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Aggregating sets of von Neumann-Morgenstern utilities*

Eric Danan† Thibault Gajdos‡ Jean-Marc Tallon§

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Abstract

We analyze the aggregation problem without the assumption that individuals and society have fully determined and observable preferences. More precisely, we endow individuals and society with sets of possible von Neumann-Morgenstern utility functions over lotteries. We generalize the classical neutrality assumption to this setting and characterize the class of neutral social welfare function. This class turns out to be considerably broader for indeterminate than for determinate utilities, where it basically reduces to utilitarianism. In particular, aggregation rules may differ by the relationship between individual and social indeterminacy. We characterize several subclasses of neutral aggregation rules and show that utilitarian rules are those that yield the least indeterminate social utilities, although they still fail to systematically yield a determinate social utility.

Keywords. Aggregation, vNM utility, indeterminacy, neutrality, utilitarianism.
JEL Classification. D71, D81.

1 Introduction

Arrovian social choice (Arrow, 1951) deals with the question of aggregating individual preference relations over a set of social alternatives into a social preference relation over this set. The main insight of Arrow's celebrated impossibility theorem is that there is no reasonable way to do so, unless one puts restrictions on either the domain or the

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†Université de Cergy-Pontoise, THEMA, 33 boulevard du Port, 95000 Cergy-Pontoise cedex, France. E-mail: Eric.Danan@u-cergy.fr.

‡CNRS, Ecole Polytechnique, CERSES. E-mail: gajdos@univ-paris1.fr.

§Paris School of Economics, Université Paris I Panthéon-Sobronne, CNRS. E-mail: jean-marc.tallon@univ-paris1.fr.

structure of individual preferences. This led Sen (1970, 1973) to move to a richer framework, replacing individual preferences by utility functions.[1] This setting allows to formulate various assumptions on the measurability and comparability of individual utility functions, the Arrovian setting corresponding to the particular "ordinal measurability, non-comparability" assumption, and to obtain possibility results under non-Arrovian assumptions (D'Aspremont and Gevers, 1977).[2] This approach of aggregating individual utility functions into a social preference relation has since then become the standard one in social choice theory.

In this paper we take issue with an assumption which is implicit in this standard approach: that individuals have fully determined and observable utility functions. Indeed, in many relevant situations, a social planner may be unable or unwilling to assign a determinate utility function to each individual. First, individuals themselves may envision more than one utility function, either because their preferences are incomplete (Aumann, 1962; Bewley, 1986; Dubra, Maccheroni, and Ok, 2004; Evren and Ok, 2010), or because they are uncertain about their tastes (Koopmans, 1964; Kreps, 1979; Dekel, Lipman, and Rustichini, 2001; Cerreia-Vioglio, 2009), or because they are driven by several "selves" or "rationales" (May, 1954; Kalai, Rubinstein, and Spiegler, 2002; Ambrus and Rozen, 2009; Green and Hojman, 2009). Second, the "individuals" under consideration may in fact be group of individuals, such as households, and the social planner may then want to remain agnostic on how individual utilities are aggregated within such groups. Third, even if all individuals have single, fully determined utility functions, the social planner may only partially observe them (Manski, 2005, 2010).

In order to account for such situations, we endow individuals with sets of utility functions. Such a set represents the possible utility functions this individual may have, according to the social planner. The particular case where this set is a singleton then corresponds to the standard setting in which the individual has a single, fully determined utility function. We shall say that the individual's utility function is *determinate* in this case and *indeterminate* otherwise, to summarize the different situations mentioned above.

1.1 The aggregation problem

How can indeterminate utilities be aggregated? Although one might find it desirable that the social planner settle for a fully determined social preference relation in all situations, one can also conceive that the indeterminacy of individual utilities sometimes prevent her to do so. In fact, even in situations where all individuals have fully determined utility functions, a social planner could leave the social preference relation indeterminate in

[1]To be precise, Sen used the term "individual welfare function". Although this terminology is more rigorous, we will follow the usual one. A discussion of the interpretation of utility in social choice can be found in Mongin and D'Aspremont (1998).

[2]See e.g. Blackorby, Donaldson, and Weymark (1984) or Roemer (1996) for surveys of these results.

order to avoid inter-personal utility comparisons. Thus, in order to allow for all these possibilities, it seems natural to consider the general problem of aggregating individual sets of utility functions into a social set of preference relations (or, equivalently, an incomplete social preference relation), the particular case where this latter set is a singleton corresponding to fully determined social preferences.

This approach, however, encounters a major difficulty that we now explain. Virtually all the possibility results for determinate utilities are obtained by means of a "neutrality" assumption, according to which the social relative ranking between two alternatives only depends on their respective utility levels for all individuals.[3] This assumption considerably simplifies the analysis because it basically boils the aggregation rule down to a social preference relation over vectors of individual utility levels. Various additional assumptions then characterize specific aggregation rules.

Now, when a utility function is indeterminate, so is, in general, the utility level of an alternative. In other words, alternatives now have sets of possible utility levels, which may or may not be singletons. This renders the neutrality assumption, now meaning that the social relative ranking between two alternatives only depends on their respective sets of utility levels for all individuals, much less reasonable. To illustrate this point, consider two alternatives x and y and the sets of utility functions $U = \{u_1, u_2, u_3\}$ and $U' = \{u_1, u_2, u_4\}$, where the utility functions u_1, u_2, u_3, and u_4 are defined by the following table.

	u_1	u_2	u_3	u_4
x	1	0	1	0
y	1	0	0	1

For both sets of utility functions, the set of utility levels of both alternatives is $\{0, 1\}$. Where the two sets of utility functions differ is in the "correlation" between the utility levels of x and y. In fact, according to U, x is clearly at least as good as y and possibly better whereas, according to U', this pattern is reversed. Nevertheless, under the neutrality assumption, the social relative ranking between x and y must be the same whether all individuals have the same set of utility functions U or all have the same set of utility functions U', for instance. More generally, since sets of utility levels do not keep track of utility "correlations", neutrality implies that social preferences cannot take them into account either, which is clearly an undesirable feature.

Our solution to this difficulty is to depart one step further from the standard approach by considering aggregation of individual sets of utility functions into a social set of utility functions rather than a social set of preference relations. In other words, we put heavier weight on the social planner's shoulders who must now not only determine the possible

[3] This assumption is usually decomposed into an "independence of irrelevant alternatives" assumption and a "Pareto indifference" assumption, see below for formal definitions.

social preference relations but also pin down the corresponding social utility functions. This setting enables us to impose the following neutrality assumption: the social set of utility levels of an alternative only depends on its sets of utility levels for all individuals. This requirement (which ignores utility "correlations" at both the individual and social level) is a reasonable one and, at the same time, makes the aggregation problem tractable by inducing a mapping from vectors of individual sets of utility levels into social sets of utility levels.

We also put restrictions on the domain and structure of individual and social sets of utility functions. Namely, we take alternatives to be lotteries and restrict attention to sets of von Neumann-Morgenstern (vNM) utility functions. This is, first, a salient setting in decision and social choice theory and, furthermore, one in which the benchmark case of determinate utilities (i.e. singleton sets of utility functions) is remarkably simple. Indeed, Coulhon and Mongin (1989) have shown that if both individuals and society are assumed to have determinate vNM utility functions then neutrality alone implies that the social utility function must be an affine transformation of the individual utility functions.[4] The intuition behind this result is that, under neutrality, the affinity property of vNM utility functions directly implies affinity of the aggregation rule. A standard "Pareto preference" assumption then suffices to make the coefficients of the affine transforamtion non-negative, i.e. yield utilitarianism.[5] Thus, for determinate vNM utilities over lotteries, the class of neutral aggregation rules "almost" reduces to that of utilitarian ones.[6]

1.2 Outline and summary of results

Section 2 introduces the formal setup. We let X denote the set of alternatives (lotteries) and $\mathscr{P}$ denote the set of compact and convex sets of (vNM) utility functions. Given such a set U of utility functions, the set $U(x)$ of possible utility levels of an alternative x is then a compact interval and we refer to it as the *utility interval* of x. Finally, we let I denote the finite set of individuals and consider a *social welfare function* F associating to each profile $(U_i)_{i\in I}$ of individual sets of utility functions a social set of utility functions $F((U_i)_{i\in I})$. We restrict attention to the case where X is the set $\Delta(Z)$ of simple lotteries over some set Z of social outcomes and the domain of F is the set of all possible profiles of sets of (vNM) utility functions, until Section 6 in which we will show that our results hold for more general alternatives and domains.

[4]Coulhon and Mongin's result is a "multi-profile" version of Harsanyi (1955)'s celebrated aggregation theorem, which is a "single-profile" result. As they discuss, the "multi-profile" approach has several advantages over the "single-profile" one, notably uniqueness of the coefficients of the affine transformation.

[5]By "utilitarianism" we mean what is sometimes referred to as "generalized utilitarianism", i.e. social utility being an affine transformation of individual utilities with non-negative coefficients.

[6]For determinate utilities over arbitrary alternatives, in contrast, additional assumptions are needed to characterize different classes of neutral aggregation rules. To characterize utilitarianism, in particular, a "cardinal measurability, unit comparability" and a "continuity" assumption must be added (Blackorby, Donaldson, and Weymark, 1984).

Our first task, in Section 3, is to provide a characterization of neutral social welfare functions in our setting of (possibly) indeterminate vNM utilities. This turns out to be substantially more difficult than in the benchmark case of determinate utilities, mainly because vNM utility intervals do not in general enjoy the affinity property of vNM utility functions. They do nevertheless satisfy (weaker) convexity properties and we are therefore able to characterize the class of neutral social welfare functions (Theorem 1). Namely, neutrality is equivalent to the following relationship between the individual and social utility intervals: for some compact and convex set $\Phi \subset (\mathbb{R}_+^I)^2 \times \mathbb{R}$,

$$F((U_i)_{i\in I})(x) = \bigcup_{(\alpha,\beta,\gamma)\in\Phi} \left(\sum_{i\in I} \alpha_i U_i(x) - \sum_{i\in I} \beta_i U_i(x) + \gamma \right).$$

This is our most general result and many interesting corollaries follow from it. It pins down the social utility interval as the union of a set of affine transformations of all individual utility intervals, each affine transformation corresponding to a weight-constant vector $(\alpha, \beta, \gamma) \in \Phi$. Each individual i's utility interval enters twice in each affine transformation, once with a non-negative coefficient α_i and once with a non-positive coefficient β_i. This duality of individual weights generalizes the benchmark case of determinate utilities in which individual utility levels enter only once in the affine transformation but coefficients have no sign restriction and, accordingly, disappears if an additional "Pareto preference" assumption (or, equivalently, a strengthening of the neutrality assumption) is imposed. Indeed, the above relationship then reduces to the following one (Corollary 1): for some compact and convex set $\Omega \subset \mathbb{R}_+^I \times \mathbb{R}$,

$$F((U_i)_{i\in I})(x) = \bigcup_{(\theta,\gamma)\in\Omega} \left(\sum_{i\in I} \theta_i U_i(x) + \gamma \right).$$

Even under these stronger assumptions, the class of neutral social welfare functions is considerably broader for indeterminate than for determinate utilities. Indeed, first, different neutral welfare functions may differ by the size of the set Ω of weight-constant vector, a larger Ω corresponding to a social planner who "generates" more indeterminacy (whether individual utilities are determinate or not). Second, the latter relationship only pins down the social utility intervals and this does not fully determine the social set of utility functions, as explained above, so that different neutral social welfare functions may differ in terms of social utility "correlations" even if they share the same Ω.

