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| Gabrielle Demange. On the influence of rankings. 2011. halshs-00589657

HAL Id: halshs-00589657

<https://shs.hal.science/halshs-00589657>

Preprint submitted on 29 Apr 2011

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PARIS SCHOOL OF ECONOMICS
ÉCOLE D'ÉCONOMIE DE PARIS

WORKING PAPER N° 2011 - 19

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JEL Codes: D71, D83

Keywords: ranking, scoring, invariant method, search



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On the influence of rankings

Gabrielle DEMANGE¹

April 28, 2011

Abstract

Ranking systems are becoming increasingly important in many areas, in the Web environment and academic life for instance. In a world with a tremendous amount of choices, rankings play the crucial role of influencing which objects are 'tasted' or selected. This selection generates a feedback when the ranking is based on citations, as is the case for the widely used invariant method. The selection affects new stated opinions (citations), which will, in turn, affect next ranking. The purpose of this paper is to investigate this feedback in the context of journals by studying some simple but reasonable dynamics. Our main interest is on the long run behavior of the process and how it depends on the preferences, in particular on their diversity. We show that multiple long run behavior may arise due to strong self enforcing mechanisms at work with the invariant method. These effects are not present in a simple search model in which individuals are influenced by the cites of the papers they first read.

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1 Introduction

Ranking systems are becoming increasingly important in many areas. In the Web environment, they are central to rank Internet sites and pages or to provide an index of trust in e-commerce. In academia, they are used to rank universities, journals, and researchers. Given their increasing impact on hiring and promoting procedures or on the determination of individuals' wages, ranking methods are the subject of a lively debate. This does not come at a surprise since a ranking method can be viewed as solving a multi-criteria problem, such as aggregating preferences for instance. As a result, various methods are meaningful, none of which dominating the other. Furthermore most of them are manipulable through strategic behavior.² This paper addresses a different issue. Presumably a

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²See for example Altman and Tennenholtz (2008) for an analysis of these issues in the context of the Web.

ranking helps individuals to make decisions by providing them with relevant information. But in a world with a tremendous amount of choices, rankings play the crucial role of influencing which objects are tasted and selected. Individuals tend to rely more on rankings rather than searching by themselves. When they search instead, readers are guided by the citations in the papers they read. Our aim is to compare some dynamics of citations in these two settings, the influence model in which readers are influenced by a journals' ranking and the search model in which they search.

The dynamics are built on a simple model linking citations to preferences and reading intensities. Reading intensities describe a possible bias, independent of the preferences, in the choice of the articles to read as a function of the journal in which they are published. In both settings -influence or search- reading intensities depend on past citations, but in a different way as described in the next paragraph. An additional key ingredient in the dynamics is that reading intensities will affect the new citations. The impact is not trivial because citations also depend on preferences, which vary across researchers. These two channels between citations and reading intensities generate a joint dynamical process: At a given date, citations affect the reading intensities, which, in turn, will affect the new citations, hence the subsequent reading intensities and so on. The differences in the reading behavior in the two settings generate different dynamics.

In the influence model, readers are influenced by the published ranking at the time they work on a paper. This describes in a simple way the assumption that, when doing research, researchers tend to read more the articles in the journals whose scores are higher (again independently of the preferences). Here it is assumed that readers use the ranking provided by the invariant method. The 'invariant' method is probably the most well-known ranking method since it serves as a basis to PageRank (Page et al. 2004), although it was introduced before by Pinski and Narin (1976) to rank journals.³ The method is based on the number of cites received and sent by each entity, say the number of citations across journals in the case of journals ranking or the link structure between pages in the case of ranking pages. Let us illustrate the method in the case of journals. The premise of the method is that citations are not equally important. The invariant method determines which journals are influential on the basis that a journal is influential if it is heavily cited by other influential journals.

Since rankings are based on past citations, a joint process on citations, reading intensities, and rankings follows along the lines described above: the current ranking, computed on previous citations, has an impact on new citations across journals, which, in turn, will influence next ranking and so on. I analyze the dynamics, especially how the long run behavior is related with preferences and whether some journals' cites become increasingly small relative to some others. A limit point

³Axiomatizations of the invariant method have been proposed in environments such as the Web, academic journals, tournaments, as in Palacios-Huerta and Volij (2004) and Slutzki and Volij (2006).

is (generically) characterized by the subset of journals that keep a positive score in the long run. Provided that, at some point in time, the journals in the subset obtain large enough scores, their received citations and scores will keep growing, eliminating the other journals. I show that multiple limit points are possible under some conditions on preferences. In that case several distinct subsets may survive in the end depending on the initial situation. This suggests that a strong self-enforcing mechanism is at play.

In a search model, individuals pick up the papers they read through a search process based on some random sample and the citations in this sample. (Of course, at the individual level, the sample may not be that random, but strongly determined by some individual specificities.) Thus reading behavior depends on citations, those in the sample. On average, reading intensities depend on some statistics on citations, specifically on an adjusted counting. The convergence of the process and the uniqueness of the limit are shown to be much more likely than in the influence model.

This paper is related to some previous studies.

An intuition of the framework developed is that, as past citations have an impact on future citations through the published ranking or through search, we might expect ‘the rich to get richer’, as indeed is the case when the scores of some journals tend to vanish. Rich-get-richer mechanisms have been recently the subject of active research in the field of ‘growing networks’, as in the preferential attachment model described by Barabasi, Albert, and Jeong (1999) or the more elaborated version of Jackson and Rogers (2005). Although our dynamics of citations have similarities with dynamics under preferential attachment, there are differences. Preferences, especially their diversity or homogeneity, influence the process and this creates a main difference with these studies. In particular, the diversity of preferences explains why journals can all keep positive scores. Also, some configurations of preferences can explain multiple limit behavior. Such types of behaviors do not arise with homogeneous preferences.

Concerns about the influence of search engines have been developing. A search engine displays the pages that are relevant to a query in the order obtained by a ranking method. A main criticism of search engines is that rankings are biased towards already popular Webpages, thus preventing the rise in popularity of recently created high quality pages. There has been some proposals to correct the bias by introducing some randomness in the rankings (Pandey, et al. 2005) or by accounting for the date of creation of a page in the computation of the ranking (Cho, Roy, and Adams 2005). Some however claim that search engines have an egalitarian effect, as Fortunato et al. (2003) in a study based on measurements of traffic to Websites. This could be explained as follows. An individual only visits a small sample of the sites and the sample would be larger with the use of a search engine rather than under simple surfing. As a result, search engines would increase the visit of the sites

that are not too popular, thereby counterbalancing the entrenchment effect.

In the field of citations, Simkin and Roychowdhury (2007) model a 'random citing scientist' in which a researcher picks up several papers at random, cites them, and *copies* a fraction of their references. Copying is taken literally, meaning that the papers are not read. In other words, neither preferences nor relevance enter into the process. According to Simkin and Roychowdhury (2003), the study of misprints propagation supports the idea that many cited papers are indeed not read. Quite clearly such a copying process favors 'old' articles. Variations can be introduced by considering for example that the first batch of papers is drawn among the new published papers. Simulations indeed show that a better fit is obtained, which could confirm that individuals tend to cite more the recent papers. Our search process is related but for the fact that researchers are assumed to read the cited papers and to cite them according to their preferences. Taking a different perspective, Chatterjee and Chowdhury (2008) propose a strategic analysis of citations and analyze its impact on knowledge diffusion.

The paper is organized as follows. Section 2 presents the framework and proposes a simple mechanism that describes citations as a function of researchers' preferences and their reading behavior. Reading behavior specifies the reading intensities, i.e., in which proportions journals are read. In the subsequent sections, reading intensities are either determined by the announced ranking or by a search process. Section 3 analyzes the dynamics on citations when the invariant ranking influences reading intensities. Section 4 considers a search process, derives the dynamics, and compares it with the influence model. Section 5 discusses the relationships with preferential attachment and concludes. Section 6 gathers some proofs.

2 The framework

There are some entities -journals, Web sites, individuals- to rank on the basis of their citations among themselves. In what follows, we refer to the situation of journals, keeping in mind other situations. Let N be a set of n journals to rank. N is fixed throughout the paper. A citation matrix C for N is a $n \times n$ non-negative matrix (c_{ij}) , in which c_{ij} represents the number of citations to i by j , or equivalently, the number of references of j to i .

