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ECOLE D'ECONOMIE DE PARIS

**WORKING PAPER N° 2007 - 24**

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# The Cournot Outcome as the Result of Price Competition *

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This version May 2008

**Abstract**

In a homogeneous product duopoly with concave revenue and convex costs we study a two stage game in which, first, firms engage simultaneously in capacity (production) and, after production levels are made public, there is sequential price competition in the second stage. Randomizing the order of play in the price subgame, we can find: (i) that the Cournot outcome can be sustained as a pure strategy subgame perfect Nash equilibrium (SPNE) of the whole game, (ii) a SPNE in which firms produce strictly more than the Cournot outcome.

KEYWORDS: Subgame Perfect Nash Equilibrium, Price, Quantity, Pure Strategies, Cournot.

JEL CLASSIFICATION: C73, D43, L13.

## 1 Introduction

In their paper of 1983, Kreps and Scheinkman state that the difference between Cournot and Bertrand competition is more than just the strategy space, but that timing of decisions is also relevant. To illustrate this, they study and solve a Bertrand like duopoly model of competition where timing of decision is inverted. Capacity decision is made simultaneously and before price decision (as opposed to Bertrand-Edgeworth models where the choice of capacity and price is interpreted as being simultaneous), and the low priced firm may not serve all the demand at her price (as it is in the Bertrand approach) due to capacity constraints. In a two stage game where firms first set simultaneously capacity and then engage in simultaneous price competition with demand rationed following the efficient rationing rule, the unique Subgame Perfect Nash Equilibrium (SPNE) has as outcome the Cournot quantities and prices. This result has been subject to criticism. Davidson and Deneckere (1986) argue that the Kreps and Scheinkman result depends strongly on the chosen rationing rule and that in fact, it is not likely that the Cournot outcome may raise as the result of the price game in most of the cases. However, their proof assumes value zero for the cost in the first (capacity-setting) stage. Moreover, for an important range of capacities the unique Nash Equilibrium of the price-setting subgame in Kreps and Scheinkman (1983) is in non degenerate mixed strategies, rendering the result hard to interpret and motivating a literature that argues that the only possible pure strategy Nash Equilibrium of price games must be the competitive outcome (see Allen and Hellwig (1986) or Dixon (1992)).

Related to the first issue Madden (1998) shows, in a slightly different framework, that for uniformly elastic demands the Kreps and Scheinkman result holds, even if proportional rationing

*This article was conceived while preparing my dissertation at Paris School of Economics (PSE) under the supervision of Roger Guesnerie and Alejandro Jofré.

†I would like to thank Roger Guesnerie and Leonardo Basso for helpful conversations; Dan Kovenock and Subir Bose for valuable suggestions; and the audience in the $8^{\text{th}}$ SAET Conference for their questions and comments.

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is used. Related to the second one, we have that departing from the original constant marginal cost and no capacity constraints hypothesis (Bertrand, 1883), we pass to models with voluntary trading constraint at the competitive supply or strictly increasing marginal costs (Edgeworth, 1925), which lead to problems with existence of equilibrium in pure-strategies. Liberating the supply constraint to any possible quantity in order to restore pure-strategy equilibria existence when convex costs are present, we end up with the competitive outcome. These Bertrand-Edgeworth approaches to oligopolistic competition, however, still feature the main difference between Bertrand and Cournot models that Kreps and Scheinkman were assessing. Namely, the production decision is actually done after prices and demands are revealed. For instance, in Dixon (1992) the equilibrium-stated-intention of serving the whole demand can be interpreted as a non credible threat or a unsustainable engagement since once a firm realizes that it has the lowest price it is not rational to serve more than the competitive supply at the announced price. Thus we are back at the timing of decisions issue.

The objective of this note is then threefold: without changing any assumptions on primitives and maintaining the timing of decisions (prices after quantities), provide a setting where the Cournot outcome can be sustained as a SPNE in pure strategies of the duopoly game, give space for the possibility of finding non-Cournot outcomes sustained as SPNE in pure strategies and give a clear condition on rationing rules that rejects Cournot as the outcome of equilibrium.

We explore a setting where in a first stage firms set capacity simultaneously and in a second stage engage in sequential price competition where the order of play is assigned randomly. The possibility of obtaining clear pure strategy subgame perfect equilibria for each subgame will help to clearly understand the strategic interaction in the duopoly framework that results in the Cournot outcome, although competition in prices is assumed. We will see that under the same assumptions on primitives as in the seting of Kreps and Scheinkman the Cournot outcome is sustained as a subgame perfect equilibrium. We will also see a simple condition on rationing rules to reject the Cournot outcome in both the Kreps and Scheinkman and the current setting. A last new feature of the approach presented in this paper is the possibility of finding non Cournot outcomes sustained as SPNE without changing any of the assumptions on demand rationing (Davidson and Deneckere, 1986; Dixon, 1992), nor on concavity or continuity of involved functions nor cost structure (Vives, 1986; Deneckere and Kovenock, 1992, 1996), but only based on uncertainty of leadership and the dynamic nature of the price subgame. We explore this issue and give conditions that allow for it to occur.

The paper provides a clear and more tractable story for the Cournot outcome as the result of price competition.

# 2 Preliminaries

Consider a market with two identical firms $a$ and $b$ that produce a homogeneous good. Aggregate demand for the firms outputs as a function of price is given by the function $D : \mathbb{R}_+ \to \mathbb{R}_+$. We assume that $0 < D(0) < +\infty$, that there is an upper bound for the set where $D(p) > 0$, and that $D$ is continuous and strictly decreasing in this set. With this we can consider the Price function $P : \mathbb{R}_+ \to [0, P(0)]$ defined by $P(q) := D^{-1}(q)$ for $q \in [0, D(0)]$ and $P(q) := 0$ for $q > D(0)$. The cost of producing is given by a function $C : \mathbb{R}_+ \to \mathbb{R}_+$.