Our goal, from then on, is to explore the class of neutral social welfare functions and characterize various interesting subclasses. Section 4 is concerned with the first of the two dimensions mentioned above, the relationship between individual and social indeterminacy. In particular, we obtain the following important consequence of Theorem 1 (Corollary 2): under neutrality, for the social utility function to be determinate, it is necessary that all individual utility functions be determinate as well (except for individuals that are "irrelevant" to the social planner). Thus, the social planner cannot "resolve" individual indeterminacy. The best she can do, then, is to "preserve" individual determinacy by adopting a determinate social utility function whenever all individuals have determinate utility functions. Adding this "determinacy preservation" requirement characterizes the particular case where the set Ω above is a singleton (Corollary 3), i.e. for some $\theta \in \mathbb{R}^I_+$ and some $\gamma \in \mathbb{R}$,

$$F((U_i)_{i\in I})(x) = \sum_{i\in I} \theta_i U_i(x) + \gamma.$$

We call such aggregation rules *locally utilitarian* since they correspond to utilitarian aggregation of individual utility intervals (but not necessarily of individual sets of utility functions, as explained above). Among neutral aggregation rules, locally utilitarian ones are those who do not avoid inter-personal utility comparisons and only exhibit social indeterminacy in situations of individual indeterminacy. In contrast, a prominent aggregation rule that is neutral but not locally utilitarian is the *unanimity rule*, which takes as social set of utility functions the (convex hull of the) union of all individual sets of utility functions. This rule corresponds to the Pareto dominance relation or, in other words, to a social planner who systematically "generates" social indeterminacy rather than making inter-personal utility comparisons. Note that this rule still satisfies a weak "determinacy preservation" property: there exists at least some profile of determinate individual utilities for which social utility is determinate as well (namely, here, any profile in which all individuals have the same, determinate utility function). Weakening the "determinacy preservation" requirement in this way characterizes the particular case where the set Ω of weight-constant vectors can be decomposed into a set of weight vectors and a single constant (Corollary 4), i.e. for some compact and convex set $\Theta \subset \mathbb{R}^I_+$ and a some number $\gamma \in \mathbb{R}$ such that, for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$,

$$F((U_i)_{i\in I})(x) = \left(\bigcup_{\theta\in\Theta} \sum_{i\in I} \theta_i U_i(x) \right) + \gamma.$$

We call such aggregation rules *locally multi-utilitarian* and, under a mild "normalization" assumption, we obtain the class of *normalized* multi-utilitarian rules for which $\Theta \subseteq \Delta(I)$. This class is bounded at one end by the normalized locally utilitarian rules corresponding to Θ being a singleton, which are those yielding the least indeterminate social utilities, and at the other end by the local unanimity rules (i.e. social welfare functions yielding the same utility intervals as the unanimity rule) corresponding to $\Theta = \Delta(I)$, which are those yielding the most indeterminate social utilities.

Section 5 tackles the second of the two dimensions mentioned above, the possibility of different aggregation rules yielding the same social utility intervals but differing in social utility "correlations". Utility intervals, as explained above, do not convey enough information to establish a relative ranking of alternatives and, for this reason, the "local" characterization results obtained so far are not sufficient to help a social planner choose an aggregation rule. In order to fully pin down the social set of utility functions, an assumption that takes utility "correlations" into account is needed. We provide such an assumption, in the form of a strengthening of the neutrality assumption, and show that, provided the set Z of outcomes is infinite, adding it to the assumptions characterizing local multi-utilitarian rules characterizes the class of rules that are fully *multi-utilitarian* (Theorem 2), i.e. defined by

$$F((U_i)_{i\in I}) = \left(\bigcup_{\theta\in\Theta} \sum_{i\in I} \theta_i U_i\right) + \gamma.$$

Similarly, adding this strengthening of the neutrality assumption to the assumptions characterizing local utilitarianism characterizes the class of rules that are fully *utilitarian* (Corollary 5), i.e. defined by

$$F((U_i)_{i\in I}) = \sum_{i\in I} \theta_i U_i + \gamma.$$

This latter characterization sheds a new light on the normative appeal of utilitarian aggregation rules: when individual and social utilities may possibly be indeterminate, utilitarianism is underlain not only by neutrality and "Pareto preference" assumptions (as in the benchmark case of determinate utilities) but also by a "determinacy preservation" assumption according to which social utility should be determinate whenever possible (i.e. whenever all individual utilities are themselves determinate).

Finally, Section 6 extends our characterization results to more general alternatives and domains. To this end, we adopt the "mixture space" framework (Herstein and Milnor, 1953) and show that our results hold for a large class of such spaces, including lotteries with continuous densities or opportunity sets of lotteries, for instance. We also show that our result apply to smaller domains than the full (vNM) domain considered so far, by identifying general properties of the domain that are sufficient for the results to holds. As a consequence of these extensions, we are able to show that our characterizations of (full) utilitarianism and multi-utilitarianism remain valid if the set Z of outcomes is finite, provided that the utility of some alternative with "full support" in Z is normalized to a determinate level for all individuals.

Section 7 concludes. Proofs are gathered in the appendix.

2 Setup

Let X be a non-empty set of social alternatives. We assume that X is the set $\Delta(Z)$ of *simple lotteries* (i.e. probability distributions with finite support) on some non-empty set Z of social (sure) outcomes. Given two alternatives $x, y \in X$ and a number $\lambda \in [0,1]$, we define the λ-*mixture* of x and y, denoted $x\lambda y$, by $x\lambda y = \lambda x + (1-\lambda)y$ (clearly, then, $x\lambda y \in X$). We will maintain this assumption on X and use this definition of mixture until Section 6, in which we will consider more general alternatives and mixtures.

A *utility function* on X is a function $u : X \to \mathbb{R}$ associating to each alternative $x \in X$ a utility level $u(x) \in \mathbb{R}$. A utility function u on X is said to be a *vNM* utility function if $u(x\lambda y) = \lambda u(x) + (1-\lambda)u(y)$ for all $x, y \in X$ and all $\lambda \in [0,1]$.[7] Let $P \subseteq \mathbb{R}^X$ denote the set of all vNM utility functions on X. P is a linear subspace of $\mathbb{R}^X$ and contains all constant functions. Given a real number $\gamma \in \mathbb{R}$, we abuse notation by also letting γ denote the corresponding constant function in P.

We consider non-empty sets of utility functions, i.e. non-empty subsets of $\mathbb{R}^X$. Our interpretation of such a set is that the utility function may possibly be any member of the set, without further information being available. We say that the utility function is *determinate* if the set is a singleton and *indeterminate* otherwise. We restrict attention to sets of vNM utility functions. Let $\mathscr{P}$ denote the set of all non-empty, compact, and convex subsets of P, where P is endowed with the subspace topology and $\mathbb{R}^X$ with the product topology. Note that $\mathscr{P}$ contains in particular all convex hulls of finite sets of vNM utility functions on X and, hence, all singletons.

When the utility function is indeterminate, an alternative does not in general have a single utility level but rather a set (in fact, a non-empty and compact interval) of possible utility levels. Given a set $U \in \mathscr{P}$ of utility functions and an alternative $x \in X$, let $U(x) = \{u(x) : u \in U\}$ denote this *utility interval*. Clearly, a set $U \in \mathscr{P}$ of utility functions is a singleton if and only if the utility interval $U(x)$ is a singleton for all $x \in X$ and, in this case, knowing the set U of utility functions is equivalent to knowing the utility interval $U(x)$ for all $x \in X$. In the general case of a non-singleton set of utility functions, however, this is no longer true: a set of utility functions determines all utility intervals but the converse does not necessarily hold, as observed in the introduction. In other words, there are distinct sets of utility functions that yield the same utility intervals for all alternatives.

Let I be a non-empty and finite set of individuals. Given a non-empty domain $\mathscr{D} \subseteq \mathscr{P}^I$, an *social welfare function* $F : \mathscr{D} \to \mathscr{P}$ associates to each profile $(U_i)_{i\in I}$ of individual sets of utility functions a social set $U = F((U_i)_{i\in I})$ of utility functions. We will maintain

[7]Given our assumption on X and definition of mixture, the two following are equivalent: (i) u is a vNM utility function, (ii) $u(x) = \sum_{z\in Z} x(z)u(z)$ for all $x \in X$. However, only (i) is still well-defined in the more general setting that we will consider in Section 6, hence the use of (i) rather than (ii) which is more usual in the current setting, for the definition of a vNM utility function.

the assumption that $\mathscr{D} = \mathscr{P}^I$ from now on, and will relax it in Section 6.

3 Neutrality

In this Section, we first introduce a neutrality assumption for indeterminate utilities and characterize the class of social welfare functions satisfying this assumption. This leads to identifying three dimensions along which this characterization generalizes the one obtained by Coulhon and Mongin (1989) in the particular case where both individual and social utilities are determinate. We then elaborate on this and study to what extent these generalizations are robust to a strengthening of the neutrality assumption.

3.1 A characterization: interval neutrality

Our first contribution is to characterize neutral social welfare functions in our setting of indeterminate utilities. To this end, we start by generalizing the classical Independence of Irrelevant Alternatives and Pareto Indifference axioms which are the two components of neutrality for determinate utilities. Considering utility intervals rather than utility levels naturally leads to the following generalizations.

Axiom 1 (Interval Independence of Irrelevant Alternatives)**.** For all $(U_i)_{i\in I}, (U_i')_{i\in I} \in \mathscr{D}$ and all $x \in X$, if $U_i(x) = U_i'(x)$ for all $i \in I$ then $F((U_i)_{i\in I})(x) = F((U_i')_{i\in I})(x)$.

Axiom 2 (Interval Pareto Indifference)**.** For all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x, y \in X$, if $U_i(x) = U_i(y)$ for all $i \in I$ then $F((U_i)_{i\in I})(x) = F((U_i)_{i\in I})(y)$.

Axiom 3 (Interval Pareto Weak Preference)**.** For all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x, y \in X$, if $U_i(x) \geq U_i(y)$ for all $i \in I$ then $F((U_i)_{i\in I})(x) \geq F((U_i)_{i\in I})(y)$.[8]

Interval Independence of Irrelevant Alternatives expresses the fact that if a given alternative has the same utility interval for all individuals according to two different profiles of individual sets of utility functions, then it also has the same utility interval according to the two corresponding social sets of utility functions. Interval Pareto Indifference, on the other hand, states that if two alternatives have the same utility interval for all individuals according to a given profile of individual sets of utility functions, then they also have the same utility interval according to the corresponding social set of utility functions. Interval Pareto Weak Preference is an obvious strengthening of Interval Pareto Indifference.

Remark 1. An alternative generalization of the classical Pareto Indifference axiom would be *Pointwise Pareto Indifference*: for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x, y \in X$, if $u_i(x) = u_i(y)$

[8] Given two compact intervals $K, K' \subset \mathbb{R}$, $K \geq K'$ means $\max K \geq \max K'$ and $\min K \geq \min K'$.

for all $i \in I$ and all $u_i \in U_i$ then $u(x) = u(y)$ for all $u \in F((U_i)_{i\in I})$. Similarly, we could have stated a pointwise rather than interval version of the Pareto Weak Preference axiom. Given Interval Independence of Irrelevant Alternatives, these pointwise versions of the Pareto axioms are stronger than the corresponding interval versions. As it turns out, the weaker interval versions that we use are sufficient for our results.

Since $\mathscr{D} = \mathscr{P}^I$, the conjunction of Interval Independence of Irrelevant Alternatives and Interval Pareto Indifference is equivalent to the following *interval neutrality* property: for all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathscr{D}$ and all $x, y \in X$, if $U_i(x) = U'_i(y)$ for all $i \in I$ then $F((U_i)_{i\in I})(x) = F((U'_i)_{i\in I})(y)$. Interval neutrality reflects the fact that the social utility interval is fully determined by all individual utility intervals, independently of the particular profile and alternative that yield these individual utility intervals. Equivalently, there exists a (unique) function $G : \mathscr{K}^I \to \mathscr{K}$, where $\mathscr{K}$ denotes the set of all non-empty and compact real intervals, such that $F((U_i)_{i\in I})(x) = G((U_i(x))_{i\in I})$ for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$. As for determinate utilities, restricting attention to vNM utility functions imposes some structure on this function G. However, since we work with sets of utility functions rather than single utility functions, this structure turns out to be weaker than affinity of G. The structure of this function is given in the following characterization of interval neutral social welfare functions.

