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Asymptotic age structures and

intergenerational trade

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Abstract

While demographers Lotka (1939) and Lopez (1961) proposed conditions on (exogenous) fertility and mortality laws under which populations with distinct initial age structures exhibit the same asymptotic age structure, this paper re-examines the issues of age structure stabilization and convergence, by considering a population whose fertility and mortality are endogenously determined in the economy. For that purpose, we develop a three-period OLG model where human capital accumulation and intergenerational trade affect fertility and longevity. It is shown that the age structure must converge asymptotically towards a stable structure, whose form depends on the structural parameters of the economy. Moreover, populations with distinct initial age structures will end up with the same long-run age structure when fertility and mortality laws are converging, which requires converging terms of trade between coexisting generations in the different populations under study.

Keywords: Age structure, OLG model, fertility, mortality, demographic transition, intergenerational trade.

JEL codes: J11, J13, J14

1 Introduction

The history of mankind is, among other things, a history of cooperations - and sometimes conflicts - between coexisting generations. At any epoch, the coexistence of individuals of different ages generated its own set of intergenerational arrangements, duties and trades, whose goal was, quite often, to take advantage of the age heterogeneity.[1] But the age heterogeneity is also an output of intergenerational relations, as fertility and longevity - and thus the age structure - depend on the precise form of intergenerational settlements. Hence, the study of the long-run dynamics of economies and populations requires to consider the joint dynamics of age structures and intergenerational relations.

This paper aims precisely at developing a model of coexisting generations, whose relative sizes are both an output and an input of intergenerational relations. Our focus on the interplay between age heterogeneity and intergenerational relations will allow us, in particular, to study major mechanisms lying behind the observed long-run dynamics of age structures.

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[1]Such intergenerational settlements include, for instance, child care, education activities, long term care, and pensions paid by the young to the old.

As this was stressed by demographers (see Lee, 2003), the age structure of populations has been evolving significantly across epochs. While human societies were, at the middle of the 18th century, mainly composed of what can be roughly called 'young' persons, the share of the 'young' started falling during the second part of the 19th century in industrialized economies, whereas the shares of middle-aged and elderly people started growing, and kept growing even more during the 20th century. That evolution is illustrated on Figure 1 by the case of Sweden.[2] The share of people younger than 35 years, which consisted of about 68 % of the population around 1850, started falling after 1850, and amounts today to only 42 % of the Swedish population.
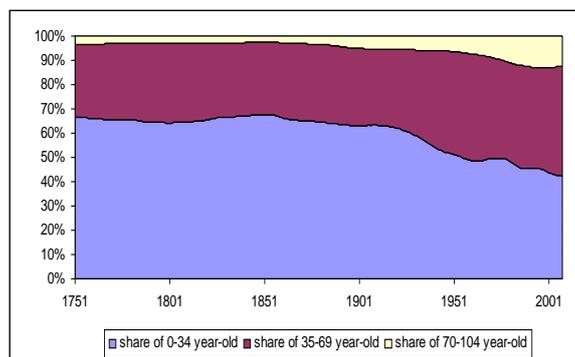

Figure 1: Age structure in Sweden, 1751-2008

This non-constancy of the age structure is mainly due to large changes in fertility and mortality over time. Demographers regard the evolution of age structures as an outcome of the demographic transition process: human societies have, during the last two centuries, switched from a regime with a high fertility and a high mortality to a regime with a low fertility and a low mortality.[3] The observed ageing of societies during the second part of the 19th century and the 20th century constitutes thus only a subproduct of those evolutions of births and deaths. Hence, all economies having experienced the demographic transition are also characterized by the evolution of age structure just described (see Figures 2 and 3 for England and Wales and France).[4]

In front of such sizeable evolutions of age structures over time, a natural question to raise is the one of the likelihood of a *stabilization* of age structures: can we expect that age structures will stabilize at some point in the future, and, if yes, what will the stable age structure look like? Another natural question is the one of the *convergence* of age structures across populations: will the long-run age structure be the same in all populations?

In what is still regarded today as a major result of theoretical demography, Alfred Lotka (1939) identified conditions that are sufficient for the asymptotic stabilization of an age structure.[5] The so-called Lotka Theorem states that,

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[2]Sources: The Human Mortality DataBase (2008). See de la Croix, Lindh and Malmberg (2009) for a study of long run demographic and economic changes in Sweden.

[3]See Lee (2003).

[4]Sources: The Human Mortality Database (2008).

[5]Note, however, that Lotka Theorem does not guarantee a stable population *size*, but only
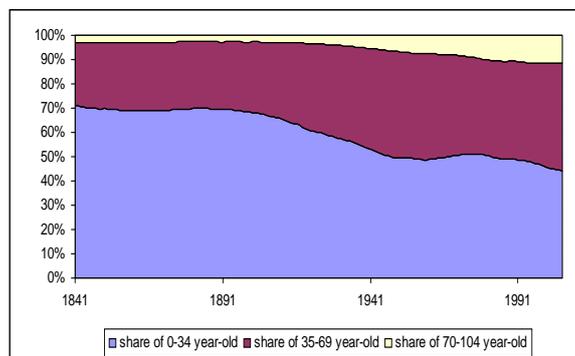
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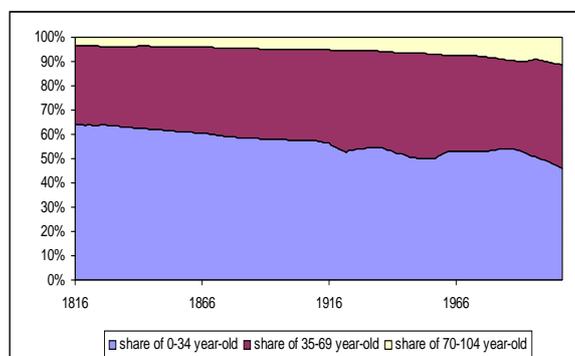Figure 2: Age structure in England and Wales, 1841-2006

Figure 3: Age structure in France, 1816-2008

provided (i) age-specific fertility rates are constant, (ii) age-specific death rates are constant, (iii) age-specific migration rates are zero, a population must necessarily end up, as time goes to infinity, with a constant age structure, which is independent from the initial age structure and size of the population. Lotka Theorem states actually a result of strong convergence: populations tend, if subject to the same, constant fertility and mortality laws, to "forget their past", which is a property known as strong ergodicity.[6] That result, which concerns stable populations (i.e. subject to constant fertility and mortality laws), was generalized by Alvaro Lopez (1961) in the context of unstable populations (i.e. subject to varying fertility and mortality laws), in what is known as a weak convergence theorem. According to Lopez, if populations are subject to the same time-varying fertility and mortality laws, populations with distinct initial age structures must exhibit, in the long-run, the same age structure.

The importance of Lotka and Lopez's results for the economic study of hu-

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a stable population structure, as whether the population size grows or falls depends ultimately on the relative strengths of mortality and fertility.

[6]Note that, as stressed by Preston *et al* (2001, p. 146), Lotka's result can be extended to the case of populations with the same age-specific migration rates, so that condition (iii) is not crucial for Lotka's result.

man societies could hardly be overemphasized.[7] On the descriptive side, the evolution of economic aggregates over time (production, consumption, savings, employment, etc.) is dependent on how the population is structured in terms of age.[8] On the normative side, it is also clear that the age heterogeneity of populations is a major source of concern for policy making. Thus, there is a strong need to know under which conditions the age heterogeneity can stabilize.

However, those two contributions, made at the highest level of generality, do not inform us about the precise *form* of the asymptotic age structure. To answer that question, one must add assumptions on how fertility and mortality are determined, and make explicit how these are influenced by, among other things, the form of intergenerational relations. But, as we shall see, endogenizing fertility and mortality is not only relevant for characterizing asymptotic age structures, but, also, for the restatement of the conditions guaranteeing the convergence of age structure across populations.

For those purposes, we develop here a three-period overlapping generations model (OLG) with human capital accumulation, where both fertility and longevity are endogenous, and analyze the dynamics of age structures under that theoretical framework.[9] In that economy, the different generations that co-exist at each period of time are characterized by trade relations, in the spirit of the seminal model by Ehrlich and Lui (1991). Young adults, as parents, transfer resources to their children, in order to raise them and educate them, but they transfer also resources to their own parents, so that intergenerational trade is here oriented both downwards and upwards. But an important difference with respect to the model of Ehrlich and Lui, in addition to the endogeneity of mortality, is that contribution rates to the children and the elderly are here not constants, but variables depending on the level of human capital.

While the literature includes several OLG models with both endogenous fertility and longevity (Blackburn and Cipriani, 2002; Gallor and Moav, 2005; de la Croix and Licandro, 2007; Strulik and Weisdorf, 2008), the specificity of this paper is that it pays a particular attention to the dynamics of the age structure, and to its relationships with intergenerational trade. The present model shares with Blackburn and Cipriani (2002) the three-period OLG structure with human capital accumulation, but introduces intergenerational trade as a motive for fertility decision, and considers the interplay between intergenerational trade and age structure dynamics. The present paper shares also with Galor and Moav (2005) the presence of a pure taste for children in the fertility decision, but concentrates on age heterogeneity within the population rather than on heterogeneity within each cohort. Finally, endogenous fertility and mortality are also treated in a model of human capital accumulation by de la Croix and Licandro (2007) and Strulik and Weisdorf (2008), but they concentrate on a society without intergenerational trade, unlike the present framework.

Thus the particularity of this study is to concentrate on the interplay between age heterogeneity, intergenerational trade and human capital accumulation, in order to examine whether Lotka and Lopez's stabilization and convergence results remain true under endogenous fertility and mortality. It is shown that,

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[7] On the relations between Lotka and Lopez' results, see Challier and Michel (1996).

[8] For a recent empirical study of the impact of age-structure on GDP growth, see Lindh and Malmberg (2009).