We make the following assumptions:

1. The function $p \to pD(p)$ is strictly concave.

2. Function $C$ is strictly increasing, convex, and $C(0) = 0$; $P$ satisfies $P(0) > \lim_{k\to 0} C'(k) > 0$.

With these elements we can define the Cournot best response mapping $r_C : \mathbb{R}_+ \to [0, D(0)]$ as:

$$r_C(k) := \begin{cases} \operatorname{argmax}_{x\in[0,D(0)-k]} \{P(k+x)\,x - C(x)\} & \text{if } k \leq D(0) \\ 0 & \text{if } k > D(0) \end{cases}$$

For simplicity we note $r(k) \equiv r_0(k)$. Assumptions 1 and 2 imply that $r_C$ is strictly decreasing when it is positive, that the function $k \to r(k) + k$ is non-decreasing (strictly increassing in the interval where $r(k) > 0$) and that if we have two cost functions $C_1$ and $C_2$ with $C_1' > C_2'$ then $r_{C_1}(y) < r_{C_2}(y)\ \forall\ y \in [0, D(0)]$ such that $0 < r_{C_2}(y)$. Thus, for each function $C$ as above there is a unique Cournot equilibrium outcome with price and production quantities $(p_C^*, k_C^*, k_C^*)$ defined by:

$$r_C(k_C^*) = k_C^* \quad D(p_C^*) = 2k_C^* \quad \text{or equivalently} \quad p_C^* = P(2k_C^*)$$

And if $C_1' > C_2'$ then $k_{C_1}^* < k_{C_2}^*$ and $p_{C_1}^* > p_{C_2}^*$. We will note $k^*$ and $p^*$ for $k_0^*$ and $p_0^*$ respectively. Finally we note $R(k)$ the value:

$$R(k) := P(r(k) + k)\, r(k),$$

the revenue associated with the best response to a quantity $k$ when there is no production cost and $p^m$ the zero cost monopoly price[1].

We will study a game in which these firms engage in a two stage competitive situation. In a first stage, firms simultaneously and independently build capacity (which can be understood as a decision in production quantities). A vector of capacity quantities $k = (k_a, k_b)$ is understood as an available supply on the market of $k_a + k_b$ to be traded after the second stage. In the second stage, after learning how much capacity their opponent installed, firms set prices sequentially.

Net payoffs in the setting of Kreps and Scheinkman (1983) are given by the function $\pi_i : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}\ i \in \{a, b\}$:

$$\pi_i(k, p) := \begin{cases} p_i \min\{k_i, D(p_i)\} - C(k_i) & \text{if } p_i < p_j \\ p_i \min\left\{k_i, \max\left\{\frac{D(p_i)}{2}, D(p_i) - k_j\right\}\right\} - C(k_i) & \text{if } p_i = p_j \\ p_i \min\{k_i, \max\{0, D(p_i) - k_j\}\} - C(k_i) & \text{if } p_i > p_j \end{cases}$$

Kreps and Scheinkman (1983) show that in this setting, the only subgame perfect Nash equilibrium outcome of a two stage game with *simultaneous* price competition in the second stage, is the Cournot outcome. Note that we are using, as in their work, the surplus maximizing rationing rule. The choice of rationing rule is not obvious. For instance, Davidson and Deneckere (1986) show that not all standard rationing rules lead to obtain Cournot price and quantity as the outcome of two stage game with simultaneous price setting. We address this issue as well in this work in a further section.

## 3 Capacity Constrained Price Competition

In this section we analyze subgame perfect equilibria of the sequential price game with capacity constraint. In this stage, each firm $i$ has installed a capacity $k_i\ i \in \{a, b\}$ and firms know the capacity installed by the rival. In this stage, since cost has already been sunk, the only relevant part of payoff is the revenue associated to sells. Firms name a price in order to maximize this revenue. Note that in capacity constrained price competition, prices smaller than $P(k_a + k_b)$ are strictly dominated and will therefore not be best replies to the rivals price nor be announced in equilibrium. This is true in sequential as well as in simultaneous price competition.

To continue, we first need to define certain elements.

**Definition 3.1.** For $k_i > 0$, $(i, j) \in \{(a, b), (b, a)\}$. We define $\underline{p}(k_i, k_j)$ as follows:

(i) If $k_i \geq r(k_j)$ then let $\underline{p}(k_i, k_j)$ be the smallest solution[2] of the equation:

$$p \min\{k_i, D(p)\} = R(k_j) \tag{3.1}$$

[1] $p^m = \operatorname{argmax} pD(p)$

[2] Concavity of $pD(p)$ and the fact that it takes value 0 in $p = 0$ and $p = P(0)$, assure that (3.1) has at most two solutions.

(ii) If $k_i < r(k_j)$ then $\underline{p}(k_i, k_j) := P(k_a + k_b)$

$R(k_j)$, the revenue for firm $i$ when playing the best response to $j$ in the zero cost Cournot game, is the maximum revenue that $i$ can receive by being the high priced firm when having enough capacity. Indeed, revenue for $i$ being the high priced firm is equal to $p_i \min\{k_i, \max\{0, D(p_i) - k_j\}\}$ which, if $k_i \geq r(k_j)$ turns into $p_i \max\{0, D(p_i) - k_j\}$ and is maximized at $p_i = P(r(k_j) + k_j)$ when $k_j \leq D(0)$ with the value $P(r(k_j) + k_j)\, r(k_j) \equiv R(k_j)$.[3]

When $k_i \geq r(k_j)$, price $\underline{p}(k_i, k_j)$ equals the payoff of being the low priced firm to $R(k_j)$, that is, price $\underline{p}(k_i, k_j)$ makes firm $i$ indifferent between being the high priced firm and the low priced firm. If $k_i < r(k_j)$, then the previous analysis does not make sense, but in this case payoff is bounded from above by $pk_i$ for any $p \leq P(k_a + k_b)$ and if $p > P(k_a + k_b)$ then it is in the decreasing part of $p \max\{0, D(p) - k_j\}$.

Roughly, if firm $j$ sets a price lower than $\underline{p}(k_i, k_j)$ then firm $i$ would rather set the highest between the residual demand monopoly price $P(r(k_j) + k_j)$ and the competitive price $P(k_a + k_b)$, and if $j$ sets a higher price then firm $i$ would rather undercut or match. Finally note that if $k_i \geq r(k_j)$ then $\underline{p}(k_i, k_j) > P(k_a + k_b)$.

