Quasi-rational solutions of the NLS equation and rogue waves

Pierre Gaillard

Quasi-rational solutions of the NLS equation and rogue waves

[+]Pierre Gaillard, [+] Université de Bourgogne, Dijon, France :
e-mail: Pierre.Gaillard@u-bourgogne.fr,

November 4, 2010

Abstract

This article contains two basic results. First we derive a wronskian representation for the well known multi phase N-periodic solutions to the focusing NLS equation. Next we perform a special passage to the limit when all the periods tend to infinity.
We claim that this provides a new systematic approach for constructing the so called higher Peregrine breathers labeled by the positive integer N. We give explicit representations of all higher Peregrine breathers for $N = 1, 2, 3, 4$ and speculate about further extensions.

1 Introduction

The nonlinear Schrödinger equation (NLS) equation plays a fundamental role in hydrodynamics, in particular for waves in deep water in the context of the rogue waves or in non-linear optics.
The basic work on the subject is due to Zakharov in 1969 [12]. The first solution, actually called breather, was discovered by Peregrine in 1983 [11]; it was the simplest solution of NLS founded.
The second order Peregrine-like solution was first constructed by Akhmediev et al. in [3].
Same families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2] using Darboux transformations.
Other Peregrine-like solutions were found for reduced self-induced transparency (SIT) integrable systems [10]. In [9], the N-phase quasi-periodic

modulations of the plane waves solutions were constructed via appropriate degeneration of the finite gap periodic solutions to the NLS equation. These results in principle provided a way to construct multi-parametric families of multi-Peregrine solutions.
Very recently, it is shown in [6] that rational solutions of NLS equation can be written as a quotient of two wronskians using modified version of [8], with some different reasoning; moreover, the link between quasi-rational solutions of the focusing NLS equation and the rational solution of the KP-I equation is established.
With this formulation we recover as particular case, Akhmediev's quasi-rational solutions of NLS equation.

In this paper, we will give a new representation of the solutions of the NLS equation in terms of wronskians. The solutions take the form of a quotient of two wronskians of even order $2N$ of elementary functions depending on a certain number of parameters.
We will call these related solutions, solutions of NLS of order N.
Then, to get quasi-rational solutions of NLS equation, we take the limit when some parameter goes to 0.
For $N = 1$, we recover the well known Peregrine's solution [11] of the focusing NLS equation. For $N = 2$, 3, we recover Akhmediev's breathers. In the case $N = 4$, we give explicit analytic solution of NLS with corresponding graphic of the modulus of the solution in the (x, t) coordinates.
Here we give a new approach to get higher order Peregrine solutions different from all previous works.

2 Expression of solutions of NLS equation in terms of Fredholm determinant

2.1 Solutions of NLS equation in terms of θ functions

The solution of the NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0, \tag{1}$$

is given in terms of truncated theta function by (see [9])

$$v(x,t) = \frac{\theta_3(x,t)}{\theta_1(x,t)} \exp(2it - i\varphi). \tag{2}$$

The functions $\theta_r(x,t)$, $(r = 1, 3)$ are the functions defined by

$$\theta_r(x,t) = \sum_{k\in\{0;1\}^{2N}} g_{r,k} \tag{3}$$

with $g_{r,k}$ given by

$$g_{r,k} = \exp\left\{ \sum_{\mu>\nu,\,\mu,\nu=1}^{2N} \ln\left(\frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu}\right)^2 k_\mu k_\nu \right. \tag{4}$$

$$\left. + \left(\sum_{\nu=1}^{2N} i\kappa_\nu x - 2\delta_\nu t + (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i} + \sum_{\mu=1,\,\mu\neq\nu}^{2N} \ln\left|\frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu}\right| + \pi i \epsilon_\nu\right) k_\nu \right\}.$$

The solutions depend on a certain number of parameters :
φ;
N parameters λ_j, satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N; \tag{5}$$

The terms ϵ_ν, $1 \leq \nu \leq 2N$ are arbitrary numbers equal to 0 or 1.
In the preceding formula, the terms κ_ν, δ_ν, γ_ν are functions of the parameters λ_ν, $\nu = 1, \ldots, 2N$, and they are given by the following equations,

$$\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu\lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}. \tag{6}$$

We also note that

$$\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1\ldots N. \tag{7}$$

2.2 Relation between θ and Fredholm determinant

It was already mentioned in [9] that the function θ_r defined in (3) can be rewritten as a Fredholm determinant.
The expression given in [9] is different from which we need in the following. We need different choices of ϵ_ν. It is the crucial point to get quasi rational solutions of NLS.
Here, we make the following choices :

$$\begin{gathered} \epsilon_\nu = 0, \quad 1 \leq \nu \leq N \\ \epsilon_\nu = 1, \quad N+1 \leq \nu \leq 2N. \end{gathered} \tag{8}$$

The function θ_r defined in (3) can be rewritten with a summation in terms of subsets of $[1,..,2N]$

$$\theta_r(x,t) = \sum_{J\subset\{1,..,2N\}} \prod_{\nu\in J}(-1)^{\epsilon_\nu} \prod_{\nu\in J,\, \mu\notin J} \left|\frac{\gamma_\nu+\gamma_\mu}{\gamma_\nu-\gamma_\mu}\right|$$

$$\times \exp\{\sum_{\nu\in J} i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu}\},$$

with

$$x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1\leq j\leq 2N, \tag{9}$$

in particular

$$x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad 1\leq j\leq N,$$
$$x_{r,N+j} = -(r-1)\ln\frac{\gamma_j - i}{\gamma_j + i} - (r-1)i\pi, \quad 1\leq j\leq N. \tag{10}$$

