Tamarit (2008) show that such a system can be solved explicitly without resorting to any dimension reduction.

3 The closed-form solution along the balanced growth path

In this section we show in closed-form the solution path for the variables of the model,[7] when we substitute the exogenous population level assuming an exponential process: $N = N_0 \exp \{nt\}$, where N_0 is the exogenous (initial or detrended) population size and n is the exogenous rate of population growth.[8] Any particular non-explosive solution to the dynamic system (12)-(15) has to satisfy the initial conditions K_0 and h_0, as well as the transversality conditions (8) and (9). These ones impose the constraints

$$(\delta + n + \pi - \theta) (\beta - \sigma) - \beta (\rho + \pi - n (\sigma + \lambda - 1) - \pi\sigma) < -\sigma (1-\beta) (\delta + n + \pi - \theta) < 0, \tag{20}$$

$$(\delta - \theta) (1-\sigma) + \lambda n - \rho < 0, \tag{21}$$

$$\frac{K_0}{{}_2F_1(0)} \left(\frac{\vartheta_1(0)}{\vartheta_2(0)} \right)^{\frac{1}{\beta}} = - \frac{\sigma \beta N_0^{\frac{\sigma+\lambda-1}{\sigma}} \vartheta_2(0)^{-\frac{1}{\sigma}} \left(\frac{\delta+n+\pi-\theta}{\epsilon} \right)^{\frac{\sigma-\beta}{\sigma(1-\beta)}}}{(\delta + n + \pi - \theta) (\beta - \sigma) - \beta (\rho + \pi - n (\sigma + \lambda - 1) - \pi\sigma)}, \tag{22}$$

$$\frac{{}_2F_1(0)}{{}_2\widetilde{F}_1(0)} = \frac{(1-\beta) \epsilon\sigma}{-((\delta - \theta) (1-\sigma) + \lambda n - \rho) \beta} \frac{K_0}{h_0} \left(\frac{\vartheta_1(0)}{\vartheta_2(0)} \right)^{\frac{1}{\beta}}, \tag{23}$$

[7] The exact solution trajectories have been obtained according to the procedure developed in Boucekkine and Ruiz-Tamarit (2008), which solve the previous dynamic system under $\lambda = 1$. In this section we only supply the long-run trajectories for the involved variables, leaving the corresponding short-run trajectories for a later section. The complete computations are available upon request.

[8] Given the dynamics assumed for N, we get identical short- and long-run trajectories, $N (t) \equiv \bar{N} (t)$, but we also get that the long-run detrended level $\bar{N}_l$ is equivalent to the initial population size N_0.

where

$$\epsilon = \beta A \left(\frac{(1-\beta) A N_0}{\delta} \right)^{\frac{1-\beta}{\beta}} > 0. \tag{24}$$

Conditions (22) and (23) make up a system of two equations with two unknowns, $\vartheta_1(0)$ and $\vartheta_2(0)$. Their values may be determined in the following way: (23) determines a unique value for the ratio $\frac{\vartheta_1(0)}{\vartheta_2(0)}$, then (22) determines the value of $\vartheta_2(0)$, which after multiplying by the value of the ratio itself gives the value of $\vartheta_1(0)$. In the above conditions we use the following hypergeometric functions written under their Euler representation form[9]

$$\begin{aligned} {}_2F_1(0) \equiv {}_2F_1(a,b;c;z_0) = \\ = {}_2F_1(a(n), b; 1+a(n); z_0(n)) = {}_2F_1(b, a(n); 1+a(n); z_0(n)) = \\ = \frac{\Gamma(1+a(n))}{\Gamma(a(n))\Gamma(1)} \int_0^1 t^{a(n)-1} (1-tz_0(n))^{-b} dt = a(n) \int_0^1 t^{a(n)-1} (1-tz_0(n))^{-b} dt = \\ = \left(1+\widetilde{a}(n)\right) \int_0^1 t^{\widetilde{a}(n)} (1-tz_0(n))^{-b} dt \end{aligned} \tag{25}$$

and

$$\begin{aligned} {}_2\widetilde{F}_1(0) \equiv {}_2F_1\left(\widetilde{a}, b; c; z_0\right) = \\ = {}_2F_1\left(\widetilde{a}(n), b; 2+\widetilde{a}(n); z_0(n)\right) = {}_2F_1\left(b, \widetilde{a}(n); 2+\widetilde{a}(n); z_0(n)\right) = \\ = \frac{\Gamma(2+\widetilde{a}(n))}{\Gamma(\widetilde{a}(n))\Gamma(2)} \int_0^1 t^{\widetilde{a}(n)-1} (1-t)(1-tz_0(n))^{-b} dt = \\ = a(n)(a(n)-1) \int_0^1 t^{a(n)-2} (1-t)(1-tz_0(n))^{-b} dt = \\ = \left(1+\widetilde{a}(n)\right)\widetilde{a}(n) \int_0^1 t^{\widetilde{a}(n)-1} (1-t)(1-tz_0(n))^{-b} dt \end{aligned} \tag{26}$$

where

$$a = -\frac{(\delta+n+\pi-\theta)(\beta-\sigma) - \beta(\rho+\pi-n(\sigma+\lambda-1)-\pi\sigma)}{\sigma(\delta+n+\pi-\theta)(1-\beta)} > 1, \tag{27}$$

$$\widetilde{a} = a - 1 = -\frac{\beta((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{\sigma(\delta+n+\pi-\theta)(1-\beta)} > 0, \tag{28}$$

$$b = -\frac{\beta-\sigma}{\sigma(1-\beta)}, \quad c = 1+a = 2+\widetilde{a}, \tag{29}$$

$$z_0 = 1 - \frac{\delta+n+\pi-\theta}{\epsilon} \left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{-\frac{1-\beta}{\beta}} \in \left]-\infty, 1\right[. \tag{30}$$

[9] Recall that $\vartheta_1(0)$ and $\vartheta_2(0)$ are both finite and different from zero, ${}_2F_1(0) = {}_2F_1(a,b;c;z_0)$ and ${}_2\widetilde{F}_1(0) = {}_2F_1\left(\widetilde{a}, b; c; z_0\right)$ are constant, ${}_2F_1(\infty) = {}_2F_1(a,b;c;0) = 1$ and ${}_2\widetilde{F}_1(\infty) = {}_2F_1\left(\widetilde{a}, b; c; 0\right) = 1$.

The long-run closed-form trajectories are[10]

$$\bar{\vartheta}_1 = \left(\frac{\delta + n + \pi - \theta}{\epsilon}\right)^{\frac{\beta}{1-\beta}} \vartheta_2(0) \exp\left\{-\left(\delta + n - \rho - \theta\right) t\right\}, \tag{31}$$

$$\bar{\vartheta}_2 = \vartheta_2(0) \exp\left\{-\left(\delta - \rho - \theta\right) t\right\}, \tag{32}$$

$$N\left(\frac{\bar{\vartheta}_1}{\vartheta_2}\right) = \frac{\delta}{(1-\beta)\, A} \left(\frac{\delta + n + \pi - \theta}{\beta A}\right)^{\frac{\beta}{1-\beta}} > 0, \tag{33}$$

$$0 < \bar{u} = -\frac{(\delta - \theta)(1-\sigma) + \lambda n - \rho}{\sigma\delta} < 1, \tag{34}$$

$$\frac{1}{N}\left(\frac{\bar{K}}{h}\right) = -\frac{(\delta - \theta)(1-\sigma) + \lambda n - \rho}{\sigma\delta} \left(\frac{\beta A}{\delta + n + \pi - \theta}\right)^{\frac{1}{1-\beta}} > 0, \tag{35}$$

$$\bar{K} = -\frac{\sigma\beta\left(\frac{(1-\beta)A}{\delta\vartheta_2(0)}\right)^{\frac{1}{\sigma}} \left(\frac{\beta A}{\delta+n+\pi-\theta}\right)^{\frac{\beta}{\sigma(1-\beta)}} N_0^{\frac{\sigma+\lambda}{\sigma}}}{(\delta+n+\pi-\theta)(\beta-\sigma)-\beta(\rho+\pi-n(\sigma+\lambda-1)-\pi\sigma)} \exp\left\{\frac{\delta + \lambda n - \theta - \rho + n\sigma}{\sigma} t\right\}, \tag{36}$$

$$\bar{h} = \frac{h_0}{{}_2\widetilde{F}_1(0)} \exp\left\{\frac{\delta + \lambda n - \theta - \rho}{\sigma} t\right\}. \tag{37}$$

The per capita narrow (market) output and broad (aggregate) output are, respectively,

$$\bar{y} = A\left(\frac{\bar{k}}{h}\right)^{\beta} \bar{u}^{1-\beta}\bar{h} = A\left(\left(\frac{\beta A}{\delta + n + \pi - \theta}\right)^{\frac{1}{1-\beta}} \frac{-\left((\delta - \theta)(1-\sigma) + \lambda n - \rho\right)}{\sigma\delta}\right)^{\beta}$$

$$\cdot \left(\frac{-\left((\delta - \theta)(1-\sigma) + \lambda n - \rho\right)}{\sigma\delta}\right)^{1-\beta}$$

$$\cdot \frac{h_0}{{}_2\widetilde{F}_1(0)} \exp\left\{\frac{\delta + \lambda n - \theta - \rho}{\sigma} t\right\} \tag{38}$$

and

$$\bar{q} = \bar{y} + \frac{1}{N}\frac{\bar{\vartheta}_2}{\bar{\vartheta}_1}\left[\delta\left(1 - \bar{u}\right)\bar{h}\right] = \bar{y}\left(1 + (1-\beta)\frac{1-\bar{u}}{\bar{u}}\right) =$$

$$= A\left(\frac{\beta A}{\delta + n + \pi - \theta}\right)^{\frac{\beta}{1-\beta}} \left(1 - \frac{\beta\left(\delta + \lambda n - \theta - \rho + \sigma\theta\right)}{\sigma\delta}\right) \frac{h_0}{{}_2\widetilde{F}_1(0)} \exp\left\{\frac{\delta + \lambda n - \theta - \rho}{\sigma} t\right\}. \tag{39}$$

Finally, the long-run rates of growth

$$\bar{g}_y = \bar{g}_q = \bar{g}_h = \frac{\delta + \lambda n - \theta - \rho}{\sigma}. \tag{40}$$

[10]All these results are general in the sense that they encompass the three different subcases arising from the relationship between the parameters representing the inverse of the intertemporal elasticity of substitution, σ, and the physical capital share, β. These subcases have drawn great attention in growth literature because they cause different patterns of dynamic behavior. However, what we supply here is a compact general solution for all of them, based on the hypergeometric function with $a > 1$, $\widetilde{a} > 0$ and $c > 2$ because of the parameter constraints (20) and (21) implied by transversality conditions, and with $b \gtreqless 0$ depending on $\sigma \gtreqless \beta$.

4 Population (size and growth) and the long-run rate of growth

In this section we start a complete study of the long-run relationship between population, per capita income, and growth. Given the assumed population process which depends on two exogenous parameters: N_0, the detrended population size, and n, the rate of population growth, we shall inquire about the impact of population, as captured by both its size and growth rate, on the economy's long-run per capita level and rate of growth. First of all, we concentrate on the consequences of demographic change on the lung-run rate of growth; that is, the **growth effect**.

Remark 1 *The common long-run rate of growth of the average human capital stock, the per capita broad output, and the per capita income do not depend on the population size.*

As in the Solow, Mankiw-Romer-Weil, and Ramsey-Cass-Koopmans models, we don't find the basic *scale effect* in the Lucas-Uzawa model. Such an effect is in contrast found in Romer's model (1986), in Barro's model (1990), and in the first wave of R & D based growth models as well. In Romer (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992), the long-run growth rate of the economy is proportional to the total amount of researchers, which depends on the population size. However, subsequent R & D based growth models, including the more general models of Dalgaard and Kreiner (2001), Strulik (2005), and Bucci (2008), do remove the scale effect, therefore producing a long-run economic growth rate independent of population size.

On the other hand, the long-run rate of growth does depend on the rate of population growth, n, as it can be inferred from equation (40). In the standard exogenous growth models of Solow, Mankiw-Romer-Weil, and Ramsey-Cass-Koopmans the rate of population growth has no impact in the long-run on the growth rate of the economy, which is given by the exogenous rate of technological progress. Moving to AK models, the growth rate of the economy is a negative function of population growth in the constant saving case. Such a negative effect also shows up in the Ramsey case under selfishness, but when $\lambda = 1$ this negative correlation vanishes. In the Lucas-Uzawa model things are sharply different as we can see in the next proposition.

Proposition 1 *When $\lambda > 0$ population growth triggers a larger long-run growth rate of the average human capital stock, as well as of per capita production in both senses, narrow and broad. Instead, for $\lambda = 0$, the minimal altruism case, the effect of population growth on the different long-run rates of growth vanishes.*

Proof: from (40) we get

$$\frac{\partial \bar{g}_y}{\partial n} = \frac{\partial \bar{g}_q}{\partial n} = \frac{\partial \bar{g}_h}{\partial n} = \frac{\lambda}{\sigma} \geqslant 0. \qquad \blacksquare \tag{41}$$

Population growth has a non-negative effect on the long-run rate of growth of the average human capital stock, per capita broad output, and per capita income. The magnitude of this growth effect is increasing with both agent's degree of intratemporal altruism and intertemporal patience (elasticity of substitution). But, if we remove altruism from the model the growth effect itself disappears.

In short, it must be highlighted that the more altruistic (selfish) and patient (impatient) is the economy, the higher (lower) is its long-run rate of growth, and the stronger (weaker) is the corresponding demographic *growth effect* associated with the rate of population growth. However, the long-run rate of growth of the economy does not depend on its demographic intensity or population density.

The relationship between the economic and demographic growth rates identified in equation (40) is consistent with the typical outcome in the class of *fully endogenous* growth models: in this framework, the growth rate of per capita income is positive even though population growth is nil, but the latter contributes positively to income per capita growth. However, there are other *fully endogenous* growth models like in Dalgaard and Kreiner (2001), Strulik (2005), and Bucci (2008), in which both technological change and human capital accumulation are endogenized and exert as engines of growth. They offer a different picture of the relationship between population growth and long-run economic growth depicted above on the enlarged Lucas model. Importantly enough, population growth has an ambiguous effect on economic growth in Strulik's paper because the economy's long-run rate of growth depends positively (negatively) on population growth if agents are altruistic (selfish). Indeed Strulik's model is built under the assumption that population growth exerts two effects on economic growth: A human capital *dilution effect* ("Since newborns enter the world uneducated they reduce the stock of human capital per capita"), which decreases economic growth, and a *time preference effect* ("A larger future size of the dynasty increases the weight assigned to consumption per capita of later generations. More patient households imply less present consumption, more investment in R & D and human capital, and hence higher growth"), which increases economic growth. The net effect determines the correlation between both growth rates, which is positive under Benthamite preferences but negative under Millian preferences because the time preference effect vanishes. In Dalgaard and Kreiner, there is only a non-positive effect: the economy's long-run rate of growth does not depend (depends negatively) on population growth if agents are altruistic (selfish). Finally according to Bucci, the effect of population growth on per capita income growth depends on the role played by agents degree of altruism as compared to the nature (skill-biased, eroding, or neutral) and the strength of the impact of technological progress on human capital investment. The growth effects of population growth are much neater in our model, which is due to the fact that growth is only generated by human capital accumulation. As we shall see hereafter the level effects are much more complex.