A very useful result regarding the limit prices $\underline{p}(k_i, k_j)$ defined in (3.1) is:

**Lemma 3.2.**

$$\left.\begin{aligned} \min\{D(0), k_i\} &> k_j \\ k_j &> r(k_i) \end{aligned}\right\} \implies k_i R(k_i) < k_j R(k_j)$$

*Moreover if $k_i, k_j > 0$,*

$$\underline{p}(k_i, k_j) > \underline{p}(k_j, k_i)$$

We will name by firm 1 the firm that sets the price first and firm 2 the one who goes second. We obtain a capacity constrained Stackelberg price game. This kind of game has been analyzed in Deneckere and Kovenock (1992), and more recently in Dastidar (2004), where capacity is assumed to be the competitive supply at the announced price. In both approaches, following Deneckere and Kovenock (1992), demand is assigned first to the follower and then to the leader in order to have no problems with upper semi-continuity of payoffs at their suppremum. We will make this same assumption on tied prices: all demand goes first to the follower (and is not equally shared as in Kreps and Scheinkman (1983)). Thus when matching a price $p_1$, firm 2 will net $\pi_2(k, p_1, p_1) = \lim_{p \nearrow p_1} \pi_2(k, p_1, p)$. Formally, tied prices revenues become,

$$\begin{aligned} \pi_1(k_1, k_2, p, p) &:= p \min\{k_1, \max\{0, D(p) - k_2\}\} \\ \pi_2(k_1, k_2, p, p) &:= p \min\{k_2, , D(p)\} \end{aligned}$$

That is, given capacities $(k_a, k_b)$ and price $p_1$, for firm 2 we use the *upper semi-continuous hull* of the payoff function of the Kreps and Scheinkman setting, and for firm 1, given $p_2$ we use the *lower semi-continuous hull* of the payoff function of the Kreps and Scheinkman setting.

With this "slight" modification the sequential price-setting subgame has always (meaning for all capacity pairs) a subgame perfect equilibrium in pure strategies. The main implication of the modification is that now player two has always a best response (since for all $p_1$ it's payoff as a function of price is *sup-compact* and upper semi-continuous). The strategy for firm 2 will simply be to match the price of firm 1 if it is greater or equal to $\underline{p}(k_2, k_1)$ and take the residual monopoly price if not. Given the capacities set in the first stage, the best response mapping $p_2^*(\cdot \mid k_1, k_2) : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ for firm 2 as a function of price $p_1$, set by firm 1, is:

$$p_2^*(p_1 \mid k_1, k_2) = \begin{cases} \mathbb{R}_+ & \text{if } p_1 = \underline{p}(k_2, k_1) = 0 \\ \max\{P(k_a + k_b), P(r(k_1) + k_1)\} & \text{if } p_1 \leq \underline{p}(k_2, k_1) \text{ and } \underline{p}(k_2, k_1) > 0 \\ \min\{p_1, \max\{P(k_2), p^m\}\} & \text{if } p_1 \geq \underline{p}(k_2, k_1) \end{cases} \quad (3.2)$$

[3]When $k_j > D(0)$, the function $p_i \to p_i \max\{0, D(p_i) - k_j\}$ is constant equal to zero and so it is maximized in $\mathbb{R}_+$ with value 0 ($\equiv R(k_j)$ since both $P(r(k_j) + k_j)$ and $r(k_j)$ equal 0).

The outcome of the game will depend on the action of firm 1, since it's profit maximizing price, given the strategy of firm 2, may depend on the first stage choice of capacity through the relative values of the prices $\underline{p}(k_1, k_2)$ and $\underline{p}(k_2, k_1)$. Proposition 3.3 below is similar to Theorems 2 and 3 of Deneckere and Kovenock (1992). The difference is that here we explicitly give the payoff of the low priced firm in the subgame where capacities are in the range where the simultaneous price-setting game has equilibrium in mixed strategies (Kreps and Scheinkman, 1983).

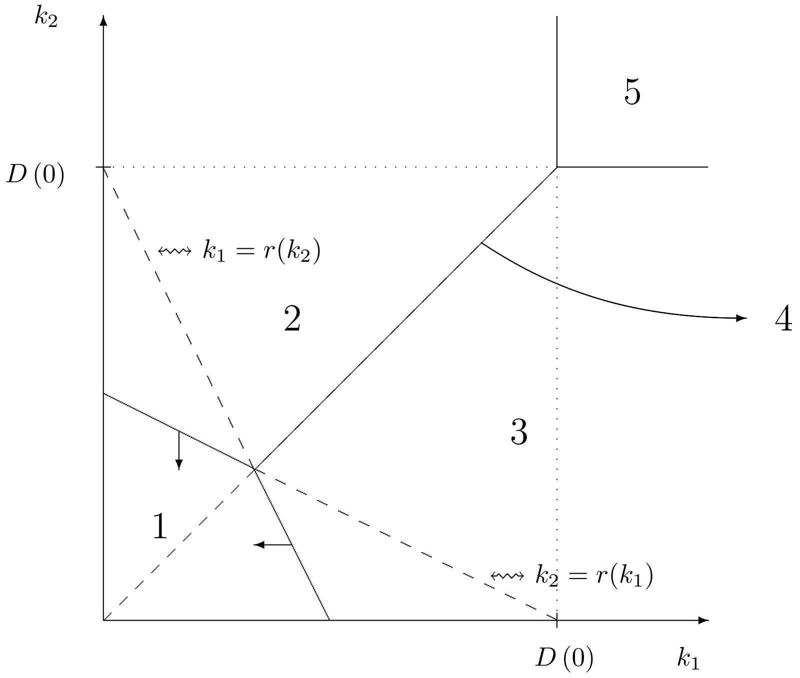


Figure 1: The five regions of interest in Proposition 3.3. The arrows indicate at what region belong the boundaries.