We consider $A_r = (a_{\nu\mu})_{1\leq\nu,\mu\leq 2N}$ the matrix defined by

$$a_{\nu\mu} = (-1)^{\epsilon_\nu}\prod_{\lambda\neq\mu}\left|\frac{\gamma_\lambda+\gamma_\nu}{\gamma_\lambda-\gamma_\mu}\right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu}). \tag{11}$$

Then $\det(I+A_r)$ has the following form

$$\det(I+A_r) = \sum_{J\subset\{1,\ldots,2N\}} \prod_{\nu\in J}(-1)^{\epsilon_\nu} \prod_{\nu\in J\ \mu\notin J} \left|\frac{\gamma_\nu+\gamma_\mu}{\gamma_\nu-\gamma_\mu}\right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu}). \tag{12}$$

From the beginning of this section, $\tilde{\theta}$ has the same expression as in (12) so, we have clearly the equality

$$\theta_r = \det(I+A_r). \tag{13}$$

Then the solution of NLS equation takes the form

$$v(x,t) = \frac{\det(I+A_3(x,t))}{\det(I+A_1(x,t))}\exp(2it - i\varphi). \tag{14}$$

3 Expression of solutions of NLS equation in terms of wronkian determinant

3.1 Link between Fredholm determinants and wronskians

We consider the following functions

$$\begin{array}{ll} \phi^r_\nu(y) = \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y), & 1 \leq \nu \leq N, \\ \phi^r_\nu(y) = \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y), & N+1 \leq \nu \leq 2N. \end{array} \tag{15}$$

For simplicity, in this section we denote them $\phi_\nu(y)$.
We use the following notations :
$\Theta_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y, \quad 1 \leq j \leq 2N.$
$W_r(y) = W(\phi_1, \ldots, \phi_{2N})$ is the wronskian

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_\nu)_{\nu,\,\mu\in[1,\ldots,2N]}]. \tag{16}$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\,\mu\in[1,\ldots,2N]}$ defined by

$$\begin{array}{l} d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\lambda\neq\mu} \left|\frac{\gamma_\lambda+\gamma_\nu}{\gamma_\lambda-\gamma_\mu}\right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu}), \\ \quad 1 \leq \nu \leq 2N, \quad 1 \leq \mu \leq 2N, \end{array}$$

with

$$x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we have the following statement

Theorem 3.1

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \ldots, \phi_{2N})(0), \tag{17}$$

where

$$k_r(y) = \frac{2^{2N}\exp(i\sum_{\nu=1}^{2N}\Theta_\nu)}{\prod_{\nu=2}^{2N}\prod_{\mu=1}^{\nu-1}(\gamma_\nu - \gamma_\mu)}.$$

Proof : We start to remove the factor $(2i)^{-1}e^{i\Theta_\nu}$ in each row ν in the wronskian $W_r(y)$ for $1 \leq \nu \leq 2N$.
Then

$$W_r = \prod_{\nu=1}^{2N} e^{i\Theta_\nu}(2i)^{-N}(2)^{-N} \times W_{r1}, \tag{18}$$

with

$$W_{r1} = \begin{vmatrix} (1-e^{-2i\Theta_1}) & i\gamma_1(1+e^{-2i\Theta_1}) & \dots & (i\gamma_1)^{2N-1}(1+(-1)^{2N}e^{-2i\Theta_1}) \\ (1-e^{-2i\Theta_2}) & i\gamma_2(1+e^{-2i\Theta_2}) & \dots & (i\gamma_2)^{2N-1}(1+(-1)^{2N}e^{-2i\Theta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ (1-e^{-2i\theta_{2N}}) & i\gamma_{2N}(1+e^{-2i\Theta_{2N}}) & \dots & (i\gamma_{2N})^{2N-1}(1+(-1)^{2N}e^{-2i\Theta_{2N}}) \end{vmatrix}$$

The determinant W_{r1} can be written as

$$W_{r1} = \det(\alpha_{jk}e_j + \beta_{jk}),$$

where $\alpha_{jk} = (-1)^k(i\gamma_j)^{k-1}$, $e_j = e^{-2i\Theta_j}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $1 \leq j \leq N$, $1 \leq k \leq 2N$,
$\alpha_{jk} = (-1)^{k-1}(i\gamma_j)^{k-1}$, $e_j = e^{-2i\Theta_j}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $N+1 \leq j \leq 2N$, $1 \leq k \leq 2N$.
Denoting $U = (\alpha_{ij})_{i,\,j\in[1,\dots,2N]}$, $V = (\beta_{ij})_{i,\,j\in[1,\dots,2N]}$, the determinant of U is clearly equal to

$$\det(U) = (i)^{\frac{2N(2N-1)}{2}} \prod_{2N\geq l>m\geq 1} (\gamma_l - \gamma_m). \tag{19}$$

Then we use the following Lemma

Lemma 3.1 *Let* $A = (a_{ij})_{i,\,j\in[1,\dots,N]}$, $B = (b_{ij})_{i,\,j\in[1,\dots,N]}$,
$(H_{ij})_{i,\,j\in[1,\dots,N]}$, *the matrix formed by replacing the jth row of* A *by the ith row of* B
Then

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}) \tag{20}$$

Proof : For $\tilde{A} = (\tilde{a}_{ji})_{i,\,j\in[1,\dots,N]}$ the transposed matrix in cofactors of A, we have the well known formula $A \times^t \tilde{A} = \det A \times I$.
So it is clear that $\det(\tilde{A}) = (\det(A))^{N-1}$.
The general term of the product $(c_{ij})_{i,j\in[1,..,N]} = (a_{ij}x_i + b_{ij})_{i,j\in[1,..,N]} \times (\tilde{a}_{ji})_{i,j\in[1,..,N]}$ can be written as
$c_{ij} = \sum_{s=1}^{N}(a_{is}x_i + b_{is}) \times \tilde{a}_{js}$
$= x_i \sum_{s=1}^{N} a_{is}\tilde{a}_{js} + \sum_{s=1}^{N} b_{is}\tilde{a}_{js}$
$= \delta_{ij}\det(A)x_i + \det(H_{ij})$.
We get