Theorem 1. Assume $X = \Delta(Z)$ with $|Z| \geq 2$ and $\mathscr{D} = \mathscr{P}^I$. Then a social welfare function F satisfies Interval Independence of Irrelevant Alternatives and Interval Pareto Indifference if and only if there exists a non-empty, compact, and convex set $\Phi \subset (\mathbb{R}^I_+)^2 \times \mathbb{R}$ such that, for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$,

$$F((U_i)_{i\in I})(x) = \bigcup_{(\alpha,\beta,\gamma)\in\Phi} \left(\sum_{i\in I} \alpha_i U_i(x) - \sum_{i\in I} \beta_i U_i(x) + \gamma \right). \qquad (1)$$

Moreover, Φ can be taken such that (1) holds for some non-empty, compact, and convex set $\Phi' \subset (\mathbb{R}^I_+)^2 \times \mathbb{R}$ if and only if $\Phi \subseteq \Phi' \subseteq \{(\alpha - \eta, \beta - \eta, \gamma) : (\alpha, \beta, \gamma) \in \Phi, \eta \in \mathbb{R}^I_+\}$.

Thus a social welfare function F is interval neutral if and only if the social utility interval is the union of a set of affine transformations of all individual utility intervals, with individual i's utility interval entering twice in each affine transformation, once with a non-negative coefficient α_i and once with a non-positive coefficient $-\beta_i$. Moreover, the set Φ of weight-constant vectors (α, β, γ) is "almost" unique in the sense that, first, there exists a unique Φ that is minimal with respect to set inclusion and, second, another set Φ' satisfies (1) if and only if it consists in the minimal Φ to which are added weight-constant vectors that are always irrelevant to the social utility interval. Indeed, it is easily checked that, for all utility interval $U_i(x)$, if $\alpha_i \geq \alpha_i - \eta_i \geq 0$ and $\beta_i \geq \beta_i - \eta_i \geq 0$ then $(\alpha_i - \eta_i)U_i(x) - (\beta_i - \eta_i)U_i(x) \subseteq \alpha_i U_i(x) - \beta_i U_i(x)$.

3.2 Discussion

How does Theorem 1 compare with the characterization of neutrality obtained by Coulhon and Mongin (1989) in the particular case where both individual and social utilities are determinate? A natural extension of their characterization to sets of utility functions would be that, for some $\theta \in \mathbb{R}^I$ and $\gamma \in \mathbb{R}$,

$$F((U_i)_{i\in I}) = \sum_{i\in I} \theta_i U_i + \gamma. \tag{2}$$

There are three dimensions along which (1) is more general than (2):

(i) whereas (2) fully pins down the social set of utility functions, (1) only pins down all social utility intervals,

(ii) in (1) the social utility interval is made of several affine transformations of all individual utility intervals, rather than a single one as in (2),

(iii) in (1) each individual i's utility interval enters twice in each affine transformation rather than once (so (2) corresponds to the particular case where either $\alpha_i = 0$ or $\beta_i = 0$).

Therefore, the class of interval neutral social welfare functions is substantially broader for indeterminate utilities than for determinate utilities.

As we will see shortly, the first two dimensions are robust to a strengthening of Interval Pareto Indifference whereas the third one is not. We thus defer the illustration of the first two points to section 3.4 and only comment on the third point now.

To illustrate this point, let $I = \{1,2\}$ and consider the two social welfare functions $F_1(U_1,U_2) = U_1 + U_2$ and $F_2(U_1,U_2) = 2(U_1+U_2) - (U_1+U_2)$, which obviously satisfy (1). F_1 uses only one weight per individual whereas F_2 uses two. These two functions agree if both $U_1(x)$ and $U_2(x)$ are singletons, but otherwise F_1 yields a smaller utility interval than F_2. For instance, $F_1([0,1],[0,1]) = [0,2] \subset [-2,4] = F_2([0,1],[0,1])$.

More generally, for any social welfare function F satisfying (1), we have

$$\begin{aligned} F((U_i)_{i\in I})(x) &= \bigcup_{(\alpha,\beta,\gamma)\in\Phi} \left\{ \sum_{i\in I} (\alpha_i u_i(x) - \beta_i v_i(x)) + \gamma : u_i, v_i \in U_i, i \in I \right\} \\ &\supseteq \bigcup_{(\alpha,\beta,\gamma)\in\Phi} \left\{ \sum_{i\in I} (\alpha_i - \beta_i)\, u_i(x) + \gamma : u_i \in U_i, i \in I \right\} \\ &= \bigcup_{(\alpha,\beta,\gamma)\in\Phi} \left(\sum_{i\in I} (\alpha_i - \beta_i)\, U_i(x) + \gamma \right), \end{aligned}$$

where, in general, equality holds if and only if $U_i(x)$ is a singleton for all $i \in I$. Thus, allowing for two weights per individual rather than a single one brings in social welfare functions yielding larger utility intervals.

3.3 Sketch of the proof

Before providing a brief sketch of the proof of Theorem 1, it is useful to gain insight into the structure of vNM utility intervals. To this end consider the example represented in Figure 1. The set of outcomes is $Z = \{z_1, z_2\}$. The left-hand side table defines utility functions $u_1, u_2, u_3, u_4 \in P$ (of course, a vNM utility function $u \in P$ is affine and, hence, fully determined by $u(z_1)$ and $u(z_2)$). These utility functions are depicted on the right-hand side graph, in which the thick horizontal segment represents the set $X = \Delta(Z)$ of alternatives and the utility level of each alternative $x \in X$ is measured along the corresponding vertical axis.

The set $U = \text{conv}(\{u_1, u_2, u_3\})$ of utility functions (i.e. all convex combinations of u_1, u_2, and u_3) fills the shaded area on the graph.[9] The utility interval $U(x)$ of an alternative $x \in X$ corresponds to the intersection of the corresponding vertical axis with this shaded area. The set $U' = \text{conv}(\{u_1, u_2, u_3, u_4\})$ of utility functions fills the same shaded area on the graph, so that we have $U'(x) = U(x)$ for all $x \in X$, although we clearly have $U' \neq U$ since $u_4 \notin U$. As explained above, this is because U and U' only differ in terms of utility "correlations": in both sets of utility functions it is possible that the utility level of z_1 be equal to 4 and it is also possible that the utility level of z_2 be equal to 1, but in U' these two possibilities may arise from the same utility function whereas in U they cannot.

Two key properties of utility intervals also appear from the graph of the set U (or, equivalently, U') of utility functions. First, although affinity of vNM utility functions does not extend to utility intervals, in the sense that one would have $U(x\lambda y) = \lambda U(x) + (1 - \lambda)U(y)$, an inclusion relation nevertheless holds, namely $U(x\lambda y) \subseteq \lambda U(x) + (1 - \lambda)U(y)$. Equivalently, the function $x \mapsto \max U(x)$ is convex and the function $x \mapsto \min U(x)$ is concave. Second, although the shaded area on the graph is not convex, it still contains all line segments joining the maximum of a utility interval with the minimum of another utility interval, i.e. $\lambda \max U(x) + (1 - \lambda) \min U(y) \in U(x\lambda y)$. This establishes a relationship between the two functions just defined. These two properties turn out to be general properties of sets of (vNM) utility functions (see Lemma 1 in the appendix).

The "if" part of Theorem 1 is straightforward. To prove the "only if" part, first note that, from interval neutrality, we know that $F((U_i)_{i\in I})(x) \in \mathscr{K}$ is a function of $(U_i(x))_{i\in I} \in \mathscr{K}^I$. Equivalently, both $\max F((U_i)_{i\in I})(x) \in \mathbb{R}$ and $\min F((U_i)_{i\in I})(x) \in \mathbb{R}$ are functions of $(\max U_i(x), \min U_i(x))_{i\in I} \in (\mathbb{R}^I)^2$. These functions are not necessarily affine but the two properties of utility intervals mentioned above nevertheless

[9]Given a set S, conv(S) denotes the convex hull of S.

	u_1	u_2	u_3	u_4
z_1	1	2	4	4
z_2	3	1	2	1

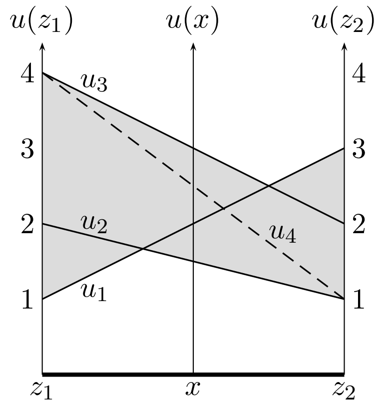

Figure 1: Example of set of utility functions

imposes some structure on them. Most importantly, the first property implies that $\max F((U_i)_{i\in I})(x)$ is convex, non-decreasing in each $\max U_i(x)$, and non-increasing in each $\min U_i(x)$. Symmetrically, $\min F((U_i)_{i\in I})(x)$ is concave, non-decreasing in each $\min U_i(x)$, and non-increasing in each $\max U_i(x)$. The second property implies that the two functions are also Lipschitzian and have "asymptotic" relationships with each other. From these and other properties, using results from convex analysis, we can construct a common, compact set $\Phi \subset (\mathbb{R}^I_+)^2 \times \mathbb{R}$ such that

$$\max F((U_i)_{i\in I})(x) = \max_{(\alpha,\beta,\gamma)\in\Phi} \left(\sum_{i\in I} \alpha_i \max U_i(x) - \sum_{i\in I} \beta_i \min U_i(x) + \gamma \right),$$
$$\min F((U_i)_{i\in I})(x) = \min_{(\alpha,\beta,\gamma)\in\Phi} \left(\sum_{i\in I} \alpha_i \min U_i(x) - \sum_{i\in I} \beta_i \max U_i(x) + \gamma \right),$$

which is equivalent to (1).

3.4 Strengthening Interval Pareto Indifference

What is the effect of strengthening Interval Pareto Indifference to Interval Pareto Weak Preference in Theorem 1? In the particular case of determinate utilities, in which neutrality yields one weight per individual, Pareto Weak Preference ensures that all weights are non-negative. Similarly, in the general case of indeterminate utilities, in which interval neutrality yields one non-negative and one non-positive weight per individual, Interval Pareto Weak Preference ensures that all non-positive weights are null. Indeed, for an interval neutral social welfare function F, Interval Pareto Weak Preference means that $\max F((U_i)_{i\in I})(x)$ and $\min F((U_i)_{i\in I})(x)$ are both non-decreasing in each $\max U_i(x)$ and each $\min U_i(x)$. In particular, the maximum of the social utility interval must be non-decreasing in the minimum of each individual's utility interval, which implies that all β coefficients must be equal to zero in (1).

To prove this implication, fix two distinct alternatives $x, y \in X$ and a real number $\mu > 0$ and consider, for each individual $i \in I$, a set $U_i \in \mathscr{P}$ of utility functions such that $U_i(x) = \{0\}$ and $U_i(y) = [-\mu, 0]$. Interval Pareto Weak Preference then implies $\max F((U_i)_{i\in I})(x) \geq \max F((U_i)_{i\in I})(y)$, i.e. $\max_{(\alpha,\beta,\gamma)\in\Phi} \gamma \geq \max_{(\alpha,\beta,\gamma)\in\Phi}(\mu \sum_{i\in i} \beta_i + \gamma)$ by (1). Since this inequality must hold for any $\mu > 0$, we must then have $\sum_{i\in i} \beta_i \leq 0$, i.e. $\beta = 0$ since $\beta \in \mathbb{R}^I_+$, for all $(\alpha, \beta, \gamma) \in \Phi$.

Given the uniqueness part of Theorem 1, setting all β coefficients to 0 fully pins down Φ. We thus obtain the following result.

