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Expectational Coordination in simple Economic Contexts : Concepts and Analysis with emphasis on Strategic Substitutabilities.

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Abstract

We consider an economic model that features : 1. a continuum of agents 2. an aggregate state of the world over which agents have an infinitesimal influence. We first review the connections between the “eductive viewpoint” that puts emphasis for example on “Strongly Rational Expectations equilibrium” and the standard game-theoretical rationalizability concepts. Besides the Cobweb tâtonnement outcomes, which mimic an ”eductive” reasoning subject to homogenous expectations, we define, characterize (and prove the convexity of) the sets of ”Rationalizable States” and “Point-Rationalizable States”, which respectively incorporate heterogenous point-expectations and heterogenous stochastic expectations.

In the case where our model displays strategic complementarities, we find unsurprisingly that all the ””eductive” criteria” under scrutiny support rather similar conclusions, particularly when the equilibrium is unique. With strategic susbstitutabilities, the success of expectational coordination, in the case where a unique equilibrium does exists, relates with the absence of cycles of order 2 of the ”Cobweb” mapping : in this case, again, heterogenity of expectations does not matter. However, when cycles of order 2 do exist, our different criteria predict different set of outcomes, although all are tied with cycles of order 2. Under differentiability assumptions, the Poincaré-Hopf method leads to global results for Strong Rationality of equilibrium.

At the local level, the different criteria under scrutiny can be adapted to the analysis of expectational coordination. They leads to the same stabilty conclusions, only when there are local strategic complementarities or strategic substitutabilities. However, so far as the analysis of local expectational coordination is concerned, it is argued and shown that the stochastic character of expectations can most often be forgotten.

Résumé :

Nous considérons une classe de modèles économiques où les agents, en grand nombre, ont une influence infinitésimale sur l'état agrégé du systéme. Nous présentons les connexions entre le point de vue ”divinatoire” qui met l'accent sur les ”Equilibres Fortement Rationnels” et les concepts de rationalisabilité standard de la théorie des jeux. A côté des issues du ”Tâtonnement du Cobweb”, qui mime un raisonnement divinatoire avec anticipations homogènes, nous définissons, caractérisons (et prouvons la convexité de) l'ensemble des ”Etats Ponctuellement Rationalisables” et des ”Etats Rationalisables”, qui incorporent l'idée que les anticipations sont encore ponctuelles mais hétérogènes puis qu'elles sont stochastiques.

Dans le cas où le modèle est à complémentarités stratégiques, on montre, sans surprise, que tous les critères divinatoires conduisent à des conclusions de stabilité identiques, particulièrement quand l'équilibre est unique. Avec des substituabilités stratégiques, le succès de la coordination des anticipations vers un équilibre unique dépend de l'absence de cycles d'ordre 2 pour l'application du Cobweb : dans ce cas à nouveau, l'hétérogénéité des anticipations n'a pas d'importance. Toutefois, si un cycle d'ordre 2 existe, l'hétérogénéité est un facteur important de l'analyse de la coordination et nos différents critères conduisent à des prédictions différentes. Des résultats globaux de Forte Rationalité peuvent dependant être obtenus, dans les cas différentiables, en utilisant la méthode de Poincaré-Hopf.

Au niveau local, les différents critères à l'examen conduisent à la même prédiction de la stabilité locale de la coordination, soit en cas de complémentarités, soit en cas de substituabilité stratégiques. De façon plus générale, on peut seulement montrer que la stochasticité des anticipations peut être oubliée pour l'analyse locale.

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1 Introduction.

This paper is concerned with the question of the quality of expectational coordination in economic contexts. It focuses attention on a class of economic models with "non-atomic" agents, i.e. agents that are too small to have a significant influence on the aggregate state of the economic system[1]. Such an assumption often fits the need of economic analysis, (in general equilibrium, macroeconomics, ..) although it excludes from our analysis [2] oligopolistic competition models in which agents, (firms) have market power. Also, our framework refers to a context of perfect information. Although a reinterpretation of the basic model allows it to encompass certain problems with imperfect information[3], a more systematic extension to general contexts of imperfect information is in progress.

We first present the skeleton of the model under scrutiny, which we interpret successively and equivalently as a game with a continuum of agents and as an economic model (Section 2). We then focus attention on equilibria and focus attention on what may be called the expectational quality (or plausibility, or robustness) of equilibria. Our viewpoint is "eductive" in the sense that it refers to the reasonings of agents attempting to guess the actions (or guesses) of others. Some of the concepts we use have purely economic underpinnings : it is the case of the Cobweb tâtonnement outcomes or the associated concept of Iterative Expectational Stability that comes from the macroeconomic literature of the eighties. Others have a general game theoretical inspiration : strategic point rationalizability or strategic plain rationalizability. We show that they can be adapted in the standard way[4], in order to take advantage of the specificities of our economic context. We then focus attention on we call State Point Rationalizability and State Rationalizability and on the corresponding stability concepts of Strong Rationality. The two just evoked concepts refer to "eductive" arguments that take full notice of the heterogeneity of expectations, when the first one (Cobweb tâtonnement outcome) refers to homogenous expectations. We also put emphasis on local counterparts to the concepts that leads to stress local rather than global expectational stability of equilibria.(global or local strong rationality in the sense of Guesnerie (1992)).

Section 3 defines and clarifies the connections between the different concepts under review within our framework. The inclusion stressed in Proposition 3.8 is unsurprising but useful : it reflects the increasing demand of the "eductive" analysis when expectations,instead of being homogenous, become heterogenous and stochastic. Another result has to be stressed : the fact that state rationalizable sets are convex.

Section 4 comes to the main application of our general analysis, which concerns economies with strategic substitutabilities. We first reformulate and prove in our setting the standard results obtained when interactions are dominated by strategic complementarities (Proposition 4.3 in Section 4.1). In such a context, a striking result is that "uniqueness is the Graal", in the sense that uniqueness of the equilibrium triggers stability of the equilibrium, for any of the criteria evoked in Section 3. However, this equivalence of expectational criteria is known to fail dramatically outside the strategic complementarities world. Our main result of Section 4, shows that in the world with strategic substitutabilities under consideration, expectational stability is still easy to analyse. Theorem 4.9 asserts that uniqueness, not of fixed point of the best response mapping, but uniqueness of its cycles of order two, is still the "Graal" : when this occurs, the unique equilibrium does fit all stability criteria under consideration. This is a (probably) surprising result but (certainly) a powerful one. It has potentially many applications, in particular, as we argue in Section 2.3, in a general equilibrium framework when strategic substitutabilities often dominate strategic complementarities. In the same way, the sufficient conditions presented trigger strong new stability results, valid in the differentiable version of the model.

We briefly conclude.

2 The canonical model

We can view our reference model either as a game with a continuum of players or as a non-atomic economic model. This Section has mainly an introductory purpose : we present the different

[1] Many existing studies on expectational coordination take place adopt such a framework with non-atomic agents and for example among others Guesnerie (1992, 2002), Evans and Guesnerie (1993, 2003, 2005), Chamley (1999, 2004) Desgranges and Heinemann (2005). The same remark applies to part of the global games literature starting from Morris and Shin (1998) and surveyed in Morris and Shin (2003)

[2] Not all, since the fashionable modelling of competition à la Dixit and Stiglitz concile market power and smallness of agents. Note also that our main result has implication for the theory of oloigopolistic competition that are examined in a forthcoming paper.

[3] See below how a simplified version of Morris and Shin (1998) can be imbedded in our model.

[4] See Guesnerie (2002)

viewpoints successively, as well as the specific notation that they call for, existence of equilibria relies on earlier results and a brief inspection of relevant literature suggests that the model is relevant for shedding light on an a priori wide range of economic questions

2.1 The model as a game with a continuum of players.

2.1.1 The setting.

The players.

Let us consider a *game with a continuum of players*[5]. In such games the set of players is the measure space $(I, \mathcal{I}, \lambda)$, where I is the unit interval of $\mathbb{R}$, $I \equiv [0,1]$, and λ is the Lebesgue measure. Each player chooses a strategy $s(i) \in S(i)$ and we take $S(i) \subseteq \mathbb{R}^n$. Strategy profiles in this setting are identified with integrable selections[6] of the set valued[7] mapping $i \rightrightarrows S(i)$. For simplicity, we will assume that all the players have the same compact strategy set $S(i) \equiv S \subset \mathbb{R}^n_+$. As a consequence, the set of meaningful strategy profiles is the set of measurable functions from I to S [8] noted from now on S^I.

The aggregator.

In a game, players have payoff functions that depend on their own strategy and the complete profile of strategies of the player $\pi(i, \cdot, \cdot) : S \times S^I \to \mathbb{R}$.

The best reply correspondence $\mathrm{Br}(i, \cdot) : S^I \rightrightarrows S$ is defined as:

$$\mathrm{Br}(i, \mathbf{s}) := \mathrm{argmax}_{y \in S} \pi(i, y, \mathbf{s}) .$$

The correspondence $\mathrm{Br}(i, \cdot)$ describes the optimal response set for player $i \in I$ facing a strategy profile $\mathbf{s}$.

In our particular framework the pay-off functions depend, for each player, on his own strategy and an average of the strategies of all the other players. To obtain this average we use the integral of the strategy profile, $\int_I s(i)\,\mathrm{di}$. This implies that all the relevant information about the actions of the opponents is summarized by the values of the integrals, which are points in the set [9]

$$\mathcal{A} \equiv \int_I S(i)\,\mathrm{di}.$$

Hypothesis over the correspondence $i \rightrightarrows S(i)$ that assure that the set $\mathcal{A}$ is well defined can be found in Aumann (1965) or in Chapter 14 of Rockafellar and Wets (1998). In this case we get, using the Liapounov theorem that $\mathcal{A}$ is a convex set (Aumann, 1965). Moreover, since $S(i) \equiv S$ we have that [10]

$$\mathcal{A} \quad \equiv \quad \mathrm{co}\,\{S\} . \tag{2.1}$$

Pay-offs $\pi(i, \cdot, \cdot)$ in this setting are evaluated from an auxiliary utility function $u(i, \cdot, \cdot) : S \times \mathrm{co}\,\{S\} \to \mathbb{R}$ such that:

$$\pi(i, y, \mathbf{s}) \equiv u\left(i, y, \int_I s(i)\ \mathrm{di}\right) \tag{2.2}$$

2.1.2 Further preliminaries.

Technical assumptions.

We assume:

[5] Games with a continuum of players have been first studied by Schmeidler (1973).

[6] A selection is a function $\mathbf{s} : I \to \mathbb{R}^n$ such that $s(i) \in S(i)$.

[7] We use the notation $\rightrightarrows$ for set valued mappings (also referred to as correspondences), and $\to$ for functions.

[8] Equivalently, the set of measurable selections of the constant set valued mapping $i \rightrightarrows S$.

[9] Following Aumann (1965) we define for a correspondence $F : I \rightrightarrows \mathbb{R}^n$ its' integral, $\int_I F(i)\ \mathrm{di}$, as:

$$\int_I F(i)\ \mathrm{di} := \left\{ x \in \mathbb{R}^n\ :\ x = \int_I f(i)\ \mathrm{di} \text{ and } f \text{ is an integrable selection of } F \right\}$$

[10] Where $\mathrm{co}\,\{X\}$ stands for the convex hull of a set X (see Rath (1992)).

C : *For all agent $i \in I$, $u(i,\ \cdot\ ,\ \cdot\ )$ is continuous.*

HM : *The mapping that associates to each agent a utility function* [11] *is measurable.*

C is standard and does not deserve special comments. **HM** is technical but in a sense natural in this setting. Adopting both assumptions on utility functions put us in the framework of Rath (1992).

Nash equilibrium.

In the general notation, a Nash equilibrium is a strategy profile $\mathbf{s}^* \in S^I$ such that, $\forall\, i \in I$ λ-a.e., $s^*(i) \in \mathrm{Br}(i, \mathbf{s}^*)$. In this setting, we write the definition as follows.

Definition 2.1. A (pure strategy) Nash Equilibrium of a game is a strategy profile $\mathbf{s}^* \in S^I$ such that:

$$\forall\, y \in S, \quad u\Big(i, s^*(i)\,, \int s^*(i)\ \mathrm{di}\Big) \geq u\Big(i, y, \int s^*(i)\ \mathrm{di}\Big), \quad \forall\, i \in I\ \lambda\text{-a.e.} \tag{2.3}$$

Under the previously mentioned hypothesis Rath shows that for every such game there exists a Nash Equilibrium.

2.2 Economies with a continuum of non-atomic agents

2.2.1 An economic reinterpretation of the game.

The aggregate states of the system.

We interpret now the model as a stylized economic model in which there is a large number of small agents $i \in I$. In this economic system, there is an aggregate variable or signal that represents the *state* of the system. Now $\mathcal{A} \subseteq I\!R^K$ is viewed as the set of all possible states of the economic system. Agents take individual actions, which determine the state of the system through an aggregation operator, A, the " mediator" of the economic interaction of the agents. The key feature of the system is that no agent, or *small* group of agents, can affect unilaterally the state of the system.

The so-called economic system is then immediately imbedded onto the just defined game with a continuum of players when we identify (individual) actions with (individual) strategies so that the aggregation operator associates to each action or strategy profile $\mathbf{s}$ a state of the model $a = A(\mathbf{s})$ in the set of states $\mathcal{A}$, the aggregation operator A being the integral[12] of the profile $\mathbf{s}$:

$$A(\mathbf{s}) \equiv \int_I s(i)\, \mathrm{di}.$$

with the state set $\mathcal{A}$ equal to co $\{S\}$

The variable $a \in \mathcal{A}$, that is now viewed as the state of the system, determines, along with each agents' own action, his payoff. Each agent $i \in I$ then, acts to maximize its payoff function $u(i,\ \cdot\ ,\ \cdot\ ) : S \times \mathcal{A} \to I\!R$ as already introduced in (2.2).

