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Pascal Billand, Christophe Bravard, Sudipta Sarangi

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GATE Groupe d'Analyse et de Théorie Économique Lyon-St Étienne

93, chemin des Mouilles 69130 Ecully – France
Tel. +33 (0)4 72 86 60 60
Fax +33 (0)4 72 86 60 90

6, rue Basse des Rives 42023 Saint-Etienne cedex 02 – France
Tel. +33 (0)4 77 42 19 60
Fax. +33 (0)4 77 42 19 50

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Local Spillovers, Convexity and the Strategic Substitutes Property in Networks

PASCAL BILLAND[a], CHRISTOPHE BRAVARD[a], SUDIPTA SARANGI[b]

[a] *Université de Lyon, Lyon, F-69003, France ; Université Jean Monnet, Saint-Etienne, F-42000, France ; CNRS, GATE Lyon St Etienne, Saint-Etienne, F-42000, France.*

email: pascal.billand@univ-st-etienne.fr, christophe.bravard@univ-st-etienne.fr.

[b] *DIW Berlin and Louisiana State University. email: sarangi@lsu.edu*

Abstract

We provide existence results in a game with local spillovers where the payoff function satisfies both convexity and the strategic substitutes property. We show that there always exists a stable pairwise network in this game, and provide a condition which ensures the existence of pairwise equilibrium networks. Moreover, our existence proof allows us to characterize a pairwise equilibrium of these networks.

JEL Classification: C70, D85

Key Words: networks, existence, spillovers.

1 Introduction

The role of networks in determining the outcome of many important social and economic relationships is now well documented. In many situations, spillovers or externalities created by links are crucial in the sense that a link between two individuals i and j has an effect on the return from the other links of i and j as well as on the return from links between other individuals. Market sharing agreements and the provision of public good are illustrative examples.

Market sharing agreements situations have been studied by Bloch and Belleflamme (2004). The authors pointed out that the European Commission has been particularly aware of the potential risk of market sharing, as firms enjoying monopoly power in certain areas seem reluctant to compete on the overall European market. In particular, it was found that for many years all producers of soda ash in Europe accepted and acted on the home market principle that each producer limited its sales to the country or countries in which it has established facilities.[1] A market sharing agreement between two producers who refrain from entering each other's market leads to lower competition and is interpreted by Bloch and Belleflamme (2004) as the existence of a link between these producers. Hence the number of competitors in each national market is decreasing in the number of links formed by the national firm. It follows that the incentive of a firm to form a market sharing agreement with another firm j, i.e. the marginal returns of firm i from a link with firm j, depends both on the number of links of i and the number of links of j in the network.

Similarly, in the context of the provision of a public good where a collaborative link between two players is an agreement to share knowledge about the production of the public good, the payoffs obtained by a player i depend on the number of links of player i and on the number of links of her neighbors (see Goyal and Joshi, 2006).[2]

[1] See Official Journal L 152, June 15, 1991, pp. 1-15.

[2] Other examples of this type can also be found in the network formation literature, see for instance Furusawa and Konishi (2002).

All these situations in which the aggregate payoff of an individual depends on the distribution of links of all the players and on the identity of neighbors in the network are referred by Goyal and Joshi (2006) as *games with local spillovers*. They assume that the payoff function of a player i is the sum of three functions which depend respectively on the number of i's links, the number of links of i's neighbors and the number of links of the remaining players.

The authors use the notion of *pairwise equilibrium network* as the equilibrium concept, to determine what network will emerge. This concept requires players using their Nash equilibrium strategy with the additional restriction that no pair of players has an incentive to add a link. It is worth noting that Goyal and Joshi (2006) deal with existence and characterization of pairwise equilibrium networks under different combinations of spillovers that arise from the formation of links in games with local spillovers. However it is not easy to obtain results when the payoff of player i satisfies convexity in own links and the strategic substitutes property with regard to links of neighbors. Convexity (Conv) means that marginal return of player i from a link is increasing in the number of collaborative links she has formed. The strategic substitutes property (SSP) means that the marginal return of a player i from a link with a player j is decreasing in the number of links formed by j.

It is striking that Goyal and Joshi do not find existence for games satisfying these properties. It follows that we do not know whether a game with these properties is consistent with a steady state solution.

Our paper provides existence results for a game with local spillovers where the payoff function satisfies both convexity and the strategic substitutes property. It fills an interesting gap in the literature since there exist several situations which can be modelled as a game with local spillovers, with a gross payoff function that simultaneously satisfies Conv and SSP. For instance, the market sharing agreements game between homogeneous firms with linear cost and linear demand in each market, which compete *à la Cournot* in national markets satisfies these properties.[3] Indeed, it

[3]Later we also consider another example based on Billand, Bravard and Sarangi (2010) in which Conv and SSP are also satisfied. This example inspired by the strategic management literature deals with firms which collaborate in benchmarking activities.

is easy to check that in this game the fewer competitors a firm has (the more numerous its links are), the greater is its incentive to have fewer competitors on its market (to form an additional link). So the game satisfies Conv. Likewise, the fewer competitors on a foreign market are, the lower is the incentive to form a market sharing agreement (a link) with the national firm. It follows that the game also satisfies SSP.