5 Population growth and the long-run level of the variables

Now we concentrate on the long-run level of the variables per capita narrow (market) output, per capita broad (aggregate) output, and human capital level of a representative worker. Notice that population growth rate does not only affect the long-run rate of growth of those variables but also, and in a separate way, their long-run levels. From the point of view of the proponents of a development theory primarily concerned with policies that raise per capital income levels but no growth rates, it is relevant to study whether a **level effect** is present in the model or, on the contrary, population growth has no impact on such long-run values of economic indicators. To study independently the effect of population growth on the levels, we remove the growth effect when it does exist by detrending trajectories from $t = 0$.

As stressed in the introduction, the standard exogenous growth models of Solow and

Mankiw-Romer-Weil yield a negative correlation between population growth and the detrended long-run level of the variables. In the altruistic ($\lambda = 1$) Ramsey-Cass-Koopmans model, the rate of population growth has no impact on detrended long-run level of the variables except consumption,[11], while in the selfish case ($\lambda = 0$) a negative dependence of such long-run levels with respect to the rate of population growth shows up. Moving to AK-like models, both the AK-Solow and the AK-Ramsey models do not generate any correlation between the long-run detrended level of output per capita and population growth rate. In the Lucas-Uzawa model things are infinitely more complex, as we can see hereafter.

We consider here the initial values of the long-run trajectories for y, q, and h, by detrending (38), (39), and (37). We get, respectively,

$$\bar{y}_l = A\left(\frac{\beta A}{\delta + n + \pi - \theta}\right)^{\frac{\beta}{1-\beta}} \left(\frac{-\left(\left(\delta - \theta\right)\left(1 - \sigma\right) + \lambda n - \rho\right)}{\sigma\delta}\right) \frac{h_0}{{}_2\tilde{F}_1(0)} \tag{42}$$

and

$$\bar{q}_l = A\left(\frac{\beta A}{\delta + n + \pi - \theta}\right)^{\frac{\beta}{1-\beta}} \left(1 - \frac{\beta\left(\delta + \lambda n - \theta - \rho + \sigma\theta\right)}{\sigma\delta}\right) \frac{h_0}{{}_2\tilde{F}_1(0)}, \tag{43}$$

with

$$\bar{h}_l = \frac{h_0}{{}_2\tilde{F}_1(0)}. \tag{44}$$

According to our expressions, there are three lines of causality arising from n. The first one is channelled through the term $\left(\frac{\beta A}{\delta+n+\pi-\theta}\right)^{\frac{\beta}{1-\beta}}$, it is associated with the optimal ratio of physical to human capital, and represents the traditional **physical capital dilution transmission mechanism**. The second one enters through the term $\frac{-((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{\sigma\delta}$, it is straightforwardly connected with the optimal fraction of non-leisure time devoted to goods production (which explains the dependence on preference parameters), and we will refer to it as the **altruism utility transmission mechanism**. Finally, the third causation line arises from the term $\frac{h_0}{{}_2\tilde{F}_1(0)}$, it is induced by human capital (this term is exactly the long-run detrended human capital level), and we shall therefore call it the **human capital transmission mechanism**. We come now to sign the impact of these three mechanisms. With respect to the first two, we can see that

$$\frac{\partial}{\partial n}\left(\frac{\beta A}{\delta + n + \pi - \theta}\right) < 0, \tag{45}$$

$$\frac{\partial}{\partial n}\left(\frac{-\left(\left(\delta - \theta\right)\left(1 - \sigma\right) + \lambda n - \rho\right)}{\sigma\delta}\right) = \frac{\partial \bar{u}}{\partial n} = -\frac{\lambda}{\sigma\delta} \leqslant 0, \tag{46}$$

$$\frac{\partial}{\partial n}\left(1 - \frac{\beta\left(\delta + \lambda n - \theta - \rho + \sigma\theta\right)}{\sigma\delta}\right) = -\frac{\lambda}{\sigma\delta} \leqslant 0. \tag{47}$$

[11]Actually, in this model the detrended long-run level of per capita consumption does depend negatively on n.

Remark 2 *A decrease (increase) in the rate of population growth increases (decreases) both the ratio physical to human capital and the fraction of non-leisure time devoted to goods production. These results may be found in the three cases: normal $\sigma > \beta$, exogenous $\sigma = \beta$, and paradoxical $\sigma < \beta$. They are also valid for $z_0 < 0$ and $z_0 > 0$,*[12] *as well as for an altruistic society (Benthamite intertemporal utility function) $\lambda = 1$. For a non-altruistic society (Millian intertemporal utility function) $\lambda = 0$, the fraction of non-leisure time devoted to goods production $\bar{u}$ is independent of n.*

It follows that the two first effects imply a negative level effect of population growth. While the negative dilution effect is standard (it is the same behind the negative level effect of population growth in exogenous growth models), the second one is specific to the Lucas-Uzawa class of models. It is important to notice that it is tightly linked to the term $N(t)^{\lambda}$ in the objective function of the optimization problem: a larger population growth increases this term in the objective function whenever $\lambda \neq 0$ featuring a kind of "quantity" bias in the preferences. In the Lucas model, such an increment is not responded by a decrease in "quality" through a drop in the non-leisure time devoted to education, $1 - u$: quality increases as well through this channel at least in the long-run, in contrast to the quantity-quality trade-off usually invoked in overlapping-generations models. As a consequence, an optimal drop in non-leisure time devoted to production occurs in response to an acceleration in population growth, which implies that the second effect, the so-called altruism utility effect, should also yield a negative correlation between long-term output per capita and population growth.

The study of the third causality line or human capital transmission mechanism is much more complicated in that it requires to analyze the term ${}_2\widetilde{F}_1(0)$ and its derivative with respect to n. This appears clearly reflected in the following derivatives

$$\frac{\partial \bar{y}_l}{\partial n} = \bar{y}_l \left[-\frac{\lambda(1-\beta)(\delta+n+\pi-\theta)-\beta((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{-((\delta-\theta)(1-\sigma)+\lambda n-\rho)(1-\beta)(\delta+n+\pi-\theta)} - \frac{\frac{\partial {}_2\widetilde{F}_1(0)}{\partial n}}{{}_2\widetilde{F}_1(0)} \right], \tag{48}$$

$$\frac{\partial \bar{q}_l}{\partial n} = \bar{q}_l \left[-\frac{\lambda(1-\beta)(\delta+n+\pi-\theta)+\beta(1-\beta)\sigma\delta-\beta^2((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{-((\delta-\theta)(1-\sigma)+\lambda n-\rho)\beta(1-\beta)(\delta+n+\pi-\theta)+(1-\beta)^2\sigma\delta(\delta+n+\pi-\theta)} - \frac{\frac{\partial {}_2\widetilde{F}_1(0)}{\partial n}}{{}_2\widetilde{F}_1(0)} \right], \tag{49}$$

$$\frac{\partial \bar{h}_l}{\partial n} = -\frac{h_0}{\left({}_2\widetilde{F}_1(0)\right)^2} \frac{\partial {}_2\widetilde{F}_1(0)}{\partial n} = -\frac{\bar{h}_l}{{}_2\widetilde{F}_1(0)} \frac{\partial {}_2\widetilde{F}_1(0)}{\partial n}. \tag{50}$$

In (48) and (49) the first terms into the brackets are both a combination of the effects transmitted by the **physical capital dilution** mechanism and the **altruism utility** mechanism. As we have seen, they push in the same direction: taken both together partially imply that the detrended long-run levels of per capita income and broad output decrease (increase) face to an increase (decrease) in the rate of population growth. The second term into the brackets is the

[12] Boucekkine and Ruiz-Tamarit (2008) proves that $0 < z_0 < 1$ corresponds with $\frac{1}{N_0}\frac{K_0}{h_0} < \frac{1}{N}\left(\frac{\bar{K}}{h}\right)$ and $z_0 < 0$ corresponds with $\frac{1}{N_0}\frac{K_0}{h_0} > \frac{1}{N}\left(\frac{\bar{K}}{h}\right)$, while if $z_0 = 0$ we have $\frac{1}{N_0}\frac{K_0}{h_0} = \frac{1}{N}\left(\frac{\bar{K}}{h}\right)$. In particular z_0 cannot be equal to unity because the Gaussian hypergeometric function has branch cuts at $z_0 = 1$. We show later the role played by these imbalances in explaining the long-run impacts of population growth and size on the remaining variables.

effect transmitted by the **human capital** mechanism but the knowledge of its sign demands a deeper study.

If we start by studying (50), notice that the derivative of the detrended long-run level of human capital depends on initial conditions, but all the complexity comes from the hypergeometric term ${}_2\widetilde{F}_1(0)$ and its derivative with respect to n. However, one can readily show that the case $\sigma = \beta$ features a long-run human capital level insensitive to any parameter of the model. We refer to it quickly in the next proposition.

Proposition 2 *When $\sigma = \beta$, we have that $\bar{h}_l = h_0$, and the population growth rate has no impact on the (detrended) human capital long-run average level. However, the initial values of the long-run trajectories for per capita narrow (market) and broad (aggregate) output, $\bar{y}_l$ and $\bar{q}_l$, depend negatively on the population growth rate. These results are independent of the degree of altruism assumed for economic agents.*

Proof: The argument simply follows from a peculiar property of hypergeometric functions. In the *exogenous growth case*, for any $\lambda \in [0,1]$ and $z_0 \in]-\infty, 1[$, it happens that $\sigma = \beta$ and the second argument of the involved hypergeometric function becomes zero. This implies the degeneracy property: ${}_2\widetilde{F}_1(0, b=0) \equiv {}_2F_1\left(\widetilde{a}, 0; c; z_0\right) = 1$, independent of n. Then,

$$\frac{\partial_2 \widetilde{F}_1(0, b=0)}{\partial n} = 0 \tag{51}$$

and consequently

$$\frac{\partial \bar{h}_l}{\partial n} = 0, \qquad \frac{\partial \bar{y}_l}{\partial n} < 0, \qquad \frac{\partial \bar{q}_l}{\partial n} < 0. \qquad \blacksquare \tag{52}$$

In the exogenous growth case the human capital effect does not play any role. Hence, we only find the combined negative effect associated with dilution and altruism. Instead, we have a different picture in the empirically relevant case in which $\sigma > \beta$. In such a *normal* case, the hypergeometric function does not degenerate into a constant, and one must compute explicitly its derivative with respect to the population growth rate. This derivative simplifies to[13]

$$\frac{\partial_2 \widetilde{F}_1(0)}{\partial n} = \left(\frac{\partial \widetilde{a}(n)}{\partial n}\right)\left[\frac{\varphi(n)}{\gamma(n)} + \frac{(1-\gamma(n))\,\psi(n)}{\gamma(n)} \frac{\beta h_0}{(1-\beta)\,\epsilon\sigma K_0}\left(\frac{\epsilon}{\delta+n+\pi-\theta}\right)^{\frac{1}{1-\beta}}\right], \tag{53}$$

where

$$\frac{\partial \widetilde{a}(n)}{\partial n} = \frac{\partial a(n)}{\partial n} = \frac{-\beta\lambda(\delta+\pi-\theta) + \beta((\delta-\theta)(1-\sigma)-\rho)}{\sigma(1-\beta)(\delta+n+\pi-\theta)^2} < 0, \tag{54}$$

$$\begin{aligned}\varphi(n) = &\left(1+2\widetilde{a}(n)\right)\int_0^1 t^{\widetilde{a}(n)-1}(1-t)(1-tz_0(n))^{-b}dt \\ &+\left(1+\widetilde{a}(n)\right)\widetilde{a}(n)\int_0^1 \ln t \; t^{\widetilde{a}(n)-1}(1-t)(1-tz_0(n))^{-b}dt,\end{aligned} \tag{55}$$

[13]See the Appendix A.

$$\psi(n)=\left(1+\widetilde{a}(n)\right)(1-z_0(n))^{\frac{1}{1-\beta}}\left[\frac{\lambda-\frac{((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{(1-\beta)(\delta+n+\pi-\theta)}}{\frac{-\lambda\beta(\delta+\pi-\theta)+\beta((\delta-\theta)(1-\sigma)-\rho)}{\sigma(1-\beta)(\delta+n+\pi-\theta)^2}}+\frac{((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{1+\widetilde{a}(n)}\right]$$

$$\cdot\int_0^1\left[1+\frac{((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{\frac{\lambda-\frac{((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{(1-\beta)(\delta+n+\pi-\theta)}}{\frac{-\lambda\beta(\delta+\pi-\theta)+\beta((\delta-\theta)(1-\sigma)-\rho)}{\sigma(1-\beta)(\delta+n+\pi-\theta)^2}}+\frac{((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{1+\widetilde{a}(n)}}\ln t\right]t^{\widetilde{a}(n)}(1-tz_0(n))^{-b}dt, \tag{56}$$

$$\gamma(n)=1-\frac{(1-\beta)\epsilon\sigma K_0}{\phi(n)\beta h_0}\left(\frac{\delta+n+\pi-\theta}{\epsilon}\right)^{\frac{1}{1-\beta}}\frac{b\widetilde{a}(n)}{2+\widetilde{a}(n)}\,{}_2F_1\left(1+\widetilde{a}(n),1+b;3+\widetilde{a}(n);z_0(n)\right), \tag{57}$$

$$\phi(n)=\left(2b+\frac{1}{1-\beta}\right)((\delta-\theta)(1-\sigma)+\lambda n-\rho)(1-z_0(n))^{\frac{\beta}{1-\beta}}\;{}_2F_1\left(1+\widetilde{a}(n),b;2+\widetilde{a}(n);z_0(n)\right)$$

$$+\left[2((\delta-\theta)(1-\sigma)+\lambda n-\rho)(1-z_0(n))^{\frac{\beta}{1-\beta}}-\frac{(1-\beta)\epsilon\sigma K_0}{\beta h_0}\left(\frac{\delta+n+\pi-\theta}{\epsilon}\right)^{\frac{1}{1-\beta}}\widetilde{a}(n)\right]$$

$$\cdot\frac{b}{2+\widetilde{a}(n)}\,{}_2F_1\left(1+\widetilde{a}(n),1+b;3+\widetilde{a}(n);z_0(n)\right). \tag{58}$$

Remark 3 *Equations (58) and (57) directly show that for any $\lambda\in[0,1]$ and $z_0\in\,]-\infty,1[$, when $b>0$ $\phi(n)<0$ and, consequently, $\gamma(n)>1$.*

Next, we shall consider the sign of both $\varphi(n)$ and $\psi(n)$ in the following three Lemmas.