**Proposition 3.3.** *With respect to the sequential price-setting subgame we can distinguish five regions of interest:*

1. *If $k_1 \leq r(k_2)$ and $k_2 \leq r(k_1)$, there is a unique pure strategy subgame perfect Nash Equilibrium (*SPNE*) in which both firms set price $P(k_1+k_2)$. Payoffs are $\pi_1(k_1, k_2) = P(k_1+k_2)k_1$ and $\pi_2(k_1, k_2) = P(k_1+k_2)k_2$.*

2. *If $k_1 < k_2$, $k_1 < D(0)$ and $k_2 > r(k_1)$ then there is a unique* SPNE *where firm 1 sets price $\underline{p}(k_2, k_1)$ and firm 2 responds by setting $P(r(k_1)+k_1)$. Payoffs are $\pi_1(k_1, k_2) = \underline{p}(k_2, k_1)\, k_1$ and $\pi_2(k_1, k_2) = R(k_1)$*

3. *If $k_1 > k_2$, $k_2 < D(0)$ and $k_1 > r(k_2)$ then there there is a unique* SPNE *where both firms set price $P(r(k_2)+k_2)$. Payoffs are $\pi_1(k_1, k_2) = P(r(k_2)+k_2)r(k_2) = R(k_2)$ and $\pi_2(k_1, k_2) = P(r(k_2)+k_2)k_2$.*

4. *If $k_1 = k_2 = k$, $k < D(0)$ and $k > r(k)$, then there are two* SPNE *in pure strategies. One where firm 1 sets $\underline{p}(k, k)$ and firm 2 responds by setting $P(r(k)+k)$ with payoffs $\pi_1(k, k) = \underline{p}(k, k)\, k = R(k)$ and $\pi_2(k, k) = R(k)$. And another where both firms set $P(r(k)+k)$ and net $\pi_1(k, k) = P(r(k)+k)r(k) = R(k)$ and $\pi_2(k, k) = P(r(k)+k)k$.*

5. *If* $\min\{k_1, k_2\} \geq D(0)$*, then there is a continuum of* SPNE *indexed by* $p_1^*$ *where firm 1 sets* $p_1^* \in \mathbb{R}_+$ *and firm 2 sets* $\min\{p_1^*, p^m\}$ *if* $p_1^* > 0$ *and sets any price* $p_2^* \in \mathbb{R}_+$ *if* $p_1^* = 0$*. Payoffs are* $\pi_1(k_1, k_2) = 0$ *for firm 1 and* $\pi_2(k_1, k_2) = \min\{p_1^*, p^m\} D(\min\{p_1^*, p^m\})$

Suppose we have $k_a > k_b$ and $k_a > r(k_b)$, so that we stand in the case were in the simultaneous price game the equilibrium is in non degenerate mixed strategies and the residual monopoly profit for firm $a$ is greater than the market clearing profit [4]. If firm $b$ is the leader, then it knows that if it sets a price higher than $\underline{p}(k_a, k_b)$ it will be undercut and then it would get the residual demand, but since $\underline{p}(k_b, k_a) < \underline{p}(k_a, k_b)$ firm $b$ would net strictly more by setting a price $\underline{p}(k_b, k_a) < p_b \leq \underline{p}(k_a, k_b)$ since then firm $a$ would rather take the residual demand. On the other hand, if firm $b$ is the follower, since it is the small firm, in order to sell it's whole production by being the low priced firm, it can set a bigger price than firm $a$ ($P(k_b) > P(k_a)$). Firm $a$ then assumes it will be undercut and thus maximizes it's payoff, as the high priced firm, setting the residual demand monopoly price $P(r(k_b) + k_b)$. Then, firm $b$ will set the same price (this is also true in the case $k_a = k_b = k > r(k)$ since the rule assigns first the demand to the follower and thus it will sell all it's capacity). Then firm $a$ will obtain a payoff $R(k_b)$ with certainty and so it is indifferent between being leader or follower in the price setting game, while firm $b$ would rather go second since it nets $\underline{p}(k_a, k_b) k_b$ if it is leader and $P(r(k_b) + k_b) k_b$ if it is follower and in this capacity region we have $P(r(k_b) + k_b) > \underline{p}(k_a, k_b)$. In Deneckere and Kovenock (1992) they address the "choice of roles" issue rising from this observation, since we could say that if leadership were endogenized, then the large firm would become leader. The small firm would wait for the announcement of the large firm, which would then set the corresponding price first. The authors focus then on games of timing of price announcement.

For the special case $k_a = k_b = k > r(k)$ we have two SPNE, one where firms set different prices and one where both firms set the residual demand monopoly price, but only the second one allows us to have upper semi-continuity of reduced payoff in the capacity setting game. Since the first firm is indifferent between the two equilibria, we turn to upper semi-continuity of payoffs to find possible non Cournot symmetric Nash equilibria of the reduced capacity setting game in the next section. Regarding the second of these equilibria, because of the rationing rule, the firm that announces it's price in second place will net a strictly greater payoff than the leader, therefore considering this SPNE we could say that both firms would prefer to wait for the announcement of the rival before announcing it's own price. The rationing rule assigns the demand first to the follower, which is the sole difference between roles for this strategy profile.

It is well known that when splitting demand on tied prices there is no subgame perfect equilibrium on pure strategies in the price setting subgame, since for many prices $p$ set by the leader, the follower does not have a well defined best response. For many of these cases we can argue that firm 2 infinitesimally undercuts firm 1 when out of the equilibrium path, allowing us to find pure strategy Nash Equilibria of the subgame. That is when the high capacity firm acts as follower. However, we still have the non existence problem when this firm is leader. There is a whole set of pairs $(k_a, k_b)$ for which no equilibrium exists. The problem rises when we analyze the game that begins at the information set described by $k_1 \geq k_2$ and $k_2 > r(k_2)$ (this region of the capacity vector plane falls into items 3 and 4 of Proposition 3.3, see Figure 2). In this price-setting subgame with equal sharing at tied prices there is no pure strategy NE, since it is not optimal for firm 2 to set the price $P(r(k_2) + k_2)$ when firm 1 sets this price, because of the downward jump in its revenue function. Nevertheless, under a subgame perfect equilibrium logic, we can think of the strategy where firm 2 undercuts if firm 1's price is strictly greater than $\underline{p}(k_2, k_1)$ and takes the residual demand monopoly price if not. Given this strategy for firm two, it is optimal for firm 1 to set the price $P(r(k_2)+k_2)$. Then $\forall\, \varepsilon > 0$, there would exist $\delta > 0$ such that firm 2 could announce a price $P(r(k_2)+k_2) - \delta$ in order to secure a revenue $u_2^\delta(k_2) := (P(r(k_2)+k_2) - \delta)k_2$ that satisfies $P(r(k_2)+k_2)k_2 - \varepsilon < u_2^\delta(k_2) < P(r(k_2)+k_2)k_2$ and $\pi_1(k_1, k_2) = P(r(k_2)+k_2)r(k_2) = R(k_2)$. That is, an "intuitive" revenue outcome for this subgame would be $\pi_1(k_1, k_2) = P(r(k_2) + k_2) r(k_2) =$

[4] The simultaneous price capacity constrained subgame has been analyzed by Levitan and Shubik (1972) for the case of linear demand, no cost and equal capacities; and with more generality by Kreps and Scheinkman (1983) and Osborne and Pitchik (1986).