Useful mappings.

In our setting, and considering the auxiliary function $u(i,\ \cdot\ ,\ \cdot\ )$, we can define the optimal strategy correspondence $B(i,\ \cdot\ ) : \mathcal{A} \rightrightarrows S$ as the correspondence which associates to each point $a \in \mathcal{A}$ the set:

$$B(i, a) \quad := \quad \mathrm{argmax}_{y \in S}\, \{u(i, y, a)\}\,. \tag{2.4}$$

[11] The set of functions for assumption **HM** is the set of real valued continuous functions defined on $S \times \mathrm{co}\,\{S\}$ endowed with the sup norm topology.

[12] The aggregation operator can as well be the integral of the strategy profile with respect to any measure $\bar{\lambda}$ that is absolutely continuous with respect to the lebesgue measure, or the composition of this result with a continuous function. That is,

$$A(\mathbf{s}) \equiv G\Big(\int_I s(i)\, f(i)\ \mathrm{di}\Big)$$

where $G : \int_I S(i)\ \mathrm{d}\bar{\lambda}(i) \to \mathcal{A}$ is a continuous function and f is the density of the measure $\bar{\lambda}$ with respect to the lebesgue measure.

However, not all the results in this work remain true if we choose such a setting.

Note that, since in this setting, $a = \int_I s(i)\,$ di, then $\mathrm{Br}(i, \mathbf{s}) = B(i, a)$.

In a situation where agents act in ignorance of the actions taken by *the others* or, for what matters, of the value of the state of the system, they have to rely on forecasts. That is, their actions must be a best response to some subjective probability distribution over the space of aggregate data $\mathcal{A}$. Mathematically, actions have to be elements of the set of points that maximize expected utility, where the expectation is taken with respect to this subjective probability. We can consider then the best reply to forecasts correspondence $I\!B(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ defined by:

$$I\!B(i, \mu) \quad := \quad \mathrm{argmax}_{y \in S} I\!E_\mu \left[u(i, y, a) \right] \tag{2.5}$$

where $\mu \in \mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ is the space of probability measures over $\mathcal{A}$. Since the utility functions are continuous, problems (2.4) and (2.5) are well defined and have always a solution, so consequently the mappings $B(i, \cdot)$ and $I\!B(i, \cdot)$ take non-empty compact values for all $a \in \mathcal{A}$. Clearly $B(i, a) \equiv I\!B(i, \delta_a)$, where δ_a is the Dirac measure concentrated in a.

2.2.2 Economic Equilibrium.

An equilibrium of this system is a state a^* generated by actions of the agents that are optimal reactions to this state. We denote $\Gamma(a) = \int_I B(i, a)\,$ di.

Definition 2.2. An *equilibrium* is a point $a^* \in \mathcal{A}$ such that:

$$a^* \in \Gamma(a^*) \equiv \int_I B(i, a^*)\,\mathrm{di} \equiv \int_I I\!B(i, \delta_{a^*})\,\mathrm{di} \tag{2.6}$$

Assumptions **C** and **HM** assure that the integrals in Definition 2.2 are well defined [13]. The equilibrium conditions in (2.6) are standard description of self fulfilling forecasts. That is, in an equilibrium a^*, agents must have a self-fulfilling point forecast (Dirac measures) over a^*, i.e in the economic terminology, if we take the model strictly speaking, a perfect foresight equilibrium.

2.3 Examples and preliminary results.

2.3.1 Examples.

Many[14] theoretical models of economic theory enter the above general framework, whenever they have no atomic agents[15]. Let us give a few examples, going from partial equilibrium, general equilibrium, finance and macro-economics. In many of these examples, the results we prove in the last section of the paper are directly useful.

The simplest example is a variant of Muth's (1961) model presented in Guesnerie (1992). In this partial equilibrium model, there is a group of "farmers" (or "firms") indexed by the unit interval. Farmers decide a positive production quantity $q(i)$ and get as payoff income sales minus the cost of production: $pq(i) - C_i(q(i))$, where p is the price at which the good is sold. The price is obtained from the inverse demand (or price) function, evaluated in total aggregate production Q. We see that this model fits our framework.

We already said that the set of agents is the unit interval $I = [0, 1]$ and we endow it with the Lebesgue measure. Strategies are production quantities, so strategy profiles are functions from the set of agents to a compact subset (individual productions are bounded) of the positive line $I\!R_+$ (i.e. $n = 1$), $\mathbf{q} : I \to S(i) \equiv S \subset I\!R_+$. The aggregate variable in this case is aggregate production Q and the aggregation operator, is the integral of the production profile $\mathbf{q}$, $Q = \int_I q(i)\,$ di. Agents evaluate their payoff from aggregate production through the price function : indeed, $u(i, q, Q) = P(Q)\,q - C_i(q)$, where $P : I\!R_+ \to I\!R_+$ is an inverse demand (or price) function that, given a quantity of good, gives the price at which this quantity is sold[16]. The model displays strategic substitutabilities in the sense that a higher (expected) aggregate production triggers a lower individual production decision.

The "three goods" general equilibrium model with three goods under study in Guesnerie (2001a) has similar features. The wage on the labour market is fixed, and individual firms must take today production decisions. They face (strategic) uncertainty on the level of total production, which

[13] See Lemma A.1 in the appendix.

[14] We might replace many by most in the present sentence, if instead of the simplified version of the aggregator under scrutiny here, we were dealing with the most sophisticated form alluded to above.

[15] Also, our results have significant implications for models of the oligopolistic competition type with atomic agents (work in progress)

[16] On this example we can make the observation that the state of the game could be chosen to be the price instead of aggregate production. In this one-dimensional setting, it is the case that most of the properties herein presented are passed on from the aggregate production to the price variable.

triggers total income available tomorrow, and then determines the market clearing price for the good. It is easy to check that the situation fits the a one-dimensional version of our model (total production bieng again the one-dimensional aggregate variable). It is shown in the just quoted article, that strategic substitutabilities, due to price effects, unambiguously dominate strategic complementarities, due to income effects of the multiplier type. One can also show that the Walrasian flexible wage version of this three-goods model enters a somewhat similar framework, although less trivially[17].

The n-commodity version of the above fixed wage model is described in Guesnerie (2001a). Production takes place in L sectors indexed by l, with N_l firms in each sector. Firms hire workers at a fixed wage w and sell at market clearing price p_l. Firm i in sector l supplies q_i^l of good l and 0 of the other goods, so that aggregate supply is the vector $\int q(i)di$. We have then a L-dimensional version of the present prototype model. It is shown in the paper that if consumers demand satisfies the gross substitutability assumption, the model displays aggregate strategic substitutabilities assumptions of Section 4. Also, the transposition of the analysis from the just described model to its (walrasian) flexible wage version [18] calls for remarks similar to those just made for the one-dimensional version of the model.

In Finance, many models of transmission of information through prices have non-atomic agents with different information, as in the pioneering work of Grossman and Stiglitz (1980). The agents'strategy, viewed in our framework, is a demand function, and the aggregate state of the system obtains by aggregation of these individual demand functions, and determines a random actual price. The study of "expectational stability" of the equilibria is done in Desgranges (2000); Desgranges and Heinemann (2005) The models under scrutiny would fit our framework, if this framework were extended to an infinite dimensional state variable (a demand function). However, special versions of the model do fit the present framework. These finance models also often satisfy the strategic substitutability assumptions of our last Section.

In macroeconomics, the analysis of expectational stability in standard OLG like infinite horizon models, leads to put emphasis on the dynamics of growth rates, (in the simplest models as those studied by Evans and Guesnerie (2003)), or on more complicated "extended growth rates" (see Evans and Guesnerie (2005), Gauthier (2002)), the dynamics of which has a reduced form that falls within the above framework. However the strategic complementarities or substitutabilities assumptions of our last section may or may not be satisfied. Infinite horizon models with infinite horizon agents, as described in macroeconomic models arising from the Real Business Cycle tradition, also enter the infinite-dimensional extension of the above framework suggested above (see Guesnerie (2008)), and this would be true of many existing macroeconomic models with infinite horizon agents, for which the substitutability assumptions often hold (work in progress).

2.3.2 Economic equilibrium and Nash equilibrium.

The next Proposition has two parts.

It states precisely (as the reader will check) that an *equilibrium* as defined in (2.6) has as a counterpart in the game-theoretical approach a *Nash Equilibrium* of the underlying game as defined in (2.3).

Also, as it is well known from Rath (1992) for example, that the game has a Nash equilibrium, the Nash existence result is also an equilibrium existence result.

Proposition 2.3. *We have*

1. *For every (pure strategy) Nash Equilibrium* $\mathbf{s}^*$ *of the system's underlying game, there exists a unique equilibrium* a^* *given by* $a^* := A(\mathbf{s}^*)$ *and if* a^* *is an equilibrium of the system, then* $\exists\ \mathbf{s}^* \in S^I$ *that is a Nash Equilibrium of the underlying game.*

2. *The stylized economic model has an equilibrium.*

We will refer equivalently then, to equilibria as points $a^* \in \mathcal{A}$, representing "economic equilibria", and $\mathbf{s}^* \in S^I$, as Nash Equilibria of the underlying game.

Such equilibria are by definition associated with "perfect" expectational coordination. However, the plausibility or "quality" of the expectational coordination that they assume, and hence

[17] The "action" or strategy of the agent in a walrasian context is no longer his production decision, but his reservation wage on the labour market (see Guesnerie, 2001b, , section 4). Strictly speaking, the aggregate state of the model is no longer an additive function of individual decisions, but the present analysis is rather easily adapted. The model has strategic substitutability whenver the best response to a higher reservation wage of others is a lower reservation wage.

[18] For an expectational analysis of a general equilibrium model along the same lines, see Ghosal (2006)

their predictive power have been challenged. The assessment of the quality of expectational coordination has been made through different approaches, coming under at least three broad headings : experimental economics, "evolutive" (real time) learning, and "eductive" assessment. "Eductive" assessment stress arguments that do not explicitly refer to real time learning, although they may sometimes be interpreted as "virtual time" learning. We focus attention here on "eductive" criteria for assessing expectational coordination in the model under consideration here.

3 "Eductive" criteria for assessing Expectational stability.

Our presentation of the "eductive" criteria that we choose to present here refers to three broad categories : purely game-theoretical criteria, "economic" criteria and mixed criteria.

3.1 The game-theoretical viewpoint : rationalizability.

3.1.1 Rationalizability.

Rationalizability is associated with the work of Bernheim (1984) and Pearce (1984). The set of Rationalizable Strategy profiles was there defined and characterized in the context of games with a finite number of players, continuous utility functions and compact strategy spaces. It has been argued that Rationalizable strategy profiles are profiles that can not be discarded as outcomes of the game based on the premises of rationality of players, independence of decision making and common knowledge (see Tan and da Costa Werlang (1988)).

First, agents only use strategies that are best responses to their forecasts and so strategies in S that are never best response will never be used; second, agents know that other agents are rational and so know that the others will not use the strategies that are not best responses and so each agent may find that some of his remaining strategies may no longer be best responses, since each agent knows that all agents know, etc. . This process continues ad-infinitum. The set of Rationalizable solutions is such that it is a "fixed point"of the elimination process, and it is the maximal set that has such a property (Basu and Weibull, 1991).

3.1.2 Point rationalizability with a continuum of players.

Rationalizability has been studied in games with finite number of players. In a game with a continuum of agents, the analysis has to be adapted. Following Jara-Moroni (2007), and coming to our setting, in a game-theoretical perspective, the recursive process of elimination of non best responses, when agents have point expectations, is associated with the mapping $Pr : \mathcal{P}\big(S^I\big) \rightarrow \mathcal{P}\big(S^I\big)$ which to each subset $H \subseteq S^I$ associates the set $Pr(H)$ defined by:

$$Pr(H) := \left\{ \ \mathbf{s} \in S^I \ \ : \ \ \mathbf{s} \text{ is a measurable selection of } i \rightrightarrows \mathrm{Br}(i, H) \right\}. \qquad (3.1)$$

The operator Pr represents the process under which we obtain strategy profiles that are constructed as the reactions of agents to strategy profiles contained in the set $H \subseteq S^I$. If it is known that the outcome of the game is in a subset $H \subseteq S^I$, with point expectations, the strategies of agent $i \in I$ are restricted to the set $\mathrm{Br}(i, H) \equiv \bigcup_{\mathbf{s}\in H} \mathrm{Br}(i, \mathbf{s})$ and so actual strategy profiles must be measurable selections of the set valued mapping $i \rightrightarrows \mathrm{Br}(i, H)$. It has to be kept in mind that strategies of different agents within a strategy profile in $Pr(H)$ describe the individual reactions to (generally) different strategy profiles in H.

We then define :

Definition 3.1. The set of Point-Rationalizable[19] Strategy Profiles is the maximal subset $H \subseteq S^I$ that satisfies:

$$H \equiv Pr(H).$$

and we note it $\mathbb{P}_S$.

3.1.3 Plain Rationalizability with a continuum of players ?

Rationalizable Strategies should be obtained from a similar exercise but considering forecasts as probability measures over the set of strategies of the opponents. A difficulty in a context

[19] Following Bernheim (1984) we refer as Point-Rationalizability to the case of forecasts as points in the set of strategies or states and plain Rationalizability to the case of forecasts as probability distributions over the corresponding set.

with continuum of players, relates with the continuity or measurability properties that must be attributed to subjective beliefs, as a function of the agent's name. It is not a priori clear whether it is better to describe stochastic individual beliefs as measures on the set of the considered (measurable) strategy profiles or as measurable profiles of distributions of beliefs on strategies. In light of the structure of our model, we "stick" to the former.