Lemma 1 *For any $\lambda\in[0,1]$ and $b>0$, if $0<z_0<1$ then $\varphi(n)>0$, if $z_0=0$ then $\varphi(n)=0$, whereas if $-\infty<z_0<0$ then $\varphi(n)<0$.*

Lemma 2 *For any $\lambda\in[0,1]$ and $z_0\in[0,1[$, when $b>0$ then $\psi(n)<0$.*

The proofs of these two Lemmas are in Appendix B. The latter result may be locally extended to include trajectories for which $z_0<0$ $\left(\frac{1}{N_0}\frac{K_0}{h_0}>\frac{1}{N}\left(\frac{\bar{K}}{h}\right)\right)$ in the proximity of $z_0=0$. This is so because the limit of the derivatives

$$\frac{\partial\mathbf{\Phi}(n)}{\partial z_0}=b\int_0^1(1+\Delta\ln t)\,t^{\tilde{a}+1}(1-tz_0)^{-b-1}dt$$

and

$$\frac{\partial^2\mathbf{\Phi}(n)}{\partial z_0^2}=b(1+b)\int_0^1(1+\Delta\ln t)\,t^{\tilde{a}+2}(1-tz_0)^{-b-2}dt$$

give us, respectively, the following definite results:

$$\lim_{z_0 \to 0^-} \frac{\partial \mathbf{\Phi}(n)}{\partial z_0} = b \int_0^1 (1 + \Delta \ln t)\, t^{\tilde{a}+1} dt = \frac{b}{\tilde{a}+2} \left(1 - \frac{\tilde{a}+1}{\tilde{a}+2} \frac{Q}{P+Q}\right) > 0 \tag{59}$$

and

$$\lim_{z_0 \to 0^-} \frac{\partial^2 \mathbf{\Phi}(n)}{\partial z_0^2} = b(1+b) \int_0^1 (1 + \Delta \ln t)\, t^{\tilde{a}+2} dt = \frac{b(1+b)}{\tilde{a}+3} \left(1 - \frac{\tilde{a}+1}{\tilde{a}+3} \frac{Q}{P+Q}\right) > 0, \tag{60}$$

which imply that near $z_0 = 0$ the continuous function $\mathbf{\Phi}(n)$ is positive, increasing, and convex.[14]

Lemma 3 *When $b > 0$, if $0 \leqslant z_0 < 1$ then $\frac{\partial_2 \widetilde{F}_1(0)}{\partial n} < 0$ for any $\lambda \in [0,1]$.*

Proof: Take equation (53) where according to (57) and (58) $\gamma(n) > 1$, if we consider the results for $\varphi(n)$ and $\psi(n)$ given in Lemmas 1 and 2, it is apparent that for $b > 0$ and $0 \leqslant z_0 < 1$ the sign of the sum into the brackets is positive. Then, given that according to (54) $\frac{\partial \widetilde{a}(n)}{\partial n} < 0$, we get $\frac{\partial_2 \widetilde{F}_1(0)}{\partial n} < 0$. ■

It's now possible to state the main result of this section.

Proposition 3 *In the normal case, $\sigma > \beta$, when $\frac{1}{N_0}\frac{K_0}{h_0} \leqslant \frac{1}{N}\left(\frac{\bar{K}}{h}\right)$ a greater (lower) rate of population growth implies a greater (lower) detrended long-run average level of human capital. Moreover, this result is independent of the degree of altruism assumed for economic agents.*

Proof: Look at equation (50) and recall the previous Lemma 3. ■

According to this proposition, in the empirically relevant cases and close to the long-run ratio of physical to human capital, population growth has a positive impact on the human capital level. This also means that, in contrast to the case analyzed in Proposition 2, the human capital effect plays here an important role in explaining the whole impact of population growth on the economy's long-run per capita production levels.

Corollary *In the normal case, $\sigma > \beta$, when $\frac{1}{N_0}\frac{K_0}{h_0} \leqslant \frac{1}{N}\left(\frac{\bar{K}}{h}\right)$ the rate of population growth impacts ambiguously on the long-run detrended levels of per capita income and per capita broad output. A greater rate of population growth may result in either a greater or a lower level of per capita production depending on the weights of two opposing forces, one negative associated with a mix of the dilution and altruism effects and the other positive associated with a pure human capital effect. The different degrees of altruism assumed for economic agents do not remove the above ambiguity.*

[14] However, we cannot globally extend the above result because for $-\infty < z_0 < 0$ the term $(1 - tz_0)^{-b}$ is a decreasing function of t, which takes the values $+1$ when $t = 0$ and $1 > \frac{1}{(1+|z_0|)^b} > 0$ when $t = 1$. This allow for an upper bound for $\mathbf{\Phi}_1(n)$ and $\mathbf{\Phi}_2(n)$ such that

$$\mathbf{\Phi}(n) < \frac{\int_0^\chi (1 + \Delta \ln t)\, t^{\tilde{a}} dt + \int_\chi^1 (1 + \Delta \ln t)\, t^{\tilde{a}} dt}{(1 - \chi z_0)^b}.$$

But, as we have shown, the right hand side of this inequality is strictly positive. So, the inequality admits both results $\mathbf{\Phi}(n) > 0$ and $\mathbf{\Phi}(n) < 0$ and, consequently, both $\psi(n) < 0$ and $\psi(n) > 0$ too.

This comes directly from equations (48) and (49), as well as from the previous Proposition 3. Even if we know the individual sign of the physical capital dilution effect, the altruism utility effect, and the human capital effect, it is not always possible to analytically specify which is exactly the sign of the aggregate level effect. Although in the exogenous case the human capital effect is nil and the negative dilution and altruism effects determine the negative total level effect, in the normal case the human capital effect, which is positive, counterbalances the two other negative effects and we cannot elucidate whether the first one more than, less than, or exactly offsets them.

In what follows we want to complement the previous analytical results with the results from a numerical exercise for the *normal* case in which we consider the two possible configurations for initial conditions. Hence, we include either the subcases studied and concluded analytically, the subcases studied analytically but not conclusive or ambiguous, and other not yet studied subcases.

The outcomes supplied under the form of different figures show the behavior of $\bar{h}_l$, $\bar{y}_l$, and $\bar{q}_l$ as n varies from zero to 0.03. According to Caballé and Santos (1993) and Mulligan and Sala-i-Martin (1993) we consider the following benchmark economy: $N_0 = 1$, $A = 1$, $\beta = 0.45$, $\sigma = 1.5$, $\pi = 0.05$, $\theta = 0.02$, $\rho = 0.05$, $\delta = 0.12$; which roughly conforms to the standard empirical evidence. Under this parameterization the long-run physical to human capital ratio varies from 3.64 to 3.28, depending on the value of λ and for the reference value $n = 0.01$.

We first show in Figures 1-4 how the detrended long-run human capital level evolves as the population growth rate continuously increases from $n = 0$ to $n = 0.03$. In these figures the black lines represent the altruistic case, that is $\lambda = 1$, and the grey lines represent the selfish case, that is $\lambda = 0$. When the economy starts below the long-run physical to human capital ratio (Figure 1), population growth rate impacts positively on the detrended long-run level of human capital. This is exactly the result shown in Proposition 3, which does not depend on the assumed degree of altruism. The remaining Figures 2-4 represent cases in which the economy starts above the long-run physical to human capital ratio. First, when the imbalance is relatively small (Figure 2), we find the same positive relationship between $\bar{h}_l$ and n, for any λ. However, as the initial imbalance becomes larger and larger (Figures 3 and 4) such a positive relationship is found only for the lowest values of n, while for higher values of n the sign reverses and, then, the rate of population growth impacts negatively on the detrended long-run level of human capital. This result involving the shape of the curve, which concerns the concavity degree as well as the position of the reversing point, is sensitive to the value of the intertemporal elasticity of substitution and to the altruism parameter value.

Next we focus on the relationship between the rate of population growth n and the detrended long-run levels of the variables per capita income and per capita broad output, $\bar{y}_l$ and $\bar{q}_l$ respectively. In the normal case and when the economy starts below the long-run physical to human capital ratio, as we have seen in the above Corollary, the sign of this relationship remains analytically undetermined. However, our numerical exercise shows (Figures 5 and 7) that, in this case, the positive effect of the *human capital* mechanism is not strong enough to reverse the stronger negative effect of the *physical capital dilution* mechanism, and less even when it is added the other negative effect of the *altruism utility* mechanism. But the numerical exercise goes beyond the case analyzed in the Corollary and also includes the situation in which the economy starts above the long-run physical to human capital ratio. We show (Figures 6 and 8) that, if the initial imbalance is not exaggeratedly large and hence the effect of the *human capital* mechanism is still positive, the impact of the population growth rate on the long-run

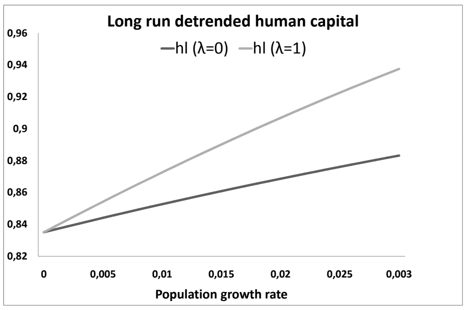

Figure 1: $K_0 = 1$, $h_0 = 1$.

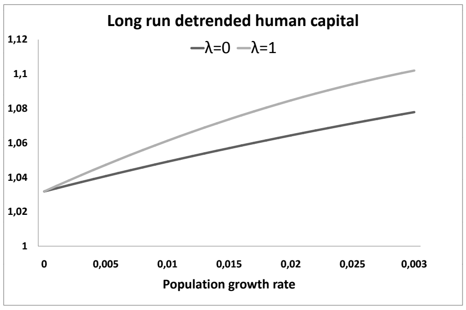

Figure 2: $K_0 = 5$, $h_0 = 1$

levels of per capita income and broad output is also unambiguously negative. The weight of the joint effect of population growth through dilution and altruism overpass the effect of population growth through human capital accumulation.

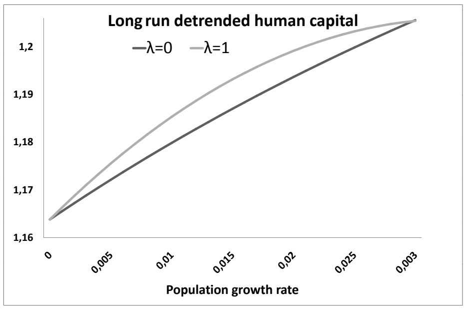

Figure 3: $K_0 = 10$, $h_0 = 1$.

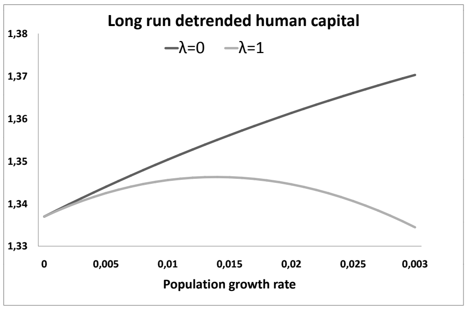

Figure 4: $K_0 = 20$, $h_0 = 1$

Consequently, for all the empirically relevant cases we may conclude that $\bar{y}_l$ and $\bar{q}_l$ decrease (increase) when n increases (decreases). Even more, numerical results show that the negative effect of the *physical capital dilution* mechanism is by itself sufficiently strong to counterbalance the positive effect of the *human capital* mechanism.

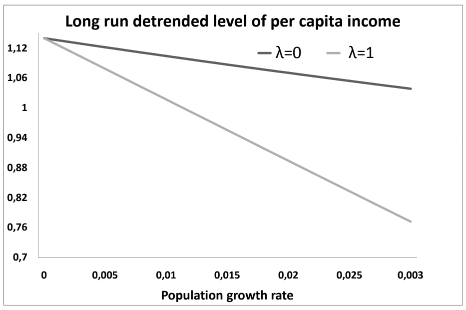

Figure 5: $K_0 = 1$, $h_0 = 1$.

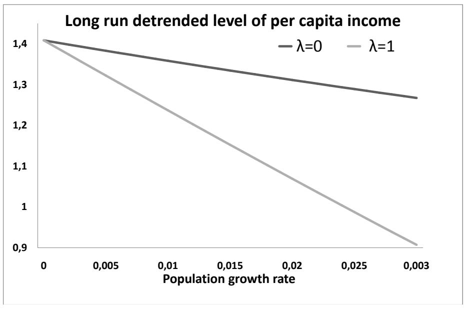

Figure 6: $K_0 = 5$, $h_0 = 1$

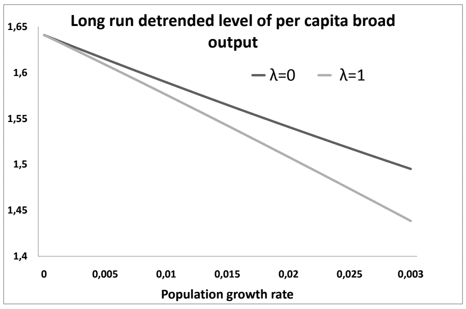

Figure 7: $K_0 = 1$, $h_0 = 1$.

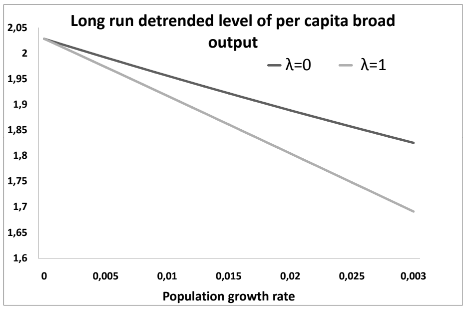

Figure 8: $K_0 = 5$, $h_0 = 1$

6 Population size and the long-run level of the variables

First of all, we find an immediate result concerning the **level effect** of population size which do not need any additional inspection. According to equations (34) and (35) we get

Remark 4 *The long-run level of both the fraction of non-leisure time devoted to goods production and the ratio physical to human capital does not depend on the population size.*

Now, we concentrate on the consequences of population size on the long-run level of the variables per capita income, per capita broad output, and human capital level of a representative worker. The impact of the exogenous detrended population size on these endogenous variables is not a complex issue, but it still depends on the relationship between preferences (σ) and technology (β). The results may be summarized in the following proposition.

Proposition 4 *In the normal (exogenous) [paradoxical] case a greater initial population size implies lower (the same) [greater] long-run detrended levels of per capita income, per capita broad output, and average human capital. This result is independent of the degree of altruism assumed for economic agents, and does not depend on the relationship between* $\frac{1}{N_0}\frac{K_0}{h_0}$ *and* $\frac{1}{N}\left(\frac{\bar{K}}{h}\right)$.

Proof: The proof of this proposition is in Appendix B.

In the study of the consequences of population size on the detrended long-run level of the variables, as it is shown in Appendix B in equations (76)-(78), there is only one causality line associated with the **human capital** mechanism. Consequently, when the detrended long-run level of average human capital decreases (increases) with population size, the detrended long-run levels of per capita income and per capita broad output decrease (increase) too.

Some concluding comments are in order here. First of all, it is important to notice that population size has no impact neither on economic growth rates nor on the long-run levels of economic variables in exogenous growth theory (including the two-sector model of Mankiw-Romer-Weil). Second, things are potentially different in endogenous growth models. For example, it is readily shown that population size reduces per capita income level under an AK production function. However, this is not a general property: for example, Dalgaard and Kreiner (2001) and Bucci (2008) find that per capita income along the balanced growth path is independent of population size. Third and more importantly, our inspection of the scale effect in the Lucas model is rather satisfactory: we find that its growth effect is zero and its level effect is negative. This is consistent with the recent empirical work of Easterly and Kraay (2000) for example.

7 The short-run effects of demographic changes

In this section, we study how demographic changes affect the main economic variables of the model along the transition to the balanced growth path. We examine the consequences of two demographic shocks: changes in the rate of population growth and changes in the initial population size, on the short-run trajectories of physical capital, human capital, income, and broad output. Along the previous sections we have studied the long-run economic effects of demographic changes applying a direct analytical method, complemented with a few numerical exercises when the latter method has led to ambiguous results. As it comes to compute transitional dynamics, and given the markedly increased complexity of the closed-form formulas giving these dynamics relative to those of balanced growth paths (see Appendix C where these formulas are reported), here we only display the outcomes of numerical simulations in the different relevant subcases and for a standard widespread parameterization.