In this paper, we have two contributions which fill the gap in the literature.

1. Our main contribution consists in providing existence results when the payoff function satisfies Conv and SSP. We use two equilibrium notions: pairwise equilibrium network and pairwise stable network. The notion of pairwise equilibrium network is a refinement of the notion of pairwise stable network, due to Jackson and Wolinsky (1996). The main difference between the two notions is the fact that in the Jackson and Wolinsky framework (1996) players can only delete one link at a time. In this paper, we show that pairwise stable networks always exist under Conv and SSP. Then, we investigate the existence of pairwise equilibrium networks. We show through an example that there does not always exist a pairwise equilibrium network when the payoff function satisfies Conv and SSP. Then, we provide a condition on the payoff function which allows for the existence of pairwise equilibrium networks.

2. Our second contribution concerns the characterization of the architecture of a pairwise stable network and the architecture of a pairwise equilibrium network. Indeed, our existence proof is based on a process which allows us to give the architecture of a pairwise stable network and the architecture of a pairwise equilibrium network when our sufficient condition is satisfied.

The paper is organized as follows. In section 2 we present the model setup. In section 3 we study pairwise stable networks when the payoff function satisfies Conv and SSP. In section 4 we examine pairwise equilibrium networks when the payoff function satisfies Conv and SSP.

2 Model setup

Our model setup uses the same notation as in Goyal and Joshi (2006).

Link formation game. Let $N = \{1, 2, \ldots, n\}$ denote a finite set of ex-ante identical players. We assume that $n \geq 3$. Every player makes an announcement of intended links. We define $s_{i,j} \in \{0, 1\}$ as follows: $s_{i,j} = 1$ means that player i intends to form a link with player j, while $s_{i,j} = 0$ means that player i does not intend to form such a link. Thus a strategy of player i is given by $s_i = \{(s_{i,j})_{j \in N \setminus \{i\}}\}$. Let S_i denote the strategy set of player i. We will frequently refer to all players other than some given subset of players X as "player X's opponents" and denote them by "$-X$". Thus $S_{-X} = \prod_{j \in N \setminus X} S_j$ denotes the space of strategies of player X's opponents, and s_{-X} is an element of S_{-X}. A link between two players i and j is formed if and only if $s_{i,j} = s_{j,i} = 1$. We denote this link by $g_{i,j} = 1$ and the absence of this link by $g_{i,j} = 0$. A strategy profile $s = \{s_1, \ldots, s_n\}$ therefore induces a network $g(s)$. For expositional simplicity we often omit the dependence of the network on the underlying strategy profile. A network $g = \{(g_{i,j})_{i \in N, j \in N \setminus \{i\}}\}$ is a formal description of the pairwise links that exist between the players. We let G denote the set of all networks, i.e. the set of all undirected networks with n vertices.

Given a network $g \in G$, $g + g_{i,j}$ denotes the network obtained by replacing $g_{i,j} = 0$ in network g with $g_{i,j} = 1$. Let $N_i(g) = \{j \in N \setminus \{i\} | g_{i,j} = 1\}$ be the set of players with whom player i has formed a link in the network g, and let $\eta_i(g) = |N_i(g)|$ be the cardinality of this set. The *empty network* g_e is the network where no player has formed links. A *complete component*, $C_{g'}$, of a network g' consists in a subset of players $N' \subset N$ such that for all $i \in N'$, $g'_{i,j} = 1$ for all $j \in N' \setminus \{i\}$ and $g'_{i,j} = 0$ for all $j \notin N'$. The *complete network* g_N is the network where there is a link between all players in N. In this paper, we construct complete networks with subsets of N. We denote by g_X the complete network associated with the set of players $X \subseteq N$.