Two assumptions simplify the presentation. First there is an identical number of articles per journal. The analysis goes through by working with the received citations per article (c_{ij}/a_i) where a_i is the number of articles in i . Second, it is assumed that each i receives citations from each j distinct from i , that is $c_{ij} > 0$ for $i \neq j$. The positivity of the diagonal elements is not required because one may want to eliminate self-citations by setting arbitrarily c_{ii} to zero. The positivity of the off-diagonal elements simplifies the presentation and the analysis carries through under the

weaker assumption of irreducibility of the matrix;⁴ irreducibility allows to 'connect' each pair of journals possibly through indirect citations.

We briefly describe ranking methods based on citations and then develop a simple model to explain how citations result from preferences and reading intensities.

Ranking methods Ranking methods based on citations assign a ranking on the basis of a citation matrix C . Here rankings evaluate the strength of the journals relative to each other. Specifically, a ranking assigns to each i a non-negative number v_i , called the *score* of i . Since only the relative values matter, the sum of the scores can be normalized to 1: a ranking of N is described by a vector v in (the simplex) Δ_N : $\Delta_N = \{v = (v_1, \dots, v_n) \in \mathbb{R}^N, v_i \geq 0 \text{ and } \sum_{i \in N} v_i = 1\}$. A *ranking method* assigns a ranking v in Δ_N to each citation matrix C .

In what follows, given a matrix (c_{ij}) we denote by c_{i+} the total in row i , $c_{i+} = \sum_{j \in N} c_{ij}$, and by c_{+j} the total in column j , $c_{+j} = \sum_{i \in N} c_{ij}$.

The paper considers two methods, the invariant method and the adjusted counting method. The invariant method is assumed to be driving readers' behavior in the influence model whereas the adjusted counting method is derived endogenously in the search model. Let us first describe the counting method and the adjustment by cite intensities.

Counting and invariant methods The counting method is the simplest method. It assigns scores proportional to the total number of cites to its articles:

$$v_i = \frac{c_{i+}}{\sum_k c_{k+}} \text{ where } c_{i+} = \sum_{j \in N} c_{ij}. \quad (1)$$

The result depends on what is called cite intensities. The cite intensity of a journal is defined as the average number of citations by an article in the journal. Journals which cite or 'vote' more than others may be considered as unduly represented. To neutralize this bias, cite intensity can be factored out so as to obtain what is called an 'intensity-invariant' method. A method is intensity-invariant if the ranking does not depend on cite intensity. Intensity-invariant methods are easy to construct by considering adjusted matrices: the adjusted matrix of C is $P = (p_{ij})$ where $p_{ij} = \frac{c_{ij}}{c_{+j}}$. Applying a method to adjusted matrices yields an intensity-invariant method in which what matters is the average number of citations to i by j relative to the average number of citations by an article of j . The adjusted counting method for example assigns scores proportional to the proportions of

⁴Consider the graph where (j, i) is a link if $c_{ij} > 0$. The matrix is said to be irreducible if any i can be reached from any j in a finite number of steps. A matrix which has all its elements strictly positive except possibly on the diagonal is irreducible.

cites to its articles:

$$v_i = \frac{p_{i+}}{n} \text{ where } p_{i+} = \sum_{j \in N} p_{ij}, p_{ij} = \frac{c_{ij}}{c_{+j}}. \quad (2)$$

A possible critic to the counting methods is that received citations are treated equally whether they are sent from a journal which ends up with a high or low score. The invariant method instead distinguishes between the senders as it weights differently the received cites by the score of the sender. There is a loopback effect because the scores are defined as proportional to the totals of the received cites weighted by the senders' scores. Specifically the aim of the method is to assign a ranking v that satisfies $v_i = \lambda \sum_{j \in N} p_{ij} v_j$ for each i for some positive λ . These equations say that v is a positive eigenvector of matrix P . Thus vector v exists because P is positive and is unique because P is irreducible and v is in Δ_N . The method is thus well defined. Furthermore the largest eigenvalue of matrix P is one, because by construction the total in each column is equal to 1.⁵ Hence the invariant method assigns the unique v in Δ_N that satisfies⁶

$$v_i = \sum_{j \in N} p_{ij} v_j \text{ or in matrix form } v = Pv \text{ where } p_{ij} = \frac{c_{ij}}{c_{+j}}. \quad (3)$$

Preferences, reading intensities, and citations A ranking method takes the citation matrix as the primitive. We propose now a simple model to explain citations as a function of two parameters, preferences and reading intensities. The premise of ranking methods is that citations are related to preferences. A citation is usually considered as a positive vote (though some may not be, a proviso especially relevant for citations on the Web). Even so a citation is a positive vote, the absence of citation to an article is not necessarily a negative one. Not citing a paper can be interpreted as a negative signal *under the condition* that the paper has been read. The absence of citation occurs under two circumstances. Thus either the paper has been read but has not been judged relevant enough to be cited or the paper has not been read. The latter case occurs more frequently as the number of alternatives (relevant papers to read) increases. To summarize this discussion, a citation matrix is likely to depend on both preferences and reading intensities. We model this as follows.

Preferences describe the chances for an article that has been read to be cited depending on where it is published and who reads it. Specifically, let π_{ij} be the probability for an author in j to cite an

⁵The equations $\sum_i p_{ij} = 1$ for each j imply that $\mathbf{1}_N$, the n -vector with components equal to 1, is a positive eigenvector of the transpose of P with eigenvalue 1: 1 is the dominant eigenvalue of the transpose of P . It suffices than to use that P and its transpose have the same eigenvalues.

⁶The vector can be computed recursively as follows. Start to assign scores as in the adjusted counting method, proportionally to the total number of received shares: $v_i^{(0)} = p_{i+}/n$ for each i . Next compute new impact by weighting the cites with $v^{(0)}$ and iterate. The sequence $(v^{(t)})$ is thus described by $v^{(t+1)} = Pv^{(t)}$. The sequence converges to the invariant ranking. Working on the citations without factoring intensity, Liebowitz and Palmer (1984) computed 'impact factors' for economic journals by performing some iterations without making explicit the method, as explained in Amir (2002).

article in i conditional on having read it. Preferences are thus represented by the matrix $\Pi = (\pi_{ij})$. All the elements π_{ij} are assumed to be positive.

Reading intensities describe in which proportions articles are read depending on the journal where they are published. They are represented by a positive n -vector (r_i) where the reading intensity of journal i , r_i , is interpreted as the average number of articles in i that are read by a researcher when working on a paper. Alternatively, normalizing the sum $\sum_i r_i = 1$, r_i gives in which proportion the read articles are published in i . As can be checked, a normalization plays no role in the following because we work on adjusted matrices (so we will choose the more convenient normalization when needed).

Preferences are taken here as primitives. Reading intensities may take various forms, depending on how individuals select the papers they read. The benchmark case arises if authors read all papers that are relevant to their work or, more generally, if authors pick up at random a limited number of relevant papers and read that sample. In this benchmark case reading intensities are all equal ($r_i = 1/n$). The adjusted citation matrix is equal to the matrix $P = (\pi_{ij}/\pi_{+j})$, where the total of each column is 1. It directly reflects preferences. A pure random selection process is however not sensible. The paper investigates two settings in which the selection of papers to read depends on citations, directly or indirectly.

In the first setting, the influence model analyzed in Section 3, individuals are influenced in their selection by the published ranking of the journals. Reading intensities at some date are function of the ranking available at that date. Since a ranking is computed on the basis of numbers of citations, citations influence reading intensities indirectly.

In the second setting, analyzed in Section 4, the selection of the papers to read is made through a search process. The search starts as the random process but, after a first batch has been drawn, an individual chooses some papers among those that are cited by the papers in the batch and read them, and possibly continues for some steps. The impact of citations here is direct since the probability for an article to be read depends on the number of articles that already cite it. Hence reading intensities, which are obtained by aggregation, are affected by citations.

Thus in both settings, existing citations have an impact on reading intensities. We now model a reverse relationship, namely how reading intensities affect new citations.

Citations in the papers being written in a current period are jointly determined by preferences and reading intensities as follows. Suppose that an author cites an article if he has read the article and finds it relevant. Then the new citations to i from an author of an article published in j are on average proportional to $r_i\pi_{ij}$. Let us describe this more precisely.

To simplify the presentation the (equal) number of articles per journal is assumed to be constant

overtime, denoted A , as well as cites' intensities, denoted by s_j for j (both assumptions can be relaxed). This gives a total number of citations by j per period equal to As_j . How these new citations are distributed to journals depend on the reading intensities prevalent when working on the papers published at t : New citations are distributed in proportion of the reading intensities and the preferences. Specifically, let $r_i^{(t)}$ denote the reading intensity of journal i , $n_{ij}^{(t)}$ the number of citations at t from j to i are proportional to $\pi_{ij}r_i^{(t)}$, which gives

$$n_{ij}^{(t)} = As_j \frac{\pi_{ij}r_i^{(t)}}{\sum_{\ell} \pi_{\ell j}r_{\ell}^{(t)}} \text{ for each } i, j. \quad (4)$$

Thus, reading intensities affect new citations which in turn will affect (possibly in addition to old citations) next reading intensities and so on. This generates a joint dynamics on citations, reading intensities, and rankings.