[9] As such, this paper complements other theoretical pieces of work concerned with the dynamics of age-structure, such as Boucekkine *et al* (2002), where mortality is exogenous.

if children are treated both as consumption goods and as investment goods by their parents, the age structure must asymptotically converge towards a stable structure, whose form depends on the structural economic and demographic parameters characterizing the population. Moreover, it is shown that populations with distinct initial age structures exhibit the same asymptotic age structure, provided fertility and mortality laws are converging, which requires converging terms of trade between coexisting generations in the populations under study.

This paper is organized as follows. Section 2 presents the model. Section 3 examines the long-run production dynamics. The dynamics of age structures is studied in Section 4. Section 5 provides a numerical illustration on the basis of France (1816-2008), and extrapolates the long-run age structure under several postulates for the survival process. Section 6 concludes.

2 The model

2.1 Environment

Let us consider a three-period OLG model. Each period of life has a length normalized to 1.

All agents live the first period of life for sure. This consists of a period of childhood, during which the child does not produce anything, and benefits from the resources of his parent.

A proportion π^1_{t+1} of the cohort born at time t will enjoy the second period of life. That period is a period of young adulthood, during which agents work, help their parents and educate their children. Reproduction is monosexual.

Finally, only a proportion π^2_{t+2} of the part of the cohort that survived to the second period will reach the third period of life. That period is a period during which agents are retired, and live thanks to the generosity of their children.

In this model, life expectancy at birth is: $(1 - \pi^1_{t+1})1 + \pi^1_{t+1}(1 - \pi^2_{t+2})2 + \pi^1_{t+1}\pi^2_{t+2}(3) = 1 + \pi^1_{t+1} + \pi^1_{t+1}\pi^2_{t+2}$.

Children born at time t inherit, during their first period of life, h_t units of human capital from their parents.

2.2 Survival conditions

Survival conditions at all ages are assumed to be shaped by the *current* stock of human capital, i.e. the stock of knowledge prevailing at the time of existence of the agents.[10] Formally, the probability of survival to the second period of life of a person born at $t - 1$, denoted by π^1_t, depends positively on the stock of human capital h_t by means of the survival function:

$$\pi^1_t \equiv \pi^1(h_t) \tag{1}$$

where $\pi^1(h_t)$ exhibits the following properties: $\pi^1(.) \geq 0$, $\pi^{1\prime}(.) > 0$ and $\pi^{1\prime\prime}(.) < 0$. We assume also that $\pi^1(h_t)$ is bounded from below and from above: $\lim_{h_t \to 0} \pi^1(h_t) = \tilde{\pi}^1 > 0$ and $\lim_{h_t \to \infty} \pi^1(h_t) = \bar{\pi}^1 < 1$.

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[10] This assumption differs from the postulate according to which individual longevity depends on the human capital stock prevailing at the *birth* of the agent. In our setting, agents do, on the contrary, benefit from the advances of knowledge during their whole life.

Similarly, the probability of survival from the second to the third period of life, π_t^2, depends positively on the stock of human capital:

$$\pi_t^2 \equiv \pi^2(h_t) \tag{2}$$

with $\pi^2(.) \geq 0$, $\pi^{2\prime}(.) > 0$, $\pi^{2\prime\prime}(.) < 0$, $\lim_{h_t \to 0} \pi^{2\prime}(h_t) < \infty$, and $\lim_{h_t \to \infty} \pi^{2\prime}(h_t) = 0$. We assume also $\lim_{h_t \to 0} \pi^2(h_t) = \tilde{\pi}^2 > 0$ and $\lim_{h_t \to \infty} \pi^2(h_t) = \bar{\pi}^2 < 1$.

2.3 Production

At the young adult age, (surviving) agents produce a good, according to a technology that is assumed, for simplicity, to be linear in human capital:

$$y_t = w h_t \tag{3}$$

where y_t denotes the output per head, while w is the wage per unit of human capital. For the sake of the presentation, this wage will be normalized to unity in the rest of this paper (i.e. $w = 1$).

The stock of human capital h_{t+1} depends on the past human capital h_t, and on the amount of education expenditures that each adult dedicates to the education of *each* child, as follows

$$h_{t+1} = A \left(\frac{e_t h_t}{n_t} \right)^\alpha h_t^{1-\alpha} \tag{4}$$

where e_t is the fraction of parental income h_t dedicated to education, n_t is the number of children, A is a productivity parameter ($A > 0$), while α is the elasticity of h_{t+1} to education spending per child (i.e. $0 < \alpha < 1$).

2.4 Agents' decisions

Young adults work during the second period, and decide the number of children n_t and the fraction of their income dedicated to education e_t.[11] But before considering those decisions, let us first specify intergenerational transfers.

Intergenerational transfers Following Ehrlich and Lui (1991), it is assumed that young adult agents must take care of their family: children (consumption and education) and old parents (consumption only). Contributions to the consumption of other family members are compulsory, and each young adult takes the family contribution rates *as given*.[12]

Each young adult must give a proportion γ_t of his resources to each of his n_t children, to cover food and clothing costs, as well as other expenditures (e.g. toys, leisure activities, travel, etc.). That fraction is a function of the level of human capital prevailing at the time of the birth of the child:

$$\gamma_t \equiv \gamma(h_t) \tag{5}$$

where $0 < \gamma_t < 1$ for any h_t. We thus allow here for a non-constancy of the (relative) child cost, which may vary with the level of human capital. The

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[11] That decision structure is close to the one developed by Erhlich and Lui (1991), where mortality is exogenous.

[12] This approach differs from the one of Ehrlich and Lui (1991) in the basic version of their model, where the contribution rate is chosen by parents under some constraints.

(relative) contribution to each child is assumed to be bounded from below and from above: $\lim_{h_t \to 0} \gamma(h_t) = \tilde{\gamma} > 0$ and $\lim_{h_t \to \infty} \gamma(h_t) = \bar{\gamma} < 1$. As far as the precise shape of the function $\gamma(h_t)$ is concerned, a recent study by Farrell and Shields (2007) on the behaviour of children as consumers suggests that the contribution rate γ_t is likely to be *non decreasing* in the level of income: $\gamma'(h_t) \geq 0$. Actually, as shown by Farrell and Shields, while some goods, which represent a small share of children's spending, such as drinks, sweets, toys and books, are normal goods, the goods that represent a larger share of their spending, such as clothes, travel, and leisure, are luxury goods (i.e. with an income elasticity larger than 1). In the present one-good model, the elasticity of expenditures per child with respect to parental income h_t, which is equal to $1 + \frac{\gamma'(h_t)}{\gamma(h_t)} h_t$, is larger than 1 if and only if $\gamma'(h_t) \geq 0$. Hence we shall assume, throughout this paper, that the contribution rate γ_t is non decreasing in h_t.[13]

Regarding the help to the elderly, it is assumed, for simplicity, that all young adults must dedicate some fixed proportion of their resources to the old, surviving, cohort, through a social insurance system that pools the risks of having surviving elderly parents on the whole young adults cohort. Thus, each young adult, whatever his parent survives or not to the old age, dedicates a proportion $\delta_t \pi_t^2$ of his resources to the old, surviving, cohort.[14] As for the help to the children, that fraction δ_t is a function of the level of human capital prevailing at the time of the retirement of the old:

$$\delta_t \equiv \delta(h_t) \tag{6}$$

where $0 < \delta_t < 1$ for any h_t. Here again, the contribution rate is assumed to be bounded from below and from above: $\lim_{h_t \to 0} \delta(h_t) = \tilde{\delta} > 0$ and $\lim_{h_t \to \infty} \delta(h_t) = \bar{\delta} < 1$. As far as the shape of $\delta(h_t)$ is concerned, it is reasonable, in the light, for instance, of the large empirical evidence on the rise of the cost of long term care (per dependent person) as the economy develops, to assume that the expenditure dedicated to the old is, like the expenditure dedicated to the young, a non decreasing function of income $h_t : \delta'(h_t) \geq 0$.[15]

Budget constraints   Each young adult earns an income h_t by his work, and uses that income to help his children and parents. Moreover, a young adult spends a fraction e_t of his resources for the higher education of his children ($0 \leq e_t \leq 1$). That fraction is, unlike γ_t, chosen by the adult agent. One can interpret that asymmetry as reflecting the fact that, as far as basic consumption is concerned, social norms and customs are strongly at work: each society produces its norms regarding how one has to treat one's children and one's elderly parents, and those norms are strictly respected. However, parents benefit from a much larger degree of freedom regarding (high) education spending.

Second-period consumption c_t is what remains of labour income once intergenerational transfers and education spending have been paid:

$$c_t = (1 - \gamma_t n_t - \delta_t \pi_t^2 - e_t) h_t \tag{7}$$

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[13]We shall also assume, in the rest of this paper, that $\lim_{h_t \to 0} \gamma'(h_t) < \infty$ and $\lim_{h_t \to \infty} \gamma'(h_t) = 0$.

[14]It is only in the special case where there is no elderly people alive (i.e. $\pi_t^2 = 0$) that young adults neglect the previous generation.

[15]On the rise of the LTC costs per dependent person, see Cutler (1996) and Norton (2000). Precise figures on the rise of the share of LTC expenditures in GDP can be found in a recent report by the European Commission (2009).

This is decreasing in the number of children, and decreasing in the proportion of survivors among the elderly, π_t^2. Note, however, that, while the number of children n_t is chosen by agents, the proportion of surviving elderly π_t^2 is not.

A young adult expects also to consume, if he is still alive at the retirement age, a consumption d_{t+1}, which comes exclusively from the contributions of the young cohort (for simplicity, there is no savings in this economy):

$$d_{t+1} = \left[\delta_{t+1}\pi_{t+1}^2 h_{t+1}\right] \frac{n_t \pi_{t+1}^1}{\pi_{t+1}^2} = n_t \pi_{t+1}^1 \delta_{t+1} h_{t+1} \tag{8}$$

The factor in brackets corresponds to what is given by each young adult at period $t+1$, while the second factor consists of the ratio of young adults over old adults. The impact of the young cohort's contribution on the old's consumption depends thus on the sizes of those two demographic groups.