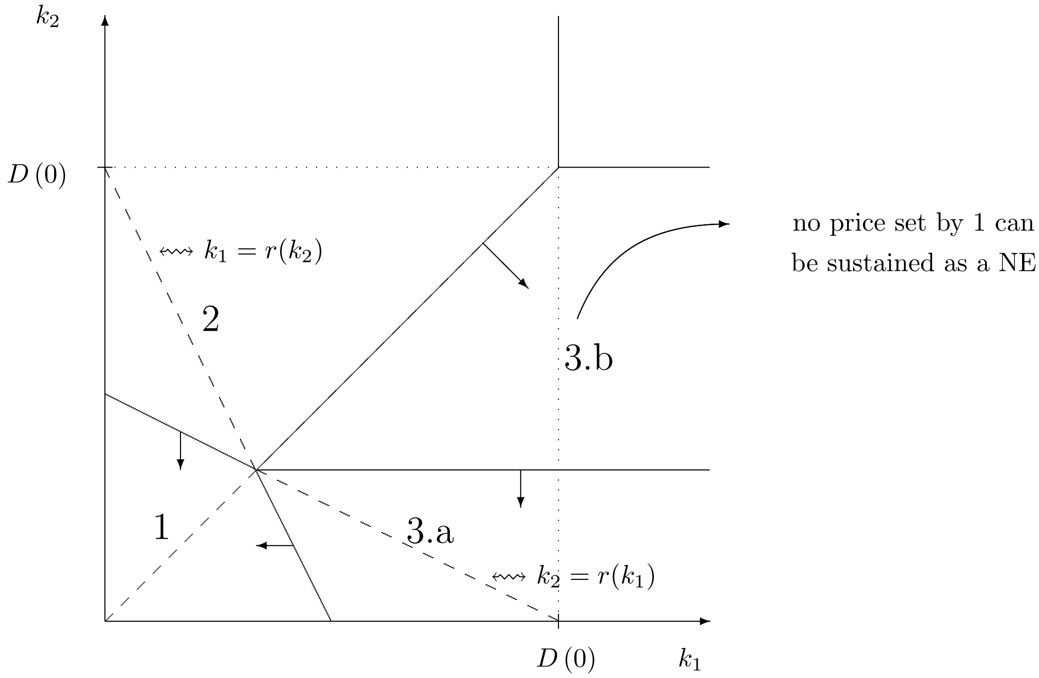


Figure 2: Equilibria with demand splitting at tied prices. In region 3.b there is no (pure strategy) Nash Equilibrium in the sequential price subgame.

$R\,(k_2)$ and $\pi_2\,(k_1, k_2) = u_2^{\delta}\,(k_2) \xrightarrow[\varepsilon \to 0]{} P(r(k_2) + k_2)k_2$. Proposition 3.3 shows that changing the splitting rule on tied prices not only helps to solve the mathematical problem of upper semi-continuity of payoffs, but also does not depart from the intuitive outcome (nor in payoffs nor in strategies) of the price game in the original setting, allowing to formally obtain these intuitive results over price competition with capacity constraints when the small firm is follower. We see that the continuity problem being solved by the Deneckere and Kovenock assumption on demand splitting on tied prices, does not affect the economic conclusions of the model and effectively helps to clarify the behavior of the agents.

## 4 Capacity Setting Reduced Game

Now that we have obtained all the SPNE of the prices-setting subgame, we can study the capacity setting stage of the game. The usual approach would be to use reduced payoff functions obtained from the outcomes of the subgame for each possible capacity pairs of the first stage generating a capacity setting reduced game. The problem is that for the case of $k_a = k_b = k > r\,(k)$ we don't have a unique payoff for the follower. So situations regarding one of the firms setting a capacity $k_i$ greater than $k^*$ have to be studied carefully. Nevertheless, let's introduce the first stage capacity-setting game and the price-setting SPNE associated payoffs.

In this stage firms set capacities simultaneously. They maximize their expected payoff of setting capacity $k_i$ knowing that the choice of the opponent will have an effect on the final profit. When setting capacity, firms do not know whether they will set their price first or second in the price-setting stage. This uncertainty may come, for instance, from the nature of the variable under scrutiny. Capacity building or stock production may occur long before price competition. Even more, once we enter the stage of price competition, and thinking in some kind of endogenous timing or role in the price game, we have that the only preference on roles is for the large firm being follower [5]. As we have said, the small firm is indifferent between announcing its price first or second. Thus it is not clear how long firms can wait before being forced to announce a price. We will assume then that there is an exogenous factor that determines the role in the price setting stage. Firms know, however, in the first stage of the game, that there is a probability $\lambda_i$ of being leader in the price game for firm $i$, $i \in \{a, b\}$. In an incomplete information type of game, these distributions could represent the probability of being impatient in the second stage of the game. They can be interpreted as well as priors or beliefs of the firms about their role in the second stage price game. Similarly, they induce priors onto the possible payoff outcomes (not necessarily roles) of this subgame. Although we allow for the probabilities to be different for each firm and they could well depend on the installed capacities, it is better, to keep track of the exercise, to think of them as being equal to 1/2. As we will see below, the values of the probabilities $\lambda_i$ are irrelevant for the purpose of telling a neat economic story behind the Cournot outcome as a result of price competition. These values become important when seeking for other equilibria of the game, an issue that we address as well in what follows. The interpretation of these distribution as priors may raise as well questions about their genesis. Questions for instance, of endogenous generation of priors, which would inspire an exercise closely related to the choice of roles (Hamilton and Slutsky, 1990; Deneckere and Kovenock, 1992; van Damme and Hurkens, 2004).