$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)})$.
Thus $\det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)})$.
$\square$
Using the previous lemma (20), we get :

$$\det(\alpha_{ij}e_i + \beta_{ij}) = \det(\alpha_{ij}) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}),$$

where $(H_{ij})_{i,\, j\in[1,\dots,N]}$ is the matrix formed by replacing the jth row of U by the ith row of V defined previously.
We compute $\det(H_{ij})$ and we get

$$\det(H_{jk}) = (-1)^{E_k}(i)^{\frac{2N(2N-1)}{2}} \prod_{2N\geq l>m\geq 1,\ l\neq k,\ m\neq k} (\gamma_l - \gamma_m) \prod_{l\neq k}(\gamma_l + \gamma_j), \quad (21)$$

with $E_k = k-1$ if $1 \leq k \leq N$ and $E_k = k$ if $N+1 \leq k \leq 2N$.
We can simplify the quotient $q = \frac{\det(H_{jk})}{\det(\alpha_{jk})}$:

$$q = \frac{(-1)^{\epsilon(k)} \prod_{l\neq k}(\gamma_l + \gamma_j)}{\prod_{l\neq k}(\gamma_l - \gamma_k)}. \quad (22)$$

So $\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})})$ can be expressed as

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + (-1)^{\epsilon(k)} \prod_{l\neq k} \left|\frac{\gamma_l + \gamma_j}{\gamma_l - \gamma_k}\right| e^{2i\Theta_j}).$$

Then the wronskian

$$W_r(\phi_1, \dots, \phi_N)(0)$$

can be written as

$$\prod_{j=1}^{2N} e^{i\Theta_j|_{y=0}} (2)^{-2N} (i)^{\frac{2N(2N-2)}{2}} \prod_{j=2}^{2N}\prod_{i=1}^{j-1}(\gamma_j - \gamma_i) \prod_{j=1}^{2N} e^{-2i\Theta_j|_{y=0}} \det(I + D_r)$$

It follows that

$$\det(I + D_r) = \frac{e^{i\sum_{j=1}^{2N}\Theta_j|_{y=0}} 2^{2N}}{\prod_{j=2}^{2N}\prod_{i=1}^{j-1}(\gamma_j - \gamma_i)} W_r(\phi_1, \dots, \phi_{2N})(0) = k_r(0) W_r(\phi_1, \dots, \phi_{2N})(0). (23)$$

$\square$
So, the solution of NLS equation takes the form

$$v(x,t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi) \quad (24)$$

3.2 Wronskian representation of solutions of NLS equation

From the previous section, we get the following result :

Theorem 3.2 *The function v defined by*

$$v(x,t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi). \tag{25}$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

Remark 3.1 *In formula (25), $W_r(y)$ is the wronskian defined in (16); the functions ϕ_ν^r are given by (15); κ_ν, δ_ν, γ_ν are defined by (6); λ_ν are arbitrary parameters given by (5).*

4 Construction of quasi-rational solutions of NLS equation

4.1 Taking the limit when the parameters $\lambda_j \to 1$ for $1 \leq j \leq N$ and $\lambda_j \to -1$ for $N+1 \leq j \leq 2N$

In the following, we show how we can obtain quasi-rational solutions of NLS equation by a rough estimation.
We consider the parameter λ_j written in the form $1 - \epsilon^2 c_j^2$. For simplicity, we denote d_j the term $\frac{c_j}{\sqrt{2}}$.
When ϵ goes to 0, we realize limited expansions at order $2N-1$ of κ_j, γ_j, δ_j, $x_{r,j}$, κ_{N+j}, γ_{N+j}, δ_{N+j}, $x_{r,N+j}$, for $1 \leq j \leq N$. For example, in the case where $N = 1$, it gives :

$$\begin{aligned}
&\kappa_j = 4d_j\epsilon + O(\epsilon^2), \quad \gamma_j = d_j\epsilon + O(\epsilon^2), \quad \delta_j = 4d_j\epsilon + O(\epsilon^2),\\
&x_{r,j} = (r-1)(2id_j\epsilon + O(\epsilon^2)),\\
&\kappa_{N+j} = 4d_j\epsilon + O(\epsilon^2), \quad \gamma_{N+j} = 1/(d_j\epsilon) - (d_j\epsilon)/2 + O(\epsilon^2), \quad \delta_{N+j} = -4d_j\epsilon + O(\epsilon^2),\\
&x_{r,N+j} = -(r-1)(2id_j\epsilon + O(\epsilon^2)),\\
&1 \leq j \leq N.
\end{aligned}$$

Then we realize limited expansions at order $2N-1$ of the functions $\phi_j^r(0)$ and $\phi_{N+j}^r(0)$, for $1 \leq j \leq N$:

$$\begin{array}{l} \phi_j^1(0) = \sin E_j, \\ \phi_j^3(0) = \sin D_j, \\ \phi_{N+j}^1(0) = \sin E_j', \\ \phi_{N+j}^3(0) = \sin D_j', \end{array}$$

with

$$\begin{array}{l} D_j = K_j(x-x_{0j})/2 + i\delta_j(t-t_{0j}) - ix_j/2, \\ E_j = K_j(x-x_{0j})/2 + i\delta_j(t-t_{0j}), \\ D_j' = K_j(x-x_{0j})/2 - i\delta_j(t-t_{0j}) + ix_j/2, \\ E_j' = K_j(x-x_{0j})/2 - i\delta_j(t-t_{0j}). \end{array}$$