Payoff function. Define $\Psi_1 : \{0, \ldots, n-1\} \to \mathbb{R}$, $\Psi_2 : \{1, \ldots, n-1\} \to \mathbb{R}$ and $\Psi_3 : \{0, \ldots, n-2\} \to \mathbb{R}$ as functions. Let $\pi_i : G \to \mathbb{R}$ be the gross payoff function of player i. Using

the gross payoff function defined by Goyal and Joshi (2006, pg.331), we have for all $g \in G$:

$$\pi_i(g) \quad = \quad \Psi_1(\eta_i(g)) + \sum_{j \in N_i(g)} \Psi_2(\eta_j(g)) + \sum_{j \notin N_i(g)} \Psi_3(\eta_j(g)). \tag{1}$$

In this class of games the payoff of each player $i \in N$ is the sum of three additive effects (i) the payoff that player i obtains due to the number of direct links she has formed, (ii) the payoff that player i obtains due to the number of links formed by players with whom i has formed a link, and (iii) the payoff that player i obtains from the number of links formed by players with whom i has not formed any link. It is worth noting that the spillovers result from the two last effects. Moreover, let $\Delta\Psi_1(x) = \Psi_1(x) - \Psi_1(x-1)$, $\varphi(x) = \Psi_2(x) - \Psi_3(x-1)$, and $\Delta\varphi(x) = \varphi(x) - \varphi(x-1)$. The marginal gross profits to player i from a link with player j in g are given by:

$$\pi_i(g + g_{i,j}) - \pi_i(g) \quad = \quad \Delta\Psi_1(\eta_i(g) + 1) + \varphi(\eta_j(g) + 1)$$

Therefore, in this class of games, the marginal return to player i from a link with player j depends only on the number of links of i and j, and is independent of the number of links of $k \neq i, j$. The (net) payoff function of player i is given by:

$$\Pi_i(g) = \Pi_i(g(s)) = \pi_i(g(s)) - c\eta_i(g(s)), \tag{2}$$

where c is the unit cost of forming a link.

We now provide an example where the payoff function satisfies Equation 2.

Example 1 (Market Sharing, Belleflamme and Bloch, 2005)[4] Consider n ex-ante symmetric firms and associate with each firm i the market i. Prior to competing in these markets, firms can form collaborative links. A link between two firms i and j is a reciprocal market sharing agreement whereby each firm refrains from entering the other firm's market. The Cournot profit earned by a firm i is given by:

$$\pi_i(g) = \Psi_1(\eta_i(g)) + \sum_{j \notin N_i(g)} \Psi_3(\eta_j(g)).$$

[4]This example is taken from Goyal and Joshi (2006, Example 4.2, pg.332).

The first term $\Psi_1(\eta_i(g))$ is the profit earned by firm i on market i when i forms $\eta_i(g)$ links, that is there are $n-\eta_i(g)$ competitors on this market. The second term $\Psi_3(\eta_j(g))$ is the profit earned by firm i on market j when i does not form a link with firm j and j has formed $\eta_j(g)$ links, that is there are $n-\eta_j(g)$ competitors on market j. Note that if a firm i forms a link with a firm j, then she does not compete on market j, so she obtains no profit from this market. Therefore $\Psi_2(x) = 0$ for all $x \in \{1, \ldots, n-1\}$.

The marginal gross payoff to firm i from forming a link with firm j is given by:

$$\pi_i(g + g_{i,j}) - \pi_i(g) = \Psi_1(\eta_i(g) + 1) - \Psi_1(\eta_i(g)) - \Psi_3(\eta_j(g)).$$

Next, we assume that the payoff function satisfies the following two properties.

1. **Convexity (Conv)**: $\Delta\Psi_1(x)$ is strictly increasing.
2. **Strategic Substitutes Property (SSP)**: $\Delta\varphi(x) < 0$.

Goyal and Joshi (2006) also use the notion of *strong monotonicity* given below. For all $x, y \in \{0, \ldots, n-2\}$:

$$\Delta\Psi_1(x+1) + \varphi(y+1) > \Delta\Psi_1(x) + \varphi(y).$$

We now show that there exist conditions such that Example 1 satisfies Conv and SSP.

Example 1 revisited. It can be checked that if Ψ_1 is convex[5] and Ψ_3 is increasing in the number of links of other firms[6], then the aggregate gross payoff function satisfies Conv and SSP.

Next we borrow an example from Billand, Bravard and Sarangi (2010) of a local spillover game that satisfies both Conv and SSP.

[5]This property is satisfied in a linear Cournot oligopoly.

[6]This property is also satisfied in a linear Cournot oligopoly: the profits of firm i on market j decreases with the number of competitors on this market.

Example 2 (Learning from others, Billand, Bravard and Sarangi, 2010) This example specifically focuses on the strategic management notion of "benchmarking" (Camp, 1989). Consider n firms. For simplicity assume that each firm i faces unit price and aims to produce a given quantity $\bar{Q}_i$ at the lowest total production cost. We assume that firms i and j can share information about their business processes if they set a partnership (in our context a link). Moreover, the more partnerships firm j has, the less firm j can spend time with firm i and so the less this partnership is valuable for firm i.