Corollary 1. Assume $X = \Delta(Z)$ with $|Z| \geq 2$ and $\mathscr{D} = \mathscr{P}^I$. Then a social welfare function F satisfies Interval Independence of Irrelevant Alternatives and Interval Pareto Weak Preference if and only if there exists a non-empty, compact, and convex set $\Omega \subset \mathbb{R}^I_+ \times \mathbb{R}$ such that, for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$,

$$F((U_i)_{i\in I})(x) = \bigcup_{(\theta,\gamma)\in\Omega} \left(\sum_{i\in I} \theta_i U_i(x) + \gamma \right). \tag{3}$$

Moreover, Ω is unique.

Thus, as for determinate utilities, Interval Pareto Weak Preference fixes the sign of individual weights. In so doing, it also yields a single weight per individual rather than two and, thereby, fills part of the identified gap between interval neutrality and utilitarianism (defined as (2) with $\theta \in \mathbb{R}^I_+$) that arises from indeterminacy of utilities.

As explained above, there remain two dimensions along which (3) is more general than utilitarianism. The first one is that (3) pins down all social utility intervals but not the social set of utility functions. To illustrate this point, consider the social welfare functions $F_1((U_i)_{i\in I}) = \sum_{i\in I} U_i$ and $F_2((U_i)_{i\in I}) = \{u \in P : u(x) \in \sum_{i\in I} U_i(x) \text{ for all } x \in X\}$. Then the two functions satisfy (1) and yield the same social utility intervals, but they yield different sets of utility functions and, in fact, only F_1 is utilitarian. For instance, if $U_i = [0, 1]$ for all $i \in I$, so that all individual sets of utility functions are made of constant functions only, then $F_2((U_i)_{i\in I}) = \{u \in P : u(z) \in [0, |I|] \text{ for all } z \in Z\}$ contains non-constant functions.[10]

The second dimension along which (3) is more general than utilitarianism is that in (3) the social set of utility functions may contain more than one affine transformation of all individual sets of utility functions. To illustrate this point, consider the *unanimity rule* $F((U_i)_{i\in I}) = \text{conv}(\bigcup_{i\in I} U_i)$, which simply corresponds to the Pareto dominance relation. For this rule, the social set of utility functions can equivalently be expressed as $F((U_i)_{i\in I}) = \bigcup_{\theta\in\Delta(I)} \sum_{i\in I} \theta_i U_i$ and, hence, is the union of a set of utilitarian rules.

[10]Note that this instance also shows that F_2 does not satisfy Pointwise Pareto Weak Preference or even Indifference, although it does satisfy Interval Pareto Weak Preference.

3.5 An alternative characterization: max-min neutrality

Corollary 1 reduces (1) to (3) by strengthening Interval Pareto Indifference while keeping Interval Independence of Irrelevant Alternatives. An alternative axiomatization of (3) consists in strengthening Interval Independence of Irrelevant Alternatives while keeping Interval Pareto Indifference. Namely, consider the following *Max-Min Independence of Irrelevant Alternatives* axiom: for all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$,

(i) if $\max U_i(x) = \max U'_i(x)$ for all $i \in I$ then $\max F((U_i)_{i\in I})(x) = \max F((U'_i)_{i\in I})(x)$,

(ii) if $\min U_i(x) = \min U'_i(x)$ for all $i \in I$ then $\min F((U_i)_{i\in I})(x) = \min F((U'_i)_{i\in I})(x)$.

Since $\mathscr{D} = \mathscr{P}^I$, the conjunction of Max-Min Independence of Irrelevant Alternatives and Interval Pareto Indifference is equivalent to the following *max-min neutrality* property: for all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathscr{D}$ and all $x, y \in X$,

(i) if $\max U_i(x) = \max U'_i(y)$ for all $i \in I$ then $\max F((U_i)_{i\in I})(x) = \max F((U'_i)_{i\in I})(y)$,

(ii) if $\min U_i(x) = \min U'_i(y)$ for all $i \in I$ then $\min F((U_i)_{i\in I})(x) = \min F((U'_i)_{i\in I})(y)$.

Max-min neutrality expresses the fact that the maximum of the social utility interval is fully determined by the maximum of all individual utility intervals whereas the minimum of the social utility interval is fully determined by the minimum of all individual utility intervals, which strengthens interval neutrality and, hence, implies (1).

It turns out that max-min neutrality is in fact equivalent to (3), i.e. to the conjunction of Interval Independence of Irrelevant Alternatives and Interval Pareto Weak Preference. Indeed, on the one hand, (3) is equivalent to

$$\max F((U_i)_{i\in I})(x) = \max_{(\theta,\gamma)\in\Omega} \left(\sum_{i\in I} \theta_i \max U_i(x) + \gamma \right),$$
$$\min F((U_i)_{i\in I})(x) = \min_{(\theta,\gamma)\in\Omega} \left(\sum_{i\in I} \theta_i \min U_i(x) + \gamma \right),$$

and, hence, implies max-min neutrality. On the other hand, recall that interval neutrality implies that $\max F((U_i)_{i\in I})(x)$ is non-decreasing in each $\max U_i(x)$ and non-increasing in each $\min U_i(x)$ whereas $\min F((U_i)_{i\in I})(x)$ is non-decreasing in each $\min U_i(x)$ and non-increasing in each $\max U_i(x)$. From there, Max-Min Independence of Irrelevant Alternatives has the same effect as Interval Pareto Weak Preference: it eliminates the dependency of the social maximum on each individual minimum as well as the dependency of the social minimum on each individual maximum and, thereby, yields (3). Thus, (3) can be derived from neutrality axioms alone, without appealing to Pareto preference axioms.

4 Individual and social indeterminacy

Allowing both individual and social utilities to be indeterminate raises the question of the relationships between individual and social indeterminacy. More precisely, we may ask the following questions:

(i) Given a social welfare function and a profile of individual sets of utility functions, does indeterminacy of individual utilities cause indeterminacy of social utility?

(ii) Given a social welfare function, if individual utilities are more indeterminate in a profile than in another, is it so for social utility as well?

(iii) Given a profile of individual utilities, when is social utility more indeterminate for a social welfare function than another?

We will examine these three questions within the class of social welfare functions identified in Corollary 1. Providing answers to these questions will lead us to identify further conditions, which strengthen the characterization obtained in (3) but still fall short of characterizing utilitarianism.

4.1 Does individual indeterminacy cause social indeterminacy?

We tackle here the first question and consider a social welfare function F satisfying (3) and a profile $(U_i)_{i\in I} \in \mathscr{D}$. For a given alternative $x \in X$, a necessary condition for the social utility interval $F((U_i)_{i\in I})(x)$ to be a singleton is that for each vector $(\theta, \gamma) \in \Omega$, the corresponding affine transformation of utility intervals $\sum_{i\in I} \theta_i U_i(x) + \gamma$ be a singleton as well. This necessary condition, in turn, can only be satisfied if each individual i's utility interval $U_i(x)$ is itself a singleton, unless $\theta_i = 0$, in which case individual i is "irrelevant" in this affine transformation. Thus, if all individuals are "relevant" then the social utility level are determinate only if all individual utility levels are themselves determinate.

To formalize this point, say that an individual $i \in I$ is *interval null* if, for all $(U_j)_{j\in I}, (U'_j)_{j\in I} \in \mathscr{D}$ and all $x \in X$, $F((U_j)_{j\in I})(x) = F((U'_j)_{j\in I})(x)$ whenever $U_j(x) = U'_j(x)$ for all $j \in I \setminus \{i\}$. This reflects the idea that individual i is completely "irrelevant" to the social planner. For a social welfare function satisfying (3), an individual $i \in I$ is interval null if and only if $\theta_i = 0$ for all $(\theta, \gamma) \in \Omega$. The "if" part of this statement is straightforward. To prove the "only if" part, assume individual $i \in I$ is interval null, fix a real number $\mu > 0$, and consider the sets of utility functions $U_i = \{\mu\}$, $U'_i = \{0\}$, and $U_j = U'_j = \{0\}$ for all $j \in I \setminus \{i\}$. Since i is interval null, we then have, for all $x \in X$, $\max F((U_i)_{i\in I})(x) = \max F((U'_i)_{i\in I})(x)$, i.e. $\max_{(\theta,\gamma)\in\Omega}(\mu\theta_i + \gamma) = \max_{(\theta,\gamma)\in\Omega} \gamma$ by (3). Since this inequality must hold for all $\mu > 0$, we must then have $\theta_i \leq 0$, i.e. $\theta_i = 0$ since $\theta \in \mathbb{R}^I_+$, for all $(\theta, \gamma) \in \Omega$.

Hence, for all $(U_i)_{i \in I} \in \mathscr{D}$ and all $x \in X$, if $F((U_i)_{i \in I})(x)$ is a singleton then, for all $i \in I$, either $U_i(x)$ is a singleton or i is interval null. Since a set U of utility functions is a singleton if and only if the utility interval $U(x)$ is a singleton for all $x \in X$, we then obtain the following result.

Corollary 2. Assume $X = \Delta(Z)$ with $|Z| \geq 2$ and $\mathscr{D} = \mathscr{P}^I$. Let F be a social welfare function satisfying (3). Then for all $(U_i)_{i \in I} \in \mathscr{D}$, if $F((U_i)_{i \in I})$ is a singleton then, for all $i \in I$, either U_i is a singleton or i is interval null.

Thus, if all individuals are "relevant" then the social utility function can only be determinate if all individual utility functions are themselves determinate. Society cannot "resolve" individual indeterminacy. For example, one may find it desirable that society selects a profile $(u_i)_{i \in I}$ of individual utility functions out of each profile $(U_i)_{i \in I}$ of individual sets of utility functions and use some affine transformation $\sum_{i \in I} \theta_i u_i + \gamma$ of the selected individual utility functions as the (determinate) social utility function, but this is incompatible with the axioms of Corollary 1, unless $\theta = 0$. This point, in fact, only relies on interval neutrality and not on Interval Pareto Weak Preference, so if one wants society to "resolve" individual indeterminacy then one must give up the assumption that the social utility interval is fully determined by all individual utility intervals.[11]

The converse to Corollary 2 does not hold: even if all individual utilities are determinate, social utility may well be indeterminate. This is the case, for example, for the unanimity rule, which yields an indeterminate social utility function whenever all individuals do not have the same, determinate utility function. A utilitarian social welfare function $F((U_i)_{i \in i}) = \sum_{i \in I} \theta_i U_i + \gamma$, on the other hand, yields a determinate social utility function whenever all individuals have determinate utility functions. The latter function always "preserves" determinacy by making inter-personal utility comparisons whereas the former, by avoiding such comparisons, sometimes "generates" indeterminacy. This raises the question of characterizing the class of all social welfare functions satisfying (3) that always "preserve" determinacy.

4.2 Local utilitarianism

The following axiom captures the fact that the social welfare function does not "generate" social indeterminacy by avoiding inter-personal utility comparisons.

Axiom 4 (Strong Determinacy Preservation)**.** For all $(\{u_i\})_{i \in I} \in \mathscr{D}$, there exists $u \in P$ such that $F((\{u_i\})_{i \in I}) = \{u\}$.

The effect of adding Strong Determinacy Preservation to the axioms of Corollary 1 is to reduce Ω to a singleton in (3). To prove this, let $(\theta, \gamma), (\theta', \gamma') \in \Omega$. Then, letting

[11] Using the same argument as above, it is easily shown that for a social welfare function satisfying (1), an individual $i \in I$ is interval null if and only if $\alpha_i = \beta_i = 0$ for all $(\alpha, \beta, \gamma) \in \Phi$.

$U_i = \{0\}$ for all $i \in I$, we have $\{\gamma, \gamma'\} \subseteq F((U_i)_{i\in I})(x)$ for all $x \in X$, so $\gamma = \gamma'$. Hence, fixing some individual $i \in I$ and letting $U_i = \{1\}$ and $U_j = \{0\}$ for all $j \in I \setminus \{i\}$, we have $\{\theta_i + \gamma, \theta'_i + \gamma\} \subseteq F((U_i)_{i\in I})(x)$ for all $x \in X$, so $\theta_i = \theta'_i$. Since this must hold for all $i \in I$, we must have $\theta = \theta'$, so that Ω is a singleton. We thus obtain the following characterization.