Note that the expression (4) gives the expected value of the new citations. Since both π_{ij} and r_i represent probabilities respectively of citing and reading a paper, fluctuations around the expected values are possible. However, if the period over which citations are counted and the rankings are released is assumed to be long enough then these fluctuations are small. In the following analysis, we neglect the fluctuations and identify the realized values with their expectation. (This is the same approach as the one taken by the mean field theory). This leads to a deterministic dynamics, affected only by the initial values and the process by which reading intensities are determined.

Notation Given I and J two non empty subsets of N , let $\Pi_{I \times J}$ denote the matrix obtained from Π by keeping the rows indexed by i in I and the columns indexed by j in J .

$\mathbf{1}$ (resp. $\mathbf{0}$) denotes the vector in \mathbb{R}^N which has all its components equal to 1 (resp. 0). Finally, for a vector x , $x \gg \mathbf{0}$ means that all components are strictly positive, $x > \mathbf{0}$ all are nonnegative and one at least is positive, $x \geq \mathbf{0}$ all are nonnegative.

Without specification, when there is no ambiguity, an index is running over N . For example we write a sum such as $\sum_{i \in N} c_{ij}$ as $\sum_i c_{ij}$.

3 The influence model

While working on a paper, a researcher who selects the papers to read by taking into account a journals' ranking tends to read more the journals the scores of which are higher. This behavior is described by larger reading intensities for journals with larger score. Specifically, let reading intensities be function of the ranking, $r(v) = (r_1(v), \dots, r_n(v))$ where $r_i(v)$ is the reading intensity for journal i . Natural assumptions are

$$v_i = v_{\ell} \Rightarrow r_i(v) = r_{\ell}(v), \text{ and } v_i > v_{\ell} \Rightarrow r_i(v) > r_{\ell}(v).$$

The first condition requires that the distortions in reading intensities only depend on the values of the scores. In particular, in the case of identical scores for all journals, there is no bias in reading intensities. The second condition requires that the larger the score of a journal the larger its reading intensity relative to other. I investigate the case where the influence function is linear, $r_i(v) = v_i$. The linear function can be thought as a benchmark. Consider the family indexed by $\gamma > 0$ where $r_i(v) = v_i^\gamma$ (recall that reading intensities are defined up to a multiplicative factor). There are diminishing or increasing returns to 'score' depending on whether γ is smaller or larger than 1, thereby generating with extremely different dynamics. The linear case is at the boundary.

In what follows I assume that researchers use the invariant ranking. So invariant ranking and linear influence of the scores are the two main assumptions in this section. Let us describe the dynamics. In period t , reading intensities are given by

$$r_i^{(t)} = v_i^{(t-1)} \text{ for each } i \quad (5)$$

in which $v^{(t-1)}$ is the ranking released at the beginning of period t . This generates the matrix of new citations ($n_{ij}^{(t)}$) as given by (4). The citation matrix which is used for computing the ranking also accounts for past citations. Typically, a ranking is computed by assigning less importance to old citations, for example by applying a discount factor or by using a window.⁷ To simplify the presentation, let us consider a window of two periods. A ranking released at the beginning of period $t + 1$ is based on the citation matrix $C^{(t)} = (c_{ij}^{(t)})$ defined by:

$$c_{ij}^{(t)} = \mu n_{ij}^{(t-1)} + (1 - \mu) n_{ij}^{(t)} \text{ for each } i, j, \quad (6)$$

where μ is a number between 0 and 1 representing the weight to the previous citations. Using the expression (4) for $n_{ij}^{(t)}$ and (5) for the reading intensities, the adjusted citation matrix $P^{(t)}$ is

$$p_{ij}^{(t)} = \frac{\mu q_{ij}^{(t-1)} + (1 - \mu) q_{ij}^{(t)}}{\sum_{\ell} [\mu q_{\ell}^{(t-1)} + (1 - \mu) q_{\ell}^{(t)}]}, \text{ where } q_{ij}^{(t)} = \frac{\pi_{ij} v_i^{(t-1)}}{\sum_{\ell} \pi_{\ell j} v_{\ell}^{(t-1)}} \text{ for each } i, j. \quad (7)$$

The ranking $v^{(t)}$ based on matrix $P^{(t)}$ is published at the beginning of period $t + 1$:

$$v^{(t)} = P^{(t)} v^{(t)}. \quad (8)$$

$P^{(t)}$ depends on the rankings at the two previous periods, $v^{(t-2)}$ and $v^{(t-1)}$, as can be seen from expression (7). Thus plugging expression (7) into equation (8) specifies the process followed by the sequence $v^{(t)}$. The processes for the citations and the reading frequencies can be straightforwardly derived from that of the rankings.

⁷Accounting for previous citations creates some inertia in the rankings, hence in the reading intensities. Although this complicates a little bit the presentation, it is important to check that the results -in particular the multiplicity of limit points- hold generally and not only when rankings are based on the new citations.

Our aim is to analyze the behavior of the sequence $v^{(t)}$. We are especially interested in the subset of journals that keep a positive score in the long run. The set of positive components of a ranking is called its *support*.

Stable points and their support To analyze the behavior of the sequence $v^{(t)}$, we first consider the fixed points of the process. Taking a constant value v^* for the sequence $v^{(t)}$, the matrix $Q^{(t)}$ as defined in (7) is constant as well, which yields a constant matrix P^* for $P^{(t)}$:

$$P^* = (p_{ij}^*) \text{ where } p_{ij}^* = \pi_{ij} \frac{v_i^*}{\sum_{\ell} \pi_{\ell j} v_{\ell}^*} \text{ for each } i, j. \quad (9)$$

Now using (8), a fixed point satisfies :

$$v_i^* = \sum_j \frac{\pi_{ij} v_i^*}{\sum_{\ell} \pi_{\ell j} v_{\ell}^*} v_j^* \text{ for each } i. \quad (10)$$

Equation (10) is surely met for i with a null v_i^* , implying that any journal may have a null score at a fixed point. This is a self-enforcing mechanism: a journal with a null score is not read hence is not cited, which justifies that its score is null. We are however interested in robust results, as described by stability.

Definition 1 *A fixed point v^* is locally asymptotically stable if the sequence $v^{(t)}$ defined by (7) and (8) converges to v^* for an open set of initial values for $v^{(0)}$. To simplify, we say that v^* is stable.*

Standard conditions characterizing local stability are difficult to interpret. For a fixed point with some null components, however, stability requires some conditions that are easy to derive and interpret. For v^* to be stable with a null i component, the sequence $v_i^{(t)}$ must converge to zero. In that case the influence of the ranking on reading intensities tends to eliminate journal i . The next proposition provides necessary conditions for a subset I to be the support of a stable point.

Proposition 1

1. *A necessary condition for subset I of N to be the support of a stable point is*

$$\text{there is } x \text{ in } \mathbb{R}^I, x \gg \mathbf{0}, \text{ such that } \Pi_{I \times I} x = \mathbf{1}_I, \Pi_{N-I \times I} x \leq \mathbf{1}_{N-I}. \quad (11)$$

There is a unique possible stable point associated to x . A subset I that satisfies (11) is called stationary.

2. *Either I is stationary or there are a subset K of I , weights α on K and β on $N - K$, $\alpha_k \geq 0$, $\sum_{k \in K} \alpha_k = 1$, $\beta_j \geq 0$ and $\sum_{j \in N-K} \beta_j = 1$ such that*

$$\sum_{k \in K} \pi_{ki} \alpha_k \leq \sum_{j \in N-K} \pi_{ji} \beta_j \text{ for each } i \in I. \quad (12)$$

The proof is in the appendix. Let us sketch the proof of (11) for a null weight μ on past citations. Taking $\mu = 0$ in (7) and using (8) the sequence $v^{(t)}$ follows the process

$$v_i^{(t+1)} = \sum_j \pi_{ij} \frac{v_i^{(t)}}{\sum_\ell \pi_{\ell j} v_\ell^{(t)}} v_j^{(t+1)} \text{ each } i. \quad (13)$$

Dividing by $v_i^{(t)}$, we obtain that the growth rate of i 's score satisfies

$$\frac{v_i^{(t+1)}}{v_i^{(t)}} = \sum_j \pi_{ij} \frac{v_j^{(t+1)}}{\sum_\ell \pi_{\ell j} v_\ell^{(t)}} \text{ each } i. \quad (14)$$

For I to be the support of a stable point, the sequence converges locally to some v^* which has positive components on I and null ones on $N - I$. (11) The right hand side of (14) converges, giving a limit to the growth rates. The growth rate must be equal to 1 for the components indexed by I and not be larger than 1 for those indexed by $N - I$. This implies (11) where x is defined on I by $x_j = v_j^* / \sum_\ell \pi_{\ell j} v_\ell^*$. The proof of 2 follows from an application of a theorem of the alternatives to linear inequalities.