The production cost of each firm i is given by:

$$C_i = \alpha_i + \beta_0\eta_i(g) + \gamma_0\eta_i(g)^2 + \sum_{j\in N_i(g)} \frac{\beta_1}{\eta_j(g)^2},$$

with $\alpha_i > 0$ for all $i \in N$, and $\beta_0, \beta_1 < 0$, $\gamma > 0$. In order to ensure that the cost is positive, we will assume that $\alpha_i + (n-1)\beta_0 + (n-1)\beta_1 > 0$. We can now write the payoff function as:

$$\pi_i(g) = \bar{Q}_i - \left(\alpha_i + \beta_0\eta_i(g) + \gamma_0\eta_i(g)^2 + \sum_{j\in N_i(g)} \frac{\beta_1}{\eta_j(g)^2}. \right).$$

Then the marginal gross profit to firm i from forming a link with firm j is:

$$\pi_i(g + g_{i,j}) - \pi_i(g) = \frac{-\beta_1}{\eta_j(g)^2} - \beta_0 - \gamma_0(2\eta_i(g) + 1),$$

The returns from a link with a firm j depends on the time firm j can spend with firm i. More precisely, the larger the number of links firm j has formed, the smaller is the payoff of forming a link with this firm ($\beta_1 < 0$). Finally, for firm i the returns from a link with a firm j depends on the number of links formed by i. In a context where more players form links, the more they become efficient in making good use of additional links, we have $\gamma_0 < 0$.

To sum up, we have $\Psi_1(\eta_i(g)) = -\beta_0\eta_i(g) - \gamma_0\eta_i(g)^2$ and $\Psi_2(\eta_j(g)) = -\beta_1/\eta_j(g)^2$ with $\gamma_0, \beta_1 < 0$. It can be easily verified that the aggregate gross payoff function satisfies Conv and SSP. Note that unlike Example 1, this example uses only the first two terms of equation 2.

We now state two definitions of network pairwise stability that will be used in the paper.

Definition 1 Pairwise stable network (Jackson and Wolinsky, 1996). *A network g is pairwise stable if (i) for all $g_{i,j} = 1$, $\Pi_i(g) \geq \Pi_i(g - g_{i,j})$ and $\Pi_j(g) \geq \Pi_j(g - g_{i,j})$, (ii) for all $g_{i,j} = 0$, if $\Pi_i(g) < \Pi_i(g + g_{i,j})$, then $\Pi_j(g) > \Pi_j(g + g_{i,j})$.*

Definition 2 Pairwise equilibrium network (Goyal and Joshi, 2006). *A strategy profile $s^* = \{s_1^*, s_2^*, \ldots, s_n^*\}$ is said to be a Nash equilibrium if $\Pi_i(g(s_i^*, s_{-i}^*)) \geq \Pi_i(g(s_i, s_{-i}^*))$, for all $s_i \in S_i$, and all $i \in N$. A network g is a pairwise equilibrium network if: (i) There is a Nash equilibrium strategy profile which supports g, and (ii) for all $g_{i,j} = 0$, if $\Pi_i(g) < \Pi_i(g + g_{i,j})$, then $\Pi_j(g) > \Pi_j(g + g_{i,j})$.*

It is clear that a pairwise equilibrium network is a pairwise stable network.

3 Pairwise stable networks

In the following example we illustrate the process we use in Proposition 1 to show existence of pairwise stable networks under Conv and SSP.

Example 3 Suppose $N = \{1, \ldots, 8\}$ and the payoff function is given by:

$$\pi_i(g) = (\eta_i(g))^2 - \frac{1}{3} \sum_{j \in N_i(g)} \eta_j(g)^3. \tag{3}$$

For simplicity let $c = 0$.

First, we construct the complete network and let players $i \in \{1, \ldots, 8\}$ in turn delete one by one the links they do not wish to preserve. We call $g^{0,1}$ the network obtained at the end of this step. Clearly, in $g^{0,1}$, players $j \in \{1, \ldots, 5\}$ have no links, and players $j' \in \{6, 7, 8\}$ form a complete component. Second, note that $8 = 3 \times 2 + 2$. We label as $g^{0,2}$ the network which contains 2 complete components with 3 players and 2 isolated players. Let players 1 and 2 be the isolated players. Third, we construct the complete network, that we call g^1, with the set of isolated players $\{1, 2\} = N_1$: $g^1 = g_{N_1}$. Then we repeat the process, that is we let players $i \in \{1, 2\}$ in turn delete the links they do not want. Clearly, no player has any incentive to remove links

in g^1. We now construct a network called $\hat{g}^1$ which contains the 2 complete components with 3 players of $g^{0,2}$ and the previous complete component with two players. It is easy to check that this network g is pairwise stable.

We begin with a lemma which is useful for proving Proposition 1.