Consider then a measurable structure in the space of strategy profiles. This is, consider the set S^I and a σ-field $\mathcal{S}^{\mathcal{I}}$. In an analogous way as above, we may consider a process of elimination of strategies that are not best response to any forecast, that are now in the form of probability measures on $\mathcal{S}^{\mathcal{I}}$. Consider then the mapping $R : \mathcal{S}^{\mathcal{I}} \to \mathcal{P}\big(S^I\big)$ which to each set H associates the set $R(H)$ given by:

$$R(H) := \big\{ \ \mathbf{s} \in S^I \ : \ \mathbf{s} \text{ is a measurable selection of } i \rightrightarrows \mathbb{B}\mathrm{r}(i, \mathcal{P}(H)) \big\} \,. \tag{3.2}$$

where $\mathcal{P}(H)$ stands for the set of probability measures over $\mathcal{S}^{\mathcal{I}}$ with support on H and $\mathbb{B}\mathrm{r}(i,\nu)$ is the set of optimal solutions of the problem of maximization of expected utility:

$$\mathbb{B}\mathrm{r}(i,\nu) \quad := \quad \mathrm{argmax}_{y\in S} \left\{ \int_{S^I} u\Big(i, y, \int_I \mathbf{s} \ \mathrm{di}\Big) \mathrm{d}\nu(\mathbf{s}) \right\} . \tag{3.3}$$

R represents the process under which we obtain strategy profiles that are constructed as the reactions of players to probabilistic forecasts contained in their supports to the set $H \in \mathcal{S}^{\mathcal{I}}$. The same reasoning as in the previous Subsection applies. We are able then to formally define a set of (correlated) Rationalizable Strategy Profiles as the maximal set of strategy profiles that is contained in its image through the process of elimination of strategies [20]:

Definition 3.2. The set of (Correlated) Rationalizable Strategy Profiles is the maximal subset $H \in \mathcal{S}^{\mathcal{I}}$ that satisfies:

$$H \equiv R(H) \, .$$

and we note it $\mathbb{R}_{S^I}$.

We present in the next section concepts of Rationalizable States and Point-Rationalizable States, where forecasts and the process of elimination are now taken over the set of states $\mathcal{A}$.

3.2 Expectational coordination from an" economic"viewpoint : Iterative expectations and Cobweb mapping.

The Cobweb mapping, which we will refer to sometimes later as the Iterative Expectational process, brings back to ideas that make sense in our simplified "economic" setting but that have not necessarily fruitful counterparts in an abstract game-theoretical framework.

3.2.1 Cobweb Mapping.

Given the optimal strategy correspondence, $B(i, \cdot)$, defined in (2.4) we can define the *cobweb mapping or the best response mapping* [21] $\Gamma : \mathcal{A} \rightrightarrows \mathcal{A}$:

$$\Gamma(a) := \int_I B(i,a) \ \mathrm{di} \tag{3.4}$$

This correspondence describes the actual possible states of the model when all agents have the same point expectations on the state of the system $a \in \mathcal{A}$.

In our framework, with the above assumptions, note that the cobweb mapping Γ is upper semi continuous as a set valued mapping, with non-empty, compact and convex[22] values $\Gamma(a)$.

In a sense, the Cobweb mapping provides a tool for assessing expectational stability while ruling out heterogenous expectations. Indeed, the concept of IE-stability that has been influential in the eighties in macroeconomics (see Lucas (1978), DeCanio (1979), Evans (1985, 1986)) in the context of infinite horizon models, refers to a cobweb-like mapping which is used for describing a mental collective process with homogenous expectations.

Next, we can define the limit points of the Cobweb mapping.

[20] What we are considering in terms of strategy profiles resembles more to "correlated strategic Rationalizability" (Brandenburger and Dekel, 1987)

[21] The name cobweb mapping comes from the familiar cobweb tâtonnement although in this general context the process of iterations of this mapping may not necessarily have a cobweb-like graphic representation.

[22] For reasons that will be explained later.

3.2.2 Aggregate Cobweb Tâtonnement Outcomes.

Definition 3.3. The set of *Aggregate Cobweb Tâtonnement Outcomes*, $\mathbb{C}_{\mathcal{A}}$, is defined by:

$$\mathbb{C}_{\mathcal{A}} := \bigcap_{t \geq 0} \Gamma^t(\mathcal{A})$$

where Γ^t is the tth iterate[23] of the correspondence Γ.

We immediately note that the equilibria of the economic system, denoted $\mathbb{E}$, identify with the fixed points of the cobweb mapping and hence belong to the set $\mathbb{C}_{\mathcal{A}}$. The same is true for the cycles of any order of the mapping Γ : a^* is a cycle of order k of $\Gamma \Leftrightarrow a^* \in \Gamma^k(a^*)$. Note that $\Gamma(a^*), ..\Gamma^{k-1}(a^*)$, are also cycles of order k. We denote $\mathbb{C}^k$ the set of cycles of order k [24] and $\Xi = \mathbb{E} \cup (\cup)_{\not{k}}^{+\infty} \mathbb{C}^{\neg}$, the set of periodic equilibria, including standard equilibria. Our previous remarks involves $\Xi \subset \mathbb{C}_{\mathcal{A}}$.

3.3 A mixed viewpoint : State Rationalizability

This mixed viewpoint refers to the conceptual references of game-theoretical inspiration, but, taking advantage of the added structure, adapt them to the "economic" context under scrutiny.

Indeed, below, is a formal presentation of *Point-Rationalizable States* and *Rationalizable States*, which exploits the simplicity of our context. For the proofs of the results herein stated and a more detailed treatment the reader is referred to Jara-Moroni (2007).

3.3.1 Point Rationalizable states.

Analogously to what is done in subsection 3.1.2, given the optimal strategy correspondence defined in equation (2.4) we can define the process of non reachable or non generated states, considering forecasts as points in the set of states, as follows:

$$\tilde{Pr}(X) := \int_I B(i, X) \, \mathrm{di}$$

If initially agents' common knowledge about the actual state of the model is a subset $X \subseteq \mathcal{A}$ and if expectations are restricted to point-expectations, agents deduce that the possible actions of each agent $i \in I$ are in the set $B(i, X) := \bigcup_{a \in X} B(i, a)$. Since all agents know this, each agent can only discard the strategy profiles $s \in S^I$ that are not selections of the mapping that assigns the above sets to each agent. Finally, they would conclude that the actual state outcome will be restricted to the set obtained as the integral of this set valued mapping.

Definition 3.4. The set of *Point-Rationalizable States* is the maximal subset $X \subseteq \mathcal{A}$ that satisfies the condition:

$$X \equiv \tilde{Pr}(X)$$

and we note it $\mathbb{P}_{\mathcal{A}}$.

We define similarly the set of Rationalizable States.

3.3.2 Rationalizable States.

The difference between Rationalizability and Point-Rationalizability is that in Rationalizability forecasts are no longer constrained to be points in the set of outcomes. To assess Rationalizability we consider the correspondence $I\!B(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ defined in (2.5). The process of elimination of non-maximizers of expected-utility is described with the mapping $\tilde{R} : \mathcal{B}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$:

$$\tilde{R}(X) := \int_I I\!B(i, \mathcal{P}(X)) \, \mathrm{di} \tag{3.5}$$

If it is common knowledge that the actual state is restricted to a borel subset $X \subseteq \mathcal{A}$, then agents will use strategies only in the set $I\!B(i, \mathcal{P}(X)) := \cup_{\mu \in \mathcal{P}(X)} I\!B(i, \mu)$ where $\mathcal{P}(X)$ stands as before for the set of probability measures whose support is contained in X. Forecasts of agents can not give positive weight to points that do not belong to X. Strategy profiles then will be selections of the correspondence $i \rightrightarrows I\!B(i, \mathcal{P}(\ X))$. The state of the system will be the integral of one of these selections.

[23]This is:

$$\Gamma^0 := \mathcal{A} \quad \Gamma^{t+1} := \Gamma(\Gamma^t(\mathcal{A}))$$

[24]Points a^* such that $a^*, \in \Gamma^k(a^*)$

Definition 3.5. The set of *Rationalizable States* is the maximal subset $X \subseteq \mathcal{A}$ that satisfies:

$$X \equiv \tilde{R}(X) \tag{3.6}$$

and we note it $\mathbb{R}_{\mathcal{A}}$.

We will make use of Proposition 3.6 below, which provides, in the continuum of agents framework, a key technical property of the set of Rationalizable States.

Proposition 3.6. *The set of Rationalizable States can be computed as*

$$\mathbb{R}_{\mathcal{A}} \equiv \bigcap_{t=0}^{\infty} \tilde{R}^t(\mathcal{A})$$

An analogous result may be obtained for the set of Point-Rationalizable States (Jara-Moroni, 2007). The sets $\mathbb{P}_{\mathcal{A}}$ and $\mathbb{R}_{\mathcal{A}}$, indeed obtain as the outcome of the iterative elimination of unreachable states.

At this stage, let us make two remarks.

First let us stress a technical point of importance i.e that for a given borel set $X \subseteq \mathcal{A}$ we have $\tilde{Pr}(X) \subseteq \tilde{R}(X)$, a fact that is used below, for example in Proposition 3.8.

Second, let us explain why, bypassing the game-theoretical difficulties occurring in games with a continuum of players, the states set approach provides a fairly convincing view of Plain Rationalizability as opposed to Point Rationalizability. To make the connection with the strategic approach developed above, let us note that agents in our economic model are unable to differentiate between strategy profiles that give the same aggregate state, in mathematical terms, strategy profiles that have the same integral. Looking at the integral as a function from the the set S^I and the values of the integrals, the set $\mathcal{A}$, we see that the measurable space structure considered in $\mathcal{A}$ may induce a measurable space structure on S^I. The σ-field $\mathcal{S}^{\mathcal{I}}$, introduced in Subsection 3.1.3 is then the σ-field of the pre-images of the Borel subsets of $\mathcal{A}$ through the integral.

$$\mathcal{S}^{\mathcal{I}} \equiv \left\{A^{-1}(X) : X \in \mathcal{B}(\mathcal{A})\right\}$$

Any probability measure, ν, defined on $\mathcal{S}^{\mathcal{I}}$ induces a probability measure in $\mathcal{P}(\mathcal{A})$ as usual:

$$\mu(\,\cdot\,) \equiv \nu\big(A^{-1}(\,\cdot\,)\big)\,.$$

Thus, when applying the process defined in (3.5), we are in fact considering at least all probability measures that can be defined over $\mathcal{S}^{\mathcal{I}}$. Reciprocally, for a given probability measure in $\mathcal{P}(\mathcal{A})$ the function $\nu : \mathcal{S}^{\mathcal{I}} \to I\!R_+$ defined by [25]:

$$\nu(H) := \mu(A(H))$$

is in fact a probability measure on $\mathcal{S}^{\mathcal{I}}$. The question rises on whether a richer σ-field would change the definition of Rationalizable Strategies in our context. This answer is given by the structure model itself. Payoff functions are defined on the values of the integrals of the strategy profiles. Consequently, when taking expectation, agents precisely only care about the probabilities of events of the form $\left\{\mathbf{s} \in S^I : \int \mathbf{s} = a\right\}$, for each $a \in \mathcal{A}$ which brings us back to the induced σ-field $\mathcal{S}^{\mathcal{I}}$.

With this in mind we can see that the process of elimination of profiles defined in (3.2) is interlaced with the process of elimination of states.

3.4 Connecting the concepts.

We have defined several concepts that all make sense in our prototype "economic" model. We first summarize the connections between the infinite agents game and the "economic" viewpoints.

Proposition 3.7. *The "game-theoretical" and the "economic" appraisal of (Point-)Rationalizability are coherent.*

$$\mathbb{P}_S \equiv \left\{ \mathbf{s} \in S^I \;:\; \mathbf{s} \textit{ is a measurable selection of } i \rightrightarrows B(i, \mathbb{P}_{\mathcal{A}})\right\} \tag{3.7}$$

$$\mathbb{P}_{\mathcal{A}} \equiv \left\{ a \in \mathcal{A} \;:\; a = \int_I s(i)\,\mathrm{d}i \textit{ and } \mathbf{s} \textit{ is a measurable function in } \mathbb{P}_S\right\}. \tag{3.8}$$

$$\mathbb{R}_{S^I} \equiv \left\{ \mathbf{s} \in S^I \;:\; \mathbf{s} \textit{ is a measurable selection of } i \rightrightarrows I\!B(i, \mathbb{R}_{\mathcal{A}})\right\} \tag{3.9}$$

$$\mathbb{R}_{\mathcal{A}} \equiv \left\{ a \in \mathcal{A} \;:\; a = \int_I s(i)\,\mathrm{d}i \textit{ and } \mathbf{s} \textit{ is a measurable function in } \mathbb{R}_{S^I}\right\}. \tag{3.10}$$

[25]By construction, if $H \in \mathcal{S}^{\mathcal{I}}$ then $A(H) \in \mathcal{B}(\mathcal{A})$.

Equations (3.7) through (3.10) stress the equivalence for (point-)rationalizability between the state approach and the strategic approach in games with continuum of players : the sets of (point-)rationalizable states can be obtained from the set of (point-)(correlated)rationalizable strategies and vice versa. For instance, in (3.7) we see that the strategy profiles in $\mathbb{P}_S$ are profiles of best responses to $\mathbb{P}_{\mathcal{A}}$. Conversely in (3.8) we get that the points in $\mathbb{P}_{\mathcal{A}}$ are obtained as integrals of the profiles in $\mathbb{P}_S$.

The next Proposition stresses the connections between the concepts of "economic" inspiration. We denote by $\mathbb{E} \subseteq \mathcal{A}$, the set of equilibria of the economic system.