We will focus on the normal case $\sigma > \beta$, and we consider the same benchmark economy as in Section 5, with $N_0 = 1$ and $n = 0.01$. For this parameterization, the long-run physical to human capital ratio is 3.28 when $\lambda = 1$, and 3.64 when $\lambda = 0$. Moreover, we need to fix the initial conditions K_0 and h_0. Proposition 3, its Corollary, and the accompanying numerical exercises show that the initial position of the economy, below or above its long-run physical to human capital ratio, is not crucial for the long-run behavior of the relevant variables when a demographic shock occurs. Of course, it does not mean that the initial position will remain unimportant for short-term dynamics, so we do study the two possible scenarios: first $\frac{1}{N_0}\frac{K_0}{h_0} < \frac{1}{N}\left(\frac{\bar{K}}{h}\right)$, or $0 < z_0 < 1$, in which case we set $K_0 = 1$ and $h_0 = 1$; and second $\frac{1}{N_0}\frac{K_0}{h_0} > \frac{1}{N}\left(\frac{\bar{K}}{h}\right)$, or $z_0 < 0$, in which case we set $K_0 = 10$ and $h_0 = 1$.

More importantly, we compare the outcomes of the benchmark economy where $N_0 = 1$ and $n = 0.01$ with the outcomes of an identical economy except for one of these two demographic parameters. This allows us to consider separately the two above-mentioned demographic shocks:

i) a change in the rate of population growth, setting $N_0 = 1$ and $n' = 0.02$; *ii*) a change in the initial population size, setting $N'_0 = 2$ and $n = 0.01$. Figures 9-12 and Figures 13-16 show both the short-run and the long-run trajectories for either per capita human capital, per capita income, per capita broad output, and aggregate physical capital, when $0 < z_0 < 1$ and $z_0 < 0$ respectively. For each of the variables we provide results corresponding to four different subcases, which arise from a combination of the two extreme values taken by the degree of altruism and the two different demographic changes considered. For each variable, dark lines represent the benchmark values and grey lines represent the new values after the shock. Moreover, solid lines correspond to the short-run trajectories and dashed lines correspond to the long-run trajectories.

In Figures 9-16, the *long-run growth effect* is represented by a slope change from the dark dashed line to the grey dashed line, while the *long-run level effect* is represented by a change of the starting point from the dark dashed line to the grey dashed line. Instead, in the case of the *short-run* trajectories our figures bring together the *growth* and *level effects* caused by demographic shocks. Hence, compared to the solid dark line, the solid grey line that depicts numerical results after the shock reaps a combination of both effects. Moreover, our transitional dynamics study cannot distinguish, for every demographic shock, the particular role played by physical capital dilution, altruism utility, and human capital as transmission mechanisms. Despite these shortcomings, as theory predicts and figures show, the solid dark and grey lines converge to the dashed dark and grey lines, respectively. Consequently, we may focus our effort on the inspection of the dynamic trajectories along the transition, and conclude about the timing of the economic consequences of demographic changes associated with rapid population growth and population size, by comparing the shapes of solid dark and grey lines represented in Figures 9-16.[15]

In the face of a greater initial population size, per capita human capital, per capita income, and per capita broad output patterns are shifted downward. Moreover, although a greater initial population size comes with a greater aggregate physical capital stock, the corresponding per capita physical capital will be lower too. These results hold unambiguously in the short term as well as in the long term regardless of the values of z_0 and λ.

Things are much more involved when the other demographic shock is considered, that's when population growth rate goes up. First, with a greater rate of population growth, the economy has a larger per capita human capital stock either in the short or in the long term regardless of the values of z_0 and λ. Second, with the proviso that $\lambda \neq 0$, in the short-run an economy with a more rapid population growth has lower per capita income and aggregate and per capita physical capital stock, while this picture is reversed in the long-run. Third, depending on the values of parameters z_0 and λ, per capita broad output in an economy with a larger rate of population growth can be either above, below or intersecting (possibly twice) the pattern corresponding to an economy with a slower population growth. In particular, one could find that, for intermediate values of λ and regardless of the values of z_0, as population growth increases per capita broad output is shifted upward in the short-run, goes below the trajectory corresponding to the initial demographic growth rate in the mid-run, and eventually shifts again upward, above the latter trajectory, in the long-run. That's to say, demographic changes can

[15]Note that for per capita income and per capita broad output, the starting values corresponding to two different rates of population growth do not coincide because beyond K_0, N_0, and h_0, $y(0)$ and $q(0)$ also depend on $u(0)$, which is strongly dependent on the value of n. Moreover, in the case of $y(0)$ and $q(0)$ corresponding to two different initial population sizes the above-mentioned differences in the starting values are due, directly, to the different N_0, but also indirectly to the different $u(0)$.

yield sophisticated dynamics even in an apparently simple model à la Lucas-Uzawa, beyond the opposition between short Vs long-term dynamics highlighted by Kelley and Schmidt (2001).

This said, our findings are essentially consistent with the point made in Kelley and Schmidt (2001) about the "complicated time relationships" between economic and demographic changes. In particular, it is usually argued, as mentioned in the introduction, that population growth may have negative economic effects in the short term (due notably to resource scarcity according to the popular stories told) versus positive effects in the long term (through growth effects originating in population growth). The above results show that the Lucas model entails such a story: more precisely, it embodies a negative effect of population growth on per capita income, which dominates in the initial periods of the transition path, and a positive effect which restores a positive correlation between population growth and economic performance, in the subsequent stages of the convergence process towards the long-run equilibrium path. Consequently, we conclude this section with the conviction that timing plays an important role in setting the linkages between demography and economic development.

8 Conclusions

In this paper we have analytically studied the short and long-run impact of two demographic variables (population size and the rate of growth of population) on two kind of economic variables (the rate of growth of the economy and the level of the essential economic indicators) in a growth model based on the accumulation of human capital. In comparison with the related existing literature, three breakthroughs have been achieved: a separate analysis of the level effects of demographic change, an inspection into the level and growth effects of population size in the context of a growing economy driven by human capital accumulation, and the study of the possible "complicated time relationships" between economic development and demographic change through the analysis of transition dynamics.

It goes without saying that many research lines are still open. One is the inclusion of feedback effects from economic growth to population change, which ultimately requires endogenizing demographics. There are several ways to undertake such a task (see for example Boucekkine and Fabbri, 2011, for a quite general one-sector model). However, it is very likely that such a step will destroy the closed-form solutions developed in this paper. Without the latter, the exercise will turn fully computational, disabling any analytical decomposition of the mechanisms at work. A second more valuable line of research is empirical and concerns the development of tools in order to identify level Vs growth effects in the data. Our paper shows that such a distinction is highly relevant.

9 Appendix

A. In this appendix we supply the different expressions which allow to obtain equation (53) by using the implicit function theorem. Consider first the function ${}_2\widetilde{F}_1(0) \equiv {}_2F_1\left(\widetilde{a}, b; c; z_0\right)$ as given by (26). Then,

$$\frac{\partial_2 \widetilde{F}_1(0)}{\partial n} = \left(1+2\widetilde{a}(n)\right)\left(\frac{\partial \widetilde{a}(n)}{\partial n}\right)\int_0^1 t^{\widetilde{a}(n)-1}(1-t)(1-tz_0(n))^{-b}dt$$

$$+\left(1+\widetilde{a}(n)\right)\widetilde{a}(n)\left(\frac{\partial \widetilde{a}(n)}{\partial n}\right)\int_0^1 \ln t\; t^{\widetilde{a}(n)-1}(1-t)(1-tz_0(n))^{-b}dt$$

$$+b\left(1+\widetilde{a}(n)\right)\widetilde{a}(n)\left(\frac{\partial z_0(n)}{\partial n}\right)\int_0^1 t^{\widetilde{a}(n)}(1-t)(1-tz_0(n))^{-b-1}dt =$$

$$= \frac{1+2\widetilde{a}(n)}{\left(1+\widetilde{a}(n)\right)\widetilde{a}(n)}\left(\frac{\partial \widetilde{a}(n)}{\partial n}\right)\ {}_2F_1\left(\widetilde{a}(n), b; 2+\widetilde{a}(n); z_0(n)\right)$$

$$-\frac{1+\widetilde{a}(n)}{\widetilde{a}(n)}\left(\frac{\partial \widetilde{a}(n)}{\partial n}\right)\ {}_3F_2\left(\widetilde{a}(n), \widetilde{a}(n), b; 1+\widetilde{a}(n), 1+\widetilde{a}(n); z_0(n)\right)$$

$$+\frac{\widetilde{a}(n)}{1+\widetilde{a}(n)}\left(\frac{\partial \widetilde{a}(n)}{\partial n}\right)\ {}_3F_2\left(1+\widetilde{a}(n), 1+\widetilde{a}(n), b; 2+\widetilde{a}(n), 2+\widetilde{a}(n); z_0(n)\right)$$

$$+b\frac{\widetilde{a}(n)}{2+\widetilde{a}(n)}\left(\frac{\partial z_0(n)}{\partial n}\right)\ {}_2F_1\left(1+\widetilde{a}(n), 1+b; 3+\widetilde{a}(n); z_0(n)\right) \tag{61}$$

So, we need to know the term $\frac{\partial z_0(n)}{\partial n}$ in view of the complete specification of $\frac{\partial_2 \widetilde{F}_1(0)}{\partial n}$. This knowledge will come from the application of the implicit function theorem. Given the definition of z_0 in (30) and the transversality condition (23) we get the expression $H(z_0, n) = 0$, or

$$(1-z_0)^{\frac{1}{1-\beta}}\ \frac{{}_2F_1(b, 1+\widetilde{a}(n), 2+\widetilde{a}(n); z_0)}{{}_2F_1(b, \widetilde{a}(n), 2+\widetilde{a}(n); z_0)} = -\frac{(1-\beta)\,\epsilon\sigma K_0}{((\delta-\theta)(1-\sigma)+\lambda n-\rho)\,\beta h_0}\left(\frac{\delta+n+\pi-\theta}{\epsilon}\right)^{\frac{1}{1-\beta}}$$

which implicitly defines the function $z_0 = z_0(n)$. Then, according to the implicit function theorem, we know that

$$\frac{\partial z_0(n)}{\partial n} = -\frac{H'_n}{H'_{z_0}} \tag{62}$$

where

$$H'_{z_0} = -\frac{((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{1-\beta}(1-z_0)^{\frac{\beta}{1-\beta}}\ {}_2F_1\left(1+\widetilde{a}(n), b; 2+\widetilde{a}(n); z_0\right)$$

$$+\frac{(1-\beta)\,\epsilon\sigma K_0}{\beta h_0}\left(\frac{\delta+n+\pi-\theta}{\epsilon}\right)^{\frac{1}{1-\beta}}\frac{b\widetilde{a}(n)}{2+\widetilde{a}(n)}\ {}_2F_1\left(1+\widetilde{a}(n), 1+b; 3+\widetilde{a}(n); z_0\right)$$

$$+\left(\left(\delta-\theta\right)\left(1-\sigma\right)+\lambda n-\rho\right)\left(1-z_0\right)^{\frac{1}{1-\beta}}\frac{b\left(1+\widetilde{a}\left(n\right)\right)}{2+\widetilde{a}\left(n\right)}\ {}_2F_1\left(2+\widetilde{a}\left(n\right),1+b;3+\widetilde{a}\left(n\right);z_0\right)$$

and

$$H_n'=\left[\lambda-\frac{\left(\left(\delta-\theta\right)\left(1-\sigma\right)+\lambda n-\rho\right)}{\left(1-\beta\right)\left(\delta+n+\pi-\theta\right)}\right]\left(1-z_0\right)^{\frac{1}{1-\beta}}\ {}_2F_1\left(1+\widetilde{a}\left(n\right),b;2+\widetilde{a}\left(n\right);z_0\right)$$

$$+\frac{\left(1-\beta\right)\epsilon\sigma K_0}{\beta h_0}\left(\frac{\delta+n+\pi-\theta}{\epsilon}\right)^{\frac{1}{1-\beta}}\frac{\partial_2\widetilde{F}_1\left(0\right)}{\partial n}$$

$$+\left(\left(\delta-\theta\right)\left(1-\sigma\right)+\lambda n-\rho\right)\left(1-z_0\right)^{\frac{1}{1-\beta}}\frac{\partial_2F_1\left(0\right)}{\partial n}$$

However, we find a new term, $\frac{\partial_2F_1(0)}{\partial n}$, which is needed for the specification of H_n'. To get it, consider the function ${}_2F_1\left(0\right)\equiv{}_2F_1\left(a,b;c;z_0\right)$ as given by (25). Then,

$$\frac{\partial_2F_1\left(0\right)}{\partial n}=\left(\frac{\partial a\left(n\right)}{\partial n}\right)\int_0^1 t^{a(n)-1}(1-tz_0\left(n\right))^{-b}dt$$

$$+a\left(n\right)\left(\frac{\partial a\left(n\right)}{\partial n}\right)\int_0^1 \ln t\ t^{a(n)-1}(1-tz_0\left(n\right))^{-b}dt$$

$$+ba\left(n\right)\left(\frac{\partial z_0\left(n\right)}{\partial n}\right)\int_0^1 t^{a(n)}(1-tz_0\left(n\right))^{-b-1}dt=$$

$$=\frac{1}{1+\widetilde{a}\left(n\right)}\left(\frac{\partial\widetilde{a}\left(n\right)}{\partial n}\right)\ {}_2F_1\left(1+\widetilde{a}\left(n\right),b;2+\widetilde{a}\left(n\right);z_0\left(n\right)\right)$$

$$-\frac{1}{1+\widetilde{a}\left(n\right)}\left(\frac{\partial\widetilde{a}\left(n\right)}{\partial n}\right)\ {}_3F_2\left(1+\widetilde{a}\left(n\right),1+\widetilde{a}\left(n\right),b;2+\widetilde{a}\left(n\right),2+\widetilde{a}\left(n\right);z_0\left(n\right)\right)$$

$$+b\frac{1+\widetilde{a}\left(n\right)}{2+\widetilde{a}\left(n\right)}\left(\frac{\partial z_0\left(n\right)}{\partial n}\right)\ {}_2F_1\left(2+\widetilde{a}\left(n\right),1+b;3+\widetilde{a}\left(n\right);z_0\left(n\right)\right) \tag{63}$$

Finally, putting all together, rearranging expressions, and gathering common terms, after using some additional algebra, as well as some standard transformations involving hypergeometric functions, we get equation (53).

B. In this appendix, we provide the proofs of the trickiest lemmas and propositions stated along the main text.

Proof of Lemma 1: Rewrite $\varphi\left(n\right)=\int_0^1 I\left(t,z_0\right)dt$ where $I\left(t,z_0\right)=t^{\tilde{a}-1}\left(1-t\right)\left(1-tz_0\right)^{-b}\left(\Omega+\Lambda\ln t\right)$, $\Omega=1+2\tilde{a}$, and $\Lambda=\left(1+\tilde{a}\right)\tilde{a}$.