Optimal fertility and education Agents' preferences are assumed to take a standard log-linear form in second-period and third-period consumptions, and in the number of children. Agents are also supposed to be expected utility maximizers, with a zero utility from being dead. Moreover, at the time of their decisions, agents form myopic anticipations regarding future survival conditions, and thus believe that the probabilities π_{t+1}^1 and π_{t+1}^2 are equal to the currently prevailing probabilities, i.e. π_t^1 and π_t^2. Similarly, agents take contributory rates γ_t and δ_t as given constants, even though these may evolve over time.[16]

Hence, if one abstracts from childhood, the expected lifetime welfare is:[17]

$$U_t = \log(c_t) + \beta \log(n_t) + \pi_t^2 \log(d_{t+1}) \tag{9}$$

where β captures the relative strength of the taste for children $(0 < \beta < 1)$. Note that this expression treats children as consumption goods *and* investment goods (through d_{t+1}), but abstracts from any altruistic concerns.[18]

Substituting for consumption and for the resources at the old age yields:

$$U_t = \log((1 - \gamma_t n_t - \delta_t \pi_t^2 - e_t)h_t) + \beta \log(n_t) + \pi_t^2 \log\left(\pi_t^1 n_t^{1-\alpha}\delta_t A e_t^\alpha h_t\right) \tag{10}$$

The first-order condition for optimal fertility is:

$$n_t = \frac{\left(\beta + \pi_t^2(1-\alpha)\right)\left(1 - \delta_t \pi_t^2\right)}{\gamma_t \left(1 + \beta + \pi_t^2\right)} \tag{11}$$

Optimal fertility is, without surprise, increasing in the taste for children β, and decreasing in the cost per child γ_t.[19] It is also decreasing in the contribution

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[16] Thus, the young adult expects to benefit, once old, from the same relative aid as the one he offered when being young.

[17] We abstract here from pure time preferences. Actually, there is not really a need for these in the present context, where the survival probability π_t^2 acts as a 'natural' discount factor, assigning lower weight to future consumption on the grounds of its risky nature.

[18] This model differs thus from Barro and Becker (1989), where altruism towards children makes parent care about the utility of the whole dynasty following them. In the literature on endogenous fertility and mortality, see de la Croix and Licandro (2007) on the quality *versus* quantity trade-off, in a context where parents care about the survival of their children, unlike the present, purely egoist context.

[19] Fertility is here strictly positive, as we have $\beta + \pi_t^2(1-\alpha) > 0$ for any level of π_t^2 under $\alpha < 1$. This interior solution differs from Ehrlich and Lui (1991), where $\beta = 0$ and $\alpha = 1$, so that a corner solution with the minimum number of surviving children prevails, as the marginal net return for investing in the quality of children always exceeds the marginal net return from investing in their quantity.

rate δ_t: as the contribution to the elderly goes up, parents' available resources are reduced, so that fewer children can be made. The chosen fertility is also decreasing in the elasticity α, which reflects the quantity *versus* quality trade-off. The higher α is, the lower the contribution of the quantity of children to the elderly's consumption is *ceteris paribus*.

The impact of the probability of survival π_t^2 on the number of children is ambiguous, and depends actually on the relative sizes of β - the taste for children - and δ_t - the contribution to the surviving old -.[20] Actually, we have

$$\frac{\partial n_t}{\partial \pi_t^2} \gtreqless 0 \iff (1-\alpha)(1-\delta_t\pi_t^2)(1+\beta+\pi_t^2) \gtreqless (\beta+\pi_t^2(1-\alpha))\left(\delta_t+\delta_t\beta+1\right)$$

It is difficult to see the sign of the derivative in general, as this depends on α, β and δ_t. It is only in some polar cases that the sign of $\partial n_t/\partial\pi_t^2$ can be identified. For instance, when α tends to 1, the LHS vanishes to 0, and the RHS remains positive, so that $\partial n_t/\partial\pi_t^2 < 0$. But for lower levels of α, things are less obvious. Note, however, that the demographic transition requires $\partial n_t/\partial\pi_t^2 < 0$, which implies, for a given level of α, some restrictions on β and δ_t (see *infra*).

The first-order condition for optimal education is:

$$e_t = \frac{\alpha\pi_t^2\left(1-\delta_t\pi_t^2\right)}{1+\beta+\pi_t^2} \tag{12}$$

The optimal fraction e_t of resources dedicated to education is increasing in α, as we expect. Note also that the optimal education is decreasing in δ_t: the higher the contribution rate to the old is, the lower the optimal education is *ceteris paribus*. The chosen education is also decreasing in the intensity of the taste for children β. Regarding the impact of π_t^2 on education, we have

$$\frac{\partial e_t}{\partial\pi_t^2} \gtreqless 0 \iff (1+\beta)(1-2\delta_t\pi_t^2) - \delta_t\left(\pi_t^2\right)^2 \gtreqless 0$$

Here again, the influence of the survival probability to the old age is ambiguous. For δ_t tending towards 0, $\partial e_t/\partial\pi_t^2$ is positive, but once some contribution is required for the elderly, things become less clear. In the extreme case where both δ_t and π_t^2 tend towards 1, the derivative is negative. While $\partial e_t/\partial\pi_t^2$ is decreasing in δ_t, it depends positively on β: the higher the taste for the number of children is, the higher is the effect of π_t^2 on the education spending.[21]

In sum, the probability of survival to the old age has here quite ambiguous effects on the individual fertility and education decisions. This is due to the twofold role of that survival probability: on the one hand, π_t^2 raises, *ceteris paribus*, the contribution to the old, which is likely to restrain fertility and education investment because of a negative income effect; on the other hand, π_t^2 raises the expected welfare gains from survival to the old age, which is a stimulus for more children and more education: this is a standard horizon effect.

Finally, note an interesting property of education spending per child $e_t h_t/n_t$: it is *independent* from the contribution rate δ_t. Thus, while a rise of the contribution rate to the old reduces optimal fertility and education, this leaves

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[20] This kind of indeterminate result is close to the one in Zhang et al (2001), where the effect of a higher π_t^2 on fertility depends on whether the taste for the "quantity" of children exceeds or not the taste for the "quality" of children, which is modelized there as guided by altruism.

[21] Note that the second-order derivative is negative, so that the influence of π_t^2 on education, if positive, must be decreasing with π_t^2.

education spending per child unchanged. On the contrary, a rise of the contribution rate to each child γ_t reduces fertility, but leaves education spending unchanged, implying a rise of education spending per child. The two intergenerational contribution rates have thus distinct effects on education spending per child, and, as we shall now see, on the dynamics of production.

3 Long-run production dynamics

Let us now characterize the long-run production dynamics of the economy under study. Given that the survival probabilities π_t^1 and π_t^2 are, like the output y_t, a function of the human capital stock h_t, and that contribution rates γ_t and δ_t are also functions of h_t, it follows that fertility and education are also a function of h_t. Hence, the constancy of the human capital stock h_t over time brings the constancy of all variables: y_t, π_t^1, π_t^2, n_t, γ_t, δ_t, e_t, c_t and d_t.

Substituting for education and fertility in the human capital accumulation equation yields:

$$h_{t+1} = A \left(\frac{\gamma(h_t)\alpha\pi^2(h_t)}{\beta + \pi^2(h_t)(1-\alpha)} \right)^\alpha h_t \equiv G(h_t) \tag{13}$$

The issue of the existence of a steady-state equilibrium amounts to studying whether the transition function $G(h_t)$ admits a fixed point, that is, a human capital level h_t such that $G(h_t) = h_t$. Proposition 1 summarizes the long-run dynamics of the economy under study, which can take three distinct forms.[22]

Proposition 1 *The long-run dynamics of the economy belongs to one of the three following cases:*

- *Case 1: if $A \left(\frac{\tilde{\gamma}\alpha\tilde{\pi}^2}{\beta+\tilde{\pi}^2(1-\alpha)} \right)^\alpha < 1$ and $A \left(\frac{\bar{\gamma}\alpha\bar{\pi}^2}{\beta+\bar{\pi}^2(1-\alpha)} \right)^\alpha < 1$, then $h^* = 0$ is the unique steady-state, which is stable: any economy with $h_0 > 0$ will converge towards $h^* = 0$.*

- *Case 2: if $A \left(\frac{\tilde{\gamma}\alpha\tilde{\pi}^2}{\beta+\tilde{\pi}^2(1-\alpha)} \right)^\alpha < 1$ and $A \left(\frac{\bar{\gamma}\alpha\bar{\pi}^2}{\beta+\bar{\pi}^2(1-\alpha)} \right)^\alpha > 1$, then there exist two steady-states: $h^* = 0$ and $h^{**} > 0$; h^* is locally stable, while h^{**} is unstable; any economy with $h_0 < h^{**}$ will converge towards $h^* = 0$, while any economy with $h_0 > h^{**}$ will exhibit perpetual growth.*

- *Case 3: if $A \left(\frac{\tilde{\gamma}\alpha\tilde{\pi}^2}{\beta+\tilde{\pi}^2(1-\alpha)} \right)^\alpha > 1$ and $A \left(\frac{\bar{\gamma}\alpha\bar{\pi}^2}{\beta+\bar{\pi}^2(1-\alpha)} \right)^\alpha > 1$, then $h^* = 0$ is the unique steady-state, which is unstable. Any economy with $h_0 > 0$ will exhibit perpetual growth.*

Proof. See the Appendix. ∎

Regarding the determinants of the long-run dynamics of human capital and output, let us first notice the crucial role played by the elasticity of productivity with respect to education spending per child, α. For instance, Case 3 can hardly occur for a low level of α. The taste for the number of children (i.e. β) plays in

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[22] Note that the case where $A \left(\frac{\tilde{\gamma}\alpha\tilde{\pi}^2}{\beta+\tilde{\pi}^2(1-\alpha)} \right)^\alpha > 1$ and $A \left(\frac{\bar{\gamma}\alpha\bar{\pi}^2}{\beta+\bar{\pi}^2(1-\alpha)} \right)^\alpha < 1$ cannot occur, as both $\pi^2(h_t)$ and $\gamma(h_t)$ are non decreasing in h_t.

the other direction, and makes Case 3 less likely, as a strong taste for children reduces education investment, and lowers human capital accumulation.