Corollary 3. Assume $X = \Delta(Z)$ with $|Z| \geq 2$ and $\mathscr{D} = \mathscr{P}^I$. Then a social welfare function F satisfies Interval Independence of Irrelevant Alternatives, Interval Pareto Weak Preference, and Strong Determinacy Preservation if and only if there exist a vector $\theta \in \mathbb{R}^I_+$ and a number $\gamma \in \mathbb{R}$ such that, for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$,

$$F((U_i)_{i\in I})(x) = \sum_{i\in I} \theta_i U_i(x) + \gamma. \tag{4}$$

Moreover, θ and γ are unique.

Call *locally utilitarian* any social welfare function satisfying (4). Corollary 3 shows that the social welfare functions satisfying (3) that always "preserve" determinacy or, equivalently, never "generate" indeterminacy, are precisely the locally utilitarian ones. This includes, in particular, all utilitarian social welfare functions, but not only: other social welfare functions belong to this class, such as the example $F((U_i)_{i\in I}) = \{u \in P : u(x) \in \sum_{i\in I} U_i(x) \text{ for all } x \in X\}$ considered in Section 3.4.

4.3 Local multi-utilitarianism

At the other end of the spectrum of social welfare functions satisfying (3) are functions that always "generate" indeterminacy, i.e. such that the social utility level is indeterminate whatever the individual utility levels. A trivial example of such a function is $F((U_i)_{i\in i}) = \sum_{i\in i} U_i + [0, 1]$, which satisfies (3) with $\Omega = \{(1, \gamma) : \gamma \in [0, 1]\}$, and for which $F((U_i)_{i\in i})(x)$ is never a singleton. The unanimity rule defined above stands somewhere in between these two extremes, since it sometimes "preserves" determinacy and sometimes "generates" indeterminacy. We shall now give a characterization of the social welfare functions that do not always "generate" indeterminacy, i.e. satisfy the following axiom

Axiom 5 (Weak Determinacy Preservation)**.** There exist $(\{u_i\})_{i\in I} \in \mathscr{D}$, $u \in P$, and $x \in X$ such that $u_i(x) = 0$ for all $i \in I$ and $F((\{u_i\})_{i\in I}) = \{u\}$.

It would in fact be sufficient for our purpose to require that there exist $(\{u_i\})_{i\in I} \in \mathscr{D}$ and $u \in P$ such that $F((\{u_i\})_{i\in I}) = \{u\}$, and we only require the existence of an alternative $x \in X$ such that $u_i(x) = 0$ for all $i \in I$ in order to simplify the exposition. Thus, the idea of this axiom is simply that there exists a profile of determinate individual utility functions for which the social utility function is determinate as well. This is clearly a weakening of Strong Determinacy Preservation. The effect of adding Weak Determinacy Preservation to the axioms of Corollary 1 is to equalize the constants of all weight-constant pairs belonging to Ω in (3) (the proof of this fact is identical to the first half of the proof of Corollary 3 and is, therefore, omitted).

Corollary 4. Assume $X = \Delta(Z)$ with $|Z| \geq 2$ and $\mathscr{D} = \mathscr{P}^I$. Then a social welfare function F satisfies Interval Independence of Irrelevant Alternatives, Interval Pareto Weak Preference, and Weak Determinacy Preservation if and only if there exist a non-empty, compact, and convex set $\Theta \subset \mathbb{R}_+^I$ and a number $\gamma \in \mathbb{R}$ such that, for all $(U_i)_{i \in I} \in \mathscr{D}$ and all $x \in X$,

$$F((U_i)_{i\in I})(x) = \left(\bigcup_{\theta \in \Theta} \sum_{i \in I} \theta_i U_i(x) \right) + \gamma. \tag{5}$$

Moreover, Θ and γ are unique.

Call *locally multi-utilitarian* any social welfare function satisfying (5). Corollary 4 shows that the social welfare functions that do not always "generate" indeterminacy are precisely the locally multi-utilitarian ones. This includes, in particular, all locally utilitarian social welfare functions, since they never "generate" indeterminacy, as well as the unanimity rule, which corresponds to $\Theta = \Delta(I)$ in (5).

4.4 The comparative statics of individual indeterminacy

The answer to the second question raised at the beginning of the section is quite simple: it is an immediate consequence of (3) that for all profiles $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathscr{D}$ and all alternatives $x, y \in X$, if $U_i(x) \subseteq U'_i(y)$ for all $i \in I$ then $F((U_i)_{i\in I})(x) \subseteq F((U'_i)_{i\in I})(y)$. So indeed, for any social welfare function satisfying the axioms of Corollary 1, more indeterminacy at the individual level translates to more indeterminacy at the social level. This implies, in particular, that Weak Determinacy Preservation can be further weakened in Corollary 4 in the following way: there exist $(U_i)_{i\in I} \in \mathscr{D}$, $u \in P$, and $x \in X$ such that $0 \in U_i(x)$ for all $i \in I$ and $F((U_i)_{i\in I}) = \{u\}$ (where, again, the requirement that $0 \in U_i(x)$ for all $i \in I$ is for simplicity only). Thus, the class of locally multi-utilitarian social welfare functions can equivalently be described as the class of social welfare functions satisfying (3) for which social utility is not always indeterminate.

4.5 Interval expansion

To provide an answer to the third question raised at the beginning of the section, we need to make precise what we mean by a social welfare function being "more indeterminate" than another one. Given two social welfare functions F_1 and F_2 on some common domain $\mathscr{D}$, say that F_2 is an *interval expansion* of F_1 if $F_1((U_i)_{i\in I})(x) \subseteq F_2((U_i)_{i\in I})(x)$ for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$. This simply means that F_2 yields a more indeterminate social utility level than F_1 for all profiles.

If F_1 and F_2 satisfy (3) then, denoting their corresponding sets of weight-constant vectors by Ω_1 and Ω_2, respectively, we have that F_2 is an interval expansion of F_1 if and only if $\Omega_1 \subseteq \Omega_2$. The "if" part of this statement is straightforward. To prove the "only if" part, assume $\Omega_1 \nsubseteq \Omega_2$ and let $x \in X$. Then since Ω_1 and Ω_2 are compact and convex subsets of $\mathbb{R}^I \times \mathbb{R}$, there exists $(\rho, \kappa) \in \mathbb{R}^I \times \mathbb{R}$ with $\kappa \neq 0$ such that $\max_{(\theta,\gamma)\in\Omega_1}(\sum_{i\in I} \theta_i \rho_i + \kappa\gamma) > \max_{(\theta,\gamma)\in\Omega_2}(\sum_{i\in I} \theta_i\rho_i + \kappa\gamma)$. If $\kappa > 0$ then dividing by κ yields $\max F_1((\frac{\rho_i}{\kappa})_{i\in I})(x) > \max F_2((\frac{\rho_i}{\kappa})_{i\in I})(x)$. If $\kappa < 0$ then dividing by κ yields $\min F_1((\frac{\rho_i}{\kappa})_{i\in I})(x) < \min F_2((\frac{\rho_i}{\kappa})_{i\in I})(x)$ for all $x \in X$. In both cases, $F_1((\frac{\rho_i}{\kappa})_{i\in I})(x) \nsubseteq F_2((\frac{\rho_i}{\kappa})_{i\in I})(x)$.

Hence, in particular, locally utilitarian social welfare functions (which correspond to Ω being a singleton) are the social welfare functions satisfying (3) that yield the smallest social utility intervals in the sense that, first, any social welfare function satisfying (3) is an interval expansion of some locally utilitarian social welfare function and, second, a locally utilitarian social welfare function is not an interval expansion of any social welfare function satisfying (3) other than itself. On the other hand, there is no social welfare function satisfying (3) that yields the largest social utility intervals, simply because there is no largest Ω. This remains true, of course, if we restrict attention to locally multi-utilitarian social welfare functions (i.e. satisfying (5)), in which case we have that F_2 is an interval expansion of F_1 if and only if $\Theta_1 \subseteq \Theta_2$ and $\gamma_1 = \gamma_2$. We may only notice that any social welfare function satisfying (3) but not (5) is an interval expansion of some social welfare function satisfying (5).

To say more, we need to impose some normalization on the weight-constant vectors. To this end, consider the following *Determinate Normalization* axiom: for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $u \in P$, if $U_i = \{u\}$ for all $i \in I$ then $F((U_i)_{i\in I}) = \{u\}$. According to this axiom, if all individuals have the same, determinate utility function then society also has this determinate utility function. Note that this axiom implies Weak Determinacy Preservation. Adding it to the axioms of Corollary 1 yields (5) with the additional normalization $\Theta \subseteq \Delta(I)$. To see this, simply observe that letting $U_i = \{0\}$ for all $i \in I$ yields $\gamma = 0$ and letting $U_i = \{1\}$ for all $i \in I$ then yields $\sum_{i\in I} \theta_i = 1$ for all $\theta \in \Theta$.

The normalized locally multi-utilitarian social welfare functions yielding the smallest social utility intervals are, of course, the normalized locally utilitarian ones. But it is now also the case that those yielding the largest social utility intervals are those corresponding to $\Theta = \Delta(I)$, i.e. such that $F((U_i)_{i\in I})(x) = \mathrm{conv}(\bigcup_{i\in I} U_i(x))$, that we shall call *local unanimity rules*. This class is not restricted to the unanimity rule but also includes, for example, the social welfare function $F((U_i)_{i\in I}) = \{u \in P : u(x) \in$

$\mathrm{conv}(\bigcup_{i\in I} U_i(x))$ for all $x \in X\}$, which is distinct from the unanimity rule (for instance, if $U_i = [0,1]$ for all $i \in I$, so that all individual sets of utility functions are made of constant functions only, then $F((U_i)_{i\in I}) = \{u \in P : u(z) \in [0,1]$ for all $z \in Z\}$ contains non-constant functions).

5 Utilitarianism

We now come to the question of characterizing utilitarianism for indeterminate utilities. The closest characterization we have obtained so far is that of local utilitarianism in Corollary 3. Utilitarianism, however, is only a particular case of local utilitarianism and, therefore, must be characterized by stronger axioms.

To introduce the issue, consider a social planner who abides by the axioms of Interval Independence of Irrelevant Alternatives, Interval Pareto Weak Preference, and Strong Determinacy Preservation and is willing to give equal weight to all individuals. By Corollary 3, this social planner should adopt a locally utilitarian social welfare function and set $\theta_i = \frac{1}{|I|}$ for all $i \in I$ and $\gamma = 0$ in (4) (we also assume Determinate Normalization for simplicity). This alone determines all social utility intervals, namely $F((U_i)_{i\in I})(x) = \sum_{i\in I} \frac{1}{|I|} U_i(x)$, so that without exactly knowing the social set of utility functions, the social planner already knows, for example, whether the social utility level of a given alternative is determinate or indeterminate.

However, if the social planner is interested in comparing different alternatives with one another rather than evaluating a single alternative in isolation, then more information is required. As a trivial example, if $U_i = [0,1]$ for each individual i then all alternatives necessarily have the same social utility interval $[0,1]$, but it might either be the case that all social utility functions deem all alternatives indifferent (if $F((U_i)_{i\in I}) = \sum_{i\in I} \frac{1}{|I|} U_i$, which is utilitarian) or that social utility functions always disagree on the relative ranking of two distinct alternatives (e.g. if $F((U_i)_{i\in I}) = \{u \in P : u(x) \in \sum_{i\in I} \frac{1}{|I|} U_i(x)$ for all $x \in X\}$, which is not utilitarian). In such cases, the social planner needs to exactly know the social set of utility functions and, in particular, whether it is utilitarian or not.