Lemma 1 *Suppose the payoff function of each player i satisfies (2) and Conv. Suppose a network g on the player set $N' = \{1, \ldots, n'\}$, $n' \leq n$, which contains a complete component C_g, with $x+1$ players, $x+1 \leq n'$. If $\Delta\Psi_1(x) + \varphi(x) < c$, then a player $i \in C_g$ has an incentive to delete all her links.*

Proof Suppose player i belongs to a complete component, C_g with $x+1$ players in g and $\Delta\Psi_1(x) + \varphi(x) < c$. We must show that player i has an incentive to delete all her links, that is $\Delta\Psi_1(k) - \varphi(x) < c$ for all $k \in \{1, \ldots, x\}$. By convexity, we have: $\Delta\Psi_1(1) < \Delta\Psi_1(2) < \ldots < \Delta\Psi_1(x)$ for all $x > 2$. It follows that $\Delta\Psi_1(1) - \varphi(x) < \Delta\Psi_1(2) - \varphi(x) < \ldots < \Delta\Psi_1(x) - \varphi(x) < c$, for all $x > 2$. The result follows. □

Proposition 1 *Suppose the payoff function of each player i satisfies (2), Conv and SSP. A pairwise stable network always exists.*

Proof In order to prove the proposition, we build an iterative process. To simplify the presentation of this process we set $N = N_0$. We start with the complete network $g^0 = g_{N_0}$ and we build a process in three steps.

Step 1. We order players in a prespecified manner: $i = 1, 2, \ldots, |N_0|$. In this step if a player, say i, has an option to revise her strategy, then we let her delete one by one the links she does not wish to preserve. Consequently, either no player has an incentive to delete links in g^0, or there is a player $i \in N$ who has an incentive to delete a link in g^0. In the former case, the complete network g^0 is a pairwise stable network, and the process stops. In the latter case, we give player 1 the option to revise her strategy. We know that players have the same payoff function and

they are in a symmetric position in g^0. It follows that player 1 has an incentive to delete links in g^0. By Lemma 1 player 1 has an incentive to delete all her links in g^0. Then we let successively players $i \in \{2, 3 \ldots, |N_0| - 1\}$ revise their strategy. Again, if player i has an incentive to delete a link, then she has an incentive to delete all her links by Lemma 1. At the end of this step, we obtain a network that we call $g^{0,1}$, where $x \in \{0, \ldots, |N_0|\}$ players have formed no links and $|N_0| - x$ players have formed a complete component. We set $x_0 = x$. •

If $x_0 \neq 0$, then x_0 players, $x_0 \in \{1, \ldots |N_0| - 2\}$, have no links in $g^{0,1}$: they are isolated players, and $|N_0| - x_0$ players have formed a complete component. Recall that if $x_0 = 0$, then the complete network is pairwise stable.

Step 2. In this step, we divide $|N_0|$ by $(|N_0| - x_0)$. Let q_0 be the quotient and r_0 be the remainder. We now construct the network $g^{0,2}$. If $x_0 = |N_0| - 1$, then $g^{0,2}$ is the empty network. Otherwise, $g^{0,2}$ contains $q_0 \geq 1$ complete components with $|N_0| - x_0$ players and r_0 isolated players, with $r_0 < |N_0| - x_0$. In that case, we let the r_0 first players of N_0 be the isolated players. Let N_1 be the set of these players: $N_1 = \{1, \ldots, r_0\}$. We denote by N_0' the set of players who are not isolated: $N_0' = \{r_0 + 1, \ldots, |N_0|\}$. Note that if $x_0 = |N_0| - 1$, then the remainder r_0 is null. •

Step 3. We now construct the network $g^1 = g_{N_1}$, that is the complete network constructed from the set of vertices in N_1. •

Then we repeat Steps 1, 2, and 3 with networks $g^1 = g_{N_1}$, $g^2 = g_{N_2}$, $g^3 = g_{N_3}$, ... till we obtain a network g^T such that $r_T = 0$.

First we show that network g^T exists. To introduce a contradiction suppose that g^T does not exist. Then for all $t \in \mathbb{N}$, we have $r_t \neq 0$. Moreover we have $r_t < r_{t-1} - x_t < r_{t-1}$. The last inequality follows from $x_t \neq 0$ (otherwise r_t is null) and the first inequality comes from the fact that the remainder is always strictly smaller than the divisor. Since $r_t < r_{t-1}$ and r_0 is a finite number, it is not possible to obtain the property $r_t \neq 0$ for all $t \in \mathbb{N}$. Hence g^T exists.

We now construct the network $\hat{g}^T$ as follows: (1) if two players i and j are such that $i \in N_t'$ and $j \in N_{t'}'$, with $t, t' \in \{0, \ldots, T\}$ and $t \neq t'$, then there is no link between i and j in $\hat{g}^T$, (2) if

players i and j belong to N'_t, $t \in \{0, \ldots, T\}$, then there is a link between i and j in $\hat{g}^T$ if and only if there is a link between i and j in $g^{t,2}$. Note that by construction, if player $i \in N'_t$, $t \leq T$, then i belongs to a complete component with $r_{t-1} - x_t$ players in $\hat{g}^T$.