Taking into account that there exists $0<\chi<1$ such that $I\left(t,z_0\right)$ is negative on the interval $[0,\chi[$ and positive on the interval $]\chi,1]$, we can decompose the integral in two parts $\varphi\left(n\right)=\varphi_1\left(n\right)+\varphi_2\left(n\right)$, where

$$\varphi_1\left(n\right)=\int_0^\chi t^{\tilde{a}-1}\left(1-t\right)\left(1-tz_0\right)^{-b}\left(\Omega+\Lambda\ln t\right)dt<0,$$

$$\varphi_2(n) = \int_{\chi}^{1} t^{\tilde{a}-1}(1-t)(1-tz_0)^{-b}(\Omega+\Lambda\ln t)\,dt > 0.$$

First, note that when $0 < z_0 < 1$ $\left(\frac{1}{N_0}\frac{K_0}{h_0} < \frac{1}{N}\left(\frac{\bar{K}}{h}\right)\right)$ the term $(1-tz_0)^{-b}$ is an increasing function of t. Then we can easily find a lower bound for $\varphi_1(n)$ and $\varphi_2(n)$ such that

$$\varphi(n) > (1-\chi z_0)^{-b}\int_0^{\chi} t^{\tilde{a}-1}(1-t)(\Omega+\Lambda\ln t)\,dt + (1-\chi z_0)^{-b}\int_{\chi}^{1} t^{\tilde{a}-1}(1-t)(\Omega+\Lambda\ln t)\,dt. \tag{64}$$

Now, trivial integration by parts and using that by definition of χ, $\Omega+\Lambda\ln\chi=0$, allow us to get

$$\int_0^{\chi} t^{\tilde{a}-1}(1-t)(\Omega+\Lambda\ln t)\,dt$$
$$= (\Omega+\Lambda\ln\chi)\left(\frac{\chi^{\tilde{a}}}{\tilde{a}} - \frac{\chi^{\tilde{a}+1}}{\tilde{a}+1}\right) - \Lambda\left(\frac{\chi^{\tilde{a}}}{\tilde{a}^2} - \frac{\chi^{\tilde{a}+1}}{(\tilde{a}+1)^2}\right) = \Lambda\left(\frac{\chi^{\tilde{a}+1}}{(\tilde{a}+1)^2} - \frac{\chi^{\tilde{a}}}{\tilde{a}^2}\right)$$

and

$$\int_{\chi}^{1} t^{\tilde{a}-1}(1-t)(\Omega+\Lambda\ln t)\,dt$$
$$= \Omega\left(\frac{1}{\tilde{a}} - \frac{1}{\tilde{a}+1}\right) - \Lambda\left(\frac{1-\chi^{\tilde{a}}}{\tilde{a}^2} - \frac{1-\chi^{\tilde{a}+1}}{(\tilde{a}+1)^2}\right).$$

After some trivial algebra we find

$$\varphi(n)(1-\chi z_0)^{b} > \Omega\left(\frac{1}{\tilde{a}} - \frac{1}{\tilde{a}+1}\right) - \Lambda\left(\frac{1}{\tilde{a}^2} - \frac{1}{(\tilde{a}+1)^2}\right). \tag{65}$$

Given that $\Omega = 1+2\tilde{a}$ and $\Lambda = \tilde{a}(1+\tilde{a})$, it follows that the right hand side of the previous inequality is equal to zero. Consequently, we get $\varphi(n) > 0$.

Second, when $-\infty < z_0 < 0$ $\left(\frac{1}{N_0}\frac{K_0}{h_0} > \frac{1}{N}\left(\frac{\bar{K}}{h}\right)\right)$ the term $(1-tz_0)^{-b}$ is a decreasing function of t. Then, we can find an upper bound for $\varphi_1(n)$ and $\varphi_2(n)$ such that

$$\varphi(n)(1-\chi z_0)^{b} < \int_0^{\chi} t^{\tilde{a}-1}(1-t)(\Omega+\Lambda\ln t)\,dt + \int_{\chi}^{1} t^{\tilde{a}-1}(1-t)(\Omega+\Lambda\ln t)\,dt. \tag{66}$$

The right hand side of the previous inequality is equal to zero. Consequently, we get $\varphi(n) < 0$.

Third, when $z_0 = 0$ $\left(\frac{1}{N_0}\frac{K_0}{h_0} = \frac{1}{N}\left(\frac{\bar{K}}{h}\right)\right)$ we get directly the expression

$$\varphi(n) = \int_0^{\chi} t^{\tilde{a}-1}(1-t)(\Omega+\Lambda\ln t)\,dt + \int_{\chi}^{1} t^{\tilde{a}-1}(1-t)(\Omega+\Lambda\ln t)\,dt, \tag{67}$$

where the right hand side is equal to zero. Hence, we get $\varphi(n) = 0$. ■

Proof of Lemma 2: Rewrite

$$\psi(n) = \left(1+\widetilde{a}\right)(1-z_0)^{\frac{1}{1-\beta}}[P+Q]\,\mathbf{\Phi}(n) \tag{68}$$

where

$$\mathbf{\Phi}(n) = \int_0^1 \Upsilon(t, z_0)\, dt = \int_0^1 (1 + \Delta \ln t)\, t^{\tilde{a}} (1 - tz_0)^{-b}\, dt \tag{69}$$

$$P = \frac{\lambda - \frac{((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{(1-\beta)(\delta+n+\pi-\theta)}}{\frac{-\lambda\beta(\delta+\pi-\theta)+\beta((\delta-\theta)(1-\sigma)-\rho)}{\sigma(1-\beta)(\delta+n+\pi-\theta)^2}} < 0 \tag{70}$$

$$Q = \frac{((\delta-\theta)(1-\sigma)+\lambda n-\rho)}{1+\tilde{a}} < 0 \tag{71}$$

$$\Delta = \frac{Q\left(1+\tilde{a}\right)}{P+Q} > 0. \tag{72}$$

Given that $z_0 < 1$, these equations also imply that $sign\ \psi(n) = -sign\ \mathbf{\Phi}(n)$.

Taking into account that it exists $0 < \chi < 1$ such that $\Upsilon(t, z_0)$ is negative on the interval $[0, \chi[$ and positive on the interval $]\chi, 1]$, we can decompose the integral in two parts $\mathbf{\Phi}(n) = \mathbf{\Phi}_1(n) + \mathbf{\Phi}_2(n)$, where

$$\mathbf{\Phi}_1(n) = \int_0^\chi (1 + \Delta \ln t)\, t^{\tilde{a}} (1 - tz_0)^{-b}\, dt < 0,$$

$$\mathbf{\Phi}_2(n) = \int_\chi^1 (1 + \Delta \ln t)\, t^{\tilde{a}} (1 - tz_0)^{-b}\, dt > 0.$$

First, note that when $0 < z_0 < 1$ $\left(\frac{1}{N_0}\frac{K_0}{h_0} < \frac{1}{N}\left(\frac{\bar{K}}{h}\right)\right)$ the term $(1 - tz_0)^{-b}$ is an increasing function of t. Then we can easily find a lower bound for $\mathbf{\Phi}_1(n)$ and $\mathbf{\Phi}_2(n)$ such that

$$\mathbf{\Phi}(n) > (1 - \chi z_0)^{-b} \int_0^\chi (1 + \Delta \ln t)\, t^{\tilde{a}} dt + (1 - \chi z_0)^{-b} \int_\chi^1 (1 + \Delta \ln t)\, t^{\tilde{a}} dt. \tag{73}$$

Now, trivial integration by parts and using that by definition of χ, $1 + \Delta \ln \chi = 0$, give us

$$\int_0^\chi (1 + \Delta \ln t)\, t^{\tilde{a}} dt = -\frac{\Delta}{(1+\tilde{a})^2} \chi^{\tilde{a}+1} < 0$$

and

$$\int_\chi^1 (1 + \Delta \ln t)\, t^{\tilde{a}} dt = \frac{1}{1+\tilde{a}} - \frac{\Delta}{(1+\tilde{a})^2} + \frac{\Delta}{(1+\tilde{a})^2} \chi^{\tilde{a}+1} > 0.$$

After some trivial algebra we find

$$\mathbf{\Phi}(n)(1 - \chi z_0)^b > \frac{1}{1+\tilde{a}} \frac{P}{P+Q} > 0. \tag{74}$$

Given that the right hand side of the previous inequality is strictly positive, we get $\mathbf{\Phi}(n) > 0$ and, consequently, $\psi(n) < 0$.

Second, when $z_0 = 0$ $\left(\frac{1}{N_0}\frac{K_0}{h_0} = \frac{1}{N}\left(\frac{\bar{K}}{h}\right)\right)$, using some of the previous calculus we get

$$\mathbf{\Phi}(n, z_0 = 0) = \int_0^1 (1 + \Delta \ln t)\, t^{\tilde{a}} dt = \frac{1}{1+\tilde{a}} \frac{P}{P+Q} > 0, \tag{75}$$

hence we obtain $\psi(n, z_0 = 0) = P < 0$. ■

Proof of Proposition 4: From equations (42), (43), and (44), given (24), (30), and (26), we get the following derivatives

$$\frac{\partial \bar{y}_l}{\partial N_0} = A\left(\frac{\beta A}{\delta + n + \pi - \theta}\right)^{\frac{\beta}{1-\beta}}\left(\frac{-((\delta - \theta)(1 - \sigma) + \lambda n - \rho)}{\sigma\delta}\right)\frac{\partial \bar{h}_l}{\partial N_0}, \tag{76}$$

$$\frac{\partial \bar{q}_l}{\partial N_0} = A\left(\frac{\beta A}{\delta + n + \pi - \theta}\right)^{\frac{\beta}{1-\beta}}\left(1 - \frac{\beta(\delta + \lambda n - \theta - \rho + \sigma\theta)}{\sigma\delta}\right)\frac{\partial \bar{h}_l}{\partial N_0}, \tag{77}$$

$$\frac{\partial \bar{h}_l}{\partial N_0} = -\frac{h_0}{\left({}_2\widetilde{F}_1(0)\right)^2}\frac{\partial_2\widetilde{F}_1(0)}{\partial N_0} = -\frac{\bar{h}_l}{{}_2\widetilde{F}_1(0)}\frac{\partial_2\widetilde{F}_1(0)}{\partial N_0}, \tag{78}$$

$$\frac{\partial_2\widetilde{F}_1(0)}{\partial N_0} = \frac{\partial_2F_1\left(\widetilde{a}, b; 2 + \widetilde{a}; z_0(N_0)\right)}{\partial N_0} = \frac{\partial_2F_1\left(\widetilde{a}, b; 2 + \widetilde{a}; z_0(N_0)\right)}{\partial z_0}\frac{\partial z_0(N_0)}{\partial N_0}, \tag{79}$$

where

$$\frac{\partial_2F_1\left(\widetilde{a}, b; 2 + \widetilde{a}; z_0(N_0)\right)}{\partial z_0} = b\frac{\widetilde{a}}{2 + \widetilde{a}}\,{}_2F_1\left(1 + \widetilde{a}, 1 + b; 3 + \widetilde{a}; z_0(N_0)\right). \tag{80}$$

The knowledge of the term $\frac{\partial z_0(N_0)}{\partial N_0}$ requires additional calculus. Given the two definitions (30) and (24), and the transversality condition (23) we get the expression $H(z_0, N_0) = 0$, or

$$\frac{\delta + n + \pi - \theta}{\beta A(1 - z_0)}\left(\frac{\delta\sigma K_0}{-((\delta - \theta)(1 - \sigma) + \lambda n - \rho)h_0}\right)^{1-\beta}\left(\frac{{}_2F_1\left(\widetilde{a}, b; 2 + \widetilde{a}; z_0\right)}{{}_2F_1\left(1 + \widetilde{a}, b; 2 + \widetilde{a}; z_0\right)}\right)^{1-\beta} = N_0^{1-\beta} \tag{81}$$

which implicitly defines the function $z_0 = z_0(N_0)$. Then, according to the implicit function theorem, we know that

$$\frac{\partial z_0(N_0)}{\partial N_0} = -\frac{H'_{N_0}}{H'_{z_0}}, \tag{82}$$

where

$$H'_{N_0} = -\frac{(1 - \beta)}{N_0^{\beta}} \tag{83}$$

and

$$H'_{z_0} = \frac{N_0^{1-\beta}}{1 - z_0}\left(1 + \frac{b(1-z_0)(1-\beta)}{2+\widetilde{a}}\left[\widetilde{a}\frac{{}_2F_1\left(1+\widetilde{a}, 1+b; 3+\widetilde{a}; z_0\right)}{{}_2F_1\left(\widetilde{a}, b; 2+\widetilde{a}; z_0\right)} - \left(1 + \widetilde{a}\right)\frac{{}_2F_1\left(2+\widetilde{a}, 1+b; 3+\widetilde{a}; z_0\right)}{{}_2F_1\left(1+\widetilde{a}, b; 2+\widetilde{a}; z_0\right)}\right]\right). \tag{84}$$

Consequently,

$$\frac{\partial z_0(N_0)}{\partial N_0} = \frac{(1 - z_0)(1 - \beta)}{N_0\left(1 - (1 - z_0)(1 - \beta)\frac{\partial}{\partial z_0}\ln\left(\frac{{}_2F_1\left(1+\widetilde{a}, b; 2+\widetilde{a}; z_0\right)}{{}_2F_1\left(\widetilde{a}, b; 2+\widetilde{a}; z_0\right)}\right)\right)}. \tag{85}$$

Putting all together, we get

$$\frac{\partial_2\widetilde{F}_1(0)}{\partial N_0} = \frac{b(1-z_0)(1-\beta)\ {}_2F_1\left(1+\widetilde{a},1+b;3+\widetilde{a};z_0\right)}{\left(1-(1-z_0)(1-\beta)\frac{\partial}{\partial z_0}\ln\left(\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};z_0\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};z_0\right)}\right)\right)N_0}\frac{\widetilde{a}}{2+\widetilde{a}}. \tag{86}$$

In the normal case ($\sigma > \beta$), when $b > 0$ we find that $\frac{\partial_2\widetilde{F}_1(0)}{\partial N_0} > 0$ because

$$1-(1-z_0)(1-\beta)\frac{\partial}{\partial z_0}\ln\left(\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};z_0\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};z_0\right)}\right) > 0 \qquad \forall\ \mathbf{z_0} < \mathbf{1}. \tag{87}$$

This means that the positive sign of the above expression does not change depending on whether $0 < z_0 < 1$ or $z_0 < 0$.

Consider first the case in which $0 < z_0 < 1$. That is, $\Omega \equiv (1-z_0)^{\frac{1}{1-\beta}}\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};z_0\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};z_0\right)} < 1$, and then $\frac{1}{N_0}\frac{K_0}{h_0} < \frac{1}{N}\left(\frac{\bar{K}}{h}\right)$. Consequently

$$\left(\frac{1}{1-z_0}\right)^{\frac{1}{1-\beta}} > \frac{{}_2F_1(0)}{{}_2\widetilde{F}_1(0)}.$$

Given that the logarithmic function is monotonically increasing, taking logarithms we get

$$\frac{1}{1-\beta}\ln\frac{1}{1-z_0} > \ln\left(\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};z_0\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};z_0\right)}\right).$$

This is equivalent to

$$\int_{\mathbf{0}}^{\mathbf{z_0}}\left[\frac{1}{(1-\beta)}\frac{1}{(1-x)} - \frac{\partial}{\partial x}\ln\left(\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};x\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};x\right)}\right)\right]dx > 0. \tag{88}$$

Then, using the monotonicity property of the definite integral, we get

$$\frac{1}{(1-\beta)(1-z_0)} - \frac{\partial}{\partial z_0}\ln\left(\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};z_0\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};z_0\right)}\right) > 0 \qquad \forall\ \mathbf{0} < \mathbf{z_0} < \mathbf{1},$$

which leads to (87).