For example, in the case where $N=1$, it gives :

$$\begin{array}{l} D_j = 2\epsilon d_j(x-x_{0j}+2i(t-t_{0j})+1) + O(\epsilon^2), \\ E_j = 2\epsilon d_j(x-x_{0j}+2i(t-t_{0j})) + O(\epsilon^2), \\ D_j' = 2\epsilon d_j(x-x_{0j}-2i(t-t_{0j})-1) + O(\epsilon^2), \\ E_j' = 2\epsilon d_j((x-x_{0j}-2i(t-t_{0j})) + O(\epsilon^2), \end{array}$$

and

$$\begin{array}{l} \phi_j^1(0) = 2\epsilon d_j(x-x_{0j}+2i(t-t_{0j})) + O(\epsilon^2), \\ \phi_j^3(0) = 2\epsilon d_j(x-x_{0j}+2i(t-t_{0j})+1) + O(\epsilon^2), \\ \phi_{N+j}^1(0) = 1 + O(\epsilon^2), \\ \phi_{N+j}^3(0) = 1 + O(\epsilon^2). \end{array}$$

Then according to the relation

$$v(x,t) = \frac{W_3(\phi_1,\ldots,\phi_N)(0)}{W_1(\phi_1,\ldots,\phi_N)(0)}\exp(2it - i\varphi).$$

the solution of NLS equation takes the form :

$$v(x,t) = \exp(2it - i\varphi)\frac{N(x,t,\epsilon)}{D(x,t,\epsilon)}$$

$$= \exp(2it-i\varphi)\frac{\begin{vmatrix} \sin D_1 & \dots & \sin D_N & \cos D'_1 & \dots & \cos D'_N \\ \gamma_1 \cos D_1 & \dots & \gamma_N \cos D_N & -\frac{1}{\gamma_1}\sin D'_1 & \dots & -\frac{1}{\gamma_N}\sin D'_N \\ \gamma_1^2 \sin D_1 & \dots & \gamma_N^2 \sin D_N & \frac{1}{\gamma_1^2}\cos D'_1 & \dots & \frac{1}{\gamma_N^2}\cos D'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1}\cos D_1 & \dots & \gamma_N^{2N-1}\cos D_N & -\frac{1}{\gamma_1^{2N-1}}\sin D'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}}\sin D'_N \end{vmatrix}}{\begin{vmatrix} \sin E_1 & \dots & \sin E_N & \cos E'_1 & \dots & \cos E'_N \\ \gamma_1 \cos E_1 & \dots & \gamma_N \cos E_N & -\frac{1}{\gamma_1}\sin E'_1 & \dots & -\frac{1}{\gamma_N}\sin E'_N \\ \gamma_1^2 \sin E_1 & \dots & \gamma_N^2 \sin E_N & \frac{1}{\gamma_1^2}\cos E'_1 & \dots & \frac{1}{\gamma_N^2}\cos E'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1}\cos E_1 & \dots & \gamma_N^{2N-1}\cos E_N & -\frac{1}{\gamma_1^{2N-1}}\sin E'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}}\sin E'_N \end{vmatrix}}$$

We use the classical developments
$\sin z = \sum_{j=1}^{N} \frac{(-1)^{j-1} z^{2j-1}}{(2j-1)!} + O(z^{2N})$ and $\cos z = \sum_{j=0}^{2N} \frac{(-1)^{j} z^{2j}}{(2j)!} + O(z^{2N+1})$,
for z in a neighborhood of 0.
Using the fact that $D_j = E_j + \alpha_j(\epsilon)$, $D'_j = E'_j - \alpha_j(\epsilon)$, and addition formulas for trigonometric functions, we can expand the numerator $N(x,t,\epsilon)$ along the columns; it takes the form of

$$\prod_{j=1}^{N} \cos^2(\alpha_j(\epsilon)) D(x,t,\epsilon) + A.$$

If we divide each term of this sum by $D(x,t)$, when ϵ tends to 0, the first term tends to 1; in each other term of the sum of A, we can multiply each fraction by some power of γ_j to get a finite limit when ϵ goes to 0.
So we get really a non trivial and non degenerate quotient of two polynomials in x and t by an exponential.
Practically, we keep only the terms of lower degree in ϵ, in numerator and denominator which expressed as powers of x and t
So we get in this way a quasi-rational solution of the NLS equation.

So we get the following result

Theorem 4.1 *The function v defined by*

$$v(x,t) = \lim_{\epsilon \to 0} \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

where
$W_r(y) = W(\phi_1, \ldots, \phi_{2N})$ *is the wronskian* $W_r(y) = \det[(\partial_y^{\mu-1}\phi_\nu)_{\nu,\,\mu\in[1,\ldots,2N]}]$, *and* ϕ_ν *the functions given by*

$$\phi_\nu(y) = \sin(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \le \nu \le N,$$

$$\phi_\nu(y) = \cos(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N+1 \le \nu \le 2N,$$

κ_ν, δ_ν, γ_ν *parameters defined by (6) for* $\nu \in [1, \ldots, 2N]$

$$\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\},$$

where $\lambda_j = 1 - \epsilon^2 c_j^2$ *is an arbitrary parameter such that* $0 < \lambda_j < 1$ *and* $\lambda_{N+j} = -\lambda_j$, c_j *an arbitrary real number, for* $j \in [1, \ldots, N]$,

If we want to recover exact solution of Akhmediev type, we must more precise in the estimations.

