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1 Introduction

If we consider the fluctuations of a stock price S, we know that when the market is complete, i.e., when all contingent claims are attainable, there exists a unique martingale measure $\mathbb{Q}$ permitting to price, and then such claim can be expressed as a stochastic integral of S, Harrison and Kreps (1979). This result underlies the insights behind the option pricing theory started with the seminal papers by Black and Scholes (1973) and Merton (1973). When the market is incomplete, the martingale measure $\mathbb{Q}$ is not unique, and a general claim is not necessarily a stochastic integral of S. On another hand, an admissible trading strategy in general must incur some intrinsic risk of the market, and this situation arises as soon as we work with discrete time models caracterized by an infinity of states of the world, and in that latter case as well there is not an unique price obtained by no-arbitrage arguments only. In other words, we face the well known problem of selecting a proper equivalent martingale measure for derivatives pricing. We address this last problem inside this short note considering a discrete time modelling including jumps under the historical measure and two alternatives strategies for obtaining a risk neutral equivalent martingale measure.

Given a financial asset whose price is represented by a stochastic process $S = (S_t)_{t\in[0,T]}$, we focus in this note on the derivatives written on the underlying asset S. We consider a filtered complete probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ associated with this stock price S_t, and we define the related discounted price $\tilde{S}_t$ as $\tilde{S}_t = e^{-rt}S_t$ where r is the risk free rate. To price this asset at time T (which corresponds to the maturity), we need to define contracts for

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the future and to choose market models. Let be introduced $(B_t)_{t\in[0,1,...T]}$ the bond price process such that $B_t = B_{t-1}e^r$. We restrict to European options which are only exercised at a fixed expiry date T, then for these call options on a stock $(S_t)_t$ with strike K at time T, the payoff equals: $(S_T - K)$, if $S_T > K$ and 0, otherwise. So, the problem of option pricing is to determine what value to assign to the option at a given date, knowing that the buyer of a European call option on a stock with price process $(S_t)_t$ will have the opportunity of receiving a payoff at time T equals to $C(T) = \max(S_T - K, 0)$, since she will exercice the option, if and only if, the final price of the stock S_T is greater than the previously agreed strike price K. The value of S_T is not known in advance, nor the time evolution of the risky asset $(S_t)_t$. To solve these previous problems, we need to specify the model under the historical measure $\mathbb{P}$, the choice of a Stochastic Discount Factor or Pricing Kernel which permits to obtain a Martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, and the existence (or not) of a close form for valuation prices under $\mathbb{Q}$.

In most markets the returns tend to leptokurticity due to presence of jumps, and the implied volatility surface displays smirk effects. Thus it appears important to incorporate these affects inside the modelling of the returns. In a first step under the historical measure $\mathbb{P}$ we introduce a model which permits to take into account both existence of volatility clustering and jumps. In a second step we discuss two different ways to obtain a martingale measure $\mathbb{Q}$ under which the valuation prices can be obtained, that is the exponential affine stochastic discount factor (Esscher and Siu, 1994) and the Minimal Entropy Martingale Measure (or extended Girsanov principle) (Elliott and Madam, 1998). The equivalence between these two approaches has been proved inside a Gaussian framework (Badescu and Kulberger, 2007), but no such equivalence is available outside this framework so the question of their impact on valuation is interesting. Finally we provide for the different models introduced under the historical measure $\mathbb{P}$, the close form for valuation purpose considering the two alternative methodologies. As soon as this close form exists, using a recursive procedure based on the computation of the characteristic function of the distribution of the underline at maturity, we get the payoff corresponding to the option. This last step appears as a semi-analytical two-steps pricing: in a first step the characteristic function of the underline at maturity is determined iteratively, and in a second step, it is inverted numerically in order to compute option prices.

In order to verify the impact of the modelling's choice under $\mathbb{P}$ in valuation terms, two models are considered under $\mathbb{P}$ including jumps effects using jump Lévy processes. First, we introduce a structural jump model: a particular model being the Normal Inverse Gaussian model. Second, we consider the compound Poisson model proposed by Merton (1976). These processes are driven by a finite number of jumps within a finite time interval, and have been referred to a finite activity jump process.