5.1 Characterization of utilitarianism

The example above also illustrates the fact that to characterize utilitarianism, we need to strengthen the axioms of Corollary 3 in a way that takes into account not only utility intervals but also utility "correlations". To this end, we first introduce the following notation: given a subset Y of X and a set $U \in \mathscr{P}$ of utility functions, we let $U|_Y$ denote the *restriction* of U to Y, i.e. $U|_Y = \{u|_Y : u \in U\}$.[12] If Y is a singleton

[12]Given a function f on a set S and a subset T of S, $f|_T$ denotes the function on T defined by $f|_T(s) = f(s)$ for all $s \in T$.

then $U|_Y$ is just the utility interval of the corresponding alternative. If Y contains more than one alternative, however, $U|_Y$ is more than the collection of utility intervals of all alternatives in Y, just the same way a set of utility functions is more than the collection of all corresponding utility intervals. We now introduce the following strengthening of the Interval Independence of Irrelevant Alternatives axiom.

Axiom 6 (Setwise Independence of Irrelevant Alternatives)**.** For all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathscr{D}$ and all finite subset Y of X, if $U_i|_Y = U'_i|_Y$ for all $i \in I$ then $F((U_i)_{i\in I})|_Y = F((U'_i)_{i\in I})|_Y$.

Obviously, Interval Independence of Irrelevant Alternatives corresponds to the particular case of Setwise Independence Alternatives where $Y = \{x\}$. What the latter adds to the former is that individual utility "correlations" determine social utility "correlations". Assuming the set Z of outcomes is infinite (as we will see in Section 6, the result also holds for specific domains $\mathscr{D} \subset \mathscr{P}^I$ if Z is finite), we obtain the following result.

Theorem 2. Assume $X = \Delta(Z)$ with $|Z| = \infty$ and $\mathscr{D} = \mathscr{P}^I$. Then a social welfare function F satisfies Setwise Independence of Irrelevant Alternatives, Interval Pareto Weak Preference, and Weak Determinacy Preservation if and only if there exist a non-empty, compact, and convex set $\Theta \subset \mathbb{R}^I_+$ and a number $\gamma \in \mathbb{R}$ such that, for all $(U_i)_{i\in I} \in \mathscr{D}$,

$$F((U_i)_{i\in I}) = \left(\bigcup_{\theta\in\Theta} \sum_{i\in I} \theta_i U_i \right) + \gamma. \tag{6}$$

Moreover, Θ and γ are unique.

Strengthening Weak Determinacy Preservation to Determinate Normalization yields the additional normalization $\Theta \in \Delta(I)$ in (6).

Thus, strengthening Interval Independence of Irrelevant Alternatives to Setwise Independence of Irrelevant Alternatives yields a characterization of multi-utilitarian (and not only locally multi-utilitarian) social welfare functions. Merely strengthening Weak Determinacy Preservation to Strong Determinacy Preservation then yields the following characterization of utilitarianism (the proof is identical to that of Corollary 3 and hence omitted).

Corollary 5. Assume $X = \Delta(Z)$ with $|Z| = \infty$ and $\mathscr{D} = \mathscr{P}^I$. Then a social welfare function F satisfies Setwise Independence of Irrelevant Alternatives, Interval Pareto Weak Preference, and Strong Determinacy Preservation if and only if there exist a vector $\theta \in \mathbb{R}^I_+$ and a number $\gamma \in \mathbb{R}$ such that, for all $(U_i)_{i\in I} \in \mathscr{D}$,

$$F((U_i)_{i\in I}) = \sum_{i\in I} \theta_i U_i + \gamma. \tag{7}$$

Moreover, θ and γ are unique.

Remark 2. Both Theorem 2 and Corollary 5 can equivalently be stated with the Pointwise Pareto Weak Preference axiom in place of Interval Pareto Weak Preference. Indeed, the pointwise version is stronger than the interval version under interval neutrality and, hence, is sufficient. Conversely, (6) (and, hence, (7)) implies Pointwise Pareto Weak Preference, which is therefore necessary as well.

5.2 Further properties

The results of Section 4 concerning individual and social indeterminacy can now be stated in terms of utility functions rather than utility levels. Define now an individual $i \in I$ to be *null* if, for all $(U_j)_{j\in I}, (U'_j)_{j\in I} \in \mathscr{D}$, $F((U_j)_{j\in I}) = F((U'_j)_{j\in I})$ whenever $U_j = U'_j$ for all $j \in I \setminus \{i\}$. We then obtain, first, that for a social welfare function satisfying (6), an individual $i \in I$ is null if and only if $\theta_i = 0$ for all $\theta \in \Theta$. Second, it is an immediate consequence of (6) that for all profiles $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathscr{D}$, if $U_i \subseteq U'_i$ for all $i \in I$ then $F((U_i)_{i\in I}) \subseteq F((U'_i)_{i\in I})$. Third, defining F_2 to be an *expansion* of F_1 if $F_1((U_i)_{i\in I}) \subseteq F_2((U_i)_{i\in I})$ for all $(U_i)_{i\in I} \in \mathscr{D}$, we obtain that if F_1 and F_2 satisfy (6) then, denoting their corresponding sets of weight vectors by Θ_1 and Θ_2, respectively, F_2 is an expansion of F_1 if and only if $\Theta_1 \subseteq \Theta_2$.

In the particular case where both individual and social utilities are determinate, utilitarianism is known to satisfy the following "cardinal measurability, full comparability" invariance property: for all $(\{u_i\})_{i\in I}, (\{u'_i\})_{i\in I} \in \mathscr{D}$, if there exist a non-negative real number a and a collection $(b_i)_{i\in I}$ of real numbers such that $u'_i = au_i + b$ for all $i \in I$ then there exists a real number b such that $F((\{u'_i\})_{i\in i}) = aF((\{u_i\})_{i\in I}) + b$. In the general case of indeterminate utilities, multi-utilitarian social welfare functions satisfy a generalization of this invariance property: for all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathscr{D}$, if there exist a non-empty subset A of $\mathbb{R}_+$ and a collection $(B_i)_{i\in I}$ of non-empty subsets of $\mathbb{R}$ such that $U'_i = AU_i + B_i$ (i.e. $U'_i = \{au + b : a \in A, b \in B_i, u \in U_i\}$) for all $i \in I$ then there exists a non-empty subset B of $\mathbb{R}$ such that $F((U'_i)_{i\in I}) = AF((U_i)_{i\in I}) + B$. Moreover, for normalized multi-utilitarian as well as for utilitarian social welfare functions, if B_i is a singleton for all $i \in I$ then B is a singleton. Hence, for such functions, taking A to be a singleton as well yields $F((U'_i)_{i\in i}) = aF((U_i)_{i\in I}) + b$ if $U'_i = aU_i + b$ for all $i \in I$, which is the same invariance property as in the particular case of determinate utilities.

5.3 Sketch of the proof

Before sketching the proof of Theorem 2, which will shed light on the role of the Setwise Independence of Irrelevant Alternatives axiom as well as the assumption that Z is infinite, it is useful to understand in more detail why a set of utility functions is only partially pinned down by all utility intervals. To this end, let us go back to the example that we considered in Section 3.3 and represented in Figure 1. The left-hand side table of

	u_1	u_2	u_3	u_4
z_1	1	2	4	4
z_2	3	1	2	1

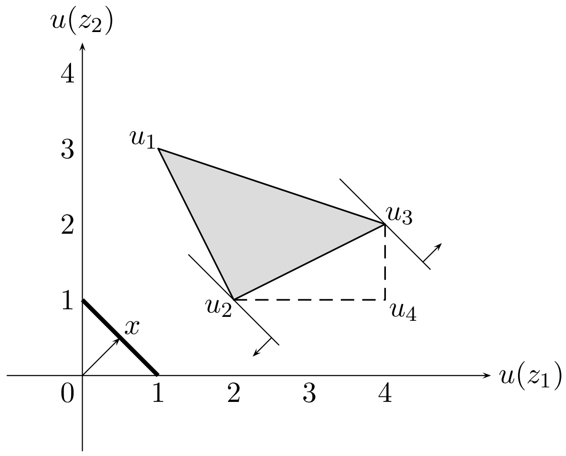

Figure 2: Example of set of utility functions (continued)

Figure 2 recalls the definitions of the four utility functions $u_1, u_2, u_3, u_4 \in P$ on the set $Z = \{z_1, z_2\}$. The right-hand side graph depicts these utility functions but this time in $\mathbb{R}^Z$ rather than $\mathbb{R}^X$ (this is possible, of course, since a vNM utility function $u \in P$ is fully determined by the vector $(u(z_1), u(z_2)) \in \mathbb{R}^Z$). The set $U = \text{conv}(\{u_1, u_2, u_3\})$ of utility functions now corresponds to the shaded triangle $u_1u_2u_3$. The utility interval $U(x)$ of alternative $x \in X$ (the set $X = \Delta(Z)$ of alternatives is represented by the thick segment on the graph) can now be visualized as follows: $\max U(x)$ corresponds to the hyperplane supporting U in the (normal) direction x whereas $\min U(x)$ corresponds to the hyperplane supporting U in the direction $-x$.

Being essentially a compact and convex subset of $\mathbb{R}^Z$, a set of utility functions is fully determined by its supporting hyperplanes in all directions of $\mathbb{R}^Z$. The utility intervals of all alternatives, however, only determine these supporting hyperplanes in the non-negative and non-positive directions and, hence, do not fully pin down the set of utility functions. Thus, the set $U' = \text{conv}(\{u_1, u_2, u_3, u_4\})$ of utility functions (corresponding to the quadrilateral $u_1u_2u_3u_4$), although strictly larger than U, only differs from U in directions with both positive and negative components and, hence, yields the same utility intervals as U for all alternatives.

Nevertheless, there are cases in which the utility intervals of all alternatives fully pin down the set of utility functions. This is true, in particular, if the utility interval of some lottery with full support in Z (such as the lottery x in the figure) is a singleton. Indeed, in this case, the set of utility functions must be entirely contained in some hyperplane in the direction x and, hence, is fully determined by the utility intervals of all alternatives in some neighborhood of x. Since x has full support, all these directions correspond to lotteries in $\Delta(X)$.

The "if" part of Theorem 1 is straightforward. To prove the "only if" part, first note that, from Corollary 4, we know that F satisfies (5). We want to strengthen (5) to (6).

In fact, given our topological assumptions, it turns out to be sufficient to show that (6) holds when restricted to all finite subsets Y of Z, i.e.

$$F((U_i)_{i\in I})|_Y = \left(\left(\bigcup_{\theta\in\Theta}\sum_{i\in I}\theta_i U_i\right)+\gamma\right)\Bigg|_Y .$$

The restriction of a set of utility functions to such a finite set Y is a compact and convex subset of the finite-dimensional Euclidean space $\mathbb{R}^Y$ so, by the argument above and Weak Determinacy Preservation, we know that (6) holds for any profile $(U_i)_{i\in I} \in \mathscr{D}$ such that, for some lottery with full support in Y, $U_i(x) = \{0\}$ for all $i \in I$. Finally, consider a profile $(U_i)_{i\in I} \in \mathscr{D}$ that does not satisfy the latter property. Since Y is finite and Z is infinite, there exists an outcome $z \notin Y$. We can then construct a profile $(U'_i)_{i\in I} \in \mathscr{D}$ such that (i) for some lottery x with full support in $Y \cup \{z\}$, $U'_i(x) = \{0\}$ for all $i \in I$ and (ii) $U'|_Y = U|_Y$ for all $i \in I$. By the latter point, (i) implies that (6) holds for $(U'_i)_{i\in I}$ restricted to $Y \cup \{z\}$ (and, hence, restricted to Y as well). By Setwise Independence of Irrelevant Alternatives, (ii) then implies that (6) also holds for $(U_i)_{i\in I}$ restricted to Y.