4.2 Quasi-rational solutions of order N

For brevity, in the following we choose $x_{0i} = 0$, $t_{0i} = 0$ for $1 \le i \le 2N$.
To compare our solutions of NLS equation written in the context of hydrodynamic (1) with recent studies in fiber optics, we can make the following changes of variables

$$\begin{aligned} t &\to X/2 \\ x &\to T. \end{aligned} \tag{26}$$

The equation (1) become

$$iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0. \tag{27}$$

We give all the solutions of (1), but the reader can easily get the analogues for (27) by using the formulas (26).

In the following we give the solutions in the form :

$$v_N(x,t) = \frac{n(x,t)}{d(x,t)} \exp(2it - i\varphi)(1 - \alpha_N \frac{G_N(2x,4t) + iH_N(2x,4t)}{Q_N(2x,4t)})e^{2it-i\varphi}$$

with

$$\begin{array}{rcl} G_N(X,T) & = & \sum_{k=0}^{N(N+1)} \mathbf{g}_k(T)X^k \\ H_N(X,T) & = & \sum_{k=0}^{N(N+1)} \mathbf{h}_k(T)X^k \\ Q_N(X,T) & = & \sum_{k=0}^{N(N+1)} \mathbf{q}_k(T)X^k \end{array}$$

4.2.1 Case N=1

If we consider the case $N = 1$, we realize an expansion at order 1 of W_3 and W_1 in ϵ. The solution of NLS equation is defined by

$$\alpha_1 = 1, \quad \mathbf{g}_2 = 0, \quad \mathbf{g}_1 = 0, \quad \mathbf{g}_0 = 4$$

$$\mathbf{h}_2 = 0, \quad \mathbf{h}_1 = 0, \quad \mathbf{h}_0 = 4\,T$$

$$\mathbf{q}_2 = 0, \quad \mathbf{q}_1 = 0, \quad \mathbf{q}_0 = T^2 + 1$$

We represent here the modulus of v in function of $x \in [-5;5]$ and $t \in [-5;5]$ by the figure 1. The maximum of modulus of v is equal to 3.

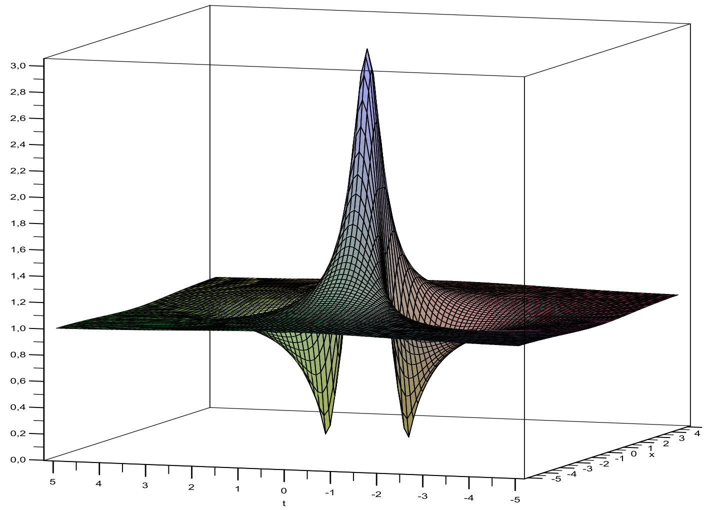

Figure 1: Solution to NLS equation for N=1.

4.2.2 Case N=2

In the case $N = 2$, we realize an expansion at order 3 in ϵ. The solution of NLS equation is defined by the polynomials G, H and Q :

$$\begin{aligned}
&\alpha_2 = 1, \quad \mathbf{g}_6 = 0, \quad \mathbf{g}_5 = 0, \quad \mathbf{g}_4 = 12, \quad \mathbf{g}_3 = 0,\\
&\mathbf{g}_2 = 72\,T^2 + 72,\\
&\mathbf{g}_1 = 0,\\
&\mathbf{g}_0 = 60\,T^4 + 216\,T^2 - 36\\
\\
&\mathbf{h}_6 = 0, \quad \mathbf{h}_5 = 0, \quad \mathbf{h}_4 = 12\,T, \quad \mathbf{h}_3 = 0,\\
&\mathbf{h}_2 = 24\,T^3 - 72\,T,\\
&\mathbf{h}_1 = 0,\\
&\mathbf{h}_0 = 12\,T^5 + 24\,T^3 - 180\,T\\
\\
&\mathbf{q}_6 = 1, \quad \mathbf{q}_5 = 0, \quad \mathbf{q}_4 = 3\,T^2 + 3,\\
&\mathbf{q}_3 = 0,\\
&\mathbf{q}_2 = 3\,T^4 - 18\,T^2 + 27,\\
&\mathbf{q}_1 = 0,\\
&\mathbf{q}_0 = T^6 + 27\,T^4 + 99\,T^2 + 9
\end{aligned}$$

(26), it can be reduced exactly at the second order Akhmediev's solution (see [1]).

We represent the modulus of v in function of $x \in [-5; 5]$ and $t \in [-5; 5]$ in figure 2. The maximum of modulus of v is equal to 5.

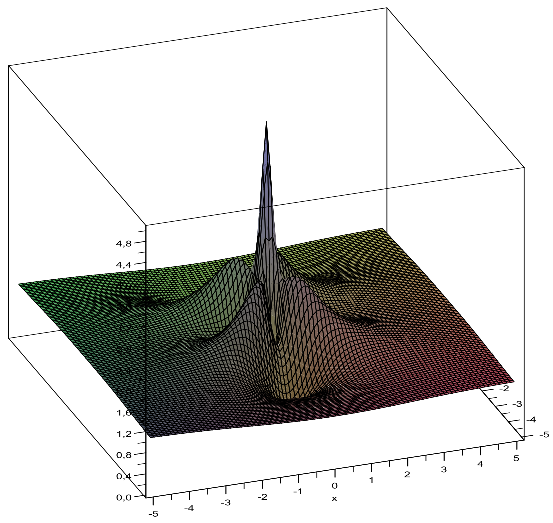

Figure 2: Solution to NLS equation for N=2.