As soon as the modelling has been defined under $\mathbb{P}$, we address the question of the choice of the pricing kernel to construct an equivalent probability measure $\mathbb{Q}$. We focus on two representations for the pricing kernel: the Esscher transform (ESS) and the Minimal Entropy Martingale Measure (MEMM). For the retained both approaches, we show that we have stability for the conditional distribution function, remaining under $\mathbb{Q}$ inside the same class of distributions choosen under $\mathbb{P}$, and we exhibit the change observed for the parameters of the distribution under $\mathbb{Q}$. We get new results corresponding to the expression under $\mathbb{Q}$ of the valuation form for pricing when we use GARCH-type process with Lévy innovations both using ESS and MEMM approaches. In the former case, our result contains the result obtained by Chorro et al (2010) as a particular case. We also provide the expression of the valuation form under $\mathbb{Q}$ for a GARCH-type model with Poisson jumps using the Esscher transform. In the case of the MEMM approach, the result has already been established by Fujiwara and Miyahara (2003). Finally, we study the impact of the choice of these methodologies to get the valuation form, considering the CAC 40 index under the period March 1990 to March 2010, in restricting to GARCH modellings with Lévy innovations for the both pricing kernels. Other modellings are provided in Guégan and Lalaharison (2010). We show that the choice of the pricing kernel is determinant and that the model chosen under the historical measure is not so important.

This note is organized as follows. In section two, we introduce the two modellings that we fit under $\mathbb{P}$. In Section three we develop the valuation form for the risk neutral work considering two different strategies. For each strategy and each model we specify new results and provide the exact form of the characteristic functions which would be used for applications. Section four is devoted to the results and some comments.

2 The Modellings under the historical measure $\mathbb{P}$

Since now a long period, the importance of asymmetry and heavy tails inside financial time series have been documented in the literature, especially for pricing issues, Bouchaud and Potters (2003) or Embrechts *et al.* (2005). The importance of the models under the historical measure is fundamental for pricing theory for several reasons we recall briefly. The combination of leverage parameters and stochastic volatility or conditional heteroskedasticity captures the stylized fact that volatility increases relatively more when the stock price drops. This fact increases the probability of a large loss and consequently the value of out-of-the-money put options. While dynamic volatility models are intuitively and theoretically appealing, existing discrete-time models may not be sufficiently flexible to explain observed option biased, even with leverage parameters included. This is particularly the case for options with short maturities. Thus it appears interesting to generate shewness in the returns distribution by modelling the conditional distribution of their innovations with an asymmetric distribution. Now, the existence of shocks having a great impact on the tails of the distribution considering a distribution with excess kurtosis bigger than 3 would increase the robustness of the options pricing. Thus recently, some authors introduce modellings with innovations whose distributions can modelled these features, Siu et al. (2004), Christoffersen et al. (2006) and Chorro et al. (2010a, b). Nevertheless, jumps are also present in the prices in period of turmoil, and the previous modelings cannot model this kind of features. Then, generalizing standard stochastic volatility models by allowing for jumps and fat-tailed negative movements in short term stock returns would be particularly useful for explaining the biased in short term options, and a model which reinforces the effects of jumps in returns seems appropriate. Even if the use of generalized hyperbolic laws appear as a flexible class of distributions for describing asset returns in finance, an alternative approach is to introduce Lévy densities rather than probability densities inside the modelling. This approach has already been developed considering pure Lévy processes, and assuming that the log-returns of the stock price S_t is such that $S_t = S_0 e^{z_t}$, where z_t be a Lévy process. Madan and Seneta (1990) propose a VG Lévy process, Schoutens (2001) develops the Meixner model, and Carr et al. (2002) introduce the CGMY model, among others.

In order to model the existence of volatility in presence of jumps under the historical measure $\mathbb{P}$, in this note we introduce two new models (for pricing purpose):

(1) - First we assume that the jumps affect mainly the volatility of the returns $(Y_t)_t$:

$$Y_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \widetilde{m}_t + X_t, \quad S_0 = s, \tag{1}$$

where $\widetilde{m}_t = r + m_t$, $X_t = \sqrt{h_t} z_t$, h_t is a related GARCH model, and z_t a Lévy process with triplet (c, σ^2, U), (Schoutens, 2003), $c \in \mathbb{R}$, $\sigma^2 \geq 0$ and U is the positive Lévy measure defined on $\mathbb{R} \setminus \{0\}$ satifying

$$\int_{\mathbb{R}} \inf\{1, z^2\} U(dz) = \int_{\mathbb{R}} (1 \wedge z^2) U(dz) < \infty. \tag{2}$$

(2) - In the second model the jumps affect the level of the returns $(Y_t)_t$:

$$Y_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \widetilde{m}_t + \sqrt{h_t} z_t + \sum_{j=1}^{N_t} V_j, \quad S_0 = s, \tag{3}$$

where z_t is a standard normal distribution, N_t is a Poisson process with intensity λ, and $V_1, V_2, \ldots$ is a sequence of identically independent distributed random variables $N(0, \sigma^2)$, independent of the Poisson Process N_t.