The capacity setting reduced game has as players the two firms, with strategy space $\mathbb{R}_+$ and payoff given by the function $\pi_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$,

$$\pi_i(k_i, k_j) := \lambda_i \pi_1(k_i, k_j) + (1 - \lambda_i)\pi_2(k_j, k_i) - C(k_i) \qquad (i, j) \in \{(a, b), (b, a)\}$$

where $\pi_1$ and $\pi_2$ are the profit functions obtained from the equilibria of the sequential price-setting subgame described in Proposition 3.3.

## 4.1 Cournot Outcome

Regarding capacity setting competition, we first notice that the Cournot outcome is a Subgame Perfect Nash equilibrium of the two-stage game with sequential price-setting, as it was in the two-stage game with simultaneous price-setting stage case. If a firm sets the Cournot capacity, then the opponent gets the Cournot payoff up to a certain capacity, namely the Cournot best response for zero cost, and for greater capacities, since the higher capacity firm nets the residual demand monopoly payoff no matter being leader or follower in the second stage game, income remains fixed while cost increases.

**Proposition 4.1.** *The Cournot outcome defined by $k_a = k_b = k_C^*$ and $p_a = p_b = p_C^*$ can be sustained as a* SPNE *of the two-stage game.*

So now we have a clear neat story behind the Cournot outcome as the result of price competition. If firms set small capacities, then price competition produces clear-market prices. Facing Cournot capacity of the rival, a firm that sets a high capacity will net an income that depends only on the capacity set by it's rival. The large firm if being leader in the price game knows it will be undercut. If being follower, the small firm will set the highest price possible at which it can sell it's whole capacity, that is, $\underline{p}(k_i, k_j)$ with $k_i > k_j$, the price that leaves the high firm indifferent between matching and taking the residual demand, so the large firm nets $R(k_j)$.

The Cournot outcome can be sustained as a subgame perfect equilibrium in pure strategies at all stages of the game.

## 4.2 Other Equilibria

Considering the possibility of other equilibria, note that setting the same capacity and it being smaller than the value of the zero cost Cournot equilibrium $k^*$, can not be an equilibrium since

[5] In the absence of naturally generated roles in sequential play (Gal-Or, 1985; Amir and Stepanova, 2006), Hamilton and Slutsky (1990) and van Damme and Hurkens (2004) study game forms where players chose when to play in a sequential game. When choosing the same timing it is assumed that the simultaneous subgame equilibrium payoffs are obtained, which we prefer to avoid in this note, as Deneckere and Kovenock (1992) do in their work.

deviating to the Cournot best response maximizes payoff given the capacity of the rival. It is the same argument as in the previous Proposition. This rules out any pure-strategy Nash equilibrium with both capacities smaller than $k^*$.

However it could be possible to find symmetric Nash equilibria where firms engage in higher capacities. Suppose that the expected reduced payoff (expectation taken over the possibility of being leader or follower) in the first stage game, given that the rival plays a capacity $\bar{k}$ strictly greater than the zero cost Cournot equilibrium $k^*$, is increasing in the interval where there is a difference between being leader or follower, and that the expected reduced payoff of matching this capacity is greater (or equal) than the maximum payoff that could be attained by setting a capacity in the indifference (between being leader or follower) interval (if not void); and consider the following strategies:

> *In the first stage set $\bar{k}$. In the second stage, once roles are assigned, follow the subgame perfect strategies given in Proposition 3.3. For the special case where $k_a = k_b = k$ let the leader set the price $P\left(r\left(k\right)+k\right)$.*

Then, using price leadership as a coordination device, the outcome $\left(\bar{k}, \bar{k}\right)$ can be sustained as a SPNE of the game.

Indeed, once capacities are set, what follows has already been discussed. So the question is: why would a firm set this capacity given that the other one did? The answer comes from the possibility that the firm was follower in the second stage game because the follower can set a relatively high price:

$$P\left(2\bar{k}\right) < P\left(r\left(\bar{k}\right)+\bar{k}\right).$$

First, notice that setting a higher capacity can not be a best response since payoff is decreasing in that zone. Given the strategies in the price-setting subgame, upper semi-continuity of reduced payoff gives the existence of the best response in the interval $\left[0, \bar{k}\right]$.

Second, note that reduced payoff for the small firm is of the form $k \to \mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right] k - C(k)$ where $\mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right]$ is the expected price faced by a small firm. Expected price $\mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right]$ for firm $a$ against capacity $\bar{k}$ is given by:

$$\mathbb{E}\left[p \mid \left(k_a, \bar{k}\right)\right] = \begin{cases} P\left(k_a + \bar{k}\right) & \text{if } k_a \leq r^{-1}\left(\bar{k}\right) \\ \lambda_a \underline{p}\left(\bar{k}, k_a\right) + \left(1-\lambda_a\right) P\left(r\left(k_a\right)+k_a\right) & \text{if } r^{-1}\left(\bar{k}\right) \leq k_a < \bar{k} \end{cases}$$

so that payoff for the small firm is continuous in $\left[0, \bar{k}\right[$ and strictly greater than the Cournot payoff for capacities in the interval $\left]r^{-1}\left(\bar{k}\right), \bar{k}\right[$.

For the small firm as a second mover, when comparing[6] the maximum profit from setting a capacity $k$ such that $k < r^{-1}\left(\bar{k}\right)$ and setting $\bar{k}$, the fall in price $\left(P\left(k+\bar{k}\right) \searrow P\left(r\left(\bar{k}\right)+\bar{k}\right) > P\left(\bar{k}+\bar{k}\right)\right)$ and the increase of the cost $\left(C(k) \nearrow C\left(\bar{k}\right)\right)$ may be compensated by the increase in the volume of sales $\left(k \nearrow \bar{k}\right)$. The second mover price function, $P\left(r\left(k\right)+k\right)$, is less steep than the standard Cournot price function, allowing the possibility that payoff was increasing in the interval $\left]r^{-1}\left(\bar{k}\right), \bar{k}\right[$.

Second mover advantage has to be such that it drags sufficiently upwards the expected payoff $k \to \mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right] k - C(k)$ and the functions $k \to \mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right]$ and $k \to -C(k)$ need not be too strongly decreasing.