4.2.3 Case N=3

In the case $N = 3$, we make an expansion at order 5 in ϵ. The solution of NLS equation (27) is defined by G, H and Q :

$$
\begin{aligned}
&\alpha_3 = 4, \quad \mathbf{g}_{12} = 0, \quad \mathbf{g}_{11} = 0, \quad \mathbf{g}_{10} = 6, \quad \mathbf{g}_9 = 0, \quad \mathbf{g}_8 = 90\,T^2 + 90, \quad \mathbf{g}_7 = 0,\\
&\mathbf{g}_6 = 300\,T^4 - 360\,T^2 + 1260,\\
&\mathbf{g}_5 = 0,\\
&\mathbf{g}_4 = 420\,T^6 - 900\,T^4 + 2700\,T^2 - 2700,\\
&\mathbf{g}_3 = 0,\\
&\mathbf{g}_2 = 270\,T^8 + 2520\,T^6 + 40500\,T^4 - 81000\,T^2 + 180\,Tb - 4050,\\
&\mathbf{g}_1 = 0,\\
&\mathbf{g}_0 = 66\,T^{10} + 2970\,T^8 + 13140\,T^6 - 45900\,T^4 - 12150\,T^2 + 4050
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{h}_{12} = 0, \quad \mathbf{h}_{11} = 0, \quad \mathbf{h}_{10} = 6\,T, \quad \mathbf{h}_9 = 0, \quad \mathbf{h}_8 = 30\,T^3 - 90\,T, \quad \mathbf{h}_7 = 0,\\
&\mathbf{h}_6 = 60\,T^5 - 840\,T^3 - 900\,T,\\
&\mathbf{h}_5 = 0,\\
&\mathbf{h}_4 = 60\,T^7 - 1260\,T^5 - 2700\,T^3 - 8100\,T,\\
&\mathbf{h}_3 = 0,\\
&\mathbf{h}_2 = 30\,T^9 - 360\,T^7 + 10260\,T^5 - 37800\,T^3 + 28350\,T,\\
&\mathbf{h}_1 = 0,\\
&\mathbf{h}_0 = 6\,T^{11} + 150\,T^9 - 5220\,T^7 - 57780\,T^5 - 14850\,T^3 + 28350T
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{q}_{12} = 1, \quad \mathbf{q}_{11} = 0, \quad \mathbf{q}_{10} = 6\,T^2 + 6, \quad \mathbf{q}_9 = 0, \quad \mathbf{q}_8 = 15\,T^4 - 90\,T^2 + 135, \quad \mathbf{q}_7 = 0,\\
&\mathbf{q}_6 = 20\,T^6 - 180\,T^4 + 540\,T^2 + 2340,\\
&\mathbf{q}_5 = 0,\\
&\mathbf{q}_4 = 15\,T^8 + 60\,T^6 - 1350\,T^4 + 13500\,T^2 + 3375,\\
&\mathbf{q}_3 = 0,\\
&\mathbf{q}_2 = 6\,T^{10} + 270\,T^8 + 13500\,T^6 + 78300\,T^4 - 36450\,T^2 + 12150,\\
&\mathbf{q}_1 = 0,\\
&\mathbf{q}_0 = T^{12} + 126\,T^{10} + 3735\,T^8 + 15300\,T^6 + 143775\,T^4 + 93150T^2 + 2025
\end{aligned}
$$

We get the following graphic for the modulus of v in function of $x \in [-5; 5]$ and $t \in [-5; 5]$, given by figure 3 : the maximum of modulus of v is equal to 7.

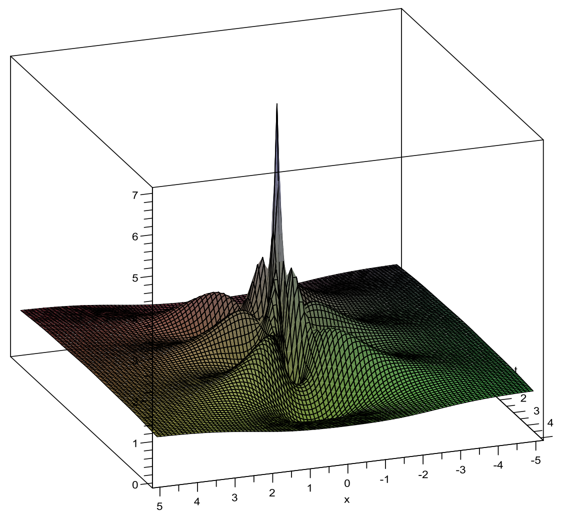

Figure 3: Solution to NLS equation for N=3.

If we make the preceding changes of variables defined by (26), we still recover the solution given recently by Akhmediev [1].
The results presented here are in accordance with the preceding work [6]; here we give moreover an explicit expression of the solutions of NLS equation for the order 3.