(3) - In both models,

$$\begin{cases} m_t = \lambda_0 \sqrt{h_t} - \frac{1}{2} h_t; \quad \lambda_0 \in \mathbb{R}, \\ h_t = G(h_{t-1}, Y_{t-1}) \ \text{ is a related GARCH model.} \end{cases} \tag{4}$$

λ_0 is the constant unit risk premium, and the process $(h_t)_t$ will be a GARCH, EGARCH, GJR or A-P-ARCH model (Bollerslev (1989), Nelson (1990), Glosten et al (1993), Ding et al (1993)).

3 The risk neutral world

As soon as the model is entirely specified under $\mathbb{P}$, we develop a valuation form under $\mathbb{Q}$ to price contingent claims. Considering the preceding economy with time horizon T with two assets, namely a risk-free bond $(B_t)_{t \leq T}$ and a risky stock $(S_t)_{t \leq T}$, under an equivalent martingale measure $\mathbb{Q}$, the price of any asset equals the expected present value of its future payoffs: thus, the price S_t at time t of an European asset paying Φ_T at T (Φ_T being $\mathcal{F}_T-$ measurable) is given by

$$S_t = E_{\mathbb{Q}}[\Phi_T e^{-r(T-t)} \mid \mathcal{F}_t] = E_{\mathbb{P}}[\Phi_T M_{t,T} \mid \mathcal{F}_t]. \tag{5}$$

The $\mathcal{F}_{t+1}-$ measurable random variable $M_{t,t+1}$ is the so-called stochastic discount factor (the quantity $M_{t,t+1}e^r$ is also known as the pricing kernel). Following Rubinstein (1976) or Cochrane (2001) we suppose that equity market returns are the only variables to deal with pricing purposes or equivalently that we may project the original stochastic discount factor onto the sigma-algebra generated by the payoffs of the risky asset.

Once the dynamics under the historical probability have been specified throughout statistical modelings, we may overcome this problem of multiplicity of stochastic discount factors due to the non-uniqueness of the Martingale measure $\mathbb{Q}$ adopting one of the two following equivalent points of view: we may impose some constraints on the form of the stochastic discount factor or choose a particular martingale measure that fulfills some economic or risk criteria (e.g the minimal martingale measure in the sense of Fôllmer and Schweiser (1991) that minimizes the variance of the hedging loss). When this choice has been made, if we know the dynamics of the risky asset under the new probability, then it is possible to price contingent claims using Monte Carlo simulations. In order to solve these two points, we first follow the work of Gerber and Shiu (1994b) choosing for the stochastic discount factor an exponential affine parametrisation and we prove that modelling the returns using (1) and (3) under $\mathbb{P}$, we obtain again under $\mathbb{Q}$ dynamics belonging to the same class of models with explicit parameters allowing for Monte Carlo simulations, and in a second step we apply the methodology of Elliott and Madan (1998) based on the existence of a Minimal Entropy Martingale Measure. In that latter case, assuming that $(Y_t)_t$ is modelled again by (1) or (3) under $\mathbb{P}$, we provide the explicit form for the valuation under the risk neutral measure with explicit parameters, permitting again Monte Carlo simulations. The two valuation forms differ by the shift observed inside the parameters of the both conditional distributions associated respectively to the models (1) and (3)

3.1 Pricing options with exponential affine stochastic discount factors

In this Section we consider that the stochastic discount factor M_t is characterized by the Esscher transform : $\forall t \in \{0, \dots, T-1\}$

$$M_{t,t+1} = \exp(\theta_{t+1} Y_{t+1} + \xi_{t+1}) \tag{6}$$

where Y_{t+1} is introduced in (1) or in (3), and where θ_{t+1} and ξ_{t+1} are $\mathcal{F}_t$-measurable random variables.

(1) - Valuation form for the returns modelled by (1). The simplest way to get this result is obtained throught the characteristic function of the process $(Y_t)_t$ which gives explicitely the conditional distributions of the log returns under $\mathbb{P}$ and $\mathbb{Q}$, and allows Monte Carlo simulation methods. These characteristic functions are respectively:

$$\begin{aligned} \log \phi_{t-1}^{\mathbb{P}}(u) &= iu\widetilde{m}_t + iuc\sqrt{h_t} - \frac{u^2}{2}\sigma^2 h_t + \\ &\int_{\mathbb{R}} (e^{iuz\sqrt{h_t}} - 1 - iuz\sqrt{h_t} 1_{\{|z|\le 1\}})\ U(dz) \end{aligned}$$