Proposition 3.8. *We have:*

1. $$\mathbb{E} \subseteq \Xi \subseteq \mathbb{C}_{\mathcal{A}} \subseteq \mathbb{P}_{\mathcal{A}} \subseteq \mathbb{R}_{\mathcal{A}}$$

2. *The sets $\mathbb{P}_{\mathcal{A}}$ and $\mathbb{R}_{\mathcal{A}}$ are convex and compact. $\mathbb{C}_{\mathcal{A}}$ is compact.*

In , the first inclusions have already been noted : Equilibria and cycles of any order do belong to $\mathbb{C}_{\mathcal{A}}$. We can obtain the two last inclusions of Proposition 3.8 noting that if a set satisfies $X \subseteq \tilde{Pr}(X)$ then it is contained in $\mathbb{P}_{\mathcal{A}}$ and equivalently if it satisfies $X \subseteq \tilde{R}(X)$ then it is contained in $\mathbb{R}_{\mathcal{A}}$. Then, the second inclusion is obtained from the fact that each point in $\mathbb{C}_{\mathcal{A}}$, as a singleton, satisfies $\{a^*\} \subseteq \tilde{Pr}(\{a^*\})$ and the third inclusion is true because the set $\mathbb{P}_{\mathcal{A}}$ satisfies $\mathbb{P}_{\mathcal{A}} \subseteq \tilde{R}(\mathbb{P}_{\mathcal{A}})$.

The above inclusions are unsurprising, in the sense that they reflect the decreasing strength of the expectational coordination hypothesis, when going from equilibria to Aggregate Cobweb outcomes, then to Point-Rationalizable States, and finally to Rationalizable States. As argued above and recalled just below, the different concepts reflect an enlargement of the complexity and diversity of expectations under scrutiny. The statement also stress that cycles of the best response mapping play a role into the analysis of expectational coordination : this role will be shown to be crucial in the next Section.

3.5 From concepts to expectational stability criteria.

The above concepts serve as a basis for assessing the expectational plausibility of the economic equilibrium of our model, from a global and a local viewpoint

3.5.1 The global criteria.

Definition 3.9. An equilibrium a^* is said to be *Globally Iteratively Expectationaly Stable* if $\forall$ $a^0 \in \mathcal{A}$ any sequence $a^t \in \Gamma\big(a^{t-1}\big)$ satisfies $\lim_{t\to\infty} a^t = a^*$. Equivalently : $\mathbb{C}_{\mathcal{A}} = \mathbb{E} = \{a^*\}$.

The concept captures the idea that virtual coordination processes, referring to *homogenous deterministic expectations*, converge globally.The terminology of Iterative Expectational Stability is adopted from the literature on expectational stability in dynamical systems (Evans and Guesnerie, 1993, 2003, 2005).

Definition 3.10. The equilibrium state a^* is (globally) *Strongly Point Rational* if

$$\mathbb{P}_{\mathcal{A}} \equiv \{a^*\} \ \ (= \mathbb{E}).$$

The idea captured by the concept is now that virtual coordination processes, referring to *heterogenous but deterministic expectations*, converge globally.

Then comes the most demanding concept referring to *heterogenous and stochastic expectations.*

Definition 3.11. The equilibrium state a^* is (globally) *Strongly Rational* if

$$\mathbb{R}_{\mathcal{A}} \equiv \{a^*\} \ \ (= \mathbb{E}).$$

The criteria and terminology used here closely follow Guesnerie (1992) and Evans and Guesnerie (1993), as well as Chamley (2004, chapter 11) [26]. With a less precise terminology, when one of the above criteria is satisfied, we say that the equilibrium is (globally) "eductively" stable.

The reader will remind here that studies on "evolutive learning" most often assume that agents have identical point expectations (that they revise according to a universally agreed upon rule).

[26] The equilibrium might also be viewed and called dominant-solvable. The terminology is intended to suggest the strength or robustness of what is generally called in an "economic" context a Rational Expectations Equilibrium.

In a similar way, Iterative Expectational Stability (IE-Stability), where iterations of the cobweb mapping Γ describe agents reactions to the same point forecast over the set of states, rules out expectational heterogeneity. And let us repeat that " eductive" coordination, as assessed from the last two definitions of Strong Point Rationality or Strong Rationality, takes into account the fact that agents [27] have *heterogenous expectations that may be point expectations or stochastic expectations.*

It is straightforward that these concepts are increasingly demanding : **Strong Rationality implies Strong Point Rationalizability that implies Iterative Expectational Stability**.

We shall turn later to the local version of these concepts.

4 Economies with strategic substitutabilities.

Games with strategic complementarities have been the focus of intensive research particularly since the end of the eighties (see Milgrom and Roberts, 1990). We adapt the standard argument and re-assess the standard findings within our "economic" framework with a continuum of agents. This is Proposition 4.3 in the next subsection. We then wonder whether the striking findings on global stability in an " economic" model with strategic complementarities can be extended. We then focus attention on another polar world dominated by strategic substitutabilities and show that we still have powerful results to analyse "eductive" global statbility.

4.1 Preliminaries : economic models with strategic complementarities.

Our economic system presents Strategic Complementarities if the individual best response mappings of the underlying game are increasing for each $i \in I$.

We could define such properties in the framework of the underlying game of Section 2.1, while directly referring to the theory of *supermodular games* as studied in Milgrom and Roberts (1990) and Vives (1990) (see as well Topkis (1998)) [28]. However, since agents can not affect the state of the system, all agents have forecasts over the same set, namely the set of states $\mathcal{A}$. and we shall directly proceed within the "economic" framework.

For that, let us make the following assumptions over the strategy set S and the utility functions $u(i, \cdot, \cdot)$.

1.B S is the product of n compact intervals in $\mathbb{R}_+$.

2.B $u(i, \cdot, a)$ is supermodular for all $a \in \mathcal{A}$ and all $i \in I$.

3.B $\forall\, i \in I$, the function $u(i, y, a)$ has increasing differences in y and a. That is, $\forall\, y,\, y' \in S$, such that $y \geq y'$ and $\forall\, a,\, a' \in \mathcal{A}$ such that $a \geq a'$:

$$u(i, y, a) - u(i, y', a) \quad \geq \quad u(i, y, a') - u(i, y', a') \tag{4.1}$$

Assumption 2.B is straightforward. Assumption 1.B implies that the set of strategies is a complete lattice in $\mathbb{R}^n$. One implication of our setting is that since S is a convex complete lattice, then $\mathcal{A} \equiv \text{co}\,\{S\} \equiv S$ is as well a complete lattice.

¿From now on we will refer to the above supermodular setting as $\mathcal{G}$. The technical results we use in the proof in the appendix are recalled as lemmas.

Lemma 4.1. *Under assumptions 1.B through 3.B, we have:*

1. *the mappings $B(i, \cdot)$ are increasing in a in the set $\mathcal{A}$, and the sets $B(i, a)$ are complete sublattices of S,*
2. *the correspondence Γ is increasing and $\Gamma(a)$ is subcomplete for each $a \in \mathcal{A}$,*
3. *$\mathbb{E}$ is a non empty complete lattice.*

[27] Even if these agents were homogeneous (with the same utility function)

[28] Supermodularity (and of course submodularity as in the next section) could be studied in the context of games with continuum of agents with a broad generality using the strategic approach (using for instance the tools available from Riesz spaces). However, the fact that we work with a continuous of agents that allows to focus on forecasts over the set of aggregate states. This does not occur in the context of "small"game since then the forecast of different agents would be in different sets, namely the set of aggregate values of "the others"which could well be a different set for each agent. Another difficulty is passing from strategies to states in terms of complementarity. An important result related with this issue is treated in Lemma A.2 in the appendix.

¿From the previous Lemma, an existence result follows, but what is most important is that the set of equilibria has a complete lattice structure. In particular we know that there exist points $\underline{a}^* \in \mathcal{A}$ and $\bar{a}^* \in \mathcal{A}$ (that could be the same point) such that if $a^* \in \mathbb{E}$ is an equilibrium, then $\underline{a}^* \leq a^* \leq \bar{a}^*$.

We also mention another intermediate result [29]. .

Lemma 4.2. *In $\mathcal{G}$, for $a' \in \mathcal{A}$ and $\mu \in \mathcal{P}(\mathcal{A})$, if $a' \leq a$, $\forall\ a \in \sup(\mu)$, then $\forall\ i \in I$*

$$B(i, a') \preceq I\!\!B(i, \mu)\,,$$

equivalently, if $a' \geq a$, $\forall\ a \in \sup(\mu)$, then $\forall\ i \in I$

$$B(i, a') \succeq I\!\!B(i, \mu)\,.$$

That is, if the forecast of an agent has support on points that are larger than a point $a' \in \mathcal{A}$, then his optimal strategy set is larger than the optimal strategy associated to a' (for the induced set ordering) and analogously for the second statement. Here is the statement that summarizes our results.

Proposition 4.3. *In the economic system with Strategic Complementarities we have:*

(i) *The set of equilibria $\mathbb{E} \subseteq \mathcal{A}$ is a complete lattice.*

(ii) *There exist a greatest equilibrium and a smallest equilibrium, that is $\exists\ \underline{a}^* \in \mathbb{E}$ and $\bar{a}^* \in \mathbb{E}$ such that $\forall\ a^* \in \mathbb{E}$, $\underline{a}^* \leq a^* \leq \bar{a}^*$.*

(iii) *The sets of Rationalizable and Point-Rationalizable States are convex sets, tightly contained in the interval $[\underline{a}^*, \bar{a}^*]$. That is,*

$$\{\underline{a}^*, \bar{a}^*\} \subseteq \mathbb{C}_{\mathcal{A}} \subseteq \mathbb{P}_{\mathcal{A}} \subseteq \mathbb{R}_{\mathcal{A}}\ \subseteq [\underline{a}^*, \bar{a}^*]\,.$$

The proof is relegated to the appendix. The intuitive interpretation of the proof is as follows. Originally, agents know that the state of the system will be greater than $\inf \mathcal{A}$ and smaller than $\sup \mathcal{A}$. Since the actual state is in the image through $\tilde{Pr}$ of $\mathcal{A}$, the monotonicity properties of the forecasts to state mappings allow agents to deduce that the actual state will be in fact greater than the image through Γ of the constant forecast $\underline{a}^0 = \inf \mathcal{A}$ and smaller than the image through Γ of the constant forecast $\bar{a}^0 = \sup \mathcal{A}$. That is, it suffices to consider the cases where all the agents have the same forecasts $\inf \mathcal{A}$ and $\sup \mathcal{A}$. The eductive procedure then can be secluded on each iteration, only with iterations of Γ. Since Γ is increasing, we get an increasing sequence that starts at $\underline{a}^0$ and a decreasing sequence that starts at $\bar{a}^0$. These sequences converge and upper semi continuity of Γ implies that their limits are fixed points of Γ [30].

Our results are unsurprising. In the context of an economic game with a continuum of agents, they mimic, in an expected way, the standards results obtained in a game-theoretical framework with a finite number of agents and strategic complementarities[31]. Additional convexity properties reflect the use of a continuum setting.

The most striking feature of the result, reinterpreted within our categories is that all the global stability criteria defined above are equivalent as soon as the equilibrium is unique. In this sense, uniqueness is the Graal, as stated formally below.

Corollary 4.4. *In $\mathcal{G}$, the four following statements are equivalent:*

(i) *an equilibrium a^* is globally* Strongly Rational.

(ii) *an equilibrium a^* is globally* Strongly Point Rational.

[29] With a given order $\geq$ on a space E, we can induce a set ordering in the set of subsets of E, as follows: for $X, Y \subseteq E$ we say that X is greater than Y, noted $X \succeq Y$, if $\forall (x, y) \in X \times Y$, $\sup_E \{x, y\} \in X$ and $\inf_E \{x, y\} \in Y$. With this definition we are able to define the concept of increasing (decreasing) set valued mapping. We will say that a mapping $F : E \rightrightarrows Y$ is increasing (decreasing) if $x \geq x'$ then $F(x) \succeq F(x')$ ($F(x) \preceq F(x')$). Note that if F is single valued we obtain the usual definition of increasing (decreasing) function.

[30] Note that there are three key features to keep in mind, that lead to the conclusion. First, the fact that there exists a set $\mathcal{A}$ that, being a complete lattice and having as a subset the whole image of the mapping A, allows the eductive process to be initiated. Second, monotonic structure of the model implies that it suffices to use Γ to seclude, in each step, the set obtained from the eductive process into a compact interval. Third, continuity properties of the utility functions and the structure of the model allow the process to converge.

[31] Note, however, that, to the best of our understanding, our results do not follow from previous results obtained in the finite agents setting. Shedding full light on the relationship requires a theory connecting the finite context and the continuum one.

(iii) *an equilibrium* a^* *is globally* IE-Stable.

(iv) *there exists a unique equilibrium* a^*.

This last statement may be interpreted as the fact that in the present setting, heterogeneity of expectations does not play any role in expectational coordination, at least when the equilibrium is unique. This is a very special feature of expectational coordination as argued in Evans and Guesnerie (1993). Surprisingly enough, the logic (and simplicity) of the analysis somewhat extends within the next class of models under consideration.

It should also be noted that this strong result requires that the mapping Γ has no cycle, whatever the order of the cycle, a fact that follows trivially from monotonicity. In the next subsection devoted to economies with strategic substitutabilities, it will be shown that the mapping Γ, may have cycles of order 2, but no cycles of other orders. As we shall see, such cycles will play a key role in the analysis.

4.2 Economies with Strategic Substitutabilities

We turn to the case of *Strategic Substitutabilities.* This is done by replacing assumption 3.B with assumption 3.B' below.

1.B S is the product of n compact intervals in $I\!R_+$.