Note that Weak Determinacy Preservation plays a crucial role in the proof: it ensures the existence of "well-behaved" profiles for which the social utility intervals of some lotteries are singletons and which, therefore, satisfy (6) (the proof is then completed by extending (6) from these profiles to arbitrary profiles). This explains, in particular, why there is no counterpart to Corollary 1, in the sense that assuming Setwise Independence of Irrelevant Alternatives and Z infinite in Corollary 1 would imply $F((U_i)_{i\in I}) = \bigcup_{(\theta,\gamma)\in\Omega}(\sum_{i\in I}\theta_i U_i + \gamma)$: without Weak Determinacy Preservation, there is no "well-behaved" profile to start from. The role of Setwise Independence of Irrelevant Alternatives and the assumption that Z is infinite is then to extend (6) from the "well-behaved" profiles to arbitrary profiles. In particular, if Z is finite (and under the axioms of Corollary 4 alone), (6) holds for the "well-behaved" profiles but not necessarily for arbitrary profiles. As we will see in the next section, some reasonable restrictions on the domain $\mathscr{D}$ ensure that all profiles are in fact "well-behaved" and, hence, that Theorem 2 and Corollary 5 also hold for finite Z.

6 General alternatives and domains

Up to now we have maintained two assumptions in order to simplify the exposition. First, the set of alternatives is the set of all simple lotteries over some set of outcomes. Second, the domain of the social welfare function is the set of all possible profiles of sets of (vNM) utility functions. Our results, however, also hold for other alternatives and domains. In this section we state general properties of the set of alternatives and the domain of the social welfare functions that are sufficient for our results and we discuss some particular settings in which these properties are satisfied.

6.1 Mixture spaces

Re now assume that X is a *mixture space*, i.e. any set endowed with a mixing operation $[0,1] \times X \times X \to X$, $(\lambda, x, y) \mapsto x\lambda y$, such that for all $x, y \in X$ and all $\lambda, \mu \in [0,1]$,

$$x1y = x, \qquad x\lambda y = y(1-\lambda)x, \qquad (x\lambda y)\mu y = x(\lambda\mu)y.$$

Note that all definitions and axioms introduced so far have been stated with this general notation and, hence, apply to any mixture space.

The set $X = \Delta(Z)$ of all simple lotteries over some set Z of outcomes, endowed with the mixing operation defined above, is an example of mixture space. Another example that has received recent attention in the literature is the set of all compact and convex sets of such lotteries, interpreted as opportunity sets. One can also consider, instead of simple lotteries, lotteries with continuous density on a set of monetary prizes or commodity bundles.

More generally, whenever X is a convex subset of some linear space, defining the mixing operation $x\lambda y = \lambda x + (1-\lambda)y$ by means of the vector addition and scalar multiplication operations turns it into a mixture space. Such a mixture space, in addition, will have the particular property that any two distinct alternatives $x, y \in X$ are *separated*, i.e. there exists $u \in P$ such that $u(x) \neq u(y)$. In fact, Mongin (2001) shows that any two distinct alternatives in X are separated if and only if there exists a mixture preserving bijection from X into a convex subset of some linear space (and that, moreover, the affine dimension of this subset is unique), and provides examples of mixture spaces in which no, or some but not all, distinct alternatives are separated.

We restrict attention to mixture spaces that are *weakly non-degenerate*, i.e. that contain at least two separated alternatives. Moreover, some of our results only hold for mixture spaces that are *strongly non-degenerate*, i.e. that contain at least two distinct alternatives and in which any two distinct alternatives are separated. By the argument above, a mixture space is strongly non-degenerate if and only if it is isomorphic to some non-singleton convex subset of some linear space.

6.2 Weakly regular domains

We now introduce a richness condition on the domain of the social welfare function. Namely, say that a domain $\mathscr{D} \subseteq \mathscr{P}^I$ is *weakly regular* if there exist two alternatives $x, y \in X$, $x \neq y$, such that:

(i) for all non-empty and finite $(W_i)_{i\in I} \subset (\mathbb{R}^{\{x,y\}})^I$, there exists $(U_i)_{i\in I} \in \mathscr{D}$ such that $U_i|_{\{x,y\}} = \text{conv}(W_i)$ for all $i \in I$,

(ii) for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $z \in X$, there exists $(U'_i)_{i\in I} \in \mathscr{D}$ such that $U_i(z) = U'_i(z) = U'_i(x)$ for all $i \in I$.

In words, there must exist a pair of distinct alternatives satisfying the two following properties. First, any profile of (convex hulls of) finite sets of pairs of utility levels corresponds to the restriction of some profile in the domain to this pair of alternatives. Second, for any profile in the domain and any alternative, there exists another profile in the domain in which, for each individual, the utility interval of this alternative is the same as in the initial profile and one alternative in the pair also has this same utility interval. Note that the first property implies that the two alternatives in the pair are separated (and, hence, that X is weakly non-degenerate).

As one can check, the full (vNM) domain $\mathscr{D} = \mathscr{P}^I$ is weakly regular. Another example is the set of all profiles of convex hulls of finite sets of utility functions. If X is a convex subset of some linear space and if the affine dimension of X is greater or equal to 2 (e.g. $X = \Delta(Z)$ with $|Z| \geq 3$), then one may also normalize the utility level of some alternative $x \in X$ (to 0, for simplicity), i.e. take $\mathscr{D}$ to be the set of all profiles of sets of utility functions such that $U_i(x) = 0$ for all $i \in I$. For all these domains, the results of Sections 3 and 4 hold.[13]

Proposition 1. Theorem 1 and Corollaries 1, 2, 3, and 4 hold provided X is a weakly non-degenerate mixture space and $\mathscr{D}$ is a weakly regular domain.

Technically, the second of the two properties in the definition of weak regularity is needed for Interval Independence of Irrelevant Alternatives and Interval Pareto Indifference to imply interval neutrality (see Lemma 2 in the appendix). The first one ensures that any profile of real intervals corresponds to the utility intervals of some alternative in some profile in the domain, and also allows the construction of some particular profiles in the proof of Theorem 1.

6.3 Strongly regular domains

For the results of Section 5, we need a stronger assumption on the domain. We first introduce some notation. A subset Y of X is said to be a *mixture subspace* of X if $x\lambda y \in Y$ for all $x, y \in Y$ and all $\lambda \in [0, 1]$. The *mixture hull* of a subset Y of X is the set $\text{mix}(Y) = \bigcup_{n\in\mathbb{N}} Y_n$, where $Y_0 = Y$ and, for all $n \in \mathbb{N}$, $Y_{n+1} = \{x\lambda y : x, y \in Y_n, \lambda \in [0, 1]\}$ (i.e. $\text{mix}(Y)$ is the smallest mixture subspace of X containing Y). The *mixture interior* of a mixture subspace Y of X is the set $\text{mint}(Y)$ of alternatives $x \in Y$ such that, for all $y \in Y$, there exist $z \in Y$ and $\lambda \in (0, 1)$ such that $x = y\lambda z$.

[13]The proof of Theorem 1 in the appendix is directly stated under these general assumptions, and that the corollaries follow from the theorem is independent of these assumptions.

To illustrate these notions, consider the case where X is a convex subset of some linear space. Then, given a subset Y of X, we have $\text{mix}(Y) = \text{conv}(Y)$ (so Y is a mixture subspace of X if and only if it is convex). Moreover, $\text{mint}(Y)$ is the relative interior of Y if Y is finite dimensional and, more generally, the pseudo relative interior of Y (Borwein and Goebel, 2003).

A domain $\mathscr{D} \subseteq \mathscr{P}^I$ is said to be *strongly regular* if:

(i) $\mathscr{D}$ is weakly regular,

(ii) for all profile $(U_i)_{i \in I} \in \mathscr{D}$ and all finite subset Y of X, there exist a finite subset Y' of X, an alternative $x \in \text{mint}(\text{mix}(Y \cup Y'))$, and a profile $(U'_i)_{i \in I} \in \mathscr{D}$ such that, for all $i \in I$, $U'_i|_Y = U_i|_Y$ and $U'_i(x) = 0$.

In words, what this adds to weak regularity is that for any profile in the domain and any finite subset of alternatives, there must exist a (weakly) larger finite subset of alternatives and a second profile in the domain that agrees with the first profile on the initial subset and for which the utility level of some lottery with "full support" in the larger subset is determinate (and normalized to 0, for simplicity).

We also make a stronger assumption on the set of alternatives by requiring that X be strongly non-degenerate. As explained above, X is then essentially a convex subset of some linear space. Moreover, if the affine dimension of X is infinite (e.g. $X = \Delta(Z)$ with Z infinite) then the full (vNM) domain $\mathscr{D} = \mathscr{P}^I$ is strongly regular. To prove this, for all $U \in \mathscr{P}$ and all finite subset Y of X, fix some maximal affinely independent subsets Z of Y and $Z \cup Z'$ of X with $Z \cap Z' = \emptyset$, and let $Y' = \{z'\}$ for some $z' \in Z'$. Then for all $u \in P$, there exists $u' \in P$ such that $u'|_Z = u|_Z$ and $u'(z') = -\sum_{z \in Z} u(z)$, so that $u'(x) = 0$, where $x = \sum_{z \in Z} \frac{1}{|Z|} z \in \text{mint}(\text{mix}(Y \cup Y'))$. Hence, letting $U' = \{u' : u \in U\}$, we have $U' \in \mathscr{P}$, $U'|_Y = U|_Y$, and $U'(x) = 0$.

As above, one may also restrict attention to the profiles of convex hulls of finite sets of utility functions. Now, if the affine dimension of X is finite (e.g. $X = \Delta(Z)$ with Z finite) then these domains are no longer strongly regular. Indeed, a finite subset Y of X may then have full affine dimension in X and, in this case, $U'_i|_Y = U_i|_Y$ implies $U'_i = U_i$ in the definition of strong regularity. This is the reason why Theorem 2 and Corollary 5 do not hold for finite Z.

However, in this case, we can obtain a strongly regular domain by normalizing the utility level of some alternative $x \in \text{mint}(X)$ (to 0, for simplicity), provided the affine dimension of X is greater or equal to 2 (i.e. $|Z| \geq 3$). Indeed, in this case, strong regularity is always verified for this x and $Y' = Z$. For all these domains, the results of Section 5 hold.[14]

[14] Again, the proof of Theorem 2 in the appendix is directly stated under these general assumptions, and that the corollary follows from the theorem is independent of these assumptions.

	Interval IIA Interval PI	Interval IIA Interval PWP	Setwise IIA Interval PWP
-	Interval neutrality	Max-min neutrality	
Weak DP		Local multi- utilitarianism	Multi- utilitarianism
Strong DP		Local utilitarianism	Utilitarianism

Figure 3: Summary of results

Proposition 2. Theorem 2 and Corollary 5 hold provided X is a strongly non-degenerate mixture space and $\mathscr{D}$ is a strongly regular domain.

Technically, the role of the additional condition in the definition of strong regularity is to ensure that the step of the proof of Theorem 2 in which arbitrary profiles are linked to "well-behaved" profiles (see the proof sketch above) can still be carried out. Indeed, whereas the possibility of such links is granted in the particular case where $X = \Delta(Z)$ with Z infinite and $\mathscr{D} = \mathscr{P}^I$, it must be explicitly assumed in the more general setting of this section.

7 Conclusion

This paper is concerned with the problem of aggregating indeterminate utilities. We have formalized the problem by endowing both individuals and society with sets of utility functions, generalized the classical neutrality assumption to this setting and characterized the class of neutral social welfare function. This class turns out to be considerably broader for indeterminate than for determinate utilities, even under an additional "Pareto preference" assumption, and in particular aggregation rules may differ by the relationship between individual and social indeterminacy. We have characterized several subclasses of neutral aggregation rules, which are summarized in Figure 3 (moving eastwards or downwards in the figure corresponds to stronger assumptions). The most specific of these subclasses is utilitarianism and, in particular, utilitarian rules are those that yield the least indeterminate social utilities, although they still fail to systematically yield a determinate social utility. Thus, an aggregation rule that systematically yields a determinate social utility must necessarily violate the neutrality assumption.

Appendix: proofs

We directly state the proofs in the general setting of Section 6.

Preliminary lemmas

Lemma 1. Let X be a non-empty mixture space. Then:

(a) For all non-empty and finite $U \subset P$, $\text{conv}(U) \in \mathscr{P}$.

(b) For all $U \in \mathscr{P}$ and $x \in X$, $U(x)$ is a non-empty and compact interval.