and

$$\begin{aligned} \log \phi_{t-1}^{\mathbb{Q}}(u) &= iu\widetilde{m}_t + iuc\sqrt{h_t} - \frac{u^2}{2}\sigma^2 h_t \theta_t + \\ &\int_{\mathbb{R}} (e^{iuz\sqrt{h_t}} - 1 - iuz\sqrt{h_t} e^{-\theta_t z\sqrt{h_t}} 1_{\{|z|\le 1\}}) e^{\theta_t z\sqrt{h_t}}\ U(dz), \end{aligned}$$

where $\widetilde{m}_t, c, \theta_t$ have been previously defined.

Proof 3.1

$$\begin{aligned} \phi_{t-1}^{\mathbb{P}}(u) &= E_{\mathbb{P}}\left[\exp(iu\widetilde{m}_t)\exp(\sqrt{h_t}iuz_t)|\mathcal{F}_{t-1}\right] \\ &= \exp(iu\widetilde{m}_t)E_{\mathbb{P}}\left[\exp(iu\sqrt{h_t}z_1)\right] \quad \textit{since } \widetilde{m}_t \textit{ is } \mathcal{F}_{t-1}\textit{-measurable} \\ &\quad \textit{and } z_t \textit{ is i.i.d.} \\ &= \exp(iu\widetilde{m}_t)\exp[\kappa(iu\sqrt{h_t})] \quad \textit{by the Lévy-Kintchine formula,} \end{aligned}$$

and the characteristic function under $\mathbb{Q}$ is derived from

$$\phi_{t-1}^{\mathbb{Q}}(u) = \frac{\phi_{t-1}^{\mathbb{P}}(\theta_t + u)}{\phi_{t-1}^{\mathbb{P}}(\theta_t)}.$$

The expression of the process under the risk neutral measure is now available:

Proposition 3.2 *If the returns $(Y_t)_t$ are modelled using (1) under $\mathbb{P}$, then under the risk neutral measure $\mathbb{Q}$, the process $(Y_t)_t$ follows the same process where X_t is a Lévy process with triplet $(c^{\mathbb{Q}}, \sigma^2 h_t, \nu^{\mathbb{Q}})$ given by:*

$$c^{\mathbb{Q}} = c\sqrt{h_t} + \sigma^2\theta_t h_t - \int_{\{|z|\leq 1\}} z\sqrt{h_t}\ U(dz) + \int_{\{|z|\leq \frac{1}{\sqrt{h_t}}\}} z\sqrt{h_t} e^{\theta_t\sqrt{h_t}z}\ U(dz)$$

and

$$\nu^{\mathbb{Q}}(dx) = e^{\theta_t x} U\left(\frac{dx}{\sqrt{h_t}}\right).$$

The proof of the proposition is provided in Guégan and Lalaharison (2010). We observe that the conditional distribution of $(Y_t)_t$ under $\mathbb{Q}$ belongs to the same family of distributions as under $\mathbb{P}$. Nevertheless, there are changes in the mean and in the variance.

(2) - Valuation form for the returns defined by (3). For this model the characteristic functions under $\mathbb{P}$ and $\mathbb{Q}$ are:

$$\phi_{t-1}^{\mathbb{P}}(u) = \exp\left[iu\widetilde{m}_t - \frac{1}{2}h_t u^2 + \lambda(e^{-\frac{1}{2}\sigma^2 u^2} - 1)\right] \quad \text{and}$$

$$\phi_{t-1}^{\mathbb{Q}}(u) = \exp\left[iu\widetilde{m}_t + h_t\left(\theta_t iu - \frac{1}{2}u^2\right) + \lambda e^{\frac{1}{2}\sigma^2\theta_t^2}\left(e^{\sigma^2(\theta_t iu - \frac{1}{2}u^2)} - 1\right)\right].$$

Proof 3.3

$$\begin{aligned}
\phi_{t-1}^{\mathbb{P}}(u) &= E_{\mathbb{P}}\left[\exp\left(iu\widetilde{m}_t + iuX_t + iu\sum_{j=1}^{N_t} V_j\right)|\mathcal{F}_{t-1}\right] \\
&= \exp\left[iu\widetilde{m}_t - \frac{1}{2}u^2 h_t + \lambda\left(e^{-\frac{1}{2}\sigma^2 u^2} - 1\right)\right],
\end{aligned}$$

and the characteristic function under $\mathbb{Q}$ is also derived from

$$\phi_{t-1}^{\mathbb{Q}}(u) = \frac{\phi_{t-1}^{\mathbb{P}}(\theta_t + u)}{\phi_{t-1}^{\mathbb{P}}(\theta_t)}.$$