The dynamics of output depends also on the contribution function $\gamma(.)$, and, more precisely, on its limit values $\tilde{\gamma}$ and $\bar{\gamma}$. The higher those limit contribution rates to the children are, the more likely a high (stationary or non-stationary) equilibrium is, because high contribution rates reduce fertility and favour education investment per child. However, the contribution rate to the old, δ_t, does not affect output dynamics, as education spending per child is invariant to δ_t.

Moreover, the long-run dynamics of output depends on the limit survival probabilities to the old age $\tilde{\pi}^2$ and $\bar{\pi}^2$. It might be the case, for instance, that the survival probability to the old age is low when the human capital is close to zero, implying that we are in either Case 1 or Case 2, but $\tilde{\pi}^2$ does not tell us everything regarding the long-run dynamics, as this depends also on $\bar{\pi}^2$. If $\bar{\pi}^2$ is large enough, we can expect to have perpetual output growth (above the unstable steady-state), whereas the economy will converge towards 0 if $\bar{\pi}^2$ is not high enough. Furthermore, it should be stressed that, although the survival probability to the old age $\pi^2(h_t)$ is a major determinant of output dynamics, the probability of survival to the young adult age, $\pi^1(h_t)$, plays here no role at all, as it is here neutral for fertility and education decisions.

Proposition 1, which describes the dynamics of human capital and output under different cases, can also be used to account for the demographic changes that took place over the last two centuries, and, in particular, for the demographic transition. Clearly, as $\pi^1(h_t)$ and $\pi^2(h_t)$ are increasing functions of human capital, a growth of h_t over time - which occurs in Case 2 for $h_0 > h^{**}$ and in Case 3 for any $h_0 > 0$ - must generate a rise in life expectancy. Moreover, provided α is sufficiently large and β is sufficiently low (which must be true under Cases 2 and 3), the rise of π_t^2 caused by the growth of h_t implies a fall of fertility. Hence, the long-run dynamics of this model is compatible with the demographic transition: as human capital accumulates, both mortality and fertility fall, in conformity with the transition.

Whereas that result is also in line with the one of Blackburn and Cipriani (2002), it should be stressed that, in our model, the connection between the mortality decline and the fertility decline depends also, unlike in Blackburn and Cipriani, on the size of intergenerational transfers. Clearly, we have $\partial n_t/\partial \pi_t^2 < 0$ only if α is large and β is low, but, also, provided the contribution rate to the elderly δ_t is sufficiently large. This condition is likely to be satisfied as human capital accumulates, but only if $\delta'(h_t) = 0$ and δ is large, or if $\delta'(h_t) > 0$. Hence the existence of a large demographic transition imposes some restrictions on the level and pattern of the upward oriented intergenerational transfers.

4 Age structure dynamics (1): theory

Let us now turn to the predictions of the model regarding the long-run dynamics of the age structure. For that purpose, we will first characterize the age structure by means of age group ratios. Then, we will examine under which conditions the age structure stabilizes over time, and characterize analytically the asymptotic age structure. Finally, we will consider the implications of this model regarding the convergence of populations with distinct initial age structures.

4.1 The population's age structure

The groups of children, young adult and old adults at time t, denoted by, respectively, N_t^c, N_t^y, and N_t^o, are equal to

$$
\begin{aligned}
N_t^c &= \left(N_{t-1}^c \pi_t^1 \right) n_t \\
N_t^y &= N_{t-1}^c \pi_t^1 \\
N_t^o &= N_{t-2}^c \pi_{t-1}^1 \pi_t^2
\end{aligned}
$$

Hence the age structure of the economy can be summarized by the ratios

$$
\begin{aligned}
\Lambda_t &\equiv \frac{N_t^c}{N_t^c + N_t^y + N_t^o} = \frac{n_{t-1}\pi_t^1 n_t}{n_{t-1}\pi_t^1 n_t + n_{t-1}\pi_t^1 + \pi_t^2} \\
\Xi_t &\equiv \frac{N_t^y}{N_t^c + N_t^y + N_t^o} = \frac{n_{t-1}\pi_t^1}{n_{t-1}\pi_t^1 n_t + n_{t-1}\pi_t^1 + \pi_t^2} \\
\Phi_t &\equiv \frac{N_t^o}{N_t^c + N_t^y + N_t^o} = \frac{\pi_t^2}{n_{t-1}\pi_t^1 n_t + n_{t-1}\pi_t^1 + \pi_t^2}
\end{aligned}
$$

where Λ_t denotes the share of children in the population, Ξ_t is the share of the middle-aged, and Φ_t is the share of the elderly in the population. In the rest of this paper, we shall thus denote an age structure by the triplet $(\Lambda_t, \Xi_t, \Phi_t)$. Each ratio, Λ_t, Ξ_t and Φ_t, is a function of h_t and h_{t-1}, as these depend on past and current fertility and mortality:

$$
\begin{aligned}
\Lambda_t &= \Lambda\left(h_t, h_{t-1}\right) \\
\Xi_t &= \Xi\left(h_t, h_{t-1}\right) \\
\Phi_t &= \Phi\left(h_t, h_{t-1}\right)
\end{aligned}
$$

Note that, as age group ratios are functions of current and past human capital stocks, the study of the stabilization of age structure requires a distinction between the different cases mentioned above concerning the dynamics of human capital. However, as we shall now see, whether the economy lies in Cases 1, 2 or 3 will not make any difference as far as the issue of stabilization is concerned, but will definitely matter for the precise *form* of the asymptotic age structure.

4.2 The asymptotic age structure

As stated in Proposition 2, the age structure of the population tends necessarily to stabilize over time, whatever the long-run dynamics of the economy falls under Cases 1, 2 or 3. In other words, the ratios Λ_t, Ξ_t and Φ_t tend, in the long-run, to converge towards some constant, stable levels.

Proposition 2 *Under Cases 1, 2 or 3, the age structure of the population $(\Lambda_t, \Xi_t, \Phi_t)$ tends asymptotically towards a stable age structure $(\Lambda^*, \Xi^*, \Phi^*)$, where $\Lambda^* \equiv \lim_{t\to\infty} \Lambda_t$, $\Xi^* \equiv \lim_{t\to\infty} \Xi_t$ and $\Phi^* \equiv \lim_{t\to\infty} \Phi_t$.*
  Under $h_0 > 0$, the asymptotic age structure $(\Lambda^*, \Xi^*, \Phi^*)$ is

Cases	Λ^*	Φ^*
1	$\dfrac{\tilde{\pi}^1\left(\left(\beta+(1-\alpha)\tilde{\pi}^2\right)(1-\tilde{\delta}\tilde{\pi}^2)\right)^2}{\Omega\left(\tilde{\pi}^1,\tilde{\pi}^2,\tilde{\gamma},\tilde{\delta}\right)}$	$\dfrac{\tilde{\pi}^2\left(\tilde{\gamma}(1+\beta+\tilde{\pi}^2)\right)^2}{\Omega\left(\tilde{\pi}^1,\tilde{\pi}^2,\tilde{\gamma},\tilde{\delta}\right)}$
2a: $h_0<h^{**}$	$\dfrac{\tilde{\pi}^1\left(\left(\beta+(1-\alpha)\tilde{\pi}^2\right)(1-\tilde{\delta}\tilde{\pi}^2)\right)^2}{\Omega\left(\tilde{\pi}^1,\tilde{\pi}^2,\tilde{\gamma},\tilde{\delta}\right)}$	$\dfrac{\tilde{\pi}^2\left(\tilde{\gamma}(1+\beta+\tilde{\pi}^2)\right)^2}{\Omega\left(\tilde{\pi}^1,\tilde{\pi}^2,\tilde{\gamma},\tilde{\delta}\right)}$
2b: $h_0=h^{**}$	$\dfrac{\pi^1(h^{**})\left(\left(\beta+(1-\alpha)\pi^2(h^{**})\right)(1-\delta(h^{**})\pi^2(h^{**}))\right)^2}{\Omega(\pi^1(h^{**}),\pi^2(h^{**}),\gamma(h^{**}),\delta(h^{**}))}$	$\dfrac{\pi^2(h^{**})\left(\gamma(h^{**})(1+\beta+\pi^2(h^{**}))\right)^2}{\Omega(\pi^1(h^{**}),\pi^2(h^{**}),\gamma(h^{**}),\delta(h^{**}))}$
2c: $h_0>h^{**}$	$\dfrac{\bar{\pi}^1\left(\left(\beta+(1-\alpha)\bar{\pi}^2\right)(1-\bar{\delta}\bar{\pi}^2)\right)^2}{\Omega\left(\bar{\pi}^1,\bar{\pi}^2,\bar{\gamma},\bar{\delta}\right)}$	$\dfrac{\bar{\pi}^2\left(\bar{\gamma}(1+\beta+\bar{\pi}^2)\right)^2}{\Omega\left(\bar{\pi}^1,\bar{\pi}^2,\bar{\gamma},\bar{\delta}\right)}$
3	$\dfrac{\bar{\pi}^1\left(\left(\beta+(1-\alpha)\bar{\pi}^2\right)(1-\bar{\delta}\bar{\pi}^2)\right)^2}{\Omega\left(\bar{\pi}^1,\bar{\pi}^2,\bar{\gamma},\bar{\delta}\right)}$	$\dfrac{\bar{\pi}^2\left(\bar{\gamma}(1+\beta+\bar{\pi}^2)\right)^2}{\Omega\left(\bar{\pi}^1,\bar{\pi}^2,\bar{\gamma},\bar{\delta}\right)}$