Consider now the case in which $z_0 < 0$. That is, $\Omega \equiv (1-z_0)^{\frac{1}{1-\beta}}\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};z_0\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};z_0\right)} > 1$, and then $\frac{1}{N_0}\frac{K_0}{h_0} > \frac{1}{N}\left(\frac{\bar{K}}{h}\right)$. Consequently

$$\left(\frac{1}{1-z_0}\right)^{\frac{1}{1-\beta}} < \frac{{}_2F_1(0)}{{}_2\widetilde{F}_1(0)}.$$

Taking logarithms in both sides we get

$$\frac{1}{1-\beta}\ln\frac{1}{1-z_0} < \ln\left(\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};z_0\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};z_0\right)}\right),$$

which is equivalent to

$$\int_0^{z_0} \frac{1}{(1-\beta)}\frac{1}{(1-x)}dx < \int_0^{z_0} \frac{\partial}{\partial x}\ln\left(\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};x\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};x\right)}\right)dx.$$

Changing the order of the integration limits we get

$$-\int_{\mathbf{z_0}}^{\mathbf{0}}\left[\frac{1}{(1-\beta)}\frac{1}{(1-x)} - \frac{\partial}{\partial x}\ln\left(\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};x\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};x\right)}\right)\right]dx < 0. \tag{89}$$

Then, the monotonicity property of the definite integral applies and we get

$$\frac{1}{(1-\beta)(1-z_0)} - \frac{\partial}{\partial z_0}\ln\left(\frac{{}_2F_1\left(1+\widetilde{a},b;2+\widetilde{a};z_0\right)}{{}_2F_1\left(\widetilde{a},b;2+\widetilde{a};z_0\right)}\right) > 0 \qquad \forall\ \mathbf{z_0}<\mathbf{0},$$

which also leads to (87).

Extending results to the paradoxical case ($\sigma < \beta$) in which $b < 0$, is immediate. Moreover, the exogenous case ($\sigma = \beta$) in which $b = 0$ is obvious given that ${}_2F_1\left(\widetilde{a}, 0; 2+\widetilde{a}; z_0\right) = 1$ and $\frac{\partial {}_2\widetilde{F}_1(0,b=0)}{\partial N_0} = 0$. ■

C. In this appendix we report the short-run closed-form trajectories corresponding to the variables of the model on which we have focused the transitional dynamics study, making explicit its dependence on the demographic parameters. To get the exact expressions we use the hypergeometric functions

$${}_2F_1(t) \equiv {}_2F_1\left(a,b;c;z\left(t\right)\right) \tag{90}$$

and

$${}_2\widetilde{F}_1(t) \equiv {}_2F_1\left(\widetilde{a},b;c;z\left(t\right)\right), \tag{91}$$

being

$$z\left(t\right) = \left(1 - \frac{\delta+n+\pi-\theta}{\epsilon}\left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{-\frac{1-\beta}{\beta}}\right)\exp\left\{-\frac{(1-\beta)(\delta+n+\pi-\theta)}{\beta}t\right\}, \tag{92}$$

and where the remaining parameters have been defined along the previous sections.

(i) Aggregate physical capital stock

$$\begin{aligned} K = &-\frac{\sigma\beta\left(\frac{(1-\beta)A}{\delta\vartheta_2(0)}\right)^{\frac{1}{\sigma}}\left(\frac{\beta A}{\delta+n+\pi-\theta}\right)^{\frac{\beta}{\sigma(1-\beta)}} N_0^{\frac{\sigma+\lambda}{\sigma}}}{(\delta+n+\pi-\theta)(\beta-\sigma)-\beta\left(\rho+\pi-n\left(\sigma+\lambda-1\right)-\pi\sigma\right)} \\ &\cdot {}_2F_1(t)\exp\left\{\frac{(\delta+n+\pi-\theta)(\beta-\sigma)-\beta\left(\rho+\pi-n\left(\sigma+\lambda-1\right)\right)}{\beta\sigma}t\right\} \\ &\cdot\left[-1+\exp\left\{\frac{(1-\beta)(\delta+n+\pi-\theta)}{\beta}t\right\}+\frac{\delta+n+\pi-\theta}{\epsilon}\left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{-\frac{1-\beta}{\beta}}\right]^{\frac{1}{1-\beta}}; \end{aligned} \tag{93}$$

(ii) Average level of the human capital stock

$$h = h_0 \frac{{}_2\widetilde{F}_1(t)}{{}_2\widetilde{F}_1(0)} \exp\left\{ \frac{\delta + \lambda n - \theta - \rho}{\sigma} t \right\}; \tag{94}$$

(iii) The shadow prices ratio

$$N\left(\frac{\vartheta_1}{\vartheta_2}\right) = \frac{\delta}{(1-\beta)A} \left(\frac{\delta + n + \pi - \theta}{\beta A}\right)^{\frac{\beta}{1-\beta}} \exp\left\{(\delta + n + \pi - \theta)\, t\right\}$$

$$\cdot \left[-1 + \exp\left\{ \frac{(1-\beta)(\delta + n + \pi - \theta)}{\beta} t \right\} + \frac{\delta + n + \pi - \theta}{\epsilon} \left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{-\frac{1-\beta}{\beta}} \right]^{-\frac{\beta}{1-\beta}}; \tag{95}$$

(iv) The flow of per capita narrow (market) output

$$y = y(0) \frac{\vartheta_1(0)}{\vartheta_2(0)} \left(\frac{\epsilon}{\delta + n + \pi - \theta}\right)^{\frac{\beta}{1-\beta}} \frac{{}_2F_1(t)}{{}_2F_1(0)} \exp\left\{ \frac{\delta + \lambda n - \theta - \rho - \sigma(\delta + n + \pi - \theta)}{\sigma} t \right\}$$

$$\cdot \left[-1 + \exp\left\{ \frac{(1-\beta)(\delta + n + \pi - \theta)}{\beta} t \right\} + \frac{\delta + n + \pi - \theta}{\epsilon} \left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{-\frac{1-\beta}{\beta}} \right]^{\frac{\beta}{1-\beta}}, \tag{96}$$

where $y(0) = \frac{A^{\frac{1}{\beta}} K_0}{N_0} \left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{\frac{1-\beta}{\beta}} \left(\frac{(1-\beta)N_0}{\delta}\right)^{\frac{1-\beta}{\beta}}$;
(v) The flow of per capita broad (aggregate) output

$$q = \left[q(0) \frac{\vartheta_1(0)}{\vartheta_2(0)} \frac{{}_2F_1(t)}{{}_2F_1(0)} + \frac{\delta h_0}{N_0} \left(\frac{{}_2\widetilde{F}_1(t)}{{}_2\widetilde{F}_1(0)} - \frac{{}_2F_1(t)}{{}_2F_1(0)} \right) \right]$$

$$\cdot \left(\frac{\epsilon}{\delta + n + \pi - \theta}\right)^{\frac{\beta}{1-\beta}} \exp\left\{ \frac{\delta + \lambda n - \theta - \rho - \sigma(\delta + n + \pi - \theta)}{\sigma} t \right\}$$

$$\cdot \left[-1 + \exp\left\{ \frac{(1-\beta)(\delta + n + \pi - \theta)}{\beta} t \right\} + \frac{\delta + n + \pi - \theta}{\epsilon} \left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{-\frac{1-\beta}{\beta}} \right]^{\frac{\beta}{1-\beta}}, \tag{97}$$

where $q(0) = y(0) + \frac{\delta h_0}{N_0} \frac{\vartheta_2(0)}{\vartheta_1(0)} \left(1 + \frac{(\delta-\theta)(1-\sigma) + \lambda n - \rho}{\sigma\delta} \frac{{}_2F_1(0)}{{}_2\widetilde{F}_1(0)}\right)$.

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Figure 9. Per capita human capital

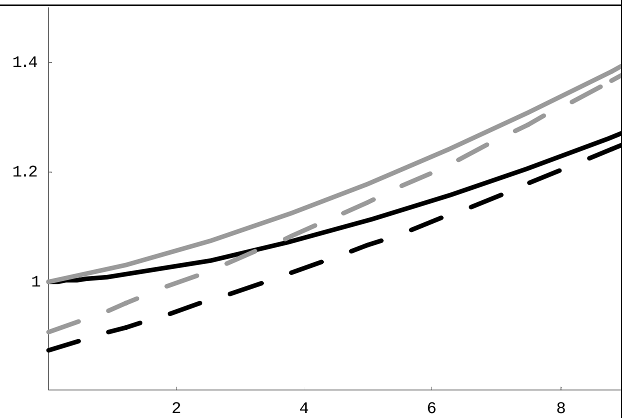
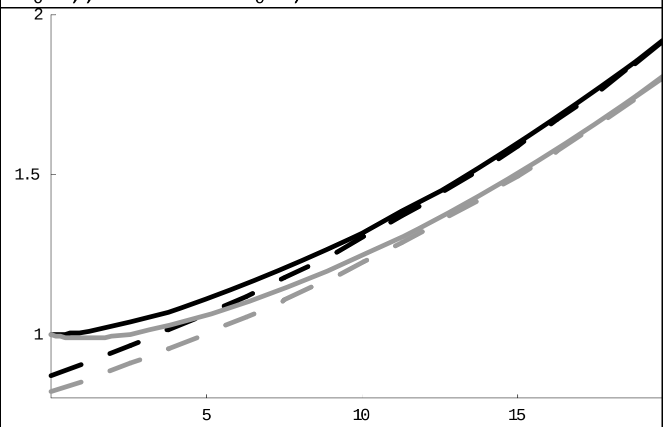
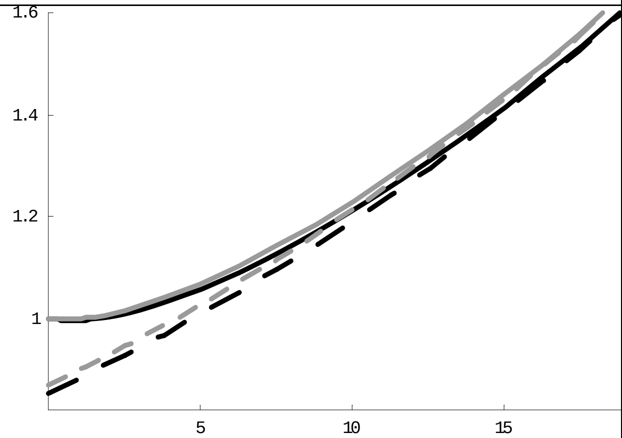
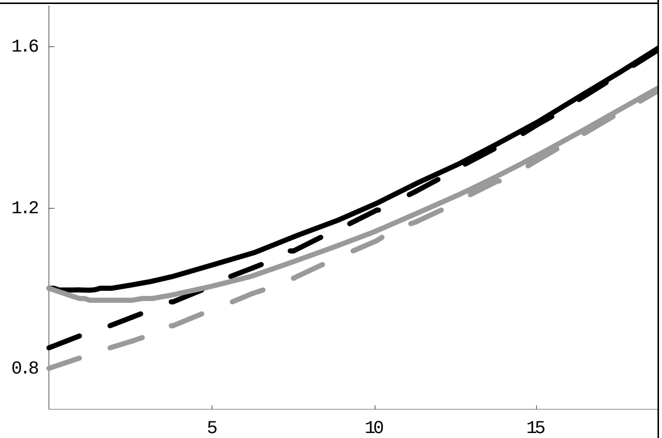
	K_0=1, h_0=1 Population growth rate change N_0=1, n =0.01 N_0=1, n'=0.02	K_0=1, h_0=1 Change in population size at *t=0* N_0=1, , n =0.01 N'_0=2, n=0.01
λ =1 0<z_0<1		
λ =0 0<z_0<1		

Figure 10. Per capita income

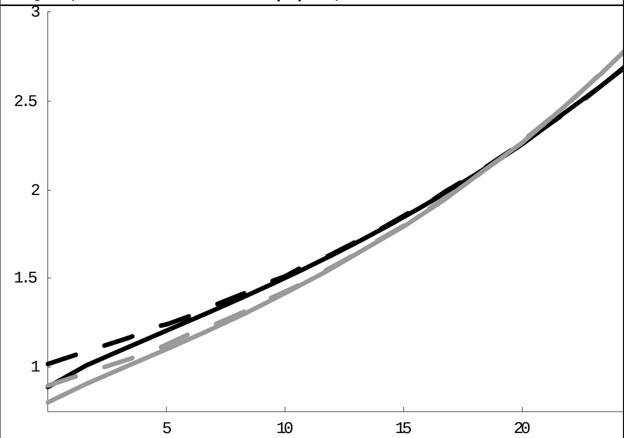
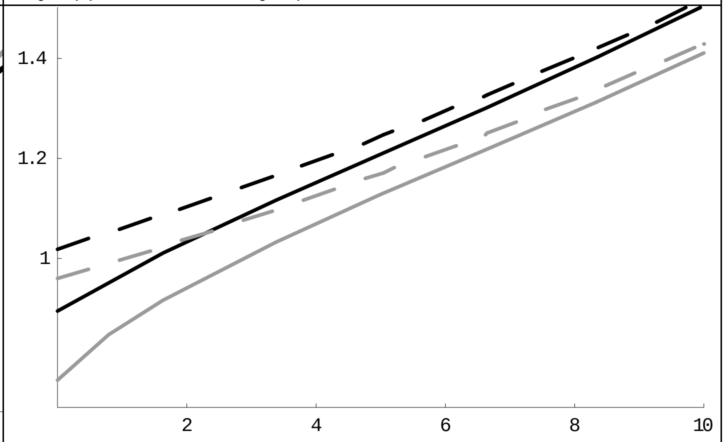
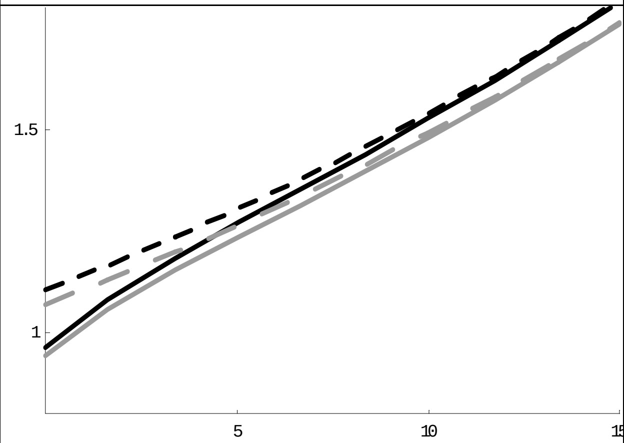
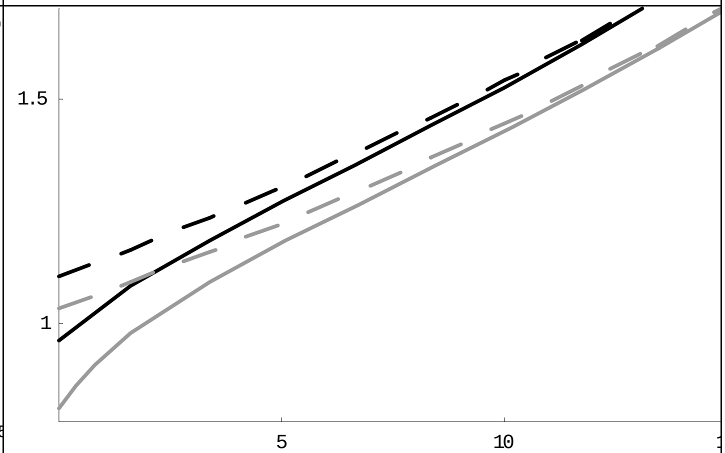
	K_0=1, h_0=1 Population growth rate change N_0=1, n =0.01 N(0)=1, n'=0.02	K_0=1, h_0=1 Change in population size at *t=0* N_0=1, , n =0.01 N'_0=2, n=0.01
λ =1 0<z_0<1		
λ =0 0<z_0<1		