Finally, we show that $\hat{g}^T$ is a pairwise stable network.

We first check that no player has any incentive to delete a link in $\hat{g}^T$. More precisely, we show that each player $i \in N'_t$, $t \in \{1, \ldots, T\}$, has no incentive to delete a link in $\hat{g}^T$. We know that at period t player $x_t + 1$ did not remove her first link with a player $j \in \{x_t + 2, \ldots, r_{t-1}\}$ when she was given the opportunity to do so. This implies that $\Delta\Psi_1(r_{t-1} - x_t - 1) + \varphi(r_{t-1} - x_t - 1) > c$. By construction of $\hat{g}^T$ player $x_t + 1$ belongs to the same complete component in $\hat{g}^T$ as she did at the time she had the opportunity to delete a link. Therefore, the marginal payoff of player x_{t+1} associated with the deletion of a link in $\hat{g}^T$ is $c - (\Delta\Psi_1(r_{t-1} - x_t - 1) + \varphi(r_{t-1} - x_t - 1)) < 0$. It follows that player x_{t+1} has no incentive to delete a link in $\hat{g}^T$. Since players $i \in N'_t \setminus \{x_t + 1\}$ and player $x_t + 1$ are in a symmetric position in $\hat{g}^T$, it follows that these other players also have no incentive to delete a link in $\hat{g}^T$. Thus no player $i \in N$ has an incentive to delete a link in $\hat{g}^T$.

We now show that pairs of players who are not linked in $\hat{g}^T$ have no incentive to form a link. The proof is in two parts.

First we show that this is true for players, $i \in N'_t$ and $j \in N'_t$, who are not linked in $\hat{g}^T$. By construction of $\hat{g}^T$, we know that this network contains a complete component with the players $\{x_t + 1, \ldots, r_{t-1}\}$. Moreover, we know that during the process player x_t removed a (first) link with a player who belongs to this component. This implies that we have $c > \Delta\Psi_1(r_{t-1} - x_t) + \varphi(r_{t-1} - x_t)$. It follows that in $\hat{g}^T$, no player $i \in N$, who belongs to a complete component with $r_{t-1} - x_t$ players, has an incentive to form an additional link with a player j who belongs to another complete component with the same number of players.

Second, by convexity we have $\Delta\Psi_1(r_{t-1} - x_t) + \varphi(r_{t-1} - x_t) > \Delta\Psi_1(k) + \varphi(r_{t-1} - x_t)$, for all $k < r_{t-1} - x_t$. Since $c > \Delta\Psi_1(r_{t-1} - x_t) + \varphi(r_{t-1} - x_t)$, it follows that for each $t \in \{1, \ldots, T\}$, no player who belongs to a complete component with k players, $k < r_{t-1} - x_t$, has an incentive to form an additional link with a player who belongs to a complete component with $r_{t-1} - x_t$

players.

To conclude, since no player in $\hat{g}^T$ has any incentive to accept a link from a player who has formed more links or as many links as she herself has formed, no additional links will be formed between a pair of players in this network if we let players revise her strategy. Since the two conditions of pairwise stability are satisfied by $\hat{g}^T$, this network is pairwise stable. □

Clearly, if the profit function given in Example 1 is such that Ψ_1 is convex and Ψ_3 is increasing, then there always exists a pairwise stable network in the market sharing game. Similarly, there always exists a pairwise stable network in Example 2.

4 Pairwise equilibrium networks

We now study the existence of pairwise equilibrium networks. Recall that the main difference between pairwise stable networks and pairwise equilibrium networks is that the latter gives players the possibility of deleting more than one link at a time.

Note that Goyal and Joshi (2006) do not find general results for pairwise equilibrium networks when the payoff function satisfies Conv and SSP. Hence the authors introduce an additional assumption: *strong monotonicity* of the payoff function. This assumption allows them to characterize pairwise equilibrium networks. However even with this assumption, the authors argue that it is not possible to obtain existence results. Indeed this is not a trivial problem. In fact as shown in the following example, a pairwise equilibrium network may fail to exist when the payoff function satisfies Conv, SSP, and strong monotonicity. It follows that a pairwise equilibrium network does not always exist when the payoff function satisfies Conv and SSP. Recall that $\Delta\varphi(x) < 0$ by SSP.