The proposition can be interpreted as follows.

Consider first the whole set N . Conditions (11) state the existence of a positive vector x for which $\Pi x = \mathbf{1}_N$, or $\sum_j \pi_{ij} x_j = 1$ for each i , that is, the journals preferences weighted by x equalize the counting scores. In words, there is a set of weights on journals (as providers of citations) under which, assuming unbiased reading, each journal receives the same weighted total of citations. Thus, N is stationary when preferences are sufficiently diverse so that they do not favor in a clear way some journals.

Let us now interpret the alternative condition 2 for N . With two journals, $N = \{1, 2\}$, alternative 2 applies to singletons only, $K = \{k\}$, and the weights, α_k and β_j , j distinct from k , are equal to 1. Condition (12) writes as $\pi_{ki} \leq \pi_{ji}$ for each i in $\{1, 2\}$. Thus either $N = \{1, 2\}$ is stationary or a journal is unanimously preferred to the other one. With more than two journals, alternative 2 for N considers a partition of the journals, K and $N - K$, and define preferences on 'composite' journals through the weights α and β according to (12): preferences for K are given by the sum of the preferences over the journals in K weighted by α , and similarly the preferences for $N - K$ are given by the sum of the preferences over the journals in $N - K$ weighted by β . With this interpretation, either N is stationary and the preferences can be aggregated so as to equalize the weighted number of cites across journals, or the journals can be partitioned into two composite groups such that everyone prefers one group to the other one.

Consider now a subset I of N . The stationarity conditions (11) require the existence of a weight vector on the preferences of I that equalizes the weighted citations totals of the journals in I and gives a lower weighted total to journals not in I (by the inequality condition). Alternative 2 is

interpreted as above in terms of two composite journals based on a partition. One composite group is formed only with journals in I (K is included I) and receives less cites from each member of I than the other composite group. Thus, either subset I is stationary and the preferences of I can be aggregated so as to equalize the weighted number of cites of the journals in I and give less to journals outside I , or it contains a composite group such that everyone in I prefers the complement composite group. It should be noted that all these conditions only depend on the preferences of journals in I . For example, a singleton $\{i\}$ is stationary if $\pi_{ii} \geq \pi_{ji}$, which says that i cites itself at least as much as any other journal (the solution x to $\pi_{ii}x = 1$, which is positive, must satisfy the inequalities $\pi_{ji}x \leq 1$). More generally conditions (11) or (12) bear on the columns indexed by I of the matrix Π , so are independent on the preferences of journals in $N - I$. This point is developed in the next section on the resilience of subsets.

Proposition 1 implies that the number of limit rankings is finite in general. The argument is as follows. According to point 1, there is a unique possible ranking with support I associated to a solution x to (11). If the matrix $\Pi_{I \times I}$ is invertible, there is at most one candidate for x to (11) because $\Pi_{I \times I}x = \mathbf{1}_I$ admits a unique solution. Hence there is at most one possible limit ranking with support I . Thus, when all diagonal submatrices are invertible, which is generically true, the number of limit rankings is finite.

Example 1 This example illustrates that the process may not converge to a unique point and that a stationary set may not be the support of a stable point. There are two journals, so adjusted matrix Π writes

$$\Pi = \begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}.$$

Take $a \neq b$ and $a + b \neq 1$. $N = \{1, 2\}$ is stationary if the solution to $\Pi x = \mathbf{1}_N$, $x = (\frac{1-2b}{1-a-b}, \frac{1-2a}{1-a-b})$, is positive. This holds for a and b both on the same side relative to $1/2$, that is, when each cites less itself than the other or, on the contrary, each cites more itself than the other. A singleton is stationary when it cites itself more than the other, i.e., $\{1\}$ is stationary for $a \leq 1/2$ and $\{2\}$ for $b \leq 1/2$.

The process converges.⁸ For a or b not both smaller than $1/2$, the stationary set is unique and

⁸The dynamics can be defined in terms of the ratio $\rho = \frac{v_1}{v_2}$:

$$\rho^{(t)} = f(\rho^{(t-1)}) \text{ where } f(\rho) = \frac{b}{a} \rho \frac{(1-a)\rho + a}{b\rho + (1-b)}.$$

The fixed points are 0, or ∞ , or $\rho^* = \frac{a}{b} \frac{1-2b}{1-2a}$ (if positive), which correspond respectively to the support $\{2\}$, or $\{1\}$, or $\{1, 2\}$. The sequence always converges. Hence, for b larger than $1/2$, 0 is not stable because $f'(0) = b/(1-b) > 1$, and similarly for a larger than $1/2$, ∞ is not stable. It follows that for a or b not both smaller than $1/2$, there is a unique stable point: 0 ($b < 1/2$) or ∞ ($a < 1/2$) or ρ^* (a and b larger than $1/2$). For a and b both smaller than $1/2$,

is the support of the limit point. For a and b smaller than $1/2$, that is, when each cites more itself than the other, N and the singletons are all stationary. In that case, both singletons are the support of stable point, but N is not although it is stationary.

Example 2 This example illustrates the possible non-monotonicity in the dynamics of rankings. There are three journals with preferences matrix

$$\Pi = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 4 & 2.1 \\ 2 & 3 & 4 \end{pmatrix}$$

The invariant ranking of Π is $v^{(1)} = (0.331, 0.337, 0.332)$. The situation is almost symmetrical, with journal 1 citing 2 more than 3, journal 2 citing 3 more than 1, and journal 3 citing 1 more than 2. Journal 2 however can be said to have a slight advantage because a totally symmetric situation between journals is obtained by lowering the value of citations journal 2 receives from journal 3 ($\pi_{23} = 2.1$) down to 2. This explains why ranking $v^{(1)}$ gives the highest score to 2, and the second rank to 3 because 2 cites 3 more than 1. Starting with $v^{(1)}$, the ranking after twelve iterations is $v^{(12)} = (0.249, 0.397, 0.353)$, and the sequence eventually converges to $(0, 0, 1)$. The score of 2 first increases and then converges to zero. The non-monotonicity can be explained as follows. Journal 1 prefers 2 to 3, hence gives more 'power' to 2. The score of journal 1 however decreases (because 2, which starts with the highest score, prefers 3 to 1). When 1's score is low enough, it is the fact that 2 cites 3 more than the opposite that matters, thereby driving the score of journal 2 to 0. ■

Resilient subsets It is unclear whether the process of rankings always converges (and it is likely not to be the case). In line with our goal, we study whether a subset of journals can dominate in the long run. This notion is defined as the resilience.

Roughly speaking, a subset is resilient if the scores of journals outside the subset vanish provided their initial scores is low enough. To make this precise, let us consider Δ_I the subset of Δ_N composed with all elements with null components in $N - I$: $\Delta_I = \{v = (v_I, \mathbf{0}_{N-I}) \geq \mathbf{0}, \sum_{i \in I} v_i = 1\}$. For $\epsilon > 0$, the ϵ -neighborhood of Δ_I is composed with the rankings the components of which are smaller than ϵ on $N - I$.

Definition 2 *Subset I of N is resilient if there is $\epsilon > 0$ for which the scores on $N - I$ converge to zero when the initial ranking is in the ϵ -neighborhood of Δ_I .*

In words, starting with rankings that assign small enough scores on $N - I$, the scores of these journals eventually converge to zero. Resilience is reminiscent to the notion of attractor for the dynamical

 both 0 and ∞ are stable and ρ^* is not.

system $v^{(t)}$ but differs. An attractor for the process of $v^{(t)}$ is a subset of Δ_N such that (a) the system evolves to that subset after a long enough time if one starts from an open neighborhood, and (b) no strict subset satisfies the same property. According to the definition, I is resilient if the subset Δ_I satisfies (a).