Cases	Ξ^*
1	$\dfrac{\tilde{\pi}^1\left(\beta+(1-\alpha)\tilde{\pi}^2\right)(1-\tilde{\delta}\tilde{\pi}^2)\tilde{\gamma}(1+\beta+\tilde{\pi}^2)}{\Omega\left(\tilde{\pi}^1,\tilde{\pi}^2,\tilde{\gamma},\tilde{\delta}\right)}$
2a: $h_0<h^{**}$	$\dfrac{\tilde{\pi}^1\left(\beta+(1-\alpha)\tilde{\pi}^2\right)(1-\tilde{\delta}\tilde{\pi}^2)\tilde{\gamma}(1+\beta+\tilde{\pi}^2)}{\Omega\left(\tilde{\pi}^1,\tilde{\pi}^2,\tilde{\gamma},\tilde{\delta}\right)}$
2b: $h_0=h^{**}$	$\dfrac{\pi^1(h^{**})\left(\beta+(1-\alpha)\pi^2(h^{**})\right)(1-\delta(h^{**})\pi^2(h^{**}))\gamma(h^{**})(1+\beta+\pi^2(h^{**}))}{\Omega(\pi^1(h^{**}),\pi^2(h^{**}),\gamma(h^{**}),\delta(h^{**}))}$
2c: $h_0>h^{**}$	$\dfrac{\bar{\pi}^1\left(\beta+(1-\alpha)\bar{\pi}^2\right)(1-\bar{\delta}\bar{\pi}^2)\bar{\gamma}(1+\beta+\bar{\pi}^2)}{\Omega\left(\bar{\pi}^1,\bar{\pi}^2,\bar{\gamma},\bar{\delta}\right)}$
3	$\dfrac{\bar{\pi}^1\left(\beta+(1-\alpha)\bar{\pi}^2\right)(1-\bar{\delta}\bar{\pi}^2)\bar{\gamma}(1+\beta+\bar{\pi}^2)}{\Omega\left(\bar{\pi}^1,\bar{\pi}^2,\bar{\gamma},\bar{\delta}\right)}$

where $\Omega\left(\pi^1,\pi^2,\gamma,\delta\right)\equiv\pi^1\left(\left(\beta+(1-\alpha)\pi^2\right)\left(1-\delta\pi^2\right)\right)^2+\pi^1\left(\beta+(1-\alpha)\pi^2\right)\left(1-\delta\pi^2\right)\gamma(1+\beta+\pi^2)+\pi^2\left(\gamma(1+\beta+\pi^2)\right)^2$

Proof. See the Appendix. ∎

The intuition behind that asymptotic stabilization result goes as follows. Fertility rates and mortality rates, although functions of human capital, are bounded from above and from below, so that even if h_t takes extreme values, this remains compatible with a stationary demography. Thus, given that ratios Λ_t, Ξ_t and Φ_t depend on fertility and mortality rates *only*, the long-run dynamics of human capital, by leading to a stabilization of fertility and mortality rates, implies also a stabilization of the age structure (Λ_t,Ξ_t,Φ_t).[23]

Note, however, that Proposition 2 does not imply that the long-run population size is constant. As in Lotka Theorem (1939), the asymptotic constancy of the age structure does not imply the asymptotic constancy of the population size: homothetic growth or reduction may occur (depending on whether the strength of fertility exceeds the one of mortality or not), but the relative sizes of all age-groups must always remain the same in the long-run.

Regarding the determinants of the asymptotic age structure (Λ^*,Ξ^*,Φ^*), these are of four distinct types: (1) the individual taste for children (i.e. β); (2) the elasticity of human capital with respect to education spending (i.e. α); (3) survival functions $\pi^1(.)$ and $\pi^2(.)$; (4) intergenerational contribution functions $\gamma(.)$ and $\delta(.)$.

As far as survival functions $\pi^1(.)$ and $\pi^2(.)$ are concerned, it is worth underlining that, except in Case 2b, only limit-values $\tilde{\pi}^1$, $\bar{\pi}^1$, $\tilde{\pi}^2$ and $\bar{\pi}^2$ matter for the asymptotic age structure, but the other characteristics of survival functions are irrelevant. Thus, in order to know the long-run proportion of old people in

---

[23] If the long-run equilibrium of human capital is a stationary equilibrium, long-run fertility and mortality rates are also constant, implying a constant asymptotic age structure. But if the long-run human capital is not stationary, the boundedness of fertility and mortality will also make the age structure stabilize over time.

the population, there is no need, in general, to know the precise shape of the survival function: only limit values matter.[24] Clearly, the limit values $\tilde{\pi}^1$ and $\bar{\pi}^1$ contribute to raise the long-run proportion of young people Λ^*, and to decrease the long-run proportion of old persons Φ^*, while the effect on the long-run proportion of middle aged agents Ξ^* is ambiguous. On the contrary, limit values $\tilde{\pi}^2$ and $\bar{\pi}^2$ have exactly the opposite effects.

Contribution rates $\gamma(.)$ and $\delta(.)$ have also significant effects on the asymptotic age structure.[25] The impacts of $\gamma(.)$ and $\delta(.)$ are quite different: whereas the former decreases the proportion of the young and raises the ones of the middle-aged and the elderly, the latter reduces the proportions of the young and the middle aged, but raises the one of the elderly. That influence of intergenerational trade is worth being underlined. True, it was often stressed, following the work by Ehrlich and Lui (1991), that the scope and form of intergenerational trade depend on the age structure. But that relation is here *two-directional*: the age structure is also influenced by intergenerational trade, that is, by what each adult gives to his children and surviving parents, because fertility and education are determined by contribution rates γ_t and δ_t. A corollary of this is that any attempt to forecast the asymptotic age structure must consider how intergenerational relations are evolving when human capital accumulates. That will be the task of the next section, but, before that, let us come back on the issue of the convergence between populations with distinct initial age structures.

4.3 The convergence between different populations

As this was stressed in Section 1, Lotka Theorem (1939) states that, under identical, constant fertility and mortality laws, populations tend to "forget their past": whatever their initial age structure was, these exhibit, in the long-run, the same age structure. This result was extended by Lopez (1961), who showed that, to obtain the convergence towards a given age structure, it is not necessary that populations are subject to the same time-invariant fertility and mortality laws, but, only, to the same (possibly time-varying) fertility and mortality laws. Proposition 3 suggests that, in general, the convergence of two populations towards the same age structure does not even require populations to be subject to the same (possibly time-varying) fertility and mortality laws: only the asymptotic *convergence* of those laws is required.

Proposition 3 *Take two populations A and B with the same structural parameters $\left\{\alpha, \beta, \tilde{\pi}^1, \bar{\pi}^1, \tilde{\pi}^2, \bar{\pi}^2\right\}$ and the same contribution functions $\gamma(h_t)$ and $\delta(h_t)$, but with different initial age structures $\left(\Lambda_0^A, \Xi_0^A, \Phi_0^A\right)$ and $\left(\Lambda_0^B, \Xi_0^B, \Phi_0^B\right)$, and with distinct initial human capital levels $h_0^A > 0$ and $h_0^B > 0$. We assume $\left(\Lambda_0^A, \Xi_0^A, \Phi_0^A\right) \neq \left(\Lambda_0^B, \Xi_0^B, \Phi_0^B\right)$ and $h_0^A < h_0^B$.*

- *Under Cases 1 and 3, A and B exhibit the same asymptotic age structure.*

- *Under Case 2,*
  - *if $h_0^A < h_0^B < h^{**}$, A and B exhibit the same asymptotic age structure.*

---

[24] On the contrary, under Case 2b, the long-run age structure depends on the level of the steady-state human capital, so that the precise shape of the survival function matters (and not only limit-values).

[25] Here again, only limit values matter, except in Case 2b.

- if $h_0^A < h^{**} < h_0^B$, A and B do not exhibit the same asymptotic age structure.

- if $h^{**} < h_0^A < h_0^B$, A and B exhibit the same asymptotic age structure.

Proof. The proof follows from Propositions 1 and 2. It is straightforward to see that, in Cases 1 and 3, h_0 does not influence the long-run level of fertility and mortality, so that the asymptotic age structure must be the same for all economies, whatever their initial conditions were. However, regarding Case 2, the convergence of age structures across populations requires the initial levels of human capital to be on the same side of the (unstable) steady-state h^{**}. Otherwise, the convergence of age structure cannot occur. ∎

Proposition 3 states that, if we exclude the Case 2 where $h_0^A < h^{**} < h_0^B$, two populations with different initial conditions will necessarily end up with the same long-run age structure. This convergence does not require that the two populations follow the same fertility and mortality rates. Clearly, under different initial levels of human capital, the two populations will exhibit different fertility rates and mortality rates, but those populations will nonetheless converge asymptotically towards the same age structure, as fertility rates and mortality rates will tend to converge as human capital evolves over time. Hence, to have a convergence of age structures, what is required is not identical (possibly time-varying) fertility and mortality laws, but, merely *converging* fertility and mortality laws. This - weaker - condition is satisfied for various initial conditions, but it is true that, if the structural parameters of the economies under study (assumed to be identical) are such that Case 2 holds, then, if we have $h_0^A < h^{**} < h_0^B$, the convergence of fertility and mortality will not take place, and the age structures of populations A and B will remain different.

Although Proposition 3 suggests that the convergence of populations towards the same age structure is likely (except for populations with extremely different initial levels of human capital under Case 2), it should be stressed, however, that this convergence remains *conditional* on populations exhibiting the same structural parameters. Actually, asymptotic age structures depend on the elasticity α, on the taste for children β, on the limit survival probabilities and on the contribution functions $\gamma(.)$ and $\delta(.)$. If, for any reason, the populations under study differ on these, then the long-run age structure will also differ.