Therefore, the risk neutral dynamics for the returns $(Y_t)_t$ belongs to the same family as under the historical measure $\mathbb{P}$, with $z_t \overset{\mathbb{Q}}{\sim} N(\sqrt{h_t}\theta_t, 1)$, $N_t \overset{\mathbb{Q}}{\sim} \mathcal{P}\left(\lambda e^{\frac{1}{2}\sigma^2\theta_t^2}\right)$, $V_j \overset{\mathbb{Q}}{\sim} N(\sigma^2\theta_t, \sigma^2)$. The characteristic triplets are $(\widetilde{m}_t + h_t\theta_t, h_t, \nu^{\mathbb{Q}})$ where $\nu^{\mathbb{Q}}(x) = \lambda e^{\theta_t x} f(x)dx$. Consequently, under $\mathbb{Q}$, the returns distribution are obtained shifting the conditional mean and the conditional variance.

3.2 Pricing options with Minimal Entropy Martingale Measure

In this Section the choice of the stochastic discount factor corresponds to the Extended Girsanov Principle introduced by Elliot and Madan (1998). We use it to get a valuation form under $\mathbb{Q}$ assuming that the returns are modelled by (1) and (3) under $\mathbb{P}$. It is based on a multiplicative Doob decomposition of the discounted stock price into a predictable finite variation component and a Martingale one. Under mild conditions Elliot and Madan (1998) build a risk neutral probability under which the conditional distribution of the discounted stock price is equal to the conditional distribution of its martingale component under the historical probability with changes inside the parameters making just a drift correction for discounted asset prices. We adapt their approch for the modellings (1) and (3).

(1) - Valuation form for returns modelled with (1), the dynamics of the prices being:

$$S_t = S_{t-1} e^{\widetilde{m}_t} \exp(X_t) \tag{7}$$

where $S_{t-1}e^{\widetilde{m}_t}$ is $\mathcal{F}_{t-1}-$measurable, and X_t is introduced in (1). Following Fujiwara and Miyahara's result (2003), we define a new probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ based on the following technical condition on the driving Lévy process (z_t).
(C): There exist constants c and $\beta \in \mathbb{R}$ that satisfies the following conditions (i) and (ii):
(i)

$$\int_{\{z>1\}} e^z e^{\beta(e^z-1)} U(dz) < \infty;$$

(ii)

$$\widetilde{m}_t + c' + \left(\frac{1}{2} + \beta\right)\sigma^2 h_t + \int_{\{|x|\le 1\}} \left[(e^x - 1)e^{\beta(e^x-1)} - x\right] U\left(\frac{dx}{\sqrt{h_t}}\right) + \int_{\{|x|>1\}} (e^x - 1)e^{\beta(e^x-1)} U\left(\frac{dx}{\sqrt{h_t}}\right) = r$$

where

$$c' = \left[c + \int_{\{|z| \leq \frac{1}{\sqrt{h_t}}\}} z\ U(dz) - \int_{\{|z| \leq 1\}} z\ U(dz)\right] \sqrt{h_t},$$

Proposition 3.4 *Let be the price process $(S_t)_t$ defined by (7) assuming that the returns follow the model (1). We assume that the condition **(C)** holds, then:*
(i) there exists a probability measure $\mathbb{Q}$ defined on $\mathcal{F}_T$:

$$L_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\beta \widehat{X}_t - bt}, \quad \text{where} \tag{8}$$

$$b = \frac{\beta}{2}(1+\beta)\sigma^2 h_t + \beta c' + \int_{\mathbb{R}\backslash\{0\}} \left[e^{\beta(e^x - 1)} - 1 - \beta x 1_{\{|x| \leq 1\}}\right] \nu(dx),$$

$$\widehat{X}_t = X_t + \frac{1}{2}\langle X^c \rangle_t + \sum_{u \in (0,t]} (e^{\Delta X_u} - 1 - \Delta X_u), \tag{9}$$

where (X_t^c) is the continuous martingal part of (X_t), and $\Delta X_u = X_u - X_{u-}$.
(ii) Under $\mathbb{Q}$ the price process $(S_t)_t$ is an exponential-Lévy process $S_t = S_{t-1}\exp(\widetilde{m}_t + X_t)$ where $\widetilde{m}_t$ is introduced in (1), and X_t is a Lévy process with Lévy triplet $(c^{\mathbb{Q}}, \sigma^2 h_t, \nu^{\mathbb{Q}})$ given by:

$$c^{\mathbb{Q}} = c\sqrt{h_t} + \sigma^2 \beta h_t - \int_{\{|z| \leq 1\}} z\sqrt{h_t}\ U(dz) + \int_{\{|z| \leq \frac{1}{\sqrt{h_t}}\}} z\sqrt{h_t} e^{\beta(e^{z\sqrt{h_t}} - 1)} U(dz),$$

$$\nu^{\mathbb{Q}}(dx) = e^{\beta(e^x - 1)}\ U\left(\frac{dx}{\sqrt{h_t}}\right).$$