To handle the behavior of the process with scores arbitrarily small, let us write the dynamics in terms of the growth rate of v 's components. Also, to simplify notation and introduce the next result, assume that the ranking is based on the citations of the previous period, as described by (13) or (14). The growth rates defined by $\delta_i^{(t+1)} = \frac{v_i^{(t+1)}}{v_i^{(t)}}$ satisfy

$$\delta_i^{(t+1)} = \sum_j \pi_{ij} \frac{v_j^{(t)}}{\sum_\ell \pi_{\ell j} v_\ell^{(t)}} \delta_j^{(t+1)} \text{ each } i. \quad (15)$$

These equations say that the vector $\delta^{(t+1)}$ is a positive eigenvector associated with the eigenvalue 1 of the matrix $(\pi_{ij} \frac{v_j^{(t)}}{\sum_\ell \pi_{\ell j} v_\ell^{(t)}})$. The eigenvector is uniquely determined thanks to the normalization on rankings: $\sum_i v_i^{(t+1)} = \sum_i \delta_i^{(t+1)} v_i^{(t)} = 1$. Let us introduce some notation to formalize this. Given a vector v , let $\Pi(v)$ be the square matrix defined by

$$\pi_{ij}(v) = \frac{v_j}{\sum_\ell \pi_{\ell j} v_\ell} \text{ each } i, j \quad (16)$$

and let $D(v)$ be the unique dominant eigenvector of $\Pi(v)$ that satisfies $D(v) \cdot v = 1$, in which $D(v) \cdot v$ denotes the scalar product. Equations (15) write

$$\delta^{(t+1)} = D(v^{(t)}). \quad (17)$$

By definition, the resilience of a subset I depends on the dynamics of the rankings in a neighborhood of Δ_I , the set of rankings with null components outside I . Consider $v = (v_I, \mathbf{0}_{N-I})$ a ranking in Δ_I . As can be seen from (16) the columns of $\Pi(v)$ with index in $N - I$ are all null and those with index in I only depend on I 's preferences and v_I . So with obvious notation

$$\Pi(v_I, \mathbf{0}_{N-I}) = \begin{pmatrix} \Pi_{I \times I}(v_I) & \mathbf{0}_{I \times N-I} \\ \Pi_{N-I \times I}(v_I) & \mathbf{0}_{N-I \times N-I} \end{pmatrix}. \quad (18)$$

The next proposition uses this expression to derive the eigenvector $D(v)$ and provide a sufficient condition for the resilience of a subset.

Proposition 2 *Let I be a subset of N .*

1. *Let $v = (v_I, \mathbf{0}_{N-I})$ in Δ_I and $D(v) = (D(v)_I, D(v)_{N-I})$ the assigned eigenvector of $\Pi(v)$. $D(v)_I$ is the unique positive eigenvector of matrix $\Pi_{I \times I}(v_I)$ that satisfies $D(v)_I \cdot v_I = 1$ and $D(v)_{N-I}$ is equal to $\Pi_{N-I \times I}(v_I) D(v)_I$.*

2. A sufficient condition for I to be resilient is: for some $k < 1$

$$D(v)_{N-I} = \Pi_{N-I \times I}(v_I)D(v)_I \text{ has all its components less than } k \text{ for any } v \text{ in } \Delta_I. \quad (19)$$

To understand point 1, observe that starting with a ranking with null components outside I , the components outside I stay null and the behavior for the components on I is dictated by the sub-matrix $\Pi(v)_{I \times I}$, which is $\Pi_{I \times I}(v_I)$ from (18). This explains why the eigenvector $D(v)$ on I is derived from that of the square matrix $\Pi_{I \times I}(v_I)$. When the initial scores outside I are small but not null, their local behavior is driven by how the journals that have large scores, those in I , cite them and grow: this explains why the growth rates of the scores in $N - I$, $D(v)_{N-I}$, is a weighted combination of the growth rates 'direction' of scores in I , the components of $D(v)_I$, in which the weights are the preferences of I towards them. An implication of point 1 is that $D(v)$ only depends on the columns of $\Pi(v)$ indexed by I for v in Δ_I , thus only on the preferences of journals in I .

The sufficient condition in 2 ensures that the growth rate of each score of journals in $N - I$ is bounded by a value smaller than 1 if the initial ranking v has low enough components on $N - I$. This condition is surely satisfied by making the probabilities for I to cite $N - I$ small enough (all elements of $\Pi_{N-I \times I}(v_I)$ decrease with these probabilities). The corollary follows.

Corollary 1 *There are preferences which admit distinct disjoint resilient subsets.*

The proof relies on the fact that conditions (19) can be satisfied simultaneously for distinct disjoint subsets. To construct matrices Π admitting multiple resilient sets, choose for example a subset I and preferences from I to $N - I$ small enough and similarly preferences from $N - I$ to I small enough. Then both I and $N - I$ are resilient. ■

Propositions 1 and 2 are both concerned with local stability, the former in a neighborhood of a fixed point and the latter in a neighborhood the set scores null outside a given subset. Technically they are related as they both rely on the growth rate of the scores. Let us draw a more precise comparison, and explain why Proposition 1 (more precisely point 1) gives only a necessary condition for stationarity whereas Proposition 2 provides a sufficient condition for resilience. Given x that satisfies (11), there is a unique candidate for a limit vector, $v^* = (v_I^*, \mathbf{0}_{N-I})$, associated to x :

$$x_j = \frac{v_j^*}{\sum_{\ell \in I} \pi_{\ell j} v_\ell^*} \text{ for each } j \text{ in } I$$

as shown in the proof of Proposition 1.

We show that the eigenvector $D(v^*)$ associated to matrix $\Pi(v^*)$ satisfies: $D(v^*)_I = \mathbf{1}_I$ and $D(v^*)_{N-I} = \Pi_{N-I \times I}x$. By definition (16), we have $\pi_{ij}(v^*) = \pi_{ij}x_j$ for each i , each j in I . Thus the equality $\sum_{j \in I} \pi_{kj}x_j = \sum_{j \in I} \pi_{kj}(v^*)$ holds for each k . Thus equation $\Pi_{I \times I}x = \mathbf{1}_I$ of (11), which writes also $\sum_{j \in I} \pi_{ij}x_j = 1$ for each i in I , implies that $\mathbf{1}_I$ is a dominant eigenvector of $\Pi_{I \times I}(v_I^*)$:

since $\mathbf{1}_I \cdot v_I^* = 1$, this proves $D(v^*)_I = \mathbf{1}_I$ by point 1 of Proposition 2. By the same point, $D(v^*)_{N-I}$ is equal to $\Pi_{N-I \times I}(v_I^*)\mathbf{1}_I$, which writes as $\Pi_{N-I \times I}x$ (since the k -th component $\sum_{j \in I} \pi_{kj}(v^*)$ is equal to $\sum_{j \in I} \pi_{kj}x_j$).

Hence conditions (11) state the equalities $D(v^*)_I = \mathbf{1}_I$ (which is not surprising since we obtain them precisely by requiring the growth rates to be equal to 1 on I at the limit) and the inequalities $D(v^*)_{N-I} \leq \mathbf{1}_{N-I}$. If these inequalities hold strictly, i.e., if $\Pi_{N-I \times I}x \ll \mathbf{1}_{N-I}$, they have a similar local implication as (19): for v in an ϵ -neighborhood of v^* and for some $k < 1$ the growth of each component of $v^{(t)}$ on $N-I$ is smaller than k . Thus if the sequence $v^{(t)}$ stays in that ϵ -neighborhood of v^* , the scores on $N-I$ converge to zero. But the sequence $v^{(t)}$ may not stay in that neighborhood, in particular because the scores on I may not converge to v_I^* (the conditions that the growth rates of the scores in I are equal to 1 at v_I^* are not sufficient to ensure convergence). This is why stationarity does not guarantee convergence.⁹ Instead condition (19) is sufficient for resilience. This is because the condition is required to hold for any ranking in Δ_I , thus independently on the behavior of the scores I .

4 A search model

This section assumes that a researcher working on a subject conducts a search for finding papers to read. Thus, instead of the previous section, a published ranking plays no role *a priori*. The search is a two-step process, in which the researcher reads a first batch of papers picked up at random and then reads other papers among those which have been cited by the papers in the first batch. We suppose that the first batch has m_r papers and that for each article in this batch, m_s papers among those cited are picked up at random and read. Note that this number m_s does not depend on the number of cites made in the paper. This implies that cite intensities are 'factored out'. The search process determines the chances for a paper to be read, and by aggregation over the articles published in a given journal, it determines the reading intensities, as computed below. The new citations follow, as a function of the preferences Π and the reading intensities as described by equation (4).

We compute first the probability for a particular article a in i to be read at t . As previously, one may consider cites that are not too old, for example those in papers that were published in the last two periods. In the expressions below, we do not need to make that precise, and only speak about the 'relevant' papers.