(c) For all $U \in \mathscr{P}$, $x, y \in X$, and $\lambda \in [0,1]$, $U(x\lambda y) \subseteq \lambda U(x) + (1-\lambda)U(y)$.

(d) For all $U \in \mathscr{P}$, $x, y \in X$, and $\lambda \in [0,1]$, $\lambda \max U(x) + (1-\lambda) \min U(y) \in U(x\lambda y)$.

Proof. (a) Let $U \subset P$ be non-empty and finite. Then $\text{conv}(U)$ is non-empty and convex by definition. Moreover, U is compact in $\mathbb{R}^X$ since it is finite. Hence $\text{conv}(U)$ is compact in $\mathbb{R}^X$ (Aliprantis and Border, 1999, Theorem 5.1, Corollary 5.15). Hence $\text{conv}(U)$ is compact in P since P is a topological subspace of $\mathbb{R}^X$, so $\text{conv}(U) \in \mathscr{P}$.

(b) Let $U \in \mathscr{P}$ and $x \in X$. Clearly, $U(x)$ is non-empty and convex since U is. Moreover, $U(x) = \text{proj}_x(U)$ by definition.[15] Hence $U(x)$ is compact since proj_x is continuous by definition of the product topology on $\mathbb{R}^X$ (Aliprantis and Border, 1999, Theorem 2.31), so $U(x)$ is a non-empty and compact interval.

(c) Let $U \in \mathscr{P}$, $x, y \in X$, and $\lambda \in [0,1]$. Then by definition,

$$\begin{aligned} U(x\lambda y) &= \{u(x\lambda y) : u \in U\} \\ &= \{\lambda u(x) + (1-\lambda)u(y) : u \in U\} \\ &\subseteq \{\lambda u(x) + (1-\lambda)v(y) : u, v \in U\} \\ &= \lambda U(x) + (1-\lambda)U(y). \end{aligned}$$

(d) Let $U \in \mathscr{P}$, $x, y \in X$, and $\lambda \in [0,1]$. Then by definition, there exist $u, u' \in U$ such that $u(x) = \max U(x)$ and $u'(y) = \min U(y)$. Suppose $\lambda \max U(x) + (1-\lambda) \min U(y) > \max U(x\lambda y)$. Then $\lambda \max U(x) + (1-\lambda) \min U(y) > u(x\lambda y) = \lambda u(x) + (1-\lambda)u(y)$. Hence, since $u(x) = \max U(x)$, it must be that $u(y) < \min U(y)$, a contradiction since $u \in U$. Similarly, suppose $\lambda \max U(x) + (1-\lambda) \min U(y) < \min U(x\lambda y)$. Then $\lambda \max U(x) + (1-\lambda) \min U(y) < u'(x\lambda y) = \lambda u'(x) + (1-\lambda)u'(y)$. Hence, since $u'(y) = \min U(y)$, it must be that $u'(x) > \max U(x)$, a contradiction since $u' \in U$. □

Lemma 2. Let X be a weakly non-degenerate mixture space and $\mathscr{D}$ be a weakly regular domain. Then a social welfare function F satisfies Interval Independence of Irrelevant Alternatives and Interval Pareto Indifference if and only if it is interval neutral.

Proof. Clearly, interval neutrality implies Interval Independence of Irrelevant Alternatives and Interval Pareto Indifference. Conversely, assume F satisfies these two axioms and

[15] Given a set $S = \Pi_{j \in J} S_j$, proj_j denotes the projection from S onto S_j.

let $(U_i)_{i\in I}, (U_i')_{i\in I} \in \mathscr{D}$ and $x, y \in X$ such that $U_i(x) = U_i'(y)$ for all $i \in I$. Then, by part (ii) of weak regularity, there exist $z \in X$ and $(V_i)_{i\in I}, (V_i')_{i\in I} \in \mathscr{D}$ such that, for all $i \in I$, $U_i(x) = V_i(x) = V_i(z) = V_i'(z) = V_i'(y) = U_i'(y)$. Hence, by successive applications of Interval Independence of Irrelevant Alternatives and Interval Pareto Indifference, we have $F((U_i)_{i\in I})(x) = F((V_i)_{i\in I})(x) = F((V_i)_{i\in I})(z) = F((V_i')_{i\in I})(z) = F((V_i')_{i\in I})(y) = F((U_i')_{i\in I})(y)$, so F is interval neutral. □

Proof of Theorem 1

Assume X is a weakly regular mixture space and $\mathscr{D}$ is a weakly regular domain (see Section 6 for definitions). Clearly, if there exists a non-empty, compact, and convex set $\Phi \subset (\mathbb{R}_+^I)^2 \times \mathbb{R}$ such that (1) holds then F satisfies Interval Independence of Irrelevant Alternatives and Interval Pareto Indifference. Conversely, assume F satisfies these two axioms. Note that for a given non-empty, compact, and convex set $\Phi \subset (\mathbb{R}_+^I)^2 \times \mathbb{R}$, (1) holds if and only if, for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$,

$$\begin{aligned}
\max F((U_i)_{i\in I})(x) &= \max_{(\alpha,\beta,\gamma)\in\Phi} \left(\sum_{i\in I} \alpha_i \max U_i(x) - \sum_{i\in I} \beta_i \min U_i(x) + \gamma \right), \\
\min F((U_i)_{i\in I})(x) &= \min_{(\alpha,\beta,\gamma)\in\Phi} \left(\sum_{i\in I} \alpha_i \min U_i(x) - \sum_{i\in I} \beta_i \max U_i(x) + \gamma \right).
\end{aligned} \tag{8}$$

Let $\mathscr{K}$ denote the set of all non-empty and compact real intervals. Since $\mathscr{D}$ is weakly regular, there exists $x \in X$ such that, for all $(K_i)_{i\in I} \in \mathscr{K}^I$, there exists $(U_i)_{i\in I} \in \mathscr{D}$ such that $U_i(x) = K_i$ for all $i \in I$. Hence $\{(U_i(x))_{i\in I} : (U_i)_{i\in I} \in \mathscr{D}, x \in X\} = \mathscr{K}^I$ so, by Interval Independence of Irrelevant Alternatives and Interval Pareto Indifference, there exists a unique function $G : \mathscr{K}^I \to \mathscr{K}$ such that, for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$,

$$F((U_i)_{i\in I})(x) = G((U_i(x))_{i\in I}).$$

Let $T = \{(r, s) \in (\mathbb{R}^I)^2 : r + s \geq 0\}$, which is clearly a half-space, and define the functions $\overline{G}, \underline{G} : (\mathbb{R}^I)^2 \to \mathbb{R} \cup \{-\infty, +\infty\}$ by, for all $(r, s) \in (\mathbb{R}^I)^2$,

$$\overline{G}(r,s) = \begin{cases} \max G(([-s_i, r_i])_{i\in I}) & \text{if } (r,s) \in T, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\underline{G}(r,s) = \begin{cases} -\min G(([-r_i, s_i])_{i\in I}) & \text{if } (r,s) \in T, \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, $\mathrm{dom}(\overline{G}) = \mathrm{dom}(\underline{G}) = T$.[16] Moreover, $\overline{G}(r,s) > -\infty$ and $\underline{G}(r,s) > -\infty$ for all $(r,s) \in (\mathbb{R}^I)^2$, so $\overline{G}$ and $\underline{G}$ are proper. Also note that, for all $(r,s) \in T$,

$$\overline{G}(r,s) + \underline{G}(s,r) = \max G(([-s_i, r_i])_{i\in I}) - \min G(([-s_i, r_i])_{i\in I}) \geq 0.$$

Finally, for all $(U_i)_{i\in I} \in \mathscr{D}$ and all $x \in X$, we have

$$\begin{aligned} \max F((U_i)_{i\in I})(x) &= \max G((U_i(x))_{i\in I}) = \overline{G}((\max U_i(x), -\min U_i(x))_{i\in I}), \\ \min F((U_i)_{i\in I})(x) &= \min G((U_i(x))_{i\in I}) = -\underline{G}((-\min U_i(x), \max U_i(x))_{i\in I}), \end{aligned}$$

so for a given non-empty, compact, and convex set $\Phi \subset (\mathbb{R}^I_+)^2 \times \mathbb{R}$, (8) holds if and only if, for all $(r,s) \in T$,

$$\overline{G}(r,s) = \max_{(\alpha,\beta,\gamma)\in\Phi} (\alpha r + \beta s - \gamma), \qquad \underline{G}(r,s) = \max_{(\alpha,\beta,\gamma)\in\Phi} (\alpha r + \beta s + \gamma). \tag{9}$$

Lemma 3. $\overline{G}$ and $\underline{G}$ are convex.

Proof. We only state the proof for $\overline{G}$, the argument for $\underline{G}$ is similar. Let $(r,s), (r',s') \in T$. Since $\mathscr{D}$ is weakly regular, there exist $x, y \in X$, $x \neq y$, and $(U_i)_{i\in I} \in \mathscr{D}$ such that $U_i|_{\{x,y\}} = \mathrm{conv}(\{w_i, w_i'\})$ for all $i \in I$, where $w_i, w_i' \in \mathbb{R}^{\{x,y\}}$ are defined by

$$\begin{aligned} w_i(x) &= -s_i, & w_i'(x) &= r_i, \\ w_i(y) &= -s_i', & w_i'(y) &= r_i'. \end{aligned}$$

Note that for all $i \in I$ and all $\lambda \in [0,1]$, we have $U_i(x\lambda y) = [-\lambda s_i - (1-\lambda)s_i', \lambda r_i + (1-\lambda)r_i']$. Hence, for all $\lambda \in [0,1]$,

$$\begin{aligned} \overline{G}(\lambda(r,s) + (1-\lambda)(r',s')) &= \overline{G}(\lambda r + (1-\lambda)r', \lambda s + (1-\lambda)s') \\ &= \max G(([-\lambda s_i - (1-\lambda)s_i', \lambda r_i + (1-\lambda)r_i'])_{i\in I}) \\ &= \max G((U_i(x\lambda y))_{i\in I}) \\ &= \max F((U_i)_{i\in I})(x\lambda y) \\ &\leq \lambda \max F((U_i)_{i\in I})(x) + (1-\lambda) \max F((U_i)_{i\in I})(y) \\ &= \lambda \max G((U_i(x))_{i\in I}) + (1-\lambda) \max G((U_i(y))_{i\in I}) \\ &= \lambda \max G(([-s_i, r_i])_{i\in I}) + (1-\lambda) \max G(([-s_i', r_i'])_{i\in I}) \\ &= \lambda \overline{G}(r,s) + (1-\lambda)\overline{G}(r',s'), \end{aligned}$$

where the inequality follows from Lemma 1(c). □

Lemma 4. $\overline{G}$ and $\underline{G}$ are non-decreasing.

[16] Given a function f, $\mathrm{dom}(f)$ denotes the effective domain of f.

Proof. We only state the proof for $\overline{G}$, the argument for $\underline{G}$ is similar. We first prove that $\overline{G}$ is non-decreasing on $T' = \{(r,s) \in (\mathbb{R}^I)^2 : r+s > 0\} \subset T$. To this end, it is sufficient to show that $\overline{G}(r',s') \geq \overline{G}(r,s)$ for all $(r,s),(r',s') \in T'$ such that $(r',s') \geq (r,s)$ and $r'+s' \leq 2(r+s)$. So let $(r,s),(r',s') \in T'$ such that $(r',s') \geq (r,s)$ and $r'+s' \leq 2(r+s)$. Since $\mathscr{D}$ is weakly regular, there exist $x,y \in X$, $x \neq y$, and $(U_i)_{i\in I} \in \mathscr{D}$ such that $U_i|_{\{x,y\}} = \text{conv}(\{w_i, w_i', w_i'', w_i'''\})$ for all $i \in I$, where $w_i, w_i', w_i'', w_i''' \in \mathbb{R}^{\{x,y\}}$ are defined by

$$\begin{aligned}
w_i(x) &= -s_i', & w_i'(x) &