4.3 Case N=4

For this case, we realize an expansion at order 7 in ϵ. The solution of NLS equation (1) can be written with polynomials G, H and Q defined by

$\alpha_4 = 4$, $\mathbf{g}_{20} = 0$, $\mathbf{g}_{19} = 0$, $\mathbf{g}_{18} = 10$, $\mathbf{g}_{17} = 0$, $\mathbf{g}_{16} = 270\,T^2 + 270$, $\mathbf{g}_{15} = 0$, $\mathbf{g}_{14} = 1800\,T^4 -3600\,T^2 + 9000$, $\mathbf{g}_{13} = 0$, $\mathbf{g}_{12} = 5880\,T^6 - 54600\,T^4 - 12600\,T^2 + 189000$, $\mathbf{g}_{11} = 0$, $\mathbf{g}_{10} = 11340\,T^8 - 176400\,T^6 + 189000\,T^4 - 378000\,T^2 - 1077300$, $\mathbf{g}_9 = 0$, $\mathbf{g}_8 = 13860\,T^{10} - 207900\,T^8 + 2356200\,T^6 + 1701000\,T^4 - 56983500\,T^2 - 4819500$, $\mathbf{g}_7 = 0$, $\mathbf{g}_6 = 10920\,T^{12} - 18480\,T^{10} + 6967800\,T^8 + 56095200\,T^6 - 342657000\,T^4 +198450000\,T^2 - 11907000$, $\mathbf{g}_5 = 0$ $\mathbf{g}_4 = 5400\,T^{14} + 163800\,T^{12} + 9034200\,T^{10} +107919000\,T^8 - 615195000\,T^6 + 178605000\,T^4 + 654885000\,T^2 + 178605000$, $\mathbf{g}_3 = 0$, $\mathbf{g}_2 = 1530\,T^{16} + 133200\,T^{14} + 5506200\,T^{12} - 116802000\,T^{10} - 1731334500\,T^8 +2532222000\,T^6 - 893025000\,T^4 + 4643730000)T^2 + 223256250$, $\mathbf{g}_1 = 0$, $\mathbf{g}_0 = 190\,T^{18} + 33150\,T^{16} + 1294200\,T^{14} + 3288600\,T^{12} + 48629700\,T^{10} -2015401500\,T^8 - 1845585000\,T^6 + 14586075000)T^4 + 2098608750T^2 - 44651250$,

$\mathbf{h}_{20} = 0$, $\mathbf{h}_{19} = 0$, $\mathbf{h}_{18} = 10\,T$, $\mathbf{h}_{17} = 0$, $\mathbf{h}_{16} = 90\,T^3 - 270\,T$, $\mathbf{h}_{15} = 0$, $\mathbf{h}_{14} = 360\,T^5 -6000\,T^3 - 5400\,T$, $\mathbf{h}_{13} = 0$, $\mathbf{h}_{12} = 840\,T^7 - 29400\,T^5 + 12600\,T^3 - 138600\,T$, $\mathbf{h}_{11} = 0$, $\mathbf{h}_{10} = 1260\,T^9 - 65520\,T^7 + 259560\,T^5 - 529200\,T^3 - 1984500\,T$, $\mathbf{h}_9 = 0$, $\mathbf{h}_8 = 1260\,T^{11} - 77700\,T^9 + 718200\,T^7 - 5329800\,T^5 - 6142500\,T^3 + 29767500\,T$, $\mathbf{h}_7 = 0$, $\mathbf{h}_6 = 840\,T^{13} - 48720\,T^{11} + 718200\,T^9 + 2973600\,T^7 - 72765000\,T^5 +436590000\,T^3 + 146853000\,T, \mathbf{h}_5 = 0$, $\mathbf{h}_4 = 360\,T^{15} - 12600\,T^{13} + 138600\,T^{11} -5859000\,T^9 - 328293000\,T^7 + 1075599000\,T^5 + 773955000\,T^3 + 535815000T$, $\mathbf{h}_3 = 0$, $\mathbf{h}_2 = 90\,T^{17} + 1200\,T^{15} - 189000\,T^{13} - 40143600\,T^{11} -307786500\,T^9 + 2085426000\,T^7 - 4465125000\,T^5 + 4405590000T^3 - 1205583750T$, $\mathbf{h}_1 = 0$, $\mathbf{h}_0 = 10\,T^{19} + 930\,T^{17} - 86040\,T^{15} - 7018200\,T^{13} - 48100500\,T^{11} - 542902500\,T^9 +6039117000\,T^7 + 12942909000T^5 + 937676250T^3$, $\mathbf{q}_{20} = 1$, $\mathbf{q}_{19} = 0$, $\mathbf{q}_{18} = 10\,T^2 + 10$, $\mathbf{q}_{17} = 0$, $\mathbf{q}_{16} = 45\,T^4 - 270\,T^2 + 405$, $\mathbf{q}_{15} = 0$, $\mathbf{q}_{14} = 120\,T^6 - 1800\,T^4 + 1800\,T^2 + 16200$, $\mathbf{q}_{13} = 0$, $\mathbf{q}_{12} = 210\,T^8 - 4200\,T^6 + 6300\,T^4 +113400\,T^2 + 425250$, $\mathbf{q}_{11} = 0$, $\mathbf{q}_{10} = 252\,T^{10} - 3780\,T^8 + 63000\,T^6 +718200\,T^4 + 3005100\,T^2 + 1644300$, $\mathbf{q}_9 = 0$, $\mathbf{q}_8 = 210\,T^{12} + 1260\,T^{10} +255150\,T^8 - 567000\,T^6 + 23388750\,T^4 - 31468500\,T^2 + 17435250$, $\mathbf{q}_7 = 0$, $\mathbf{q}_6 = 120\,T^{14} + 5880\,T^{12} + 476280\,T^{10} + 16443000\,T^8 + 162729000\,T^6 -154791000\,T^4 + 130977000\,T^2 + 130977000$, $\mathbf{q}_5 = 0$, $\mathbf{q}_4 = 45\,T^{16} + 5400\,T^{14} +459900\,T^{12} + 19845000\,T^{10} + 153798750\,T^8 + 702513000\,T^6 - 89302500\,T^4 +1250235000T^2 + 111628125, \mathbf{q}_3 = 0$, $\mathbf{q}_2 = 10\,T^{18} +2250\,T^{16} + 225000\,T^{14} + 4422600\,T^{12} - 99508500\,T^{10} - 224248500\,T^8 +9704205000\,T^6 + 15181425000T^4 - 1920003750T^2 + 223256250$, $\mathbf{q}_1 = 0$, $\mathbf{q}_0 = T^{20} + 370\,T^{18} + 44325\,T^{16} + 2208600\,T^{14} + 62795250\,T^{12} + 693384300\,T^{10} +6641129250\,T^8 + 4346055000T^6 + 14042818125)T^4 + 2902331250T^2 + 22325625$