Example 4 Let $N = \{1, 2, 3, 4\}$ and assume the following parameters for the payoff function.

$$\begin{array}{lll} \Psi_1(1) = 10 & \Psi_1(2) = 21 & \Psi_1(3) = 33 \\ \Psi_2(1) = -9.9 & \Psi_2(2) = -10.7 & \Psi_2(3) = -11.1 \end{array}$$

Moreover suppose $\Psi_1(0) = \Psi_3(x) = 0$ for $x \in \{0, 1, 2\}$. Clearly, the payoff function satisfies Conv, SSP and strong monotonicity. It is easy to check that given these parameters no network can be a pairwise equilibrium network.

This non-existence result motivates the introduction of a stronger condition on gross payoffs which ensures that there always exists a pairwise equilibrium network. This condition can be stated as

$$\textbf{Condition 1:} \quad (\Delta\Psi_1(x) - c + \varphi(x)) \notin [0, -(x-1)\Delta\varphi(x)],$$

for all $x \in \{2, \ldots, n-2\}$. Recall that the marginal payoff that each player i obtains from forming an additional link with a player j, when both players have formed $x-1$ links, is $\Delta\Psi_1(x)+\varphi(x)-c$. Moreover, $-(x-1)\Delta\varphi(x)$ captures the difference in the payoff due to the spillover effect of an isolated player (i) when she forms $x-1$ links with players who have formed $x-1$ links, and (ii) when she forms $x-1$ links with players who have formed x links.

Note that Condition 1 can also be rewritten as follows: $\Delta\Psi_1(x) - \Delta\Psi_1(x-1) \notin [c - \varphi(x) - \Delta\Psi_1(x-1), c - \varphi(x) - (x-1)\Delta\varphi(x) - \Delta\Psi_1(x-1)]$. Hence, the higher SSP ($\Delta\varphi(x)$) is, the higher the convexity ($\Delta\Psi_1(x) - \Delta\Psi_1(x-1)$) must be in order to have $\Delta\Psi_1(x) - \Delta\Psi_1(x-1) > c-\varphi(x)-(x-1)\Delta\varphi(x)-\Delta\Psi_1(x-1)$. To sum up for Condition 1 to be satisfied, either convexity has to be low enough or convexity has to be high enough relative to SSP.

We now use the market sharing game to provide some economic intuition for Condition 1. Recall that in the market sharing game, $\Psi_2(x) = 0$, for all $x \in \{0, \ldots, n-1\}$. To simplify the interpretation we assume that $c = 0$. It follows that Condition 1 can be rewritten as:

$$\textbf{Condition 1':} (\Delta\Psi_1(x) - \Psi_3(x-1)) \notin [0, (x-1)(\Psi_3(x-1) - \Psi_3(x-2))].$$

Clearly, $\Delta\Psi_1(x) - \Psi_3(x-1)$ is the marginal payoff that firm i obtains from forming a market sharing agreement with a competitor j which has the same number of competitors on its market as i. Consequently, Condition 1 means that for $x \geq 2$,

1. either this marginal payoff is negative, that is the payoff that firm i obtains on a market with $n-(x-1)$ competitors is higher than the additional payoff it obtains on its own

market when the number of competitors goes from $n-(x-1)$ to $n-x$;

2. or this marginal payoff is higher than $(x-1)(\Psi_3(x-1)-\Psi_3(x-2))$, that is the marginal payoff of firm i is greater than the difference between (i) the payoff i obtains if it enters in $x-1$ markets with $n-(x-1)$ competitors and (ii) the payoff i obtains if it enters in $x-1$ markets with $n-x$ competitors.

Proposition 2 *Suppose the payoff function of each player satisfies equation (2), Conv and SSP. Moreover, suppose Condition 1 is satisfied. Then a pairwise equilibrium network always exists.*

Proof In this proof, we use the same process and the same notation as in the proof of Proposition 1. More precisely, we use Step 1, Step 2 and Step 3 given in Proposition 1 and we repeat these steps with networks $g^1 = g_{N_1}$, $g^2 = g_{N_2}$, $g^3 = g_{N_3}$, ... till we obtain a network g^T such that $r_T = 0$. By the same arguments used in the proof of Proposition 1 we can show that network g^T exists in the process. Then we construct the network $\hat{g}^T$ as in the proof of Proposition 1.
We now show that $\hat{g}^T$ is a pairwise equilibrium network.
First by the same reasoning as in the proof of Proposition 1, it is easy to show that no player has any incentive to delete links in $\hat{g}^T$. We now show that no pair of players, $i \in N'_t$ and $j \in N'_t$, who are not linked in $\hat{g}^T$ have an incentive to form a link. By construction of the process a player who has formed no link in $\hat{g}^T$ has no incentive to form a link with another player in $\hat{g}^T$. Moreover, by construction of $\hat{g}^T$ we know that this network contains a complete component with the players $\{x_t+1,\ldots,r_{t-1}\}$. Moreover, we know that during the process player x_t removed her links with players $i \in \{x_t+1,\ldots,r_{t-1}\}$, while player x_t+1 maintained her links with players $i \in \{x_t+2,\ldots,r_{t-1}\}$. It follows that we have both:

$$\Psi_1(r_{t-1}-x_t)+(r_{t-1}-x_t)\varphi(r_{t-1}-x_t) \;<\; (r_{t-1}-x_t)c+\Psi_1(0),$$

and

$$\Psi_1(r_{t-1}-x_t-1)+(r_{t-1}-x_t-1)\varphi(r_{t-1}-x_t-1) \;>\; (r_{t-1}-x_t-1)c+\Psi_1(0).$$