⁹This discussion however suggests the following decomposition result: if starting with null components on $N-I$, the sequence converges when the initial value for v_I belongs to a neighborhood of v_I^* , then inequality condition in (11) satisfied strictly ensures the convergence of the sequence when the initial value for v belongs to a well chosen neighborhood of $v^* = (v_I^*, \mathbf{0}_{N-I})$.

Let $A^{(t)}$ denote the total number of relevant articles published up to the beginning of t (stock), and $A_j^{(t)}$ those published in j . Let $c_{ij}^{(t)}$ denote the number of cites from j to i and $c_a^{(t)}$ denote the number of cites to a in these relevant papers at the beginning of $t + 1$. Assume that the cites to an article a in i are distributed as among the whole population of cites to an article in i . This assumption is on average valid, and can be relaxed as explained in the remarks below. We show that the probability for an author to read article a in i in period $t + 1$ is

$$\frac{m_r}{A^{(t)}} + \frac{m_r}{A^{(t)}} \sum_j \frac{m_s}{s_j} \frac{c_{ij}^{(t)}}{c_{i+}^{(t)}} c_a^{(t)}. \quad (20)$$

The first term is the probability that a is read among the first batch of m_r articles chosen at random. The second term is the probability that a is read because it is cited by one of the papers in the first batch (up to a small error), as we explain now. Under the assumption that the cites to a are distributed as among the whole population of cites to an article in i , a receives $\frac{c_{ij}^{(t)}}{c_{i+}^{(t)}} c_a^{(t)}$ cites from articles in j . This gives a proportion of articles in j citing a equal to $\frac{1}{A_j^{(t)}} \frac{c_{ij}^{(t)}}{c_{i+}^{(t)}} c_a^{(t)}$. Hence, the probability for an article among the m_r drawn at random in the first step to be in j and to cite a is $m_r \frac{A_j^{(t)}}{A^{(t)}} \frac{1}{A_j^{(t)}} \frac{c_{ij}^{(t)}}{c_{i+}^{(t)}} c_a^{(t)}$ (neglecting the case where several articles are in j and cite a). For such an article, m_s cites are read among the s_j cites, which gives a chance m_s/s_j that a is read. So finally the probability for a to be read because it is cited by an article in j in the first batch is equal to $\frac{m_r}{A^{(t)}} \frac{m_s}{s_j} \frac{c_{ij}^{(t)}}{c_{i+}^{(t)}} c_a^{(t)}$. Summing over j gives the probability that a is read because it has been cited by an article in the first batch, the second term in (20).

Reading intensities are equal to the expected number of articles read by a typical researcher in each journal up to a scalar. The expected number of articles in i that are read is given by the sum of (20) over all articles published in journal i . Since by definition the sum of the $c_a^{(t)}$ over all articles a in i is the number of cites received by i , $c_{i+}^{(t)}$, this expected number is

$$\frac{m_r A_i^{(t)}}{A^{(t)}} + \frac{m_r}{A^{(t)}} \sum_j \frac{m_s}{s_j} c_{ij}^{(t)}. \quad (21)$$

Since there is the same number of articles per journal, $\frac{A_j^{(t)}}{A^{(t)}} = \frac{1}{n}$. Also by definition of the cite intensity, $s_j A_j^{(t)} = c_{+j}^{(t)}$, (21) can also be written as

$$\frac{m_r}{n} \left[1 + m_s \sum_j \frac{c_{ij}^{(t)}}{c_{+j}^{(t)}} \right]. \quad (22)$$

Observe that $\sum_j \frac{c_{ij}^{(t)}}{c_{+j}^{(t)}}$ is the sum of the adjusted cites received by i ; hence, it is the score of i assigned by the adjusted counting method to the citation matrix $C^{(t)}$ as defined in (2). Normalizing by the number of articles read by a researcher, $m_r(1 + m_s)$, we obtain the expression (23) for the reading intensities in the following proposition.

Proposition 3 *In the two-step search process the reading intensities are described by a combination of a pure random process and the adjusted counting method:*

$$r_i^{(t+1)} = \frac{1}{n} [\alpha + (1 - \alpha) \sum_j \frac{c_{ij}^{(t)}}{c_{+j}^{(t)}}] \text{ with } \alpha = \frac{1}{1 + m_s}. \quad (23)$$

The process of citations and intensities converge to a unique limit point for m_s not too large, whatever the starting point.

The expression for reading intensities has been proved in the text. The statement on convergence follows from well known results because the process is defined by a contraction mapping for α close enough to 1 (see the proofs section). We conjecture that the process converges for any value of α between 0 and 1.

According to (23), reading intensities depend on statistics on citations although individuals are not assumed to have access to such information. The search is local through the citations in the first batch of papers. The dynamics depends on m_s , the (average) number of articles read through citations per article read by chance, and not on the number m_r of articles read at random at the first step. Note that m_s is bounded by the number of articles cited per article, hence cannot be too large, so that the weight due to the random part α is bounded below by a strictly positive number. The smaller m_s is, the closer to a random process the search is, and the less important the influence of the cites.

Thus the fact that there is an initial random choice in the individuals' search process plays an important role. The impact of the realized initial draws disappear at the journal level, because we assumed the draws independent through readers (and we neglect fluctuations around expected values as explained at the end of Section 2). However the draws give a minimal value for the reading intensity to each journal, thereby preventing each one to disappear, in contrast with the influence model. At the unit level, the draws give chances to each article. But for articles, the precise realized initial draws may matter and influence the subsequent citations.

Reading intensities are still given by the same expression (23) when some assumptions are relaxed. For example the expression is valid if the numbers m_r and m_s are an average per researcher, under the proviso that the variations are uncorrelated with preferences. It is also valid if the cites to an article in i are distributed on average only as that of the whole journal provided that the total number of citations to an article in i is uncorrelated with the distribution of its cites (conditional of being published in i). The expression is also valid in a completely different setup than a search, as we explain now.

Assume that individuals read some papers at random (say because they are influenced by some idiosyncratic factors linked with their environment) and some others as a function of the number

of their citations (not necessarily cited by the papers in the random draw). This supposes the statistics to be known. In that interpretation, there is no lower bound on the weight α . Taking α to be null gives an influence model in which the influence is driven by the adjusted numbers of citations. The process also converges for the adjusted counting method, α equal to zero, as shown in Demange (2010) and the technics developed there might be useful for proving the convergence of the process for any value of α between 0 and 1.

The length of the search has been arbitrarily bounded by two. An individual stops searching once he has read some articles cited in the first batch. The search process extends easily to more than two steps, in which an individual continues to read articles cited in the second batch and so on. The convergence result stated in Proposition 3 is likely to hold for a finite, possibly small enough, number of steps. If instead the search continues and authors are able to read an (almost) infinite number of papers, the reading intensities *within* a period converge to the invariant score, independently of the initial random choice. This property has been used to justify the invariant ranking (see Brin and Page 1998 for example who consider a 'random surfer' who clicks indefinitely on the hyperlinks uniformly at random). Presumably, the dynamics with 'infinite' search is likely to be very close to the one generated by the influence model studied in Section 3. However, it is important to keep in mind a main difference between the infinite search and the influence model. Relying on a ranking avoids the cost associated with search, which can be very large with an infinite search. Compared with the more sensible case of a long but finite search, the availability of a ranking eliminates the impact of the 'noise' generated in a finite search by the first random draw.

Example 1 continued With two journals, $\Pi = \begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}$, the process followed by 1's reading intensity is $r_1^{(t)} = \frac{1}{2}[\alpha + (1-\alpha)f(r_1^{(t-1)})]$ with

$$f(r) = \left[\frac{(1-a)r}{(1-a)r + a(1-r)} + \frac{br}{br + (1-b)(1-r)} \right]$$

and that for 2's reading intensity is derived from the relation $r_1^{(t)} + r_2^{(t)} = 1$. For $\alpha > 0$, there is a unique possible limit point which is always interior, given by the solution to $r_1^* = \frac{1}{2}(\alpha + (1-\alpha)f(r_1^*))$, and the process converges to r_1^* .

The impact of random attachment α on the reading intensities is easily obtained by differentiating the equation defining r_1^* . We get that: $\frac{\partial r_1^*}{\partial \alpha} = \frac{(\frac{1}{2} - f(r_1^*))}{1 - (1-\alpha)f'(r_1^*)}$ which is positive if $r_{1*} < \frac{1}{2}$ and negative if the reverse inequality holds. By separating the cases $b > a$ ($\Leftrightarrow r_1^* > \frac{1}{2}$) and $b < a$, we obtain that as randomness α increases, r_1^* converges monotonically to 1/2. The adjusted number of citations, which is given by $f(r_1^*)$ for 1, also converges monotonically to their fair number. In other words, more randomness lead to reading intensities closer to the "fair" number 1/2.