2.B $u(i,\,\cdot\,,a)$ is supermodular for all $a \in \mathcal{A}$

3.B' $u(i,y,a)$ has decreasing differences in y and a. That is, $\forall\ y,\ y' \in S$, such that $y \geq y'$ and $\forall$ $a,\ a' \in \mathcal{A}$ such that $a \geq a'$:

$$u(i,y,a) - u(i,y',a) \quad \leq \quad u(i,y,a') - u(i,y',a') \tag{4.2}$$

Assumptions 1.B through 3.B' turn the underlying game of our model into a *submodular game* with a continuum of agents. The relevant difference with the previous section is that now the monotonicity of the mapping A along with assumption 3.B' implies that the best response mappings are decreasing on the strategy profiles (note that two examples of economic models fitting the above assumptions are mentioned in Section 2).

The following Lemmas and Corollary are the counterparts of Lemmas 4.1 and 4.2.

Lemma 4.5. *Under assumptions 1.B, 2.B and 3.B', the mappings* $B(i,\,\cdot\,)$ *are decreasing in a in the set* $\mathcal{A}$*, and the sets* $B(i,a)$ *are complete sublattices of* S*. The correspondence* Γ *is decreasing and* $\Gamma(a)$ *is subcomplete for each* $a \in \mathcal{A}$*.*

We denote Γ^2 for the second iterate of the cobweb mapping, that is $\Gamma^2 : \mathcal{A} \rightrightarrows \mathcal{A}$, $\Gamma^2(a) := \cup_{a' \in \Gamma(a)} \Gamma(a')$.

Corollary 4.6. *In* $\mathcal{G}$ *the correspondence* Γ^2 *is increasing and* $\Gamma^2(a)$ *is subcomplete for each* $a \in \mathcal{A}$*.*

Proof. Is a consequence of Γ being decreasing.

■

The correspondence Γ^2 will be our main tool for the case of strategic substitutabilities. This is because, in the general context, the fixed points of Γ^2 are point-rationalizable just as the fixed points of Γ are. Actually, one checks immediately that the fixed points of any iteration of the mapping Γ are as well point-rationalizable. The relevance of strategic substitutabilities is that under their presence it suffices to use the second iterate of the cobweb mapping to seclude the set of point-rationalizable states. Using Lemma 4.1 we get that under assumptions 1.B, 2.B and 3.B', the set of fixed points of Γ^2, shares the properties that the set of equilibria $\mathbb{E}$ had under strategic complementarities.

Lemma 4.7. *The set of fixed points of* Γ^2 *is a non empty complete lattice.*

Proof. Apply Lemma 4.1 to Γ^2.

■

The relevance of Lemma 4.7 is that, as in the case of strategic complementarities, under strategic substitutabilities it is possible to seclude the set of Point-Rationalizable States into a tight compact interval. This interval is now obtained from the complete lattice structure of the set of fixed points of Γ^2, which can be viewed, in a multi-period context, as cycles of order 2 of the system.

We also need, as above :

Lemma 4.8. *In $\mathcal{G}'$, for $a' \in \mathcal{A}$ and $\mu \in \mathcal{P}(\mathcal{A})$, if $a' \leq a$, $\forall\ a \in \sup(\mu)$, then $\forall\ i \in I$*

$$B(i, a') \succeq I\!B(i, \mu)\,,$$

equivalently, if $a' \geq a$, $\forall\ a \in \sup(\mu)$, then $\forall\ i \in I$

$$B(i, a') \preceq I\!B(i, \mu)\,.$$

which can be proved by adapting the proof of Lemma 4.2, in the appendix, to the decreasing differences case.

We are now able to state the main result of the strategic substitutabilities case, which, together with its corollaries, is also the main result of the paper.

Theorem 4.9. *In economies with Strategic Substitutabilities we have:*

(i) *There exists at least one equilibrium a^*.*

(ii) *There exist a greatest and a smallest rationalizable states, that is $\exists\ \underline{a} \in \mathbb{R}_{\mathcal{A}}$ and $\bar{a} \in \mathbb{R}_{\mathcal{A}}$ such that $\forall\ a \in \mathbb{R}_{\mathcal{A}}$, $\underline{a} \leq a \leq \bar{a}$, where $\underline{a}$ and $\bar{a}$ are cycles of order 2 of the Cobweb mapping.*

(iii) *The sets of Rationalizable and Point-Rationalizable States are convex.*

(iv) *The sets of Rationalizable and Point-Rationalizable States are tightly contained in the interval $[\underline{a}, \bar{a}]$. That is,*

$$\{\underline{a}, \bar{a}\} \subseteq \mathbb{C}_{\mathcal{A}} \subseteq \mathbb{P}_{\mathcal{A}} \subseteq \mathbb{R}_{\mathcal{A}} \subseteq [\underline{a}, \bar{a}]\,.$$

The proof is relegated to the appendix. Keeping in mind the proof of Proposition 4.3, we can follow the idea of the proof of Theorem 4.9. As usual, common knowledge says that the state of the system will be greater than $\inf \mathcal{A}$ and smaller than $\sup \mathcal{A}$. In first order basis then, the actual state is known to be in the image through $\tilde{Pr}$ of $\mathcal{A}$. Since now the cobweb mapping is decreasing, the structure of the model allows the agents to deduce that the actual state will be in fact smaller than the image through Γ of the constant forecast $\underline{a}^0 = \inf \mathcal{A}$ and greater than the image through Γ of the constant forecast $\bar{a}^0 = \sup \mathcal{A}$. That is, again it suffices to consider the cases where all the agents having the same forecasts $\inf \mathcal{A}$ and $\sup \mathcal{A}$ and this will give $\underline{a}^1$, associated to $\bar{a}^0$, and $\bar{a}^1$, associated to $\underline{a}^0$. However, now we have a difference with the strategic complementarities case. In the previous section the iterations started in the lower bound of the state set were lower bounds of the iterations of the eductive process. As we see, this is not the case anymore. Nevertheless, here is where the second iterate of Γ gains relevance. In a second order basis, once we have $\underline{a}^1$ and $\bar{a}^1$ obtained as above, we can now consider the images through Γ of these points and we get new points $\bar{a}^2$, from $\underline{a}^1$, and $\underline{a}^2$, from $\bar{a}^1$, that are respectively upper and lower bounds of the second step of the eductive process. This is, in two steps we obtain that the iterations started at the upper (resp. lower) bound of the states set is un upper (resp. lower) bound of the second step of the eductive process. Moreover, the sequences obtained by the second iterates are increasing when started at $\underline{a}^0$ and decreasing when started at $\bar{a}^0$. The complete lattice structure of $\mathcal{A}$ again implies the convergence of the monotone sequences while Γ^2 inherits upper semi continuity from Γ. This implies that the limits of the sequences are fixed points of Γ^2.

The three key features that lead to the conclusion are analogous to the strategic complementarity case. First, $\mathcal{A}$ is a complete lattice that has as a subset its image through the function A and thus allows the eductive process to be initiated. Second, monotonic structure of the model implies that it now suffices to use Γ^2 to seclude, every second step, the set obtained from the eductive process into a compact interval. Third, continuity properties of the utility functions and the monotonic structure of the model allow the process to converge.

Note that, also as in the case of strategic complementarities, since the limits of the interval in Theorem 4.9 are point-rationalizable, this is the smallest interval that contains the set of point-rationalizable states.

The full equivalence of Global Stability criteria obtains

Corollary 4.10. *If in $\mathcal{G}'$, Γ^2 has a unique fixed point a^*, then*

$$\mathbb{E} = \mathbb{C}_{\mathcal{A}} = \mathbb{R}_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}} \equiv \{a^*\}\,.$$

This can be reassessed as :

Corollary 4.11. *The four following statements are equivalent:*

(i) *an equilibrium* a^* *is globally* Strongly Rational.

(ii) *an equilibrium* a^* *is globally* Strongly Point Rational.

(iii) *an equilibrium* a^* *is globally* IE-Stable.

(iv) *there exists a unique cycle of order two* a^*.

Proof. Note that if Γ^2 has a unique fixed point, it is necessarily a degenerate cycle i.e an equilibrium (since there exists at least an equilibrium and equilibria are cycles of any order). Observe that both limits of the interval presented in Theorem 4.9, $\underline{a}$ and $\bar{a}$, are fixed points of Γ^2. Hence the result.

■

Under strategic substitutabilities, the strong equivalence result of Corollary 4.4 does not obtain any longer. If the sequences $\bar{b}^t$ and $\underline{b}^t$ defined in the proof of Theorem 4.9 (in the appendix) converge to the same point, i.e. $\underline{b}^* = \bar{b}^* = a^*$, then a^* is the unique equilibrium of the system, it is strongly rational and IE-stable. However, uniqueness of equilibrium does not imply this situation, there could well be a unique equilibrium that is not necessarily strongly rational. Think of the case of $\mathcal{A} \subset I\!R$, where a continuous decreasing function Γ has a unique fixed point, that could well be part of a bigger set of Point-Rationalizable States (see figure 1).

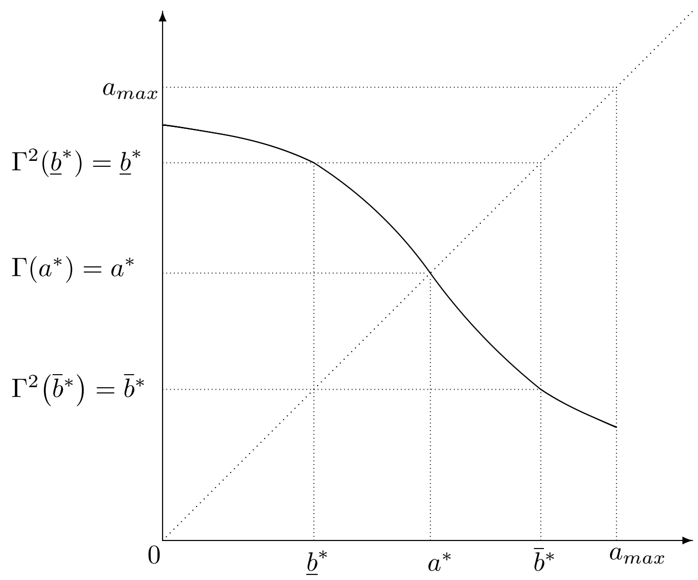

Figure 1: Strategic substitutes for $\mathcal{A} \equiv [0, a_{max}] \subset I\!R$. There exists a unique equilibrium and multiple fixed points for Γ^2

The Graal is no longer uniqueness of the equilibrium, as in the strategic complementarities case, but existence of a unique cycle of order two. Uniqueness of cycles of order two or uniqueness of equilibrium are equally demanding in the case of strategic complementarities but the former is more demanding here. Note that the results immediately apply to the models studied before and presented in Section 2 and that it enriches the known results[32].

In particular, the diagram below, that displays a cycle or order two, in a one-dimensional model with strategic substitutabilities, may be viewed as visualizing the Rationlizable set of the Muth model, (a fact that was not known to the best of our understanding).

[32]The fact that, in a partial equilibrium à la Muth, as studied by Guesnerie (1992), the absence of cycles of order 2 is sufficient for the global "eductive" stability of the equilibrium was not known, (to the best of our knowledge). The same remark applies to the general equilibrium models mentioned in the introduction. Theorem enriches immediately the results obtained in the fixed wage version of Guesnerie (2001b,a) and less immediately those obtained in the flexible wage version of these models.

It is less immediate, but extremely enligthening to use the findings of the Theorem in order to discuss the transmission of information through prices. Let us finally note that early insights on the role of cycles in the mechanics of expectational coordination can be found in an unpublished paper of the nineties by Jess Benhabib.

4.3 The differentiable case.

Here, we add an assumption concerning the cobweb mapping Γ:

H1 $\Gamma : \mathcal{A} \to \mathcal{A}$ is a $\mathcal{C}^1$-differentiable function.

Remark 4.12. Note that from the definition of Γ, in both cases, the vector-field $(a - \Gamma(a))$ points outwards on $\mathcal{A}$: formally, this means that if $p(a)$ is a supporting price vector at a boundary point of $\mathcal{A}$ $(p(a) \cdot \mathcal{A} \leq 0)$, then $p(a) \cdot (a - \Gamma(a)) \geq 0$. When, as in most applications $\mathcal{A}$ is the product of intervals for example $[0, M_h]$, this means $\Gamma_h(a) \geq 0$, whenever $a_h = 0$, and $M_h - \Gamma_h(a) \geq 0$, whenever $a_h = M_h$.

The jacobian of the function Γ, $\partial\Gamma$, can be obtained from the first order conditions of problem (2.4) along with (3.4).

$$\partial\Gamma(a) = \int_I \partial B(i, a) \text{ di}$$

where $\partial B(i, a)$ is the jacobian of the optimal strategy (now) function. This jacobian is equal to:

$$\partial B(i, a) \quad \equiv \quad -[Du_{ss}(i, B(i, a), a)]^{-1} Du_{sa}(i, B(i, a), a) \tag{4.3}$$

where $Du_{ss}(i, B(i, a), a)$ is the matrix of second derivatives with respect to s of the utility functions and $Du_{sa}(i, B(i, a), a)$ is the matrix of cross second derivatives, at the point $(B(i, a), a)$.

4.3.1 The strategic complementarities case.

Under assumptions 1.B to 3.B, along with $\mathcal{C}^2$ differentiability of the functions $u(i, \cdot, \cdot)$, we get from (4.3) that the matrices $\partial B(i, a)$ are positive[33], and consequently so is $\partial\Gamma(a)$.

The properties of positive matrices are well known. When there exists a positive vector x, such that $Ax < x$, the matrix A is said "productive": its eigenvalue of highest modulus is positive and smaller than one. When a is one-dimensional, the condition says that the slope of Γ is smaller than 1.

In this special case, as well as in our more general framework, the condition has the flavor that actions do not react too wildly to expectations..