Figure 11. Per capita broad output

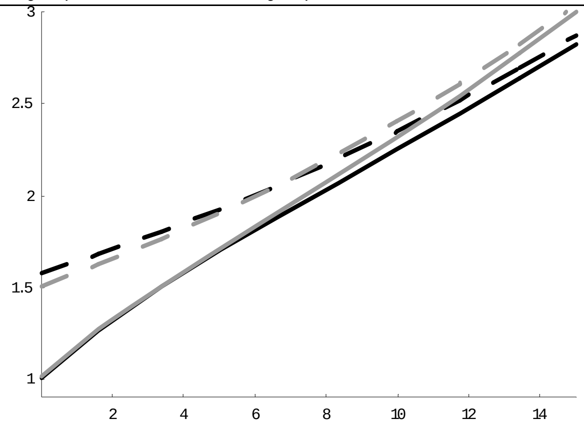
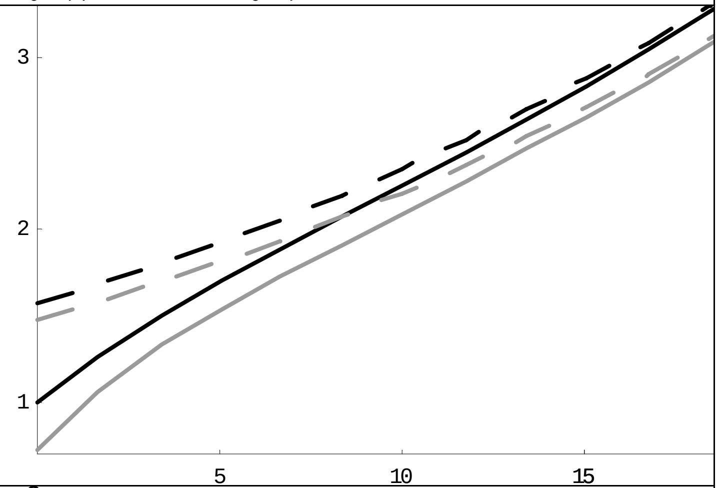
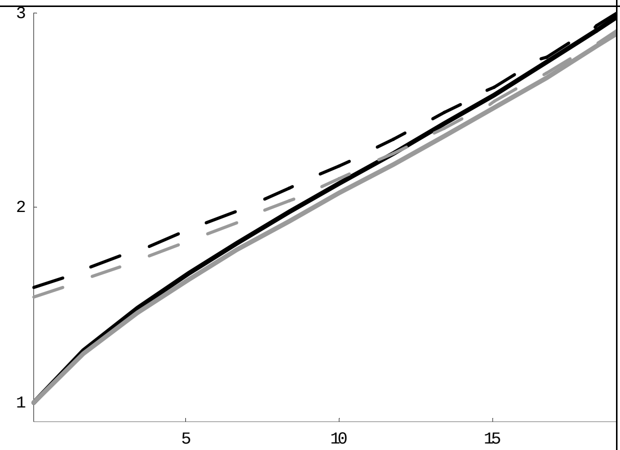
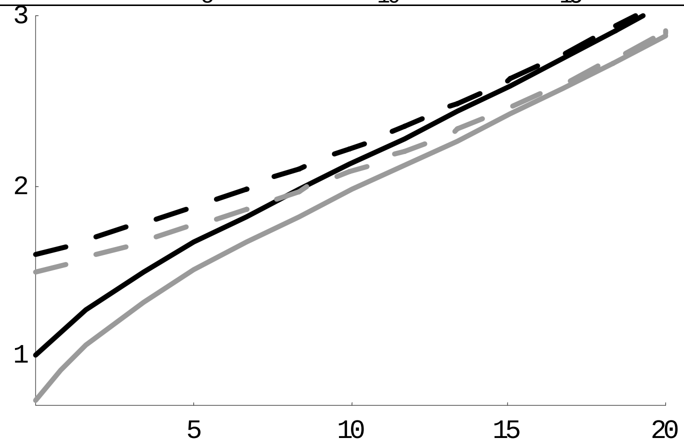
	$K_0=1$, $h_0=1$ Population growth rate change $N_0=1$, $n=0.01$ $N_0=1$, $n'=0.02$	$K_0=1$, $h_0=1$ Change in population size at *t=0* $N_0=1$, , $n=0.01$ $N'_0=2$, $n=0.01$
$\lambda=1$ $0<z_0<1$		
$\lambda=0$ $0<z_0<1$		

Figure 12. Aggregate physical capital

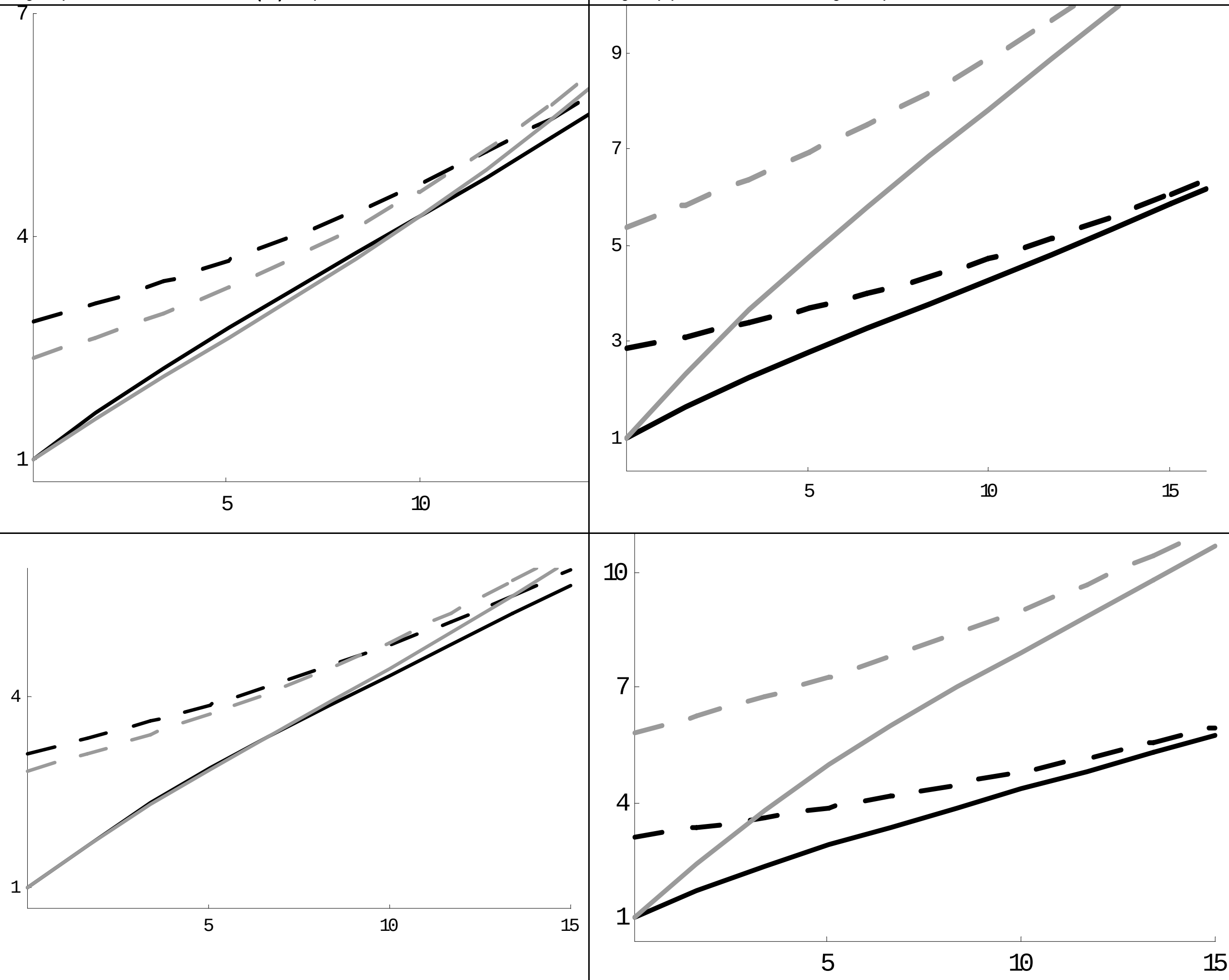
	K_0=1, h_0=1 Population growth rate change N_0=1, n =0.01 N(0)=1, n'=0.02	K_0=1, h_0=1 Change in population size at *t=0* N_0=1, , n =0.01 N'_0==2, n=0.01
λ =1 0<z_0<1		
λ =0 0<z_0<1		

Figure 12b. Per capita physical capital

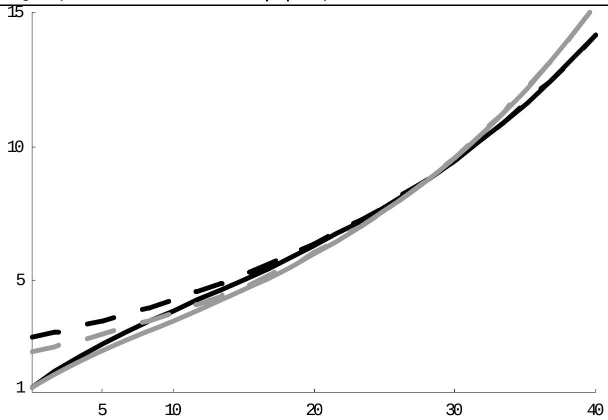
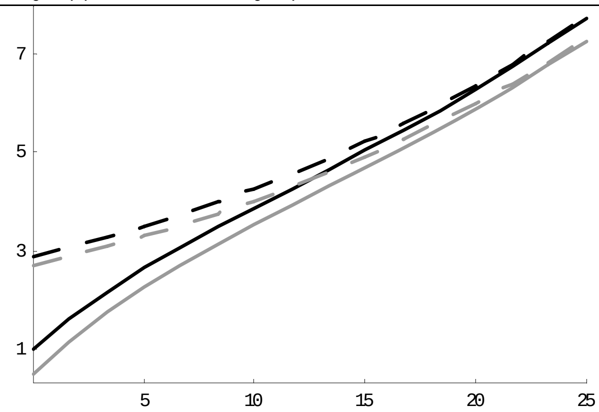
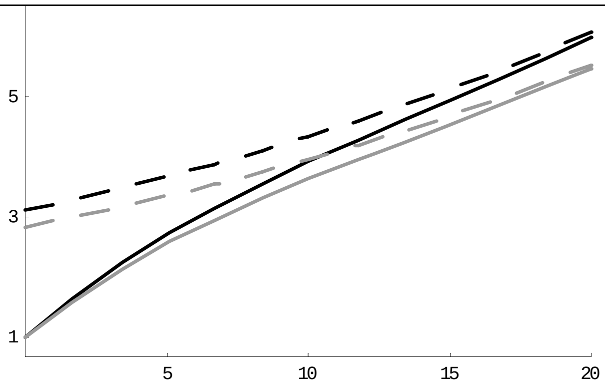
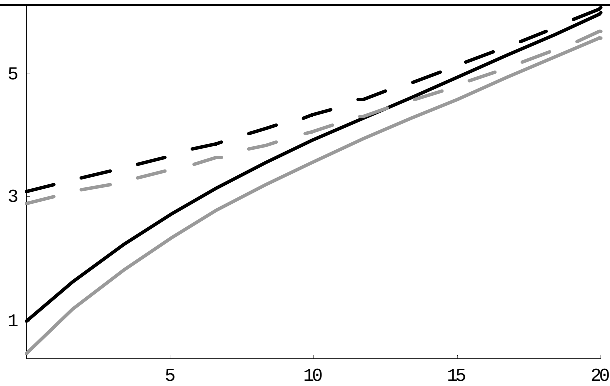
	K_0=1, h_0=1 Population growth rate change N_0=1, n =0.01 N(0)=1, n'=0.02	K_0=1, h_0=1 Change in population size at *t=0* N_0=1, , n =0.01 N'_0=2, n=0.01
λ =1 0<z_0<1		
λ =0 0<z_0<1		

Figure 13. Per capita human capital

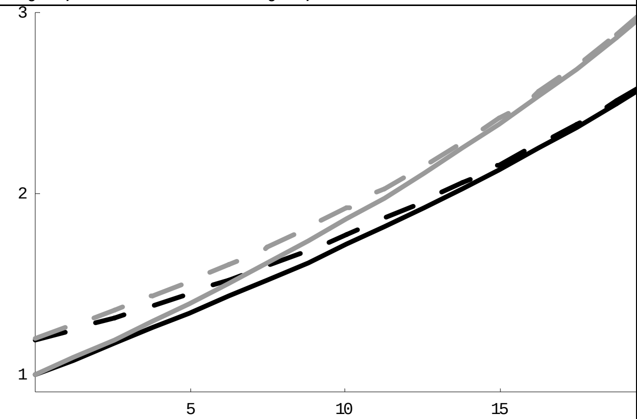
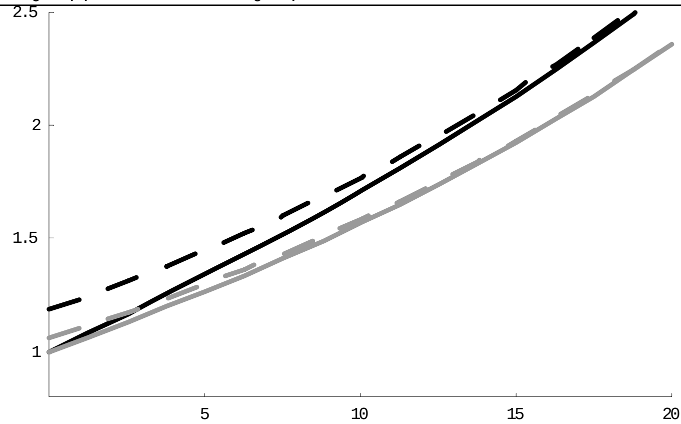
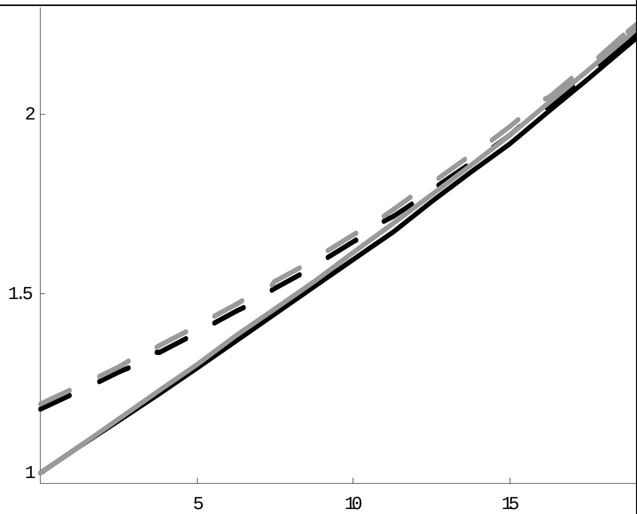
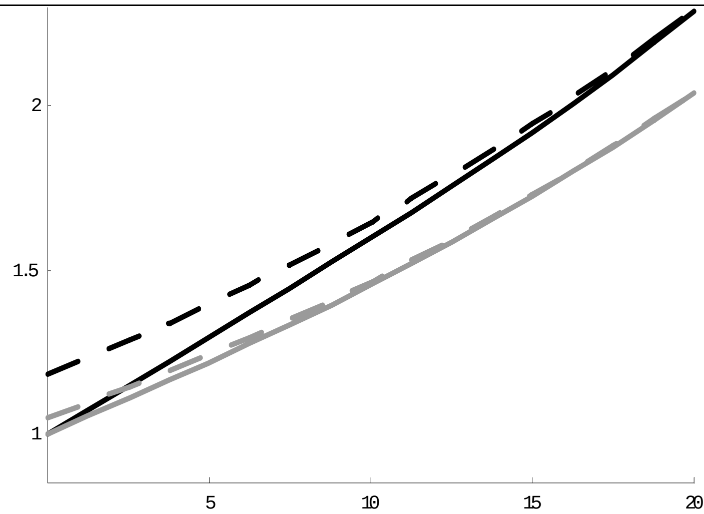
	K_0=10, h_0=1 Population growth rate change N_0=1, n =0.01 N_0=1, n'=0.02	K_0=10, h_0=1 Change in population size at *t=0* N_0=1, , n =0.01 N'_0=2, n=0.01
λ =1 z_0<0		
λ =0 z_0<0		