To complete the analysis, take α equal to zero. This case surely does not occur in the search model but as seen before, the process can be interpreted as a pure influence model in which individuals are influenced by the adjusted counting scores. Figure 1 represents the support of the limit points according to the values of a and b . (The two parametric curves are derived from equations $f'(0) = 1$ and $f'(1) = 1$.)

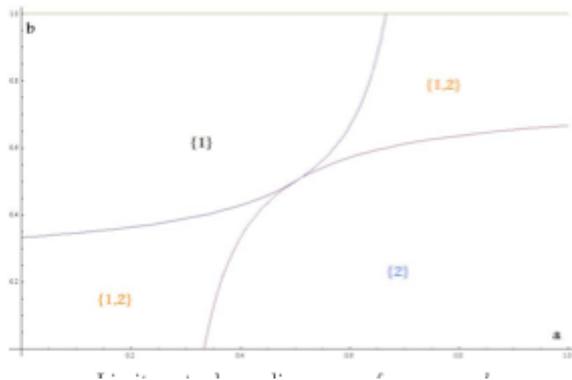


Figure 1: Limits sets depending on preferences a, b

The extreme values 1 and 0 are always fixed points. However the process always converges to a unique point, which is the interior point $r_1^* = \frac{1}{2} + \frac{b-a}{(1-2a)(1-2b)}$ when it is between 0 and 1. This is to be contrasted with the process driven by influence in which several stable points are possible.

A comparison between the two processes shows that the limits differ and it cannot be said that one process is more discriminant and tends to eliminate more journals: The values of a and b for which both journals survive differ in the two processes and are not nested. On one hand, both journals can stay alive simultaneously in the search model for values of a and b smaller than $1/2$ whereas only one survives in the influence model. On the other hand, for a and b both larger than $1/2$, one journal may be eliminated in the search model but not in the influence model.

It is worth pointing that in the counting method self-citations are not enough to make oneself 'stable', for example $\{1\}$ is not necessarily stable for a small enough. Thus the resilience of a subset does not only depend on the preferences of that subset. This is to be contrasted with the invariant method, as followed by Proposition 2.

5 Links with preferential attachment and concluding remarks

The dynamics of the influence model have similarities with models of 'preferential attachment' in growing networks, as described by Barabasi, Albert, and Jeong (1999) for instance. At each period,

a new node is born and chooses to attach itself to existing nodes with a probability proportional to their degree. This process leads to a time-invariant scale-free distribution of degrees in the network. A more micro-founded point of view for justifying preferential attachment is presented in Jackson and Rogers (2005). No precise nor broad knowledge of the degree distribution is needed for the process: the new node attaches itself to a number of nodes at random, and then to a number of these random nodes' neighbors. A node with a greater number of neighbors (degree) is thus more likely to be found via the search process and to be attached to, leading again to a 'rich-get-richer' phenomenon. The model may thus provide a foundation for preferential attachment in contexts where information is only local, such as collaboration networks. Our influence model (in which reading intensities are proportional to some aggregate statistics) and search model (in which reading intensities are derived from aggregation of individuals' behavior) bear similarities with these two approaches. There are two main differences.

The first difference is that the statistical properties in a growing network -especially the power law for the degree of nodes- drastically differ with those obtained with a fixed number of nodes. This can be seen by considering (generalized) Polya's urn problem, also called balls and bins as described by Chung et al. (2003). In the finite problem, there is a finite number of bins. Balls are introduced one at a time, and each new ball is placed in a bin with a probability proportional to m where m is the number of balls in that bin (or more generally proportional to m^γ). In the infinite problem there is a probability that a new bin is formed at each date, like in a growing network. The statistical properties of the finite and infinite problems differ, partly because one must look at different statistics. In a finite problem, as the number of balls in each bin is typically going to infinity, one looks at the proportions of the balls in the bins, and in particular whether one bin gets an increasingly large proportion of the balls, thanks to chance at the initial draws and the reinforcement mechanism. In an infinite problem, the number of balls in each bin is typically finite and one is interested at the distribution of this number per bins and in particular whether it follows a power law.

Although in a setting a little more complex, our model is one in which 'balls' (citations) are placed into a finite number of bins ('journals') and we look similarly at the behavior of the proportions of the cites, or some transformed versions of these proportions as given by the scores. According to our result, multiplicity may arise in the influence model but not in the search model, provided α is small enough. If instead of considering journals, we considered articles, then the problem would become one with a growing set and one could investigate the distribution of the number of citations per article, in particular under which condition it follows a power law.

The second and important difference is that (some form of) preferences influence the 'attachment'

process (the throwing of the balls). Whereas our modeling of reading intensities is similar to a preferential attachment model, new citations also depend on preferences. Our aim is precisely to study the interaction between these two elements. The diversity of preferences is crucial for allowing several journals to survive. As results from Proposition 1 for example, the whole set of journals N can survive only if preferences can be aggregated so as to equalize all scores.

Concluding remarks The paper develops a model in which researchers are influenced by the invariant ranking on journals in order to choose the papers they read when conducting research. This generates a joint dynamical process on citations and rankings. We show that the impact is strong. In particular the invariant method induces strong self-enforcing effects, which explain the possibility of multiple stable rankings. These effects are compared with those induced by a finite search process in which the cites of the read papers influence subsequent readings. On average over the population of researchers and articles, we show that it is the number of citations obtained by a journal, properly adjusted by cites intensities, that affects reading behavior. Although citations matter, self-enforcing mechanisms are not at work, and a unique limit reading behavior obtains under plausible conditions.

This work is a preliminary investigation on the influence of a ranking. Several questions could be addressed.

The invariant ranking is defined in the context of the 'judgment by peers' because it ranks the entities on the basis of their citations/votes: the entities to be ranked are also the ones which judge. There are other relevant ranking methods in such a situation. For example, in contrast to the invariant method, Kleinberg (1999) proposes to distinguish between the capacity of an entity to judge from that to perform. de Clippel *et al.* (2008) analyze a family of methods in which the score of an entity is not affected by its own citations. It would be interesting to conduct the same type of analysis as in this paper to see whether self-enforcing mechanisms are still present. Some preliminary results suggest that the multiplicity of stable rankings that is due to a self-enforcing mechanism in the invariant method does not arise in some other methods. For example, when influence is driven by the counting method, the process converges to a unique limit (Demange 2010).

There is a variety of available rankings, which differ not only by the method used to compute them, but also by the journals on which cites are drawn. Presumably, researchers are influenced by the ranking that fits better their needs and their preferences, or they are influenced by several rankings, or they are influenced by the most 'popular' rankings. This would call for a more elaborate model to explain the choice of the ranking. More generally, the influence of a ranking is modeled in a simple but crude way. A natural line of research would be to give some micro-foundations to that influence.

6 Proofs

PROOF OF PROPOSITION 1.

PROOF OF 1. The dynamics writes for each i

$$v_i^{(t+1)} = \sum_j p_{ij}^{(t)} v_j^{(t+1)} \quad (24)$$

in which $p_{ij}^{(t)}$ is given by (7):

$$p_{ij}^{(t)} = \frac{\mu q_{ij}^{(t-1)} + (1-\mu)q_{ij}^{(t)}}{\sum_\ell [\mu q_{\ell}^{(t-1)} + (1-\mu)q_{\ell}^{(t)}]}, \text{ where } q_{ij}^{(t)} = \frac{\pi_{ij} v_i^{(t-1)}}{\sum_\ell \pi_{\ell j} v_\ell^{(t-1)}}.$$

Hence

$$\frac{v_i^{(t+1)}}{\mu v_i^{(t-1)} + (1-\mu)v_i^{(t)}} = \sum_j \pi_{ij} \frac{v_j^{(t+1)}}{\sum_\ell \pi_{\ell j} [\mu v_\ell^{(t-1)} + (1-\mu)v_\ell^{(t)}]}. \quad (25)$$

Let v^* be a limit point of the sequence $(v^{(t)})$ and I its support (i.e., the set of indices for which v_i^* is strictly positive). Define the positive vector $x = (x_j)_{j \in I}$ in \mathbb{R}^I by

$$x_j = \frac{v_j^*}{\sum_{\ell \in I} \pi_{\ell j} v_\ell^*} \text{ for each } j \text{ in } I. \quad (26)$$

For each i in N , the right hand side of (25) tends to $\sum_{j \in I} \pi_{ij} x_j$.

For each i in I , since $v_i^* > 0$, the left hand side of (25) tends to 1, and we obtain $1 = \sum_{j \in I} \pi_{ij} x_j$ for i in I , or in matrix form $\Pi_{I \times I} x = \mathbf{1}_I$.