We recover a result of Akhmediev formulated in [1] in the case where $x = 0$. Here we give the complete solution in x and t.
We can make the following conjecture : at the order N, we get with this method a quasi rational solution v quotient of two polynomials of degree

$N(N+1)$.
There is $N(N+1)-1$ local maximum for the modulus and the global maximum modulus is equal to $2N+1$.

We get the following graphic for the modulus of v in function of $x \in [-5;5]$ and $t \in [-5;5]$: the maximum of modulus of v is equal to 9. It is given in figure 4.

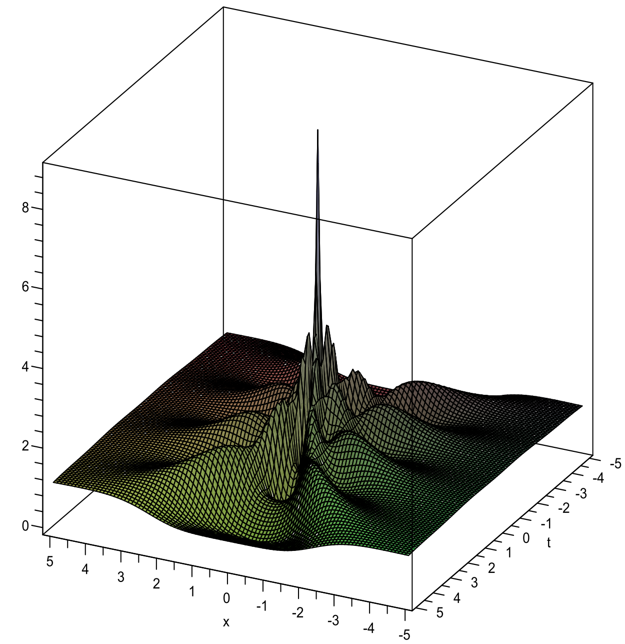

Figure 4: Solution to NLS equation for N=4.

This method can be extended to get an infinite family of quasi-rational solutions of NLS equation at any order.

5 Conclusion

Here is given a new formulation of solutions of NLS equation that provides as particular case, Akhmediev's solutions. These solutions expressed as a quotient of wronskians are similar to a previous result of Eleonski, Krichever and Kulagin [8]. It has been rewritten recently and their formulation can also be seen as a quotient of two wronkians (see [6]); the method describe in [6] can be also used to get solutions of NLS equation, but it is more complicated to find explicit solution in x and t; the computation gives complicated results. This method described in the present paper provides a powerful tool to get explicitly solutions in analytical form. This new formulation gives an infinite set of non singular solution of NLS equation.

References

[1] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogue waves and rational solutions of nonlinear Schrödinger equation, Physical Review E, V. **80**, N. 026601, (2009).

[2] N. Akhmediev, V. Eleonskii, N. Kulagin, Exact first order solutions of the nonlinear Schrödinger equation, Th. Math. Phys., V. **72**, N. 2, 183-196, (1987).

[3] N. Akhmediev, V. Eleonsky, N. Kulagin, Generation of periodic trains of picosecond pulses in an optical fiber : exact solutions, Sov. Phys. J.E.T.P., V. **62**, 894-899, (1985).

[4] N. Akhmediev, A. Ankiewicz, P.A. Clarkson, Rogue waves, rational solutions, the patterns of their zeros and integral relations, J. Phys. A : Math. Theor., V. **43**, 122002, 1-9, (2010).

[5] E.D. Belokolos, A.i. Bobenko, A.R. its, V.Z. Enolskij and V.B. Matveev, Algebro-geometric approach to nonlinear integrable equations, Springer series in nonlinear dynamics, Springer Verlag, 1-360, (1994).

[6] P. Dubard, P. Gaillard, C. Klein, V.B. Matveev, On multi-rogue waves solutions of the NLS equation and positon solutions of the KdV equation, Eur. Phys. J. Special Topics, V. **185**, 247-258, (2010).

[7] P. Dubard, V.B. Matveev, Multi-rogue waves solutions of the focusing NLS equation and the KP-i equation, Nat. Hazards Earth Syst. Sci., V. **11**, 667-672, (2011).

[8] V. Eleonskii, I. Krichever, N. Kulagin, Rational multisoliton solutions to the NLS equation, Soviet Doklady 1986 sect. Math. Phys., V. **287**, 606-610, (1986).

[9] A.R. Its, A.V. Rybin, M.A. Salle, Exact integration of nonlinear Schrödinger equation, Teore. i Mat. Fiz., V. **74**., N. 1, 29-45, (1988).

[10] V.B. Matveev, M.A. Salle, Darboux transformations and solitons, Series in Nonlinear Dynamics, Springer Verlag, Berlin, (1991).

[11] D. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, J. Austral. Math. Soc. Ser. B, V. **25**, 16-43, (1983).

[12] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Of Appl. Mech. And Tech. Physics, V. **9**, N. 2, 190-194, (1969)