Figure 14. Per capita income

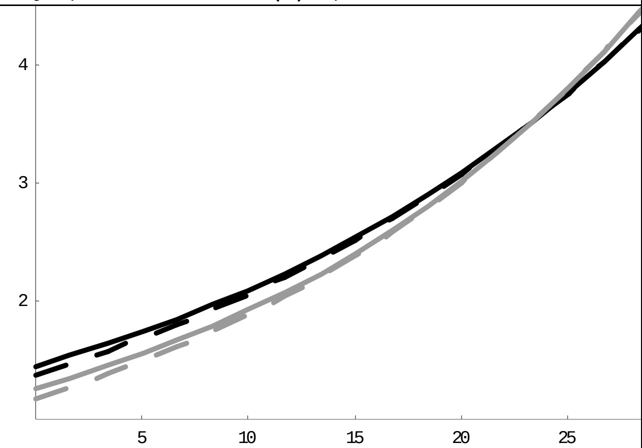
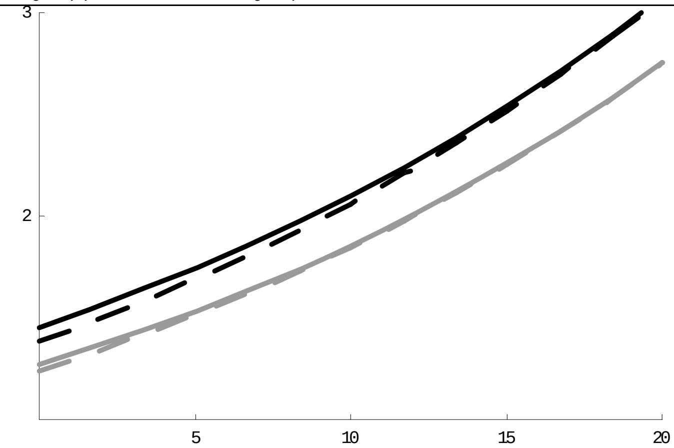
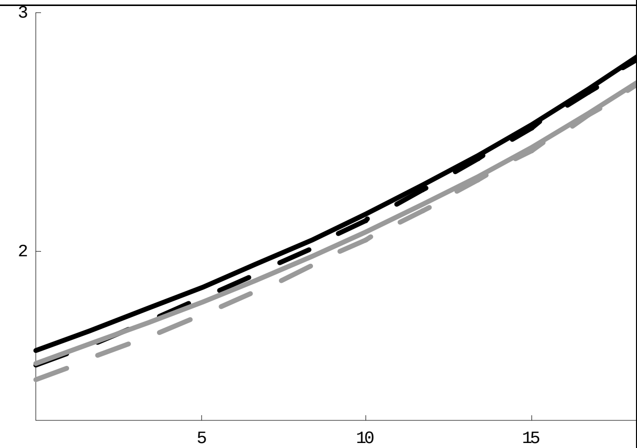
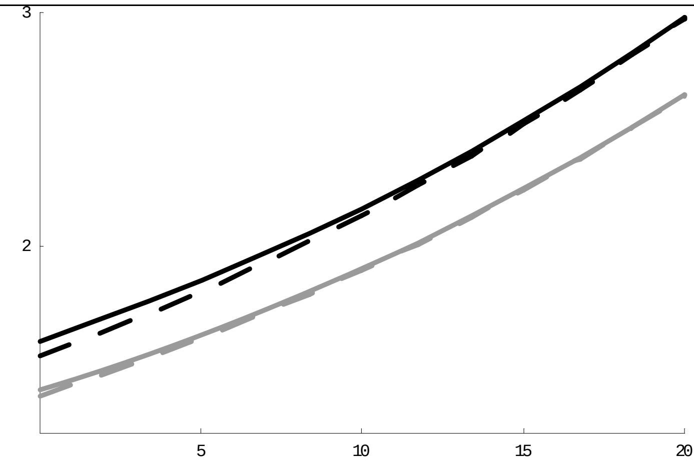
	K_0=10, h_0=1 Population growth rate change N_0=1, n =0.01 N(0)=1, n'=0.02	K_0=10, h_0=1 Change in population size at *t=0* N_0=1, , n =0.01 N'$_0$=2, n=0.01
λ =1 z_0<0		
λ =0 z_0<0		

Figure 15. Per capita broad output

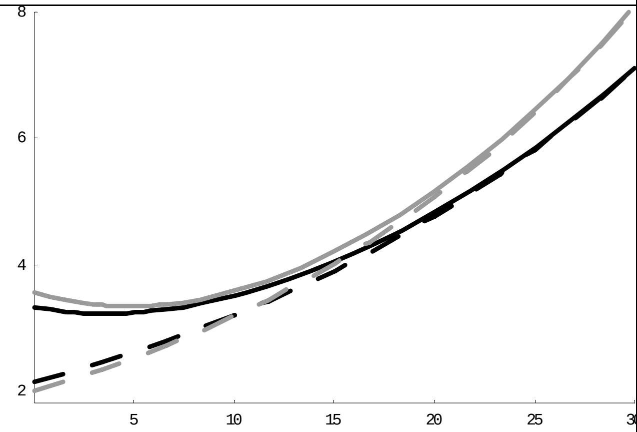
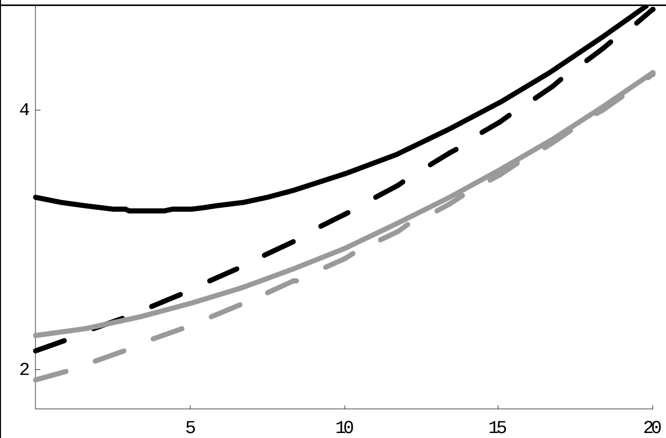
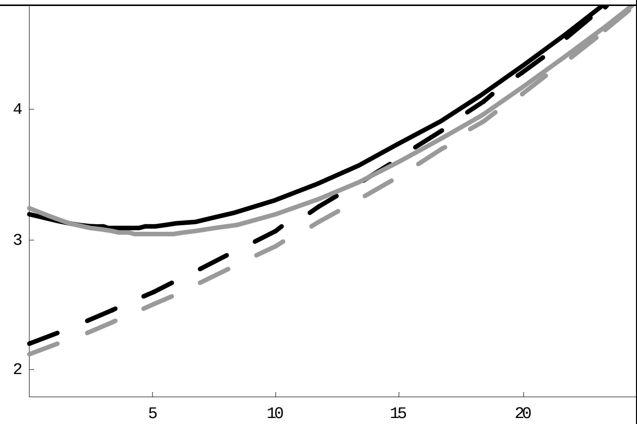
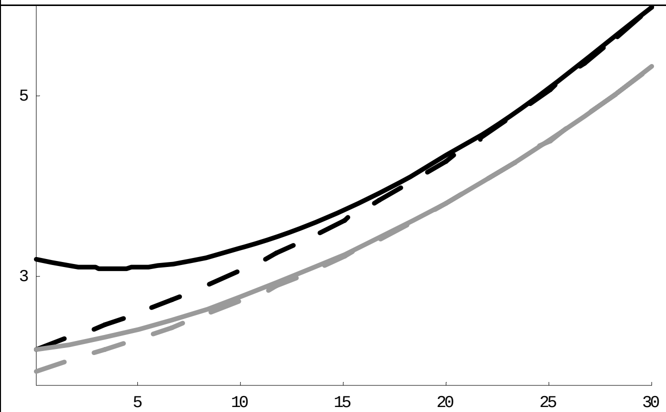
	K_0=10, h_0=1 Population growth rate change N_0=1, n =0.01 N(0)=1, n'=0.02	K_0=10, h_0=1 Change in population size at *t=0* N_0=1, , n =0.01 N'_0=2, n=0.01
λ =1 z_0<0		
λ =0 z_0<0		

Figure 16. Aggregate physical capital

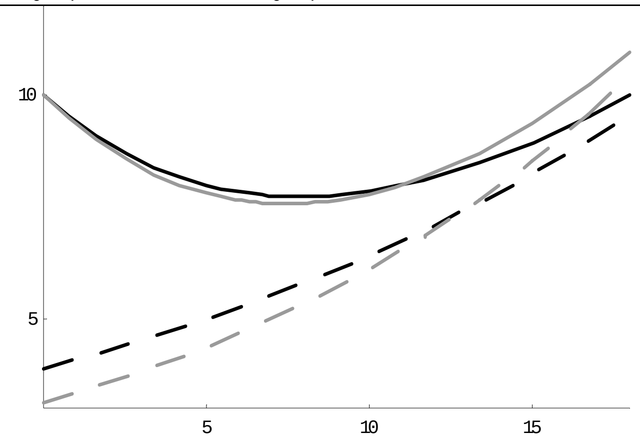
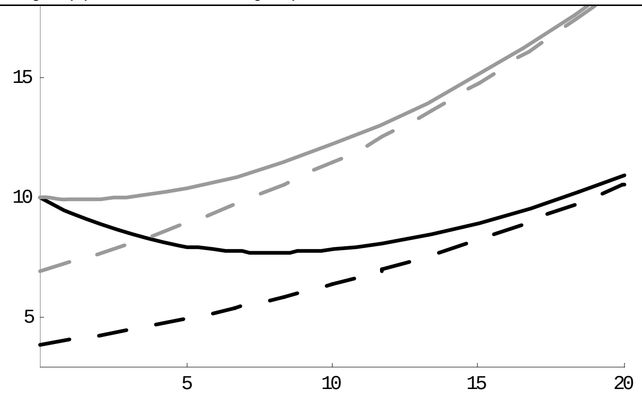
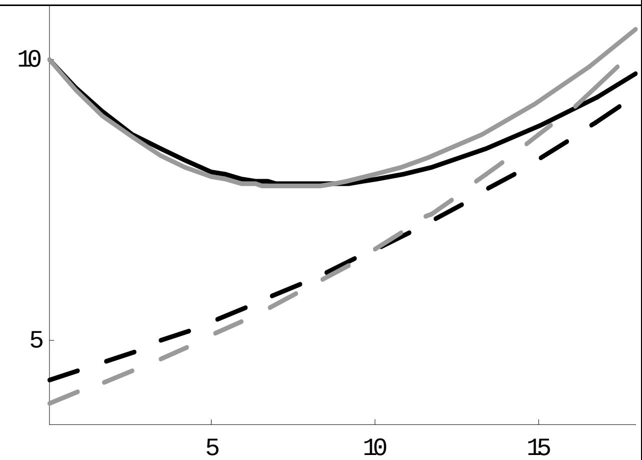
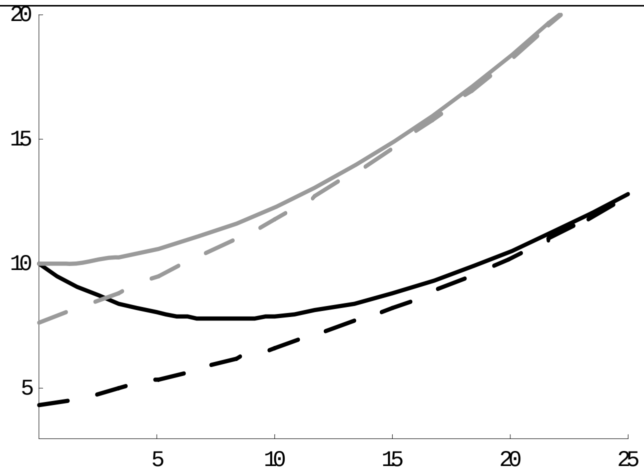
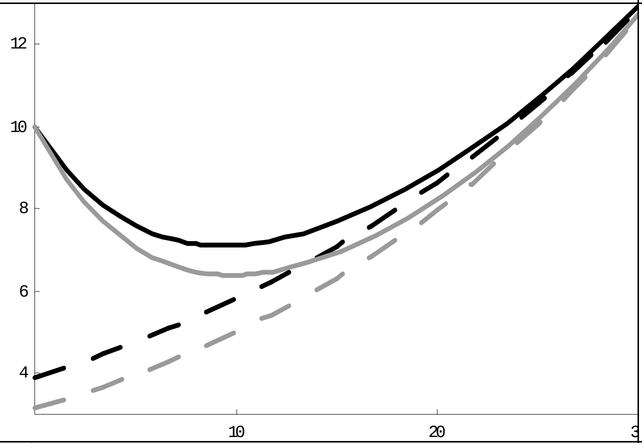
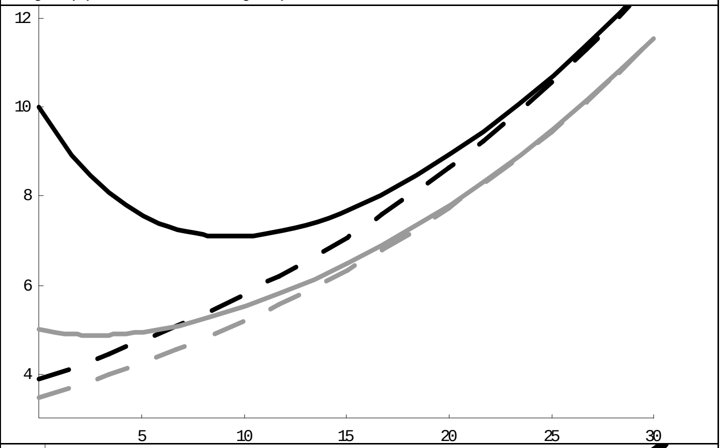
	K_0=10, h_0=1 Population growth rate change N_0=1, n =0.01 N_0=1, n'=0.02	K_0=10, h_0=1 Change in population size at *t=0* N_0=1, , n =0.01 N'_0=2, n=0.01
λ =1 z_0<0		
λ =0 z_0<0		

Figure 16b. Per capita physical capital

	K_0=10, h_0=1 Population growth rate change N_0=1, n =0.01 N_0=1, n'=0.02	K_0=10, h_0=1 Change in population size at *t=0* N_0=1, , n =0.01 N'_0=2, n=0.01
λ =1 z_0<0		
λ =0 z_0<0