5 Age structure dynamics (2): back to history

Let us now turn back to the empirical data on the evolution of age structures across epochs, to investigate to what extent the present, simple framework can replicate the observed dynamics of demographic ratios over time. Moreover, turning back to the data will also be an opportunity to cast a new light on the precise form of the long-run equilibrium age structure under plausible assumptions on the structural parameters of the economy.

5.1 Functional forms

For that purpose, we need to postulate some functional forms for survival functions. For simplicity, we shall assume that $\pi^1(h_t)$ and $\pi^2(h_t)$ take the forms:

$$\pi^1(h_t) = \tilde{\pi}^1 + \frac{(\bar{\pi}^1 - \tilde{\pi}^1)\rho h_t}{1 + \rho h_t}$$

$$\pi^2(h_t) = \tilde{\pi}^2 + \frac{(\bar{\pi}^2 - \tilde{\pi}^2)\sigma h_t}{1 + \sigma h_t}$$

where ρ and σ are positive parameters. Those functional forms satisfy the properties stated in Section 2: $\lim_{h_t \to 0} \pi^1(h_t) = \tilde{\pi}^1 > 0, \lim_{h_t \to \infty} \pi^1(h_t) = \bar{\pi}^1 < 1, \lim_{h_t \to 0} \pi^2(h_t) = \tilde{\pi}^2 > 0$ and $\lim_{h_t \to \infty} \pi^2(h_t) = \bar{\pi}^2 < 1$.

Regarding the contribution rates to each child and to each old, we assume

$$\gamma(h_t) = \tilde{\gamma} + \frac{(\bar{\gamma} - \tilde{\gamma})\mu h_t}{1 + \mu h_t}$$

$$\delta(h_t) = \tilde{\delta} + \frac{(\bar{\delta} - \tilde{\delta})\eta h_t}{1 + \eta h_t}$$

where we have $0 < \tilde{\gamma} \leq \bar{\gamma} < 1$ and $0 < \tilde{\delta} \leq \bar{\delta} < 1$, while μ and η are non-negative parameters, implying that the parental contribution rate to each child and each elderly is non-decreasing in human capital.

5.2 Calibration

In this subsection, we calibrate our model in such a way as to fit the data concerning the economic growth process and the demographic evolutions in France (1816-2008). Clearly, this emphasis on a particular economy involves a significant simplification, as economic and demographic evolutions do not have exactly the same timing and the same size across countries. This simulation exercise has thus no pretension to exhaustiveness.

As far as the lower bounds of survival probabilities are concerned, we shall assume that $\tilde{\pi}^1 = 0.018$ and $\tilde{\pi}^2 = 0.005$, which implies a life expectancy at birth equal to $1 + 0.018 + (0.018 * 0.005) = 1.018$, which is close to 36 years. Regarding the upper bounds of survival probabilities, demographers disagree on the existence or non-existence of some maximum age at death, and on its level (see Lee, 2003). We shall assume here that $\bar{\pi}^1 = 0.95$ and $\bar{\pi}^2 = 0.6$, which implies a maximum life expectancy at birth equal to $1 + 0.95 + (0.95 * 0.6) = 2.52$, which is close to 88 years. This calibration, which can be regarded as based on a pessimistic scenario, will serve only as a benchmark.

Regarding the calibration of β, we rely on fertility estimates in the early 19th century. These estimates point to about 6 children per women. However, France started its demographic transition far before other countries, and fertility started falling there already at the end of the 18th century. Actually, at the beginning of the 19th century, the total fertility rate in France was about 4.5 children per women, so that $n = 2.25$.[26]

From the optimal fertility, we have, under $\pi^2 = \tilde{\pi}^2$,

$$2.25 = \frac{(\beta + 0.005(1 - 0.23))(1 - \delta_t(0.005))}{\gamma_t(1 + \beta + 0.005)}$$

---

[26]See Binion (2000).

We know that γ_t must take a low value, given that transfers in developing economies are generally ascendants (i.e. from children to parents), while δ_t should be higher. Fixing $\gamma_t = \tilde{\gamma} = 0.03$ and $\delta_t = \bar{\delta} = 0.1$ yields $\beta = 0.069$.

Regarding the calibration of production parameters A and α, note that, according to Maddison (2008), real GDP per capita was equal to about 1,135 \$ in 1820 in France (in international Geary-Khamis 1990 \$), and to 22,675 \$ in 2008. Hence the annual average compound growth rate of real GDP per capita is equal to $\left(\frac{22675}{1135}\right)^{\frac{1}{186}} - 1 = 0.0162 = 1.62\%$ per year. Given that periods are here 35 years long, this coincides with a periodic growth factor of $(1.0162)^{35} = 1.75498$. Hence we have

$$\frac{h_{t+1}}{h_t} = A \left(\frac{\gamma(h_t)\alpha\pi^2(h_t)}{\beta + \pi^2(h_t)(1-\alpha)} \right)^{\alpha} = 1.75498$$

Regarding the calibration of the parameter α, it should be first reminded that α can be interpreted as the elasticity of output per capita growth with respect to the share of income per head dedicated to education per child.[27] Given that, according to Barro and Sala-i-Martin (1995), the elasticity of output per capita growth with respect to the share of income dedicated to education is close to 0.23 when controlling for fertility, we shall assume here that α equals 0.23.

Assuming average π^2 equal to 0.15 and average $\gamma = 0.1$ and substituting for α and β yields:

$$A = 1.75498 \left(\frac{0.069 + (0.15)(0.77)}{0.1(0.23)(0.15)} \right)^{0.23}$$

which yields $A = 4.383$. Thus we shall, throughout this numerical exercise, use this value for the productivity parameter A, and use also, for convenience, $h_0 = 1$ as a starting value for the stock of human capital.

The table below summarizes the calibration in the benchmark case.

Parameters	α	A	h_0	β	$\tilde{\gamma}$	$\bar{\delta}$	$\tilde{\pi}^1$	$\bar{\pi}^1$	$\tilde{\pi}^2$	$\bar{\pi}^2$
Values	0.230	4.383	1.000	0.069	0.030	0.100	0.018	0.950	0.005	0.600

5.3 Results

In order to replicate the past trend of the age structure in France (1816-2008), various combinations of the - so far non calibrated - parameters ρ, σ, $\bar{\gamma}$, $\tilde{\delta}$, μ and η can be used. Given that there is no space here to provide an exhaustive study of the age structure patterns associated to all sets of parameters, we shall confine ourselves here to select some parameters values allowing us to replicate the observed trend of the age structure over 1816-2008, and, then, to extrapolate from those parameters the future age structure pattern.

If we set $\rho = 0.5$ and $\sigma = 0.12$ in the survival functions $\pi^1(.)$ and $\pi^2(.)$, it is possible to replicate the age structures of the early 19th century and the late

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[27]Indeed, we have $\frac{h_{t+1} - h_t}{h_t} = A \left(\frac{e_t}{n_t} \right)^{\alpha} - 1$, where e_t is the fraction of y_t dedicated to the education of all children, so that the ratio in brackets is the share of income given to the education of a child.

20th century, under a light growth of the contribution rate to each child (i.e. $\bar{\gamma} = 0.230$, $\mu = 0.15$), and under a constancy of the contribution rate to the elderly ($\tilde{\delta} = 0.1$, $\eta = 0$). The result of that simulation is shown on Figure 4.[28]
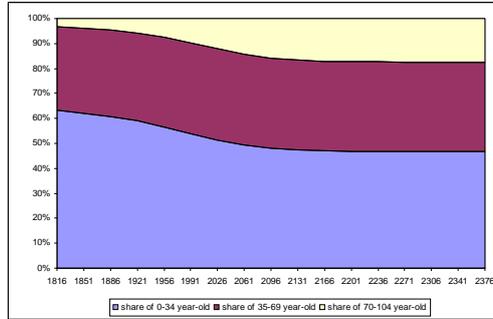

Figure 4: Age structure in France: simulated long-run dynamics

Figure 4 shows, for France, the past, present and future age structures that are associated with the above calibration of the model. It is the equivalent of Figure 3 in Section 1, but simulated from the model, and extending the time horizon beyond the actual data (stopping in 2008), until 2400. Note first that, in comparison with Figure 3, Figure 4 shows a dynamics of age structure that is smoother than the actual one for the 19th and the 20th century. Given that our model is concerned only with long-run evolutions, this does not come as a surprise, but is not problematic for the issue at stake.

Regarding the future, the model predicts a stabilization of the age structure around year 2150, at a level that is significantly different from the one prevailing today. Actually, the current proportion of the French population older than 70 years is about 11 percent, while the relative size of that group will stabilize at about 17.4 percent. That rise of the proportion of the elderly is naturally made at the cost of the two other age groups, and, in particular, of the young.

How plausible is the above picture as a predictor of future age structure in France? Actually, Figure 4 derives its plausibility not only from its ability to fit the past trend of the age structure, but, also, from the fact that the predicted trends are not a simple empirically-based extrapolation from past data, but are rooted in a theoretical model. However, despite this, one may argue that there exist various combinations of parameters that may fit with the past data, so that there is no unique possible future picture. In particular, the above picture was based on a pessimistic scenario concerning maximum longevity prospects, which may be questioned, as we shall now discuss.

5.4 Pessimistic *versus* optimistic scenarios

The simulations carried out above relied on $\bar{\pi}^1 = 0.95$ and $\bar{\pi}^2 = 0.6$, which implied a maximum life expectancy at birth equal to about 88 years. That assumption is fully compatible with the views defended by the least optimistic demographers (see Olshanksy and Carnes, 2001). While such a life expectancy

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[28]Note that, under those parameters values, the economy falls under Case 2 of Section 3.

is significantly higher than the one prevailing today, one may argue that such a limitation of longevity is hardly plausible, and that there is no obvious reason why the accumulation of knowledge would not allow for further expansions of the human lifespan. That questioning of "pessimistic" scenarios on longevity limits has been made, among others, by Oeppen and Vaupel (2002).