Thus the dynamics of $(Y_t)_t$ under $\mathbb{Q}$ follows the same class of models than under $\mathbb{P}$ with change inside the parameters of the Lévy process. The proof is provided in Guégan and Lalaharison (2010).

(2) - Valuation form for returns following the model (3).

Proposition 3.5 *Let be the price process $(S_t)_t$ defined by (3) and assuming that the condition **(C)** holds, and there exists $\beta \in \mathbb{R}$ verifying:*

$$\widetilde{m}_t + \left(\frac{1}{2} + \beta\right) h_t + \lambda \int_{\{|z| \leq 1\}} \left[(e^z - 1)e^{\beta(e^z - 1)} - z\right] f(z) dz$$
$$+ \lambda \int_{\{z > 1\}} (e^z - 1) e^{\beta(e^z - 1)} f(z) dz = r,$$

then the following holds:

1. *There exists a unique constant b such that* e^{L_t} *is a* $\mathbb{P}$*-martingale, where* $L_t = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{\beta\hat{X}_t - b}$, $\hat{X}_t$ *given by (9) and b verifies:*

$$b = \frac{\beta}{2}(1+\beta)h_t + \beta\widetilde{m}_t + \lambda\int_{\mathbb{R}\setminus\{0\}}\left[e^{\beta(e^z-1)} - 1 - \beta z 1_{\{|z|\leq 1\}}\right] f(z)dz.$$

2. *Under* $\mathbb{Q}$ *the stochastic process* $(Y_t)_t$ *is still a Lévy process, and the characteristics associated with the truncation function* $\tau(z) := z1_{\{|z|\leq 1\}}$ *is given by*

$$\left(\beta h_t + \widetilde{m}_t + \lambda\int_{\{|z|\leq 1\}} z(e^{\beta(e^z-1)} - 1)f(z)dz, h_t, \nu^{\mathbb{Q}}\right) \text{ where,} \quad (10)$$

$$\nu^{\mathbb{Q}}(dz) = \lambda e^{\beta(e^z-1)}f(z)dz. \quad (11)$$

The proof follows Fujiwara and Miyahara's result (2003). As soon as we know the characteristic triplets of the returns Y_t under $\mathbb{Q}$, we get the moments, and the prices are obtained shifting the mean and the variance.

4 Carrying out and Discussion

In order to illustrate the previous methodology, we provide now a simple exercise showing the impact of the choice of the models under $\mathbb{P}$ for the returns and the choice of the stochastic discount factor to compute the discount price under $\mathbb{Q}$. A more detailed analysis can be found in Guégan and Lalaharison (2010). We denote $\overline{P}$ the prices computed under $\mathbb{Q}$ using Monte Carlo simulations based on the previous methods, they are obtained in the following way:
(i) The price of the European call option under $\mathbb{Q}$ obtained using an exponential affine Stochastic Discount Factor is

$$\overline{P}(t,T,K) = \frac{1}{N}\sum_{i=1}^{N} e^{-r(T-t)} f(Y_T^i) \quad (12)$$

where $f(Y_T^i) = (S_T^i - K)^+$ is the payoff, T is the maturity, K is the strike price, and (Y_T^i) for $i = 1, \dots, N$ are independent realizations of Y_T under $\mathbb{Q}$.
(ii) The price of the European call option under $\mathbb{Q}$ obtained using the Minimal Entropy Martingale Measure is given by:

$$\overline{P}(t,T,K) = \frac{1}{N}\sum_{i=1}^{N} e^{-r(T-t)} L_T^i f(Y_T^i) \quad (13)$$

where (Y_T^i) for $i = 1, \dots, N$ are independent realizations of Y_T under $\mathbb{P}$ and L_T^i are realizations of L_T introduced in (8).

The data set we consider correspond to the closing values of the French CAC 40 daily index from March 1st, 1990 to March 1st, 2010, (corresponding to 5052 observations), and we assume that the risk-free interest rate is zero (this assumption also simplifies the definition of moneyness $\widetilde{m}_{t+1}$).