In this case, we obtain :

Theorem 4.13 (Uniqueness). *If $\forall$ $a \in \mathcal{A}$, $\partial\Gamma(a)$ is a "productive" matrix, then there exists a unique Strongly Rational Equilibrium.*

Proof.

Compute in any equilibrium a^* the sign of $\det [I - \partial\Gamma(a^*)]$. If $\partial\Gamma(a^*)$ is productive, its eigenvalue of highest modulus is real positive and smaller than 1. Hence the real eigenvalues of $[I - \partial\Gamma(a^*)]$ are all positive[34]. It follows that the sign of $\det [I - \partial\Gamma(a^*)]$ is the sign of the characteristics polynomial $\det\{[I - \partial\Gamma(a^*)] - \lambda I\}$ for $\lambda \to -\infty$, i.e is plus. The index of $\varphi(a) = a - \Gamma(a)$ is then $+1$. The Poincaré-Hopf theorem for vector fields pointing inwards implies that the sum of indices must be equal to $+1$, hence the conclusion of uniqueness. Strong Rationality follows from Corollary 4.4.

■

The above statements generalize in a reasonable way the intuitive findings easily obtainable from the one-dimensional model. It is a clearly unsurprising statement, altough it does not seem to have been stressed in previous literature.

4.3.2 The strategic substitutabilities case.

Let us go to the Strategic Substitutabilities case.

When passing from 3.B to 3.B' we get that now the matrix $\partial\Gamma$ has negative[35] entries. And $I - \partial\Gamma(a)$ is a positive matrix. Again, it has only positive eigenvalues, whenever the positive eigenvalue of highest modulus of $-\partial\Gamma$ is smaller than 1.

[33]It is a well know fact that increasing differences implies positive cross derivatives on $Du_{sa}(i, B(i, a), a)$ and it can be proved that for a supermodular function the matrix $-[Du_{ss}(i, \cdot, \cdot)]^{-1}$ is positive at $(B(i, a), a)$

[34]It has at least one real eigenvalue, associated with the eigenvalue of highest modulus of $\Gamma(a^*)$.

[35]Since the matrix $Du_{sa}(i, B(i, a), a)$ has only non-positive entries under strategic substitutabilities (see note 33).

Theorem 4.14. *Let us assume that $\forall\, a_1,\, a_2 \in \mathcal{A}$, $\partial\Gamma(a_1)\,\partial\Gamma(a_2)$ is "productive", then*
1- There exists a unique equilibrium.
2- It is globally Strongly Rational.

Proof.
The assumption implies that $\forall a \in \mathcal{A}$, $-\partial\Gamma(a)$ is productive.
Hence $I - \partial\Gamma$ is a positive matrix, and whenever the positive eigenvalue of highest modulus of $-\partial\Gamma$ is smaller than 1, it has only positive eigenvalues. Then its determinant is positive.
Then the above Poincaré-Hopf argument applies to the first and second iterate of Γ.
It follows that there exists a unique equilibrium and no two-cycle.
Then, Theorem 4.9 applies.

■

Again, this seems a potentially most useful results for assessing expectional stability in the context of strategic substitutabilities.

5 The local viewpoint.

5.1 The local stability criteria and their general connections.

We now give the local version of the above stability concepts. Outside the cases where global stability results may be expected (and they may be expected for well understood reasons, as just precisley shown in the economies under consideration, either with strategic complementarities and substitutabilities), the local analysis becomes essential, and in a sense, the present Section may have more bite on general expectational coordination problems than the previous one.

Again, the definition of (local) IE-Stability, stated below is similar to the one given in Evans and Guesnerie (1993).

Definition 5.1. An equilibrium a^* is said to be *Locally Iterative Expectationaly Stable* if there is a neighborhood $V \ni a^*$ such that $\forall\, a^0 \in V$ any sequence $a^t \in \Gamma\big(a^{t-1}\big)$ satisfies $\lim_{t\to\infty} a^t = a^*$.

Definition 5.2. An equilibrium state a^* is *Locally Strongly Point Rational* if there exists a neighborhood $V \ni a^*$ such that the process governed by $\tilde{Pr}$ started at V generates a nested family that leads to a^*. This is, $\forall\; t \geq 1$,

$$\tilde{Pr}^{\,t}(V) \subset \tilde{Pr}^{\,t-1}(V)$$

and

$$\bigcap_{t\geq 0} \tilde{Pr}^{\,t}(V) \equiv \{a^*\}\,.$$

Definition 5.3. An equilibrium state a^* is *Locally Strongly Rational* (Guesnerie, 1992) if there exists a neighborhood $V \ni a^*$ such that the eductive process governed by $\tilde{R}$ started at V generates a nested family that leads to a^*. This is, $\forall\; t \geq 1$,

$$\tilde{R}^{t+1}(V) \subset \tilde{R}^{t}(V)$$

and

$$\bigcap_{t\geq 0} \tilde{R}^{t}(V) \equiv \{a^*\}\,.$$

There are straightforward connections between the local concepts concepts stressed above.

Proposition 5.4. *We have:*

(i) *a^* is (Locally) Strongly Rational $\implies$ a^* is (Locally) IE-Stable.*

(ii) *a^* is Locally Strongly Rational $\implies$ a^* is Locally Strongly Point Rational.*

A sufficient condition for the converse to be true is that there exist a neighborhood V of a^* such that for almost all $i \in I$, for any borel subset $X \subseteq V$:

$$\mathbb{B}(i, \mathcal{P}(X)) \qquad \subseteq \mathrm{co}\,\{B(i,X)\} \tag{5.1}$$

At a first glance the hypothesis in the second part of Proposition 5.4 appears to be very restrictive, however it involves only local properties of the agents' utility functions. It intuitively states that given a restriction on common knowledge (subsets of the set V), when agents evaluate all the possible actions to take when facing probability forecasts with support "close" to the equilibrium, these actions are somehow "not to far" and "surrounded" by the set of actions that could be taken when facing point forecasts ($\mathbb{B}(i,\mu) \subseteq \text{co}\,\{B(i,X)\}$ if $\sup(\mu) \subseteq X$). The assumption is true in a number of applications where it can be derived from standard assumptions over utility functions.

5.2 More on the general connections between the local stability criteria.

In general contexts and outside the case stressed in Evans and Guesnerie (2003), the three local concepts are unlikely to be equivalent : Local Iterative Expectational Stability is in general a weaker requirement than Local Strong Rationality (see Guesnerie, 2002; Evans and Guesnerie, 2005).

Concerning the connection between Local Point Strong Rationality and Local Strong Rationality, we have noted that Condition (5.1) above, which relates the individual reactions of agents facing non degenerate subjective forecasts, with their reactions when facing point (Dirac) forecastsis, is sufficient for the equivalence. An alternative approach consists in focusing attention on the convergence of the process generated by point forecasts. When this convergence is sufficiently fast, then we shall say that the equilibrium is *Strictly* Locally Point Rational, and we shall argue it drags the the convergence to equilibrium of the stochastic eductive process, as well.

For a positive number $\alpha > 0$ and a set $V \subseteq \mathcal{A}$ that contains a unique equilibrium a^* we will denote by V_α the set:

$$V_\alpha := \{x \in \mathcal{A} \;:\; x = \alpha(v - a^*)\,,\; v \in V\}$$

Definition 5.5. We say that an equilibrium state a^* is *Strictly Locally Point Rational* if it is Locally Strongly Point Rational and there is a number $\bar{k} < 1$ such that, $\forall\ 0 < \alpha \leq 1$,

$$\sup_{v \in \tilde{Pr}(V_\alpha)} \parallel v - a^* \parallel < \bar{k} \sup_{v' \in V_\alpha} \parallel v' - a^* \parallel .$$

Strict Locally Point Rationality assesses the idea of fast convergence of the point forecast process. Under this property, we have that $\tilde{Pr}(V) \subset V_{\bar{k}}$, with $\bar{k} < 1$, and so $\tilde{Pr}^t(V) \subset V_{\bar{k}^t}$.

Theorem 5.6. *If the utility functions are twice continuously differentiable, $a^* \in \text{int}\mathcal{A}$, $\mathbb{B}(i,\mu)$ is single valued for all μ with support in a neighborhood of a^* and $Du_{ss}(s,a)$ is non singular in an open set $V \ni a^*$, then a^* is Locally Strongly Rational $\iff$ a^* is Strictly Locally Point Rational.*

The idea of the theorem is that if the process $\tilde{Pr}$ associated with point forecasts is sufficiently fast, then, although the eductive process $\tilde{R}$ may be slower, it is anyhow dragged to the equilibrium state.

We have called the result theorem, since t*he local analysis of stability is considerably simplified when the rather weak assumption stated above holds. Being allowed to forget about stochastic expectations and concentrating on (local) point-rationalizability, makes an operationally decisive difference for the easiness of the analysis.*

5.3 Local Stability in economies with Strategic Complementarities and Strategic Substitutabilities.

We now stress that the equivalence of the three local criteria, generally hopeless, is easy to assess whenever we are in the cases of strategic complementarities and substitutabilities. In both cases, the assessment of local stability can be made without explicit reference to the heterogeneity of expectations, i.e by referring only to the map Γ.

Proposition 5.7. *Assume either strategic substitutabilities or strategic complementarities. Consider an equilibrium a^*. If there exists a neigbourhood V of a^*, such that $\Gamma(V) \subset \text{int}V$, then the equilibrium is locally stable in the three different senses of stability just introduced.*

The proof is left to the reader who will show that, under the condition stressed, $\Gamma(V) \subset \text{int}V$, the model *restricted to the neighbourhood V*, fits the assumption of the theorem. It is easy to check then that there is no cycle in V : the conclusion follows from the above theorem. The reader will also notice that the condition is almost necessary for stability in the three cases, but only almost since in border cases, the strict equivalence of the three criteria is not warranted.

Note that in fact the local version of our theorems relying on differentiability allows to prove the next corollary to Theorem 5.6 (the proof again left to the reader, immediate with Strategic Complementarities, easy with Strategic Substitutabilities).

Corollary 5.8. *Under Strategic Complementarities or Substitutabilities, if* $\partial\Gamma(a^*)$ *is "productive", for some equilibrium* a^**, then* a^* *is locally Strongly Rational.*

This Corollary is nothing else than the differentiable version of the previous Proposition : since $\partial\Gamma(a^*)$ productive implies the existence of V such that $\Gamma(V) \subset \text{int}V$.

6 Comments and Conclusions

The Rational Expectations Hypothesis has been subject of scrutiny in recent years through the assessment of Expectational Coordination. Although the terminology is still fluctuating, the ideas behind what we call here *Strong Rationality* or *Eductive Stability* have been at the heart of the study of diverse macroeconomic and microeconomic models of standard markets with one or several goods, see Guesnerie (1992, 2001a), models of information transmission (Desgranges (2000), Desgranges and Heinemann (2005), Ben-Porath and Heifetz (2006)).

This paper comes back on the conceptual basis of previously used *eductive stability* concepts, by connecting a now standard line of research in game-theory- games with a continuous of players - with what may be called the *economic* viewpoint The framework under consideration, the so-called *economic* model with non-atomic agents, is potentially useful in broad economic contexts. The paper explores in a pedestrian way, the connections between the different concepts that have been proposed for analyzing the expectational or *plausibility* of equilibria : Cycles, Cournot tâtonnement outcomes, Point Rationalizable States, Ratioanalizable States. (the two latter sets being proved to be convex) and the corresponding *stability criteria :* Iterative Expectatioanl Stability, Strong Point Rationality, Strong Rationality.

In Section 4, we first derive, in a world with strategic complementarities, results that reformulate, in our setting, the classical game-theoretical findings of Milgrom and Roberts (1990) and Vives (1990). In the opposite polar case of strategic substitutabilities, (much less documented although economically most relevant), we are still able to exhibit results that are still strong although somewhat less strong. The flavour of those results is however strikingly different. For example, when in the strategic complementarities case, uniqueness triggers stability along the lines of all expectational stability criteria under scrutiny, this is no longer the case with strategic substitutabilities : uniqueness does not imply "expectational stability", whatever the exact sense given to the assertion. We give however simple and appealing conditions (absence of cycle of order 2) implying both uniqueness of equilibria and stability in the most demanding sense of Strong Rationality. Hence, in the strategic substituabilities setting, the analysis, particularly when differentiability assumptions are introduced, leads to analytically tractable sufficient conditions for global stability.

Local stability is in itself a subject of high interest, particularly, when we fall outside the polar worlds under consideration in Section 4, so that global stability is unlikely[36] Theorem 5.6 provides an operationally important simplification for the analysis, when Proposition 5.7 shows that again the case of complementarities and substitutabilities is local analysis is strikingly simpler.

Our results can be read in a different way : we have stressed in the text how the different concepts under scrutiny reflect different assesments of the heterogeneity of expectations : homogenous expectations associated with "Cobweb tâtonnement", heterogenous and deterministic expectations or heterogenous and stochastic expectations associated with the ratioanlizability process. In a sense, the above results provide a useful benchmark for a better understanding of the role of the heterogeneity of beliefs in expectational coordination.

Three different lines of research are likely to provide improvements on our present findings. Incorporating incomplete information in our setting, along the line of argument of the global games literature (Carlsson and van Damme, 1993; Morris and Shin, 1998, 2003), looks a potentially fruitful undertaking. Indeed, when the informational random variable can take a finite number of values,

[36]As stressed earlier, many economic models that fit our framework, such as the one associated with the analysis of expectational stability in a class of general dynamical systems (Evans and Guesnerie, 2005) have neither strategic complementarities nor substitutabilities. The complexity of the findings that has increased when going from the first case to the second one, will still increase. Hence, the local Stability analysis that is presented in the last Section, may turn out to provide the most important operational insights,

our basic setting can be reinterpreted to describe an incomplete information world. In general, however, the reinterpretation requires that an inifinite-dimensional version of our basic model be considered. As we already argued, such an infinite-dimensional variant would also be most useful in the context of models theoretical finance.