These two conditions can be written as find $\bar{k}$ such that:

$$P\left(r\left(\bar{k}\right)+\bar{k}\right)\left[\lambda_i r\left(\bar{k}\right) + \left(1-\lambda_i\right)\bar{k}\right] - C\left(\bar{k}\right) \geq \max_{k \in \left[0, r^{-1}\left(\bar{k}\right)\right]} P\left(k+\bar{k}\right) k - C(k) \tag{4.1}$$

$$\lim_{k \nearrow \bar{k}} \frac{\partial}{\partial k}\left[\mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right] k - C(k)\right] \geq 0 \tag{4.2}$$

Condition (4.1) states that the payoff of matching is greater than the payoff of setting a capacity in the zone where there is no difference between being leader or follower, and condition (4.2) states

[6] This comparison is relevant only if $\bar{k} \leq r(0)$.

that the left hand derivative[7] of the payoff function at the point $\bar{k}$ is positive which, along with concavity of payoff in the region where there is difference between being leader and follower and continuity at the point $k = r^{-1}(\bar{k})$, assures that payoff is maximized at $k = \bar{k}$.

We illustrate this situation with the following example:

**Example 4.2.** Consider the duopoly with price function $P(q) = 1 - q$, and cost $C(q) = cq$ for both firms. The Cournot best response function is then:

$$r_C(k) = \max\left\{\frac{1-c-k}{2}, 0\right\}.$$

The zero-cost Cournot equilibrium is given by $k^* \equiv 1/3$ and $k_C^* = (1-c)/3$.

If $\lambda_a = \lambda_b = 1/2$ and $c$ is sufficiently small then we can find a continuum of equilibria along the line $k_a = k_b$. For instance for $c \equiv 0.05$, $\bar{k}_a = \bar{k}_b = \bar{k} = 0.3375$ can be sustained by a SPNE of the game and there is a continuum of Nash Equilibria along the line $k_a = k_b$ in a neighborhood of $\bar{k}$. Of course, $k_C^*$ is not in this neighborhood.

## 4.3 Cournot Outcome and Contingent Demands

In Davidson and Deneckere (1986) Theorem 1 argues that the Cournot outcome can not emerge as an equilibrium of the two stage game discussed by Kreps and Scheinkman considering almost any rationing rule, in a certain class, different from the *surplus maximizing* rationing rule. This is relevant as well for this work since the central argument states that if we change the rationing rule then, given that in the first stage the set capacities were the Cournot quantities, the Cournot price is not a best response to itself, ruling out as well the Cournot outcome as the equilibrium of the sequential subgame. However the proof is made only for the case of zero cost of capacity. As we will see below their result, as stated in their work, is in fact only valid for this case and once there are costs a finer treatment becomes necessary. Madden (1998) addresses this matter for a special setting that is not completely compatible with the Kreps and Scheinkman or Davidson and Deneckere assumptions on the demand function, but that enlightens what really is relevant when trying to answer the question about how dependent are the results on the chosen rationing rule. Consider the class of contingent demands that satisfy:

1. $D_i(p_i|p_j)$ downward-sloping twice differentiable except in $p_i = p_j$.
2. $D_i(p_i|p_j) = D(p_i)$ for $p_i < p_j$
3. $D_i(p_i|p_j) = \max\left\{\frac{D(pi)}{2}, D(p_i) - k_j\right\}$ when $p_i = p_j$
4. $\max\{0, D(p_i) - k_j\} \le D_i(p_i|p_j) \le \max\{0, \min\{D(p_j) - k_j, D(p_i)\}\}$ for $p_i > p_j$

Both rationing rules: *proportional* and *surplus maximizing* satisfy points 1 to 4.

Suppose that contingent demand for the high priced firm is given by a $\mathcal{C}^1$ function $CD(\cdot|p_j, k_i, k_j) : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$D_i(p_i|p_j) = \begin{cases} D(p_i) & \text{if } p_i < p_j \\ \max\left\{\frac{D(p_i)}{2}, D(p_i) - k_j\right\} & \text{if } p_i = p_j \\ \max\{0, CD(p_i|p_j, k_j, k_j)\} & \text{if } p_i > p_j \end{cases}$$

and $D_i$ satisfies conditions 1 to 4. Note that for $D_i$ to satisfy conditions 1 to 4, we must have that $CD(p_j|p_j, k_j, k_j) = D(p_j) - k_j$. Using the notation $e_{q,p}^{CD}$ for the elasticity of demand $CD(\cdot|p_C^*, k_C^*, k_C^*)$ we can state the following proposition.

[7]From the definition of $\underline{p}(k_i, k_j)$ when $k_j \nearrow k_i$.

**Proposition 4.3.** *If:*

$$e_{q,p}^{CD}\left(p_C^*\right) > -1 \tag{4.3}$$

*then the Cournot outcome does not rise as the equilibrium of the two stage game (sequential or simultaneous).*

The proof follows the same ideas behind Davidson and Deneckere's proof. The condition implies that payoff as a function of price, is increasing at the Cournot price. In their result, the condition of $D_i(p_i|p_j)$ being locally distinct to the right from the surplus maximizing contingent demand at the point $p_i = p_j = p_C^*$ when $k_i = k_j = k_C^*$, is not sufficient when costs are not equal to zero and is a special case of our condition (4.3). Condition (4.3) is also in the line of Madden (1998), if the residual demand is not sufficiently elastic at the Cournot price, then the Cournot outcome can not emerge as an equilibrium of the two stage game.

# 5 Concluding Remarks

We have studied a three stage duopoly game where firms first set simultaneously capacities and then engage in sequential price competition. We have related the outcome of the sequential price game (Deneckere and Kovenock, 1992) to the simultaneous price game (Kreps and Scheinkman, 1983) and we have obtained that the Cournot outcome of the market can be sustained as a SPNE in pure strategies

Mixed strategies are a solution to the problem of existence in simultaneous price games. However, there is still not a consensus about what a mixed strategy outcome represents in a price game. Usual justifications for mixed strategy equilibria such as a population distribution from where individuals are randomly selected to play the game, as in evolutionary game theory do note seem appealing. Moreover, the fact that the pure strategies that are in the support of the mixed strategy distributions are not necessarily best responses to the pure strategies that are in the support of the rival's mixed strategy distribution, gives rise to the regret property of mixed strategies equilibria: once a realization of the distribution is played, players may not be satisfied with the outcome, which is somehow incoherent with Nash behavior or perfectness of equilibria (see for instance Vives (1999); Friedman (1988) for a discussion on Mixed strategies Nash equilibria in oligopoly games).