Within the present model, that questioning can be captured by raising the limit probabilities $\bar{\pi}^1$ and $\bar{\pi}^2$, that is, the values of $\pi^1(h_t)$ and $\pi^2(h_t)$ when h_t tends to infinity. In order to explore the impact of raising $\bar{\pi}^1$ and $\bar{\pi}^2$, we computed the evolution of age structure, still for France, under the assumptions $\bar{\pi}^1 = 0.99$ and $\bar{\pi}^2 = 0.8$, corresponding to a limit life expectancy at birth of 97 years, and contrasted the age structure dynamics with the one under $\bar{\pi}^1 = 0.99$ and $\bar{\pi}^2 = 0.99$, corresponding to a limit life expectancy at birth of 104 years.

Figures 5 and 6 show the age structure dynamics resulting from those two alternative assumptions on limit survival conditions.[29]
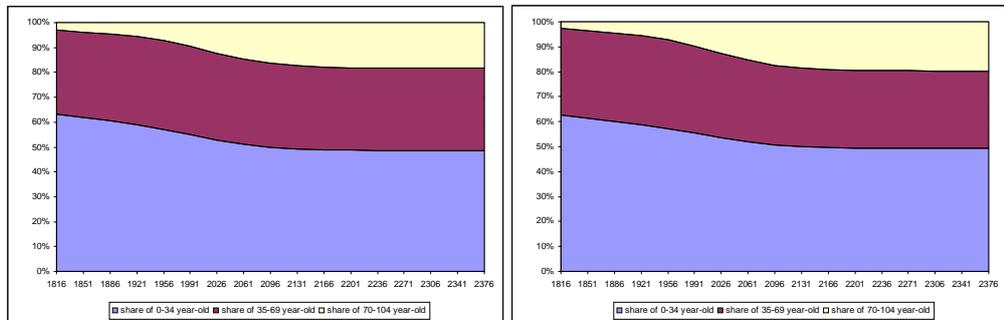

Figure 5: Max life expectancy = 97 years Figure 6: Max life expectancy = 104 years

When comparing Figure 5 with Figure 4, there does not seem to be a major change, except that the long-run proportion of agents older than 70 years is significantly increased under $\bar{\pi}^1 = 0.99$ and $\bar{\pi}^2 = 0.8$: this grows from about 17.4 percent of the population to 18.2 percent, which is a statistically significant change. Under the most optimistic scenario, i.e. $\bar{\pi}^2 = 0.99$, the long-run proportion of people older than 70 years is also larger, and equal to 19.6 percent.

Those significant changes show the sensitivity of the forecasts to the assumptions made on maximum life expectancy. That point being stressed, the sensitivity of forecasts should not be exaggerated: the changes in the long-run composition of the population induced by variations in $\bar{\pi}^1$ and $\bar{\pi}^2$ remain of relatively small sizes with respect to the benchmark case. Thus, although the long-run age structure depends on limit survival probabilities, and, as such, relies on some non-observable assumptions, there are no extreme differentials between the forecasts obtained under various scenarios. True, the optimistic one makes the elderly represent about 20 % of the population, which is enormous in comparison with what prevails today. But even in the pessimistic scenario, with

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[29]Note that, in each case, the parameter σ in $\pi^2(h_t)$ had to be modified, in order to still have a compatibility of the simulated age structures with the observations for the 19th and 20th centuries. σ is now assumed to be equal to 0.08 under $\bar{\pi}^2 = 0.80$, and equal to 0.06 under $\bar{\pi}^2 = 0.99$.

a maximum life expectancy of 88 years, the old will represent more than 17.4 percent of the whole population near 2150, which is much larger than today.

Moreover, large changes in limit survival probabilities do not seem to affect strongly the dynamics of the adjustment towards the long-run age structure. Under each scenario, the age structure will stabilize in the middle of the 22nd century, and this timing is invariant to changes in the limit survival probabilities.

6   Concluding remarks

The goal of this paper was to study the dynamics of age heterogeneity, and, in particular, to examine whether the asymptotic stabilization results of Lotka (1939) and Lopez (1961) still hold when fertility and mortality are endogenous. We wanted also to characterize the *form* of the asymptotic age structure.

For those purposes, we studied a three-period OLG economy with human capital accumulation and endogenous fertility and mortality, and where coexisting generations are linked through intergenerational trade. In that model, children are treated both as a consumption good and as an investment good. We showed that the long-run dynamics of output is determined by the intensity of the taste for children, the elasticity of future human capital with respect to education spending, the limit values of the survival probability to the old age and of the cost of raising a child.

It was also shown that the age structure will necessarily converge asymptotically towards some particular form, determined by the fundamentals of the economy. Regarding the factors determining the asymptotic age structure, this model highlighted the crucial roles played by limit values of the survival functions to the young adulthood and old adulthood, as well as the influence of contribution rates to the children and to the elderly. What the asymptotic age heterogeneity will be depends on what the intergenerational terms of trade are.

Note that the addition of postulates on the determinants of fertility and mortality does not only allow us to cast a new light on the precise form of the asymptotic age structure, but allows us also to reconsider the assumptions that guarantee the convergence of populations with distinct age structures towards a unique age structure. While Lopez (1961) argued that such a convergence holds when populations are subject to the same time-varying fertility and mortality laws, we show that this convergence holds as long as fertility and mortality laws are merely *converging* across nations, convergence which requires converging terms of trade between generations in all populations.

The model was then used to replicate numerically the past evolution of the age structure in France (1816-2008), and to extract, from that evolution, the future evolution of the age structure over 2008-2400. We showed that the model can approximate the long-run trend of the age structure over the period. Applying the model to the future yields also some forecasts, under various scenarios regarding the maximum life expectancy that can be reached. The age structure at which the French population will stabilize is significantly sensitive to the postulated limit survival probabilities, but that sensitivity is of limited size, and the timing of the adjustment - around 2150 - is robust to the different scenarios.

To conclude, it should be stressed that this model suffers from some restrictions, which invite further research. First, the microfoundations of fertility and education are here purely egoistic, and, as such, complement the works inspired

by Barro and Becker (1989). This way of looking at the fertility decision is not neutral, and it would be worth reconsidering the age structure dynamics in a Barro-Becker type of economy, where intergenerational relations involve altruistic concerns. Secondly, this paper still relies on some postulate on the maximum longevity, fixed to three periods (i.e. equivalent to 105 years). This limitation plays a major role in the possibility to demonstrate the asymptotic stabilization of age structures. Moreover, replacing it by some other, weaker postulate would introduce new parameters in the asymptotic age structure. Hence much work remains to be done on long-run age structure dynamics.

7   References

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8 Appendix

8.1 Long-run output dynamics

The question of the existence of a steady-state equilibrium can be reformulated as whether the transition function $G(h_t)$ admits a fixed point. Let us thus study the properties of $G(h_t)$.

Note first that, given $A > 0$, we have $G(h_t) \geq 0$. We can also see that $G(0) = 0$.

The ratio $G(h_t)/h_t$, equal to

$$G(h_t)/h_t = A \left(\frac{\gamma(h_t)\alpha\pi^2(h_t)}{\beta + \pi^2(h_t)(1-\alpha)} \right)^\alpha$$

is increasing in h_t, as the ratio in brackets is increasing in h_t under $\gamma'(h_t) \geq 0$.

Moreover, we have, under $\lim_{h_t \to \infty} \pi^2(h_t) = \bar{\pi}^2 < 1$ and $\lim_{h_t \to \infty} \gamma(h_t) = \bar{\gamma} < 1$:

$$\lim_{h_t \to \infty} G(h_t)/h_t = A \left(\frac{\bar{\gamma}\alpha\bar{\pi}^2}{\beta + \bar{\pi}^2(1-\alpha)} \right)^\alpha$$

which can be larger or smaller than unity, depending on whether:

$$\lim_{h_t \to \infty} G(h_t)/h_t \lesseqgtr 1 \iff A \left(\frac{\bar{\gamma}\alpha\bar{\pi}^2}{\beta + \bar{\pi}^2(1-\alpha)} \right)^\alpha \lesseqgtr 1$$

The derivative $G'(h_t)$ is:
$$G'(h_t) = A \left(\frac{\gamma(h_t)\alpha\pi^2(h_t)}{\beta + \pi^2(h_t)(1-\alpha)} \right)^\alpha$$
$$+ h_t A\alpha \left(\frac{\gamma(h_t)\alpha\pi^2(h_t)}{\beta + \pi^2(h_t)(1-\alpha)} \right)^{\alpha-1} \left(\frac{\gamma(h_t)\alpha\pi^{2'}(h_t)\beta + \gamma'(h_t)\alpha\pi^2(h_t)\left(\beta + \pi^2(h_t)(1-\alpha)\right)}{[\beta + \pi^2(h_t)(1-\alpha)]^2} \right)$$

so that $G'(h_t) \geq 0$.

Note that, under $\lim_{h_t \to 0} \pi^2(h_t) = \tilde{\pi}^2 > 0$, $0 < \lim_{h_t \to 0} \pi^{2'}(h_t) < \infty$, and $\lim_{h_t \to 0} \gamma'(h_t) < \infty$, we have:

$$\lim_{h_t \to 0} G'(h_t) = A \left(\frac{\tilde{\gamma}\alpha\tilde{\pi}^2}{\beta + \tilde{\pi}^2(1-\alpha)} \right)^\alpha$$

We are now in position to prove Proposition 1.

In Case 1, $G(h_t)$ is below the 45° line in the neighborhood of 0 (as $G(0) = 0$ and $\lim_{h_t \to 0} G'(h_t) < 1$), and remains below the 45° line when h_t tends to $+\infty$ (as $\lim_{h_t \to \infty} G(h_t)/h_t < 1$). Thus, given that $G'(h_t) \geq 0$, and that $G(h_t)/h_t$ is increasing in h_t, $G(h_t)$ always remains below the 45° line, so that no positive steady-state exists.