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A Technical Lemmas

Lemma A.1. *Under assumptions **C** and **HM**, for a closed set $X \subseteq \mathcal{A}$ the correspondence $i \rightrightarrows B(i, X)$ is measurable and has compact values.*

Proof.
We show first that the mapping $\mathrm{G} : I \rightrightarrows \mathcal{A} \times S$, that associates with each agent $i \in I$ the graph of the best response mapping $B(i,\ \cdot\ )$, $\mathrm{G}(i) := \mathrm{gph}B(i,\ \cdot\ )$, is measurable.

Take a closed set $C \subseteq \mathcal{A} \times S$. We need to prove that the set

$$\mathrm{G}^{-1}(C) \equiv \{i \in I \ :\ C \cap \mathrm{gph}B(i,\ \cdot\ ) \neq \emptyset\}$$

is measurable. Consider the subset $U \subseteq \mathcal{U}_{S\times\mathcal{A}}$ defined by:

$$U := \{\ g \in \mathcal{U}_{S\times\mathcal{A}}\ :\ \exists\,(a,s) \in C \text{ such that } g(s,a) \geq g(y,a)\ \forall\ y \in S\}$$

note that $\mathbf{u}^{-1}(U) \equiv \mathrm{G}^{-1}(C)$ and so, from the measurability assumption over $\mathbf{u}$, it suffices to prove that U is closed. That is, we have to show that for any sequence $\{g^\nu\}_{\nu\in I\!N} \subset U$, such that $g^\nu \to g^*$ uniformly $g^* \in U$.

Since the functions g^ν are finite and continuous in $S \times \mathcal{A}$, from Weierstrass' Theorem g^* is continuous and so it belongs to $\mathcal{U}_{S\times\mathcal{A}}$. Moreover, g^ν converges continuously to g^*, that is, for any convergent sequence (a^ν, s^ν) with limit (a^*, s^*), the sequence $g^\nu(s^\nu, a^\nu)$ converges to $g^*(s^*, a^*)$. Indeed, consider any $\varepsilon > 0$. By the continuity of g^* there exists $\bar{\nu}_1 \in I\!N$ such that $\forall\ \nu > \bar{\nu}_1$,

$$|g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \frac{\varepsilon}{2}.$$

¿From the uniform convergence of g^ν we get that there exists $\bar{\nu}_2 \in I\!N$ such that,

$$|g^\nu(s,a) - g^*(s,a)| < \frac{\varepsilon}{2} \text{ for all } \nu \geq \bar{\nu}_2 \text{ and } \forall\,(s,a) \in S \times \mathcal{A},$$

in particular this is true for all the elements of the sequence of points. We get then that $\forall$ $\nu \geq \max\{\bar{\nu}_1, \bar{\nu}_2\}$,

$$|g^\nu(s^\nu, a^\nu) - g^*(s^*, a^*)| \leq |g^\nu(s^\nu, a^\nu) - g^*(s^\nu, a^\nu)| + |g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \varepsilon.$$

We have to show then that there exists a point $(a,s) \in C$ such that $g^*(s,a) \geq g^*(y,a)\ \forall\ y \in S$. Since $g^\nu \in U$, we have for each $\nu \in I\!N$, points $(a^\nu, s^\nu) \in C$ such that $g^\nu(s^\nu, a^\nu) \geq g^\nu(y, a^\nu)\ \forall\ y \in S$. Let $(a^*, s^*) \in C$ be the limit of a convergent subsequence of $\{(a^\nu, s^\nu)\}_{\nu\in I\!N}$, which without loss of generality we can take to be the same sequence. We see that (a^*, s^*) is the point we are looking for since for a fixed $y \in S$, continuous convergence implies that in the limit

$$g^*(s^*, a^*) \geq g^*(y, a^*)\,.$$

We conclude then that $g^* \in U$. Thus, U is closed and since $\mathbf{u}$ is a measurable mapping, $\mathbf{u}^{-1}(U)$ is measurable.

With this in mind, consider a closed set $X \subseteq \mathcal{A}$ and the mapping $i \rightrightarrows B(i, X)$. Applying Theorem 14.3 in Rockafellar and Wets (1998) to the constant mapping $i \rightrightarrows X$ along with G above, we get that $i \rightrightarrows B(i, X)$ is measurable and has closed values (hence compact since S is compact). ■

Lemma A.2. *If $S \subset I\!R^n$ is a complete lattice for the product order in $I\!R^n$, then for a measurable correspondence $F : I \rightrightarrows S$ with nonempty, closed and subcomplete values, the functions $\underline{\mathbf{s}} : I \to S$ and $\bar{\mathbf{s}} : I \to S$, defined by*

$$\begin{aligned}\underline{s}(i) &:= \inf_S F(i)\,,\\ \bar{s}(i) &:= \sup_S F(i)\,,\end{aligned}$$

are measurable selections of F.

Proof.
Since $F(i)$ is subcomplete, $\underline{s}(i)$ and $\bar{s}(i)$ belong to $F(i)$. We have to prove that $\underline{\mathbf{s}}$ and $\bar{\mathbf{s}}$ are measurable.

Since F is measurable, it has a Castaing representation. That is, there exists a countable family of measurable functions $\mathbf{s}^\nu : I \to I\!R^n$, $\nu \in I\!N$, such that $s^\nu(i) \in F(i)$ and,

$$F(i) \equiv \mathrm{cl}\{\{s^\nu(i)\ :\ \nu \in I\!N\}\}\,. \tag{A.1}$$

For $\underline{\mathbf{s}}$, consider then for each $\nu \in I\!N$ the set valued mappings $F^\nu : I \rightrightarrows I\!R^n$, defined by [37]

$$F^\nu(i) := F(i) \bigcap \,]-\infty, s^\nu(i)]$$

Since F is measurable and closed valued, and we can write $]-\infty, s^\nu(i)] = s^\nu(i) - I\!R^n_+$ which is as well measurable and closed valued, the correspondences F^ν are measurable and closed valued [38].

Note that $\forall\, \nu \in I\!N$, $\underline{s}(i) \in F^\nu(i)$. Defining the closed valued correspondence $\underline{F} : I \rightrightarrows I\!R^n$:

$$\underline{F}(i) := \bigcap_{\nu \in I\!N} F^\nu(i)$$

we get then that $\underline{s}(i) \in \underline{F}(i)$. The correspondence $\underline{F}$ is as well measurable [38].

We now prove that actually $\underline{F}(i) \equiv \{\underline{s}(i)\}$, which completes the proof. Indeed, suppose that $y \in \underline{F}(i)$. Then, by definition of $\underline{F}$, $y \in F(i)$ and $y \leq s^\nu(i)$, $\forall\, \nu \in I\!N$. From equality (A.1) we get that any point in $F(i)$ can be obtained as the limit of a subsequence of $\{s^\nu(i) \;:\; \nu \in I\!N\}$, so in the limit the inequality is maintained, this is $\forall\, s \in F(i)$, $y \leq s$. That is, y is a lower bound for $F(i)$. This implies, by the definition of $\inf_S F(i)$, that $y \leq \inf_S F(i)$, but $y \in F(i)$, so $\inf_S F(i) \leq y$. Thus, $y \leq \underline{s}(i) = \inf_S F(i) \leq y$.

Analogous arguments applied to the mapping $\bar{F} : I \rightrightarrows I\!R^n$:

$$\bar{F}(i) := F(i) \bigcap \left(\bigcap_{\nu \in I\!N} [s^\nu(i)\,, +\infty\,[\right)$$

prove the statement for $\bar{\mathbf{s}}$.

■

B Proof of Lemma 4.1

Proof.

In (i) the first part is a consequence of Theorem 2 in Milgrom and Roberts (1990) and the second part we apply Theorem 2.8.1 in Topkis (1998) considering the constant correspondence $S_a \equiv S \;\forall\, a \in \mathcal{A}$

Part (ii) is a consequence of Lemma A.2 above.

Finally, as a consequence of the Theorem 2.5.1 in Topkis (1998) the set of fixed points of Γ is a non-empty complete lattice.

■

C Proof of Lemma 4.2

Proof. Observe first that supermodularity of $u(i, \,\cdot\,, a)$ is preserved[39] when we take expectation on a.

Now consider $y' \in B(i, a')$ and $y \in I\!B(i, \mu)$ we show that $\min\{y, y'\} \in B(i, a')$ and $\max\{y, y'\} \in I\!B(i, \mu)$. Since $y' \in B(i, a')$ we have that:

$$0 \leq u(i, y', a') - u(i, \min\{y, y'\}, a')\,.$$

Increasing differences of $u(i, y, a)$ in (y, a) implies that $\forall\, a \in \sup(\mu)$,

$$u(i, y', a') - u(i, \min\{y, y'\}, a') \leq u(i, y', a) - u(i, \min\{y, y'\}, a)$$

and so if on the right hand side we take expectation with respect to μ we get

$$u(i, y', a') - u(i, \min\{y, y'\}, a') \leq I\!E_\mu\left[u(i, y', a)\right] - I\!E_\mu\left[u(i, \min\{y, y'\}, a)\right].$$

[37] The *interval* $]-\infty, x]$ is the set of points of $I\!R^n$ that are smaller than $x \in I\!R^n$, similarly $[x, +\infty\,[$ is the set of points in $I\!R^n$ that are greater than x.

[38] See Proposition 14.11 in Rockafellar and Wets (1998).

[39] If $u(i, \,\cdot\,, a)$ is supermodular, then for $s, s' \in S$, we have for each $a \in \mathcal{A}$:

$$u(i, \min\{s, s'\}, a) + u(i, \max\{s, s'\}, a) - \left(u(i, s, a) + u(i, s', a)\right) \geq 0$$

Taking expectation we get the result.

Supermodularity of $u(i,\ \cdot\ ,a)$ implies that

$$\mathbb{E}_\mu\left[u(i,y',a)\right]-\mathbb{E}_\mu\left[u(i,\min\left\{y,y'\right\},a)\right]\le\mathbb{E}_\mu\left[u(i,\max\left\{y,y'\right\},a)\right]-\mathbb{E}_\mu\left[u(i,y,a)\right]$$

and the last term is less or equal to 0 since $y\in\mathbb{B}(i,\mu)$.

All these inequalities together imply that $\max\left\{y,y'\right\}\in\mathbb{B}(i,\mu)$ and $\min\left\{y,y'\right\}\in B(i,a')$

The second statement is proved analogously.

■

D Proof of Proposition 4.3

Proof.

Let us denote the fixed points of Γ by E_Γ. We will first prove the following statement:

In $\mathcal{G}$ we have

$$\mathbb{P}_{\mathcal{A}}\subseteq\left[\inf_{E_\Gamma}\left\{E_\Gamma\right\},\sup_{E_\Gamma}\left\{E_\Gamma\right\}\right]$$

and $\inf_{E_\Gamma}\left\{E_\Gamma\right\}$ and $\sup_{E_\Gamma}\left\{E_\Gamma\right\}$ are equilibria.

¿From Lemma 4.1 we get that E_Γ is non empty and has a greatest and a smallest element.

Following the structure of the proof of Theorem 5 in Milgrom and Roberts we prove that $\tilde{Pr}^t(\mathcal{A})$ is contained in some interval $[\underline{a}^t,\bar{a}^t]$ and that the sequences $\underline{a}^t$ and $\bar{a}^t$ satisfy $\underline{a}^t\to\inf_{E_\Gamma}\left\{E_\Gamma\right\}$ and $\bar{a}^t\to\sup_{E_\Gamma}\left\{E_\Gamma\right\}$.

Define $\underline{a}^0$ and $\underline{a}^t$ as:

$$\begin{aligned}\underline{a}^0 &:= \inf\mathcal{A}\\ \underline{a}^t &:= \inf_{\mathcal{A}}\Gamma\big(\underline{a}^{t-1}\big)\qquad\forall\ t\ge 1\end{aligned}$$

- $\tilde{Pr}^t(\mathcal{A})\subseteq[\underline{a}^t,+\infty[$

  Clearly it is true for $t=0$.

  Suppose that it is true for $t\ge 0$. That is, $\underline{a}^t\le a\ \forall\ a\in\tilde{Pr}^t(\mathcal{A})$. Since $B(i,\ \cdot\ )$ is increasing, we get that $B(i,\underline{a}^t)\preceq B(i,a)\ \forall\ a\in\tilde{Pr}^t(\mathcal{A})$. In particular $\forall\ y\in B(i,a)$ and $\forall\ a\in\tilde{Pr}^t(\mathcal{A})$, we have $\inf_S B(i,\underline{a}^t)\le y$. From Lemma A.2, the correspondence $i\rightrightarrows\ \inf_S B(i,\underline{a}^t)$ is measurable. This implies that for any measurable selection $\mathbf{s}\in S^I$ of $i\rightrightarrows B\Big(i,\tilde{Pr}^t(\mathcal{A})\Big)$,

  $$\int\inf_S B\big(i,\underline{a}^t\big)\,\mathrm{di}\quad\le\quad\int s(i)\,\mathrm{di}.\tag{D.1}$$

  Since $B(i,\underline{a}^t)$ is subcomplete, $\inf_S B(i,\underline{a}^t)\in B(i,\underline{a}^t)$ and so we get that:

  $$\begin{aligned}\inf_{\mathcal{A}}\Gamma\big(\underline{a}^t\big)\quad&\equiv\quad\inf_{\mathcal{A}}\big\{\ b\in\mathcal{A}\ :\ \exists\ \mathbf{s}\text{ measurable selection of } i\rightrightarrows B\big(i,\underline{a}^t\big)\text{ such that, } b=A(\mathbf{s})\big\}\\ &\le\quad\int\inf_S B\big(i,\underline{a}^t\big)\end{aligned}\tag{D.2}$$

  We conclude then that

  $$\underline{a}^{t+1}\equiv\inf_{\mathcal{A}}\Gamma\big(\underline{a}^t\big)\le\int\inf_S B\big(i,\underline{a}^t\big)\le a\qquad\forall\ a\in\tilde{Pr}^{t+1}(\mathcal{A}).$$

  The equality is the definition of $\underline{a}^{t+1}$, the first inequality comes from (D.2) and the second one is obtained from (D.1) and the definition of $\tilde{Pr}$.