We restrict our study to the model (1). For the GARCH-type modelling $(h_t)_t$, we use two modellings

(1) - An EGARCH modelling that ensures positivity for the conditional variance without restrictions on the coefficients (Nelson, 1990):

$$\log(h_{t+1}) = \alpha_0 + \beta_1 \log(h_t) + \alpha_1 \left(\frac{|\varepsilon_t|}{\sqrt{h_t}} - E\left[\frac{|\varepsilon_t|}{\sqrt{h_t}} \right] \right) + \gamma_1 \frac{\varepsilon_t}{\sqrt{h_t}}, \quad (14)$$

(2) - The GJR model (Glosten et al., 1993)

$$h_{t+1} = K + G h_t + A \varepsilon_t^2 + L 1_{\{\varepsilon_t < 0\}} \varepsilon_t^2, \quad (15)$$

with nonnegative coefficients such that $G + A + \frac{1}{2}L < 1$ and $A + L \geq 0$.

(3) - For both models we restrict the innovations to the NIG-Lévy process which has no Brownian component and whose Lévy triplet is given by $(c, 0, U)$ with

$$c = \mu + \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\beta z) K_1(\alpha z) dz, \quad (16)$$

$(\alpha, \beta, \delta, \mu) \in \mathbb{R}^4$ with $\delta > 0$ and $0 < |\beta| < \alpha$, K_1 is the modified Bessel function of the third kind, and the Lévy measure which determines the jump behavior of discontinuous Lévy processes is:

$$U(dz) = u(z)dz = \frac{\delta\alpha}{\pi|z|} \exp(\beta z) K_1(\alpha|z|) dz.$$

In order to compare the two pricing methods, we use the average absolute relative pricing errors criterion which is given by:

$$AARPE(\%) = \frac{1}{M} \sum_{j=1}^{M} \left| \frac{P_j(t,T,K) - \overline{P}_j(t,T,K)}{P_j(t,T,K)} \right| \times 100$$

where M is the total number of options, P is the average option price and $\overline{P}$ is defined in (12) and (13).

For the two previous models, to get the expression of the price $\overline{P}$ computed under $\mathbb{Q}$, we use a particular result of the Propositions 3.2 and 3.4.

(1) - Particular case of the Proposition 3.2. Conditionally to $\mathcal{F}_{t-1}$, under $\mathbb{P}$, the process z_t which appears in relationship (1) follows a $NIG(\mu, \alpha, \beta, \delta)$ distribution with triplet $(c, 0, U)$ and the process $(X_t)_t$ in (1) is a Lévy process with triplet $(c', 0, \nu)$ with

$$c' = c\sqrt{h_t} + \int_{\mathbb{R}} z\sqrt{h_t} \left(1_{\{|z| \le \frac{1}{\sqrt{h_t}}\}}(z) - 1_{\{|z|\le 1\}}(z) \right) u(z)dz$$

Under $\mathbb{Q}$, conditionally to $\mathcal{F}_{t-1}$, X_t in (1) is again a Lévy process with triplet $(c^{\mathbb{Q}}, 0, \nu^{\mathbb{Q}})$ where

$$c^{\mathbb{Q}} = c\sqrt{h_t} - \int_{\{|z|\le 1\}} z\sqrt{h_t}\; u(z)dz + \int_{\{|z| \le \frac{1}{\sqrt{h_t}}\}} z\sqrt{h_t} e^{\theta_t \sqrt{h_t} z}\; u(z)dz$$

and

$$\nu^{\mathbb{Q}}(dx) = e^{\theta_t x} u\left(\frac{x}{\sqrt{h_t}}\right) \frac{dx}{\sqrt{h_t}}.$$

It is easy to check that under $\mathbb{Q}$, the conditional mean return is not $\widetilde{m}_t$ but $\widetilde{m}_t + \widetilde{m}_t^{\text{shift}}$ with $\widetilde{m}_t^{\text{shift}} = \int_{\mathbb{R}} z\sqrt{h_t}(e^{\theta_t\sqrt{h_t}z} - 1)U(dz)$. Moreover, the process $(Y_t)_t$ is no longer centered and its variance is not h_t but $var(Y_t) = h_t\left(\sigma^2 + \int_{\mathbb{R}} z^2 e^{\theta_t\sqrt{h_t}z} U(dz)\right)$.