In Case 2, $G(h_t)$ is also below the 45° line in the neighborhood of 0, but lies above the 45° line when h_t tends to $+\infty$ (as $\lim_{h_t \to \infty} G(h_t)/h_t > 1$). As a consequence, given the continuity of $G(h_t)$, it must be the case that $G(h_t)$ crosses the 45° line at least once at a positive $h = h^{**}$. Regarding the uniqueness of that non-trivial steady-state, one cannot *a priori* rule out the existence of several steady-states, as no functional form is imposed on the survival function.

The uniqueness of a positive steady-state equilibrium under Case 2 can be proved by *reductio ad absurdum*. Let us assume that h^{**} is not the unique strictly positive steady-state equilibrium. As $G(h_t)$ first crosses the 45° line from below, the multiplicity of steady-states would imply that $G(h_t)$ would have to cross the 45° line from above at least once, and, then, again, from below, as we know that $\lim_{h_t \to \infty} G(h_t)/h_t > 1$. Let us denote by h^+ the intermediate steady-state, which is obtained when $G(h_t)$ crosses the 45° line from above. Given that $G(h_t)$ crosses the 45° line from above at h^+, it must be the case that this equilibrium is locally stable. However, at that steady-state, $G'(h^+)$ is

$$G'(h^+) = A \left(\frac{\gamma(h^+)\alpha\pi^2(h^+)}{\beta + \pi^2(h^+)(1-\alpha)} \right)^\alpha$$
$$+ h^+ A\alpha \left(\frac{\gamma(h^+)\alpha\pi^2(h^+)}{\beta + \pi^2(h^+)(1-\alpha)} \right)^{\alpha-1} \left(\frac{\gamma(h^+)\alpha\pi^{2\prime}(h^+)\beta + \gamma'(h^+)\alpha\pi^2(h^+)\left(\beta + \pi^2(h^+)(1-\alpha)\right)}{[\beta + \pi^2(h^+)(1-\alpha)]^2} \right)$$

At that steady-state, we have, by definition, $G(h^+) = h^+$, so that the first term, equal to $G(h^+)/h^+$, equals 1. But given that the second term of $G'(h^+)$ is, under $\gamma'(h^+) \geq 0$, strictly positive, we have $|G'(h^+)| > 1$, which is incompatible with the stability of the steady-state. Thus a contradiction is reached. If a non-zero steady-state exists, this must be unstable. Hence, if $G(h_t)$ first crosses the 45° line from below, it cannot cross it again from above. But given that $\lim_{h_t \to \infty} G(h_t)/h_t > 1$, the transition function must end up above the 45° line, so that there must be a unique steady-state equilibrium. Hence h^{**} is the unique positive steady-state. That steady-state is clearly unstable, as $G(h_t)$ crosses the 45° line from below, so that an economy starting with $h_0 < h^{**}$ will converge towards 0, while an economy with $h_0 > h^{**}$ will exhibit perpetual growth. That instability appears clearly if one substitutes h^+ for h^{**} in the above formula. We necessarily have $|G'(h^{**})| > 1$, in conformity with instability.

In Case 3, $G(h_t)$ is above the 45° line in the neighborhood of 0, and remains above the 45° line when h_t tends to $+\infty$. Thus, given that $G'(h_t) \geq 0$, and that $G(h_t)/h_t$ is increasing in h_t, it always remains above the 45° line, so that no positive steady-state exists: the economy exhibits eternal growth.

8.2   Long-run age structure dynamics

To demonstrate Proposition 2, let us consider the 3 cases studied in Section 3, and show that, in each case, the demographic ratios Λ_t, Ξ_t and Φ_t tend towards constant levels Λ^*, Ξ^* and Φ^*.

Under Case 1, we know that $\lim_{t\to\infty} h_t = 0$. Hence, we have:

$$\lim_{t\to\infty} n_t = \frac{\left(\beta + (1-\alpha)\tilde{\pi}^2\right)\left(1 - \tilde{\delta}\tilde{\pi}^2\right)}{\tilde{\gamma}(1 + \beta + \tilde{\pi}^2)}$$

which is, under $\tilde{\pi}^2 > 0$, a positive constant. Note that also $\lim_{t\to\infty} n_{t-1} = \lim_{t\to\infty} n_t = \frac{\left(\beta + (1-\alpha)\tilde{\pi}^2\right)\left(1 - \tilde{\delta}\tilde{\pi}^2\right)}{\tilde{\gamma}(1 + \beta + \tilde{\pi}^2)}$.

Hence, the ratio Λ_t tends, in the long-run, towards:

$$\lim_{t\to\infty} \Lambda_t = \frac{\tilde{\pi}^1\left(\frac{\left(\beta+(1-\alpha)\tilde{\pi}^2\right)\left(1-\tilde{\delta}\tilde{\pi}^2\right)}{\tilde{\gamma}(1+\beta+\tilde{\pi}^2)}\right)^2}{\tilde{\pi}^1\left(\frac{\left(\beta+(1-\alpha)\tilde{\pi}^2\right)\left(1-\tilde{\delta}\tilde{\pi}^2\right)}{\tilde{\gamma}(1+\beta+\tilde{\pi}^2)}\right)^2 + \tilde{\pi}^1\frac{\left(\beta+(1-\alpha)\tilde{\pi}^2\right)\left(1-\tilde{\delta}\tilde{\pi}^2\right)}{\tilde{\gamma}(1+\beta+\tilde{\pi}^2)} + \tilde{\pi}^2} = \Lambda^*$$

which is a positive constant, as $\tilde{\pi}^1, \tilde{\pi}^2, \tilde{\gamma}$ and $\tilde{\delta}$ are positive constants.

The same kind of rationale can be used to show that Φ_t and Ξ_t tend towards constant levels Φ^* and Ξ^*. As a consequence, the entire age structure $(\Lambda_t, \Xi_t, \Phi_t)$ tends towards an equilibrium age structure $(\Lambda^*, \Xi^*, \Phi^*)$.

Under Case 2a, i.e., Case 2 with $h_0 < h^{**}$, we have $\lim_{t\to\infty} h_t = 0$. Hence the same conclusions as in Case 1 hold.

Under Case 2b, that is, Case 2 with $h_0 = h^{**}$, we know that $\lim_{t\to\infty} h_t = h^{**}$. Thus we have

$$\lim_{t\to\infty} n_t = \frac{\left(\beta + (1-\alpha)\pi^2(h^{**})\right)\left(1 - \delta(h^{**})\pi^2(h^{**})\right)}{\gamma(h^{**})(1 + \beta + \pi^2(h^{**}))}$$

which is a positive constant. Note that also $\lim_{t\to\infty} n_{t-1} = \lim_{t\to\infty} n_t$. Hence, the ratio Λ_t tends, in the long-run, towards:

$$\lim_{t\to\infty} \Lambda_t = \frac{\pi^1(h^{**})\left(\frac{\left(\beta+(1-\alpha)\pi^2(h^{**})\right)\left(1-\delta(h^{**})\pi^2(h^{**})\right)}{\gamma(h^{**})(1+\beta+\pi^2(h^{**}))}\right)^2}{\left(\frac{\left(\beta+(1-\alpha)\pi^2(h^{**})\right)\left(1-\delta(h^{**})\pi^2(h^{**})\right)}{\gamma(h^{**})(1+\beta+\pi^2(h^{**}))}\right)^2\pi^1(h^{**}) + \pi^1(h^{**})\frac{\left(\beta+(1-\alpha)\pi^2(h^{**})\right)\left(1-\delta(h^{**})\pi^2(h^{**})\right)}{\gamma(h^{**})(1+\beta+\pi^2(h^{**}))} + \pi^2(h^{**})}$$
$$= \Lambda^*$$

which is, given that $\pi^1(h^{**})$, $\pi^2(h^{**})$, $\gamma(h^{**})$ and $\delta(h^{**})$ are positive constants, a positive constant.

The same kind of rationale can be used to show that Φ_t and Ξ_t tend towards constant levels Φ^* and Ξ^*.

Under Case 2c, that is, Case 2 with $h_0 > h^{**}$, we know that $\lim_{t\to\infty} h_t = +\infty$. Thus we have

$$\lim_{t\to\infty} n_t = \frac{\left(\beta + (1-\alpha)\bar{\pi}^2\right)\left(1 - \bar{\delta}\bar{\pi}^2\right)}{\bar{\gamma}(1 + \beta + \bar{\pi}^2)}$$

which is a positive constant. Note that also $\lim_{t\to\infty} n_{t-1} = \lim_{t\to\infty} n_t$.

Hence, the ratio Λ_t tends, in the long-run, towards:

$$\lim_{t\to\infty} \Lambda_t = \frac{\bar{\pi}^1\left(\frac{\left(\beta+(1-\alpha)\bar{\pi}^2\right)\left(1-\bar{\delta}\bar{\pi}^2\right)}{\bar{\gamma}(1+\beta+\bar{\pi}^2)}\right)^2}{\bar{\pi}^1\left(\frac{\left(\beta+(1-\alpha)\bar{\pi}^2\right)\left(1-\bar{\delta}\bar{\pi}^2\right)}{\bar{\gamma}(1+\beta+\bar{\pi}^2)}\right)^2 + \bar{\pi}^1\frac{\left(\beta+(1-\alpha)\bar{\pi}^2\right)\left(1-\bar{\delta}\bar{\pi}^2\right)}{\bar{\gamma}(1+\beta+\bar{\pi}^2)} + \bar{\pi}^2} = \Lambda^*$$

which is, given that $\bar{\pi}^1$, $\bar{\pi}^2$, $\bar{\gamma}$ and $\bar{\delta}$ are positive constants, a positive constant.

The same kind of rationale can be used to show that Φ_t and Ξ_t tend towards constant levels Φ^* and Ξ^*.

Finally, regarding Case 3, we know that $\lim_{t \to \infty} h_t = +\infty$, so that the same rationale as in Case 2c applies.