- The sequence is increasing:

  By definition of $\underline{a}^0$, $\underline{a}^0\le\underline{a}^1$. Suppose that $\underline{a}^{t-1}\le\underline{a}^t$, then from Lemma 2.4.2 in Topkis (1998), $\underline{a}^t\equiv\inf_{\mathcal{A}}\Gamma\big(\underline{a}^{t-1}\big)\le\inf_{\mathcal{A}}\Gamma(\underline{a}^t)\equiv\underline{a}^{t+1}$.

- The sequence has a limit and $\lim_{t\to+\infty}\underline{a}^t$ is a fixed point of Γ:

  Since the sequence is increasing and $\mathcal{A}$ is a complete lattice, it has a limit $\underline{a}^*$. Furthermore, since Γ is subcomplete, upper semi-continuity of Γ implies that $\underline{a}^*\in\Gamma(\underline{a}^*)$.

- $\underline{a}^* \equiv \inf_{E_\Gamma} \{E_\Gamma\}$:

  According to the previous demonstration, since the fixed points of Γ are in the set $\mathbb{P}_{\mathcal{A}}$, all fixed points must be in $[\underline{a}^*, +\infty[$ and so $\underline{a}^*$ is the smallest fixed point.

Defining $\bar{a}^0$ and $\bar{a}^t$ as:

$$\begin{aligned} \bar{a}^0 &:= \sup \mathcal{A} \\ \bar{a}^t &:= \sup_{\mathcal{A}} \Gamma\big(\bar{a}^{t-1}\big) \qquad \forall\, t \geq 1 \end{aligned}$$

In an analogous way we obtain that $\mathbb{P}_{\mathcal{A}} \subseteq \quad]-\infty, \bar{a}^*]$, with $\bar{a}^*$ being the greatest fixed point of Γ.

This proves the statement. Proposition 4.3 obtains then using Lemma 4.2 we can see that $\tilde{R}^t(\mathcal{A}) \subseteq [\underline{a}^t, \bar{a}^t]$ and so we get the result.

■

E Proof of Theorem 4.9

Proof.

Following the proof of Proposition 4.3, consider the sequence $\{\underline{a}^t\}_{t=0}^{\infty}$ therein defined, but let us change the definition of $\underline{a}^t$ when t is odd to:

$$\underline{a}^t := \sup_{\mathcal{A}} \Gamma\big(\underline{a}^{t-1}\big)\,.$$

By the definition of $\underline{a}^0$, we know that $\forall\, a \in \mathcal{A}$, $a \geq \underline{a}^0$. Since the mappings $B(i,\ \cdot\ )$ are decreasing we have $B\big(i, \underline{a}^0\big) \succeq B(i,a)\ \forall\, a \in \mathcal{A}$ and in particular

$$\sup_S B\big(i, \underline{a}^0\big) \geq y \quad \forall\, y \in B(i,a)\ \ \forall\, a \in \mathcal{A}$$

Since $B\big(i, \underline{a}^0\big)$ is subcomplete $\sup_S B\big(i, \underline{a}^0\big) \in B\big(i, \underline{a}^0\big)$ and from Lemma A.2 the function $i \to \sup_S B\big(i, \underline{a}^0\big)$ is measurable, so $\int \sup_S B\big(i, \underline{a}^0\big) \in \Gamma\big(\underline{a}^0\big)$, thus

$$\sup_{\mathcal{A}} \Gamma\big(\underline{a}^0\big) \geq \int \sup_S B\big(i, \underline{a}^0\big)\, \mathrm{di} \geq \int s(i)\, \mathrm{di}$$

for any measurable selection $\mathbf{s}$ of $i \rightrightarrows B(i, \mathcal{A})$. That is $\underline{a}^1 \geq a\ \forall\, a \in \tilde{Pr}^1(\mathcal{A})$; or equivalently,

$$\tilde{Pr}^1(\mathcal{A}) \subseteq \big]-\infty, \underline{a}^1\big]\,.$$

A similar argument leads to conclude that $\tilde{Pr}^2(\mathcal{A}) \subseteq \big[\underline{a}^2, +\infty\big[$.

Let us define then the sequence $\underline{b}^t := \underline{a}^{2t}$, $t \geq 0$. This sequence satisfies:

1. $\tilde{Pr}^{2t} \subseteq \big[\underline{b}^t, +\infty\big[$. This can be obtained as above by induction over t.
2. $\big\{\underline{b}^t\big\}_{t\geq 0}$ is increasing.

As before, we get that $\big\{\underline{b}^t\big\}_{t\geq 0}$ has a limit $\underline{b}^*$. Since Γ is u.s.c. and $\mathcal{A}$ is compact, we obtain that the second iterate of Γ, Γ^2 is as well u.s.c.. Moreover, from Proposition 4.6, we get that $\underline{b}^t \in \Gamma^2\big(\underline{b}^{t-1}\big)$. This implies that $\underline{b}^*$ is a fixed point of Γ^2 and so it is a point-rationalizable state. Consequently we get

1. $\mathbb{P}_{\mathcal{A}} \subseteq [\underline{b}^*, +\infty[$
2. $\underline{b}^* \in \Gamma^2(\underline{b}^*)$ and $\underline{b}^*$ is a point-rationalizable state.

Considering the analogous sequence to obtain the upper bound for $\mathbb{P}_{\mathcal{A}}$:

$$\begin{aligned} \bar{a}^0 &:= \sup \mathcal{A} \\ \bar{a}^t &:= \inf_{\mathcal{A}} \Gamma\big(\bar{a}^{t-1}\big) \qquad \text{when } t \text{ is odd} \\ \bar{a}^t &:= \sup_{\mathcal{A}} \Gamma\big(\bar{a}^{t-1}\big) \qquad \text{when } t \text{ is even} \end{aligned}$$

We generate a decreasing sequence $\big\{\bar{b}^t\big\}_{t\geq 0}$ defined by $\bar{b}^t := \bar{a}^{2t}$, $t \geq 0$, who's limit $\bar{b}^*$, is a point-rationalizable state and an upper bound for $\mathbb{P}_{\mathcal{A}}$, that is:

1. $\mathbb{P}_{\mathcal{A}} \subseteq \left]-\infty, \bar{b}^*\right]$
2. $\bar{b}^* \in \Gamma^2(\bar{b}^*)$. Which implies that $\bar{b}^*$ is a point-rationalizable state.

As a summary, we get:

$$\mathbb{P}_{\mathcal{A}} \subseteq \bigcap_{t\geq 0} \tilde{Pr}^t(\mathcal{A}) \subseteq \bigcap_{t\geq 0} \tilde{Pr}^{2t}(\mathcal{A}) \subseteq \left[\underline{b}^*, \bar{b}^*\right]$$

Then again using Lemma 4.8 we can see that $\tilde{R}^{2t}(\mathcal{A}) \subseteq \left[\underline{a}^{2t}, \bar{a}^{2t}\right]$ and so we get (ii) and (iv).

Assertion (iii) is a consequence of the general setting of Rath (1992). Proposition 2.3 gives the existence of equilibrium.

■

F Proof of Proposition 5.4

Proof.

For (i): note that

$$\Gamma(a) \equiv \int_I B(i,a)\,\mathrm{di} \equiv \begin{cases} \tilde{Pr}(\{a\}) \\ \int_I I\!B(i, \delta_{a^*})\,\mathrm{di} \equiv \tilde{R}(\{a\}) \end{cases}$$

and use Proposition 3.8.

For (ii) from Proposition 3.8 we see that we only need to prove that under condition (5.1):

a^* is Locally Strongly Point Rational $\quad\Longrightarrow\quad$ a^* is Locally Strongly Rational.

For a subset $X \subseteq \mathcal{A}$ call $\mathbb{P}(X) := \bigcap_{t\geq 0} \tilde{Pr}^t(X)$ and note that if $\mathbb{P}(X) \equiv \{a^*\}$ then $\forall\ X' \subseteq X$, $\mathbb{P}(X') \equiv \{a^*\}$.

Take V, the neighborhood of the Proposition. For a borel subset $X \subseteq V$ the hypothesis implies that the integral of $i \rightrightarrows I\!B(i, \mathcal{P}(X))$ is contained in the integral of $i \rightrightarrows \mathrm{co}\,\{B(i,X)\}$. From Aumann (1965) we know that:

$$\int_I \mathrm{co}\,\{B(i,X)\}\,\mathrm{di} \equiv \int_I B(i,X)\,\mathrm{di}$$

and so

$$\tilde{R}(X) \equiv \int_I I\!B(i, \mathcal{P}(X))\,\mathrm{di} \subseteq \int_I B(i,X)\,\mathrm{di} \equiv \tilde{Pr}(X) \tag{F.1}$$

If a^* is Locally Strongly Point Rational then there exists a neighborhood V' such that $\mathbb{P}(V') = \{a^*\}$. So now take an open ball of radius $\varepsilon > 0$ around a^* that is contained in both V and V'. To ensure that the process for probability forecasts is well defined, we can take the closed ball of radius $\varepsilon/2$, $\mathrm{B}(a^*, \frac{\varepsilon}{2})$, that is strictly contained in the previous ball and of course in the intersection of both neighborhoods. In particular, we have that $\mathbb{P}(\mathrm{B}(a^*, \frac{\varepsilon}{2})) \equiv \{a^*\}$ and that $\tilde{R}^t(\mathrm{B}(a^*, \frac{\varepsilon}{2}))$ is well defined and closed for all $t \geq 1$. The last assertion, along with (F.1), imply that for all $t \geq 1$ $\tilde{R}^t(\mathrm{B}(a^*, \frac{\varepsilon}{2})) \equiv \tilde{Pr}^t(\mathrm{B}(a^*, \frac{\varepsilon}{2}))$. We conclude that,

$$\bigcap_{t\geq 0} \tilde{R}^t\left(\mathrm{B}\left(a^*, \frac{\varepsilon}{2}\right)\right) \equiv \mathbb{P}\left(\mathrm{B}\left(a^*, \frac{\varepsilon}{2}\right)\right) \equiv \{a^*\}$$

■

G Proof of Theorem 5.6

Proof.

We give the proof for the case where all the agents have the same utility function $u : S \times \mathcal{A} \to I\!R$.

Consider then a convex neighborhood V of a^* and the space of probability measures $\mathcal{P}(V)$. Take a probability measure with finite support, μ, in this space, this is $\mu = \sum_{l=1}^{L} \mu_l \delta_{a_l}$, with

$\{a_l\}_{l=1}^{L} \subset V$. For this measure, under the differentiability hypothesis, we can prove that if the support of μ, $\{a_1, \ldots, a_L\}$, is contained in a ball [40] $\mathrm{B}(a^*, \varepsilon_1)$, then

$$\| \mathbb{B}(\mu) - B(\mathbb{E}_\mu [a]) \| < \varepsilon_1^2.$$

Since $\mathbb{E}_\mu [a] \in V$ we get that $B(\mathbb{E}_\mu [a]) \in B(V)$. Using a density argument we may conclude that $\mathbb{B}(\mu)$ is "close" to $B(V) \subseteq \mathrm{co}\,\{B(V)\} \equiv \tilde{Pr}(V)$ for any measure in $\mathcal{P}(V)$. We can take then $\varepsilon_1 > 0$ small, related to the neighborhood V, such that,

$$\tilde{R}(V) \subset \quad \tilde{Pr}(V) + \mathrm{B}\big(0, \varepsilon_1^2\big) \tag{G.1}$$

¿From the hypothesis we get that we can choose a number $\bar{k} < k' < 1$ such that the following inclusions hold:

$$\tilde{Pr}(V) \subset \quad V_{\bar{k}} \subset V_{k'} \subset V \tag{G.2}$$

$$\tilde{R}(V) \subset \quad \tilde{Pr}(V) + \mathrm{B}\big(0, \varepsilon_1^2\big) \subset V_{k'} \tag{G.3}$$

Moreover, taking the second iterate of $\tilde{R}$ starting at V, using (G.3) and (G.1) on $V_{k'}$,

$$\tilde{R}^2(V) \quad \subset \tilde{Pr}(V_{k'}) + \mathrm{B}\big(0, \varepsilon_2^2\big)$$

where ε_2 depends on k'. However it can be chosen in such a way that the following inclusions hold. Using (G.2) we get

$$\begin{aligned} \tilde{Pr}(V_{k'}) + \mathrm{B}\big(0, \varepsilon_2^2\big) \subset &\quad V_{\bar{k}k'} + \mathrm{B}\big(0, \varepsilon_2^2\big) \\ \subset &\quad V_{k'^2}. \end{aligned}$$

We have then,

$$\tilde{R}^2(V) \subset \quad V_{k'^2}$$

Using the same argument, choosing ε_t related to the powers of k', ${k'}^{t-1}$, we get that for all t,

$$\tilde{R}^t(V) \subset \quad \tilde{Pr}(V_{k'^{t-1}}) + \mathrm{B}\big(0, \varepsilon_t^2\big) \subset V_{\bar{k}k'^{t-1}} + \mathrm{B}\big(0, \varepsilon_t^2\big) \subset V_{k'^t}$$

We conclude then that the eductive process converges to the equilibrium a^*.

■

[40]Since $\mathcal{A}$ is compact V is bounded.