For i not in I , we prove by contradiction that inequality $1 \geq \sum_j \pi_{ij} x_j$ must hold. Otherwise, for some $k > 1$ $\sum_j \pi_{ij} x_j^{(t)} \geq k$ for t large enough. This implies

$$\frac{v_i^{(t+1)}}{\mu v_i^{(t-1)} + (1-\mu)v_i^{(t)}} \geq k > 1, \quad t \geq t_0 \quad (27)$$

We show that the sequence $v_i^{(t)}$ cannot converge to zero. For that consider sequences defined by the linear second order equation in which inequality (27) is replaced by an equality:

$$u^{(t+1)} - k(1-\mu)u^{(t)} - k\mu u^{(t-1)} = 0. \quad (28)$$

The characteristic equation $\rho^{(2)} - k(1-\mu)\rho - k\mu = 0$ has two real solutions ρ_1 strictly larger than 1 (because $k > 1$), the other one ρ_2 negative.

All solutions to (28) are linear combinations of ρ_1^t and ρ_2^t , $a\rho_1^t + b\rho_2^t$. Let u be the solution that takes on the same values as $v_i^{(t)}$ at t_0 and $t_0 + 1$. Since the $v_i^{(t)}$ are positive, $u^{(t)}$ is positive at the two consecutive dates t_0 and $t_0 + 1$ so the weight a on ρ_1^t must be positive. Now, using (28) and (28), a simple induction argument implies $v_i^{(t+1)} \geq u^{(t+1)}$ for $t \leq t_0$. The sequence $u^{(t+1)}$ is unbounded, a contradiction.

Let us prove now that given a vector x that satisfies (11), there is a unique positive ranking vector v_I on I that satisfies (26) for all j in I . The unique candidate limit point is then obtained by setting the components outside I to zero. To see this, write equations (26) as

$$\sum_{\ell \in I} \pi_{\ell j} x_j v_\ell = v_j \text{ for each } j \in I.$$

They state that v_I is a positive left eigenvector associated to eigenvalue 1 of the $I \times I$ matrix B with elements $b_{\ell j} = \pi_{\ell j} x_j$. Equation $\Pi_{I \times I} x = \mathbf{1}_I$ writes $\sum_j b_{\ell j} = 1$ for each ℓ in I , hence $\mathbf{1}_I$ is a positive (right) eigenvector with eigenvalue 1 of the positive matrix B . This implies the existence of the positive (left) eigenvector v_I . This vector is unique (up to the normalization) because B is irreducible (since off diagonal elements of Π are assumed all positive and $x \gg \mathbf{0}$). ■

PROOF OF 2.

Point 2 is proved by using a theorem of the alternative, see e.g. Gale (1960). Condition (11) can be written as a set of linear inequalities by introducing an additional positive variable x_0 :

$$\text{there is } x_0 > 0, x \text{ in } \mathbb{R}^I, x \gg \mathbf{0}_I, \text{ such that } \Pi_{I \times I} x = x_0 \mathbf{1}_I \Pi_{N-I \times I} x \leq x_0 \mathbf{1}_{N-I}$$

or

$$\text{there is } \begin{pmatrix} x_0 \\ x \end{pmatrix} \gg \mathbf{0}_{0 \cup I} \text{ such that } (\mathbf{1}_I - \Pi_{I \times I}) \begin{pmatrix} x_0 \\ x \end{pmatrix} = \mathbf{0}_I, (\mathbf{1}_{N-I} - \Pi_{N-I \times I}) \begin{pmatrix} x_0 \\ x \end{pmatrix} \geq \mathbf{0}_{N-I} \quad (29)$$

From a theorem of the alternatives, either there is a solution to (29) or

$$\text{there are } y_I \text{ and } y_{N-I}, y_{N-I} \geq \mathbf{0}_{N-I}, \text{ such that} \quad (30)$$

$$y'_I (\mathbf{1}_I - \Pi_{I \times I}) + y'_{N-I} (\mathbf{1}_{N-I} - \Pi_{N-I \times I}) < \mathbf{0}, \quad (31)$$

where a ' denotes the transpose of a vector. The vector on the left hand side of (31) is a $I + 1$ row vector with first component $\sum_{j \in N} y_j$ and with other components indexed by i in I given by $-\sum_{j \in N} \pi_{ji} y_j$. Thus (31) writes as

$$\sum_{j \in N} y_j \leq 0 \text{ and } -\sum_{j \in N} \pi_{ji} y_j \leq 0, i \in I \text{ with at least one strict } <. \quad (32)$$

This implies that not all y_j are null. Let K be the set of i with y_i negative. We prove that K is a non empty subset of I , and is never the whole set N (a possibility only if I is N). If K was empty, the sum $\sum_{j \in N} y_j$ would be positive (since y is not null), a contradiction. Furthermore since $y_{N-I} \geq 0$ by (30), K is a subset of I . Also K is a strict subset of N because otherwise some term $\sum_{j \in J} \pi_{ji} y_j$ would be positive. Using set K (33) can be rewritten as

$$\sum_{k \in K} |y_k| \geq \sum_{j \in N-K} y_j \text{ and } \sum_{k \in K} \pi_{ki} |y_k| \leq \sum_{j \in N-K} \pi_{ji} y_j \text{ for each } i \in I. \quad (33)$$

The inequality on the left can be replaced by an equality: if it is not binding, the non-negative y_j for j on $N - K$ can be slightly increased; this increases each sum $\sum_{j \in N-K} \pi_{ji} y_j$ so surely the inequalities on the right of (33) still hold. Hence one may assume $\sum_{k \in K} |y_k| = \sum_{j \in N-K} y_j$. There is surely a non-null component for y in each set K and $N - K$. Thus defining $\alpha_k = \frac{|y_k|}{\sum_{k \in K} |y_k|}$, $k \in K$ and $\beta_j = \frac{y_j}{\sum_{j \in N-K} y_j}$, $j \in N - K$, the inequalities on the right of (33) divided by the common sum $\sum_{k \in K} |y_k|$ or $\sum_{j \in N-K} |y_j|$ yields

$$\sum_{k \in K} \pi_{ki} \alpha_k \leq \sum_{j \in N-K} \pi_{ji} \beta_j \text{ for each } i \in I$$

the desired inequalities (12). Furthermore α and β are non-negative and their components sum to 1. ■

PROOF OF PROPOSITION 3.

Point 1. Recall that $D(v)$ is the dominant eigenvector of $\Pi(v)$ that satisfies $D(v) \cdot v = 1$. For a vector v in Δ_I , $v = (v_I, \mathbf{0}_{N-I})$, a dominant eigenvector of $\Pi(v)$ satisfies

$$\begin{pmatrix} \Pi_{I \times I}(v_I) & \mathbf{0}_{I \times N-I} \\ \Pi_{N-I \times I}(v_I) & \mathbf{0}_{N-I \times N-I} \end{pmatrix} \begin{pmatrix} \delta_I \\ \delta_{N-I} \end{pmatrix} = \begin{pmatrix} \delta_I \\ \delta_{N-I} \end{pmatrix}.$$

This is equivalent to $\Pi_{I \times I}(v_I) \delta_I = \delta_I$ and $\Pi_{N-I \times I}(v_I) \delta_I = \delta_{N-I}$. By the first equation δ_I is a positive eigenvector of $\Pi_{I \times I}(v_I)$ associated to the eigenvalue 1. Such an eigenvector exists because the total of each column of $\Pi_{I \times I}(v_I)$ is equal to 1. Furthermore, since v is null outside I , $\sum_{i \in N} \delta_i v_i = \sum_{i \in I} \delta_i v_i = 1$. Thus $D(v)$ is found in a unique way by choosing for $D(v)_I$ the dominant eigenvector δ_I of $\Pi_{I \times I}(v_I)$ that satisfies $\sum_{i \in I} \delta_i v_i = 1$ and by setting $D(v)_{N-I} = \Pi_{N-I \times I}(v_I) \delta_I$.

Point 2. Conditions (19) require that all components of $D(v)_{N-I}$ are less than k , $k < 1$ for each v in Δ_I . The function that assigns the eigenvector $D(v)$ of $\Pi(v)$ is continuous. Hence for $k' < 1$, $k < k'$, there is an ϵ -neighborhood \mathcal{V} of Δ_I such that the components of $D(v)_{N-I}$ are smaller than k' for v is in \mathcal{V} , meaning that the growth rates of the components indexed by $N - I$ of v are smaller than k' . Thus once the ranking v enters into \mathcal{V} , it stays there. Therefore, the components indexed by $N - I$ converge to zero. This proves the resilience of I . ■

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