We have proposed in this note an alternative approach to the problem of existence of equilibrium in pure strategies that is more in the line of Harsanyi (1973). Firms are uncertain of their role in the price stage of the game. They know that sequential price competition will take place, but since this competition is so far in the future, price leadership is unknown. As in an incomplete information game, players know a probability distribution over their roles in the price game and maximize expected payoff in the first stage. Price leadership serves then as a coordination device that allows to obtain equilibrium outcomes in pure strategies. It is important to notice that this result does not depend on the value of the probability distribution, but only in the characteristics of the involved functions. That is, on the structure of the market: cost of production, elasticity of demand. We have seen that suitably and slightly modifying the sharing rule on tied prices it is possible to formally reproduce the intuitively expected behavior of firms in the original setting (equal share at tied prices).

Finally we have made space for the possibility of finding other symmetric quantity outcomes sustained as SPNE depending only on primitives and on the probabilities of being leader in the price game. No changes with respect to the Kreps and Scheinkman (1983) setting have been made on cost or demand functions. The discussion is opened then on how to treat such probabilities (endogenously generated, parameters of the model, firm independent, etc.). Many lines of exploration can be taken: general $n$ firm oligopoly, non symmetric cost structures of firms (Deneckere and Kovenock, 1996) and effects of changes on contingent demand (Davidson and Deneckere, 1986), to give some examples.

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# A Proofs

*Proof of Proposition 3.3.* Consider the best response mapping for firm 2 presented in 3.2 and solve by backward induction.

For 1 $\underline{p}(k_2, k_1) = P(k_a + k_b)$ so we know that for any possible price set by firm 1 in equilibrium, firm 2 will match or undercut. Thus firm 1 gets the residual demand payoff which is maximized at $p_1^* = P(k_a + k_b)$.

For 2 given the strategy of firm 2, firm 1 nets the low priced firm payoff when setting $p_1 < \underline{p}(k_2, k_1)$ and the residual demand payoff when setting $p_1 > \underline{p}(k_2, k_1)$. Since $\underline{p}(k_1, k_2) < \underline{p}(k_2, k_1)$, the only possibility for firm 1 to have a maximizer is to have upper semi-continuity at $p_1 = \underline{p}(k_2, k_1)$. And this only obtains if firm 2 sets $P(r(k_1) + k_1)$.

In 3 we have $\underline{p}(k_1, k_2) > \underline{p}(k_2, k_1)$ and thus firm 1 maximizes it's payoff by setting $P(r(k_2) + k_2)$ at which firm 2 responds by matching.

In 5, $\underline{p}(k_2, k_1) = 0$, so if $p_1 > 0$ firm two will match or undercut and firm 1 will net 0. Then, firm 1 can set, in equilibrium, any price $p_1^* > 0$, and firm 2 will match or undercut depending on the value of $p_1^*$. For the special case of $p_1 = 0$, note that then firm 1 still nets 0 (so it is a possible strategy in equilibrium) and firm 2 nets 0 as well, so then firm 2 can actually set any price in the price set. Thus any pair $(0, p_2^*)$, $p_2^* \in \mathbb{R}_+$ can be sustained as a SPNE of the price game.

$\blacksquare$

*Proof of Proposition 4.1.* Set $k_b^* = k_C^*$. Then $k_b^* < k^*$ and thus Proposition 3.3 gives a unique reduced payoff for firm $a$:

$$\pi_a(k_a, k_C^*) = \begin{cases} P(k_a + k_C^*)\, k_a - C(k_a) & \text{if } k_a \leq r(k_C^*) \\ R(k_C^*) - C(k_a) & \text{if } k_a > r(k_C^*) \end{cases}$$

Clearly $\pi_a(\,\cdot\,, k_C^*)$ is continuous and if $k_a > r(k_C^*)$ it is strictly decreasing. Moreover, from what we said in section 2, the function $k_a \to k_a P(k_a + k_C^*) - C(k_a)$, the Cournot payoff function, attains it's maximum at $k_a^* = r_C(k_C^*) < r(k_C^*)$ and thus $k_a = k_b^*$ is the best response to $k_b^* = k_C^*$ in the reduced form capacity-setting game. Since the (unique) best response to $k_C^*$ in the reduced form game is to set the same capacity and the game is symmetric, we conclude.

$\blacksquare$

*Proof of Proposition 4.3.* Suppose the two players set $k_C^*$ in the first stage. If player $j$ sets price $p_C^*$ the payoff for player $i$ as a function of price is:

$$\pi_i(p, p_C^*, k_C^*, k_C^*) = \begin{cases} p k_C^* & p \leq p_C^* \\ p \max\{0, CD(p|p_C^*, k_C^*, k_C^*)\} & p > p_C^* \end{cases}$$

This function is then continuous[8] and increasing to the left of $p_C^*$. The derivative to the right of $p_C^*$ is

$$\frac{\partial}{\partial p}\pi_i(p, p_C^*, k_C^*, k_C^*) = p\ CD'(p|p_C^*, k_C^*, k_C^*) + CD(p|p_C^*, k_C^*, k_C^*)$$

thus we have:

$$\begin{aligned}\lim_{p \downarrow p_C^*} \frac{\partial}{\partial p}\pi_i(p, p_C^*, k_C^*, k_C^*) &= p_C^* CD'(p_C^*|p_C^*, k_C^*, k_C^*) + CD(p_C^*|p_C^*, k_C^*, k_C^*) \\ &= p_C^* CD'(p_C^*|p_C^*, k_C^*, k_C^*) + k_C^* \\ &= k_C^* \left( \frac{p_C^*}{k_C^*} CD'(p_C^*|p_C^*, k_C^*, k_C^*) + 1 \right) \\ &> 0 \end{aligned}$$

[8]Since $CD(p|p_C^*, k_C^*, k_C^*)$ satisfies 2, $\lim_{p \downarrow p_C^*} CD(p|p_C^*, k_C^*, k_C^*) = k_C^*$.

Then $p_C^* \notin \text{argmax}_p \left\{\pi_i(p, p_C^*, k_C^*, k_C^*)\right\}$.

$\blacksquare$