(2) - Particular case of the Proposition 3.4. If z_t follows a Lévy process (with the same notation as before), then the process $(X_t)_t$ is a Lévy process with triplet $(c^{\mathbb{Q}}, 0, \nu^{\mathbb{Q}})$:

$$c^{\mathbb{Q}} = c\sqrt{h_t} - \int_{\{|z|\le 1\}} z\sqrt{h_t}\; u(z)dz + \int_{\{|z| \le \frac{1}{\sqrt{h_t}}\}} z\sqrt{h_t} e^{\beta(e^{z\sqrt{h_t}} - 1)}\; u(z)dz,$$

$$\nu^{\mathbb{Q}}(dx) = e^{\beta(e^x - 1)}\; u\left(\frac{x}{\sqrt{h_t}}\right) \frac{dx}{\sqrt{h_t}}.$$

The characteristic functions of the process $(Y_t)_t$ given $\mathcal{F}_{t-1}$ under $\mathbb{P}$ and $\mathbb{Q}$ are:

$$\begin{aligned} \log \phi_{t-1}^{\mathbb{P}}(u) &= iu\widetilde{m}_t + iuc\sqrt{h_t} - \frac{u^2}{2}\sigma^2 h_t + \\ &\int_{\mathbb{R}} (e^{iuz\sqrt{h_t}} - 1 - iuz\sqrt{h_t} 1_{\{|z|\le 1\}})\; U(dz) \quad \text{and} \end{aligned}$$

$$\begin{aligned} &\log \phi_{t-1}^{\mathbb{Q}}(u) = iu\widetilde{m}_t + iuc\sqrt{h_t} - \frac{u^2}{2}\sigma^2 h_t \beta + \\ &\int_{\mathbb{R}} (e^{iuz\sqrt{h_t}} - 1 - iuz\sqrt{h_t} e^{-\beta(e^{z\sqrt{h_t}}-1)} 1_{\{|z|\le 1\}}) e^{\beta(e^{z\sqrt{h_t}}-1)}\; U(dz) \end{aligned}$$

Conditionnally to $\mathcal{F}_{t-1}$, the characteristic function of $(Y_t)_t$ under $\mathbb{Q}$ has the same expression as the historical characteristic function with shift $\sigma^2 h_t \beta$ in the coefficient of iu and integrand $(e^{iuz\sqrt{h_t}} - 1)e^{\beta(e^{z\sqrt{h_t}}-1)}$ over $\mathbb{R}$.

It is important to note that we get the same representation for the conditional mean as the one we obtained using the exponential affine Stochastic Discount Factor but with $\widetilde{m}_t^{\text{shift}} = \int_{\mathbb{R}} z\sqrt{h_t}(e^{\beta(e^{z\sqrt{h_t}}-1)} - 1)U(dz)$ and $var(Y_t) = h_t \left(\sigma^2 + \int_{\mathbb{R}} z^2 e^{\beta(e^{z\sqrt{h_t}}-1)} U(dz)\right)$.

We now provide the pricing errors for different moneyness for the CAC 40 using the two previous models respectively with the Esscher transform (ESS) and with the Minimal Entropy Martingal Measure (MEMM). The computations have been done at a 2 years maturity.

Moneyness	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
Egarch ESS	3.44	4.74	5.59	6.06	8.43	8.36	9.94	12.7	12.3
GJR ESS	**3.40**	**4.63**	**5.39**	**5.75**	**8.02**	**7.82**	**9.26**	**11.9**	**11.3**
Egarch MEMM	3.49	6.13	9.78	14.2	16.6	22.4	26.6	29.3	37.3
GJR MEMM	3.76	6.56	10.1	14.6	17.0	22.9	27.2	30.0	38.3

Table: Absolute average pricing errors for the Cac 40 French index, using the NIG-EGARCH and NIG-GJR under an exponential affine stochastic discount factor (ESS) or under a minimal entropy martingal measure (MEMM). We put the minimal errors in bold face.

We observe that the Esscher transform methodology associated with the GJR model provides the better results for the option valuation. The results with the EGARCH or GJR models are competitive inside a same methodology to compute the prices under $\mathbb{Q}$. With this example, it appears that the MEMM approach exhibits a largest bias than the ESS method. It is important to consider other data sets and other periods of study in order to understand the impact of the choice of the stochastic discount factor in a practical valuation under $\mathbb{Q}$. Nevertheless, the choice of the dynamics under $\mathbb{P}$ seems not so important comparing with the choice of the conditional distributions.

In this exercise, we do not investigate the impact of the estimation method used to estimate the parameters of the models. Our approach is based on a one-step maximum likelihood approach. It could be interesting to extend also this work using the methodology developed and discussed in Chorro et al (2010) where they compare recursive estimation procedures and the one step maximum likelihood method.

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