Clearly, the two inequalities above are simultaneously satisfied only if $\Delta\Psi_1\ (r_{t-1} - x_t) < c - \varphi(r_{t-1}-x_t)-(r_{t-1}-x_t-1)\Delta\varphi(r_{t-1}-x_t)$. By Condition 1 we have $\Delta\Psi_1(r_{t-1}-x_t) < c-\varphi(r_{t-1}-x_t)$, for all $r_{t-1} - x_t \in \{2, \ldots, n-2\}$. Consequently, suppose that player x_t belongs to a component with $r_{t-1} - x_t$ players in $\hat{g}^T$. In that case, since $\Delta\Psi_1(r_{t-1} - x_t) < c - \varphi(r_{t-1} - x_t)$, player x_t has no incentive to form a link with a player $i \in \{x_t + 1, \ldots, r_{t-1}\}$ in $\hat{g}^T$. It follows that in $\hat{g}^T$ no pair of players who belong to two distinct components with the same number of players, have an incentive to form a link. Using convexity, it is straightforward that for each $t \leq T$ no player who belongs to a complete component with k players, $k < r_{t-1} - x_t$, has an incentive to form an additional link with a player who belongs to a complete component with $r_{t-1} - x_t$ players. It follows that in $\hat{g}^T$ no additional links will be formed if we let players revise their strategy. Since the two conditions for pairwise equilibrium are satisfied by $\hat{g}^T$, this network is a pairwise equilibrium network. □

We now use the above result in a market sharing game example. In this example, (i) each firm obtains a higher profit on its own market than on the market of another firm when the number of competitors is the same, and (ii) the profit obtained by a firm i on each market k is increasing and convex with the number of agreements made by firm k. It follows that the gross profit function in this example satisfies Conv and SSP.

Example 1 revisited. Suppose $N = \{1, \ldots, 7\}$ and $\varepsilon \in]0, 1/10[$. To simplify, let the cost of setting links be null. Let Ψ_1 and Ψ_3 in the profit function given in Example 1 be such that

	$\Psi_1(x)$	$\Psi_3(x)$
$x = 0$	1	$2/5$
$x = 1$	$3/2$	$3/5$
$x = 2$	$5/2 + \varepsilon$	$3/2$
$x = 3$	$7/2 + 3\varepsilon$	$5/2 + 3\varepsilon$,

and for $x \in \{3, \ldots, 6\}$, $\Psi_1(x+1) = 2\Psi_1(x) - \Psi_1(x-1) + \varepsilon$, and for $x \in \{3, 4, 5\}$, $\Psi_3(x+1) = 2\Psi_3(x) - \Psi_3(x-1) + \varepsilon$. Clearly, we have for all $x > 2$, $\Delta\Psi_1(x) - \Psi_3(x-1) < 0$ and for $x = 2$,

$\Delta\Psi_1(x) - \Psi_3(x-1) > \Psi_3(x-1) - \Psi_3(x-2)$. Consequently, there exists a pairwise equilibrium network. Moreover, by using the process given in the proof of Proposition 2, we can check that the network with two complete components with three players and an isolated player is a pairwise equilibrium network.

Example 2 revisited. If c, γ_0, β_0 and β_1 in this example are such that $c > -\beta_0 - \gamma_0(1-2x) - \beta_1/x^2$ or $c < -\beta_0 - \gamma_0(1-2x) + \beta_1/x(x-1)$, for $x \in \{2, ..., n-1\}$, then there always exists a pairwise equilibrium network.

5 Conclusion

In this paper, we show that there always exists a pairwise stable network in a game with local spillovers where the payoff function satisfies convexity and the strategic substitutes property. Through an example we show that pairwise equilibrium networks however do not always exist. Finally, we establish that under a stronger condition on the payoff function provided in the paper, a pairwise equilibrium network always exists. Thus, in this paper, we show that there are at least two ways in which a local spillovers game that satisfies both Conv and SSP is consistent with a steady state solution. Additionally, our existence proof also allows us to characterize the set of stable network allowing us to obtain both existence and characterization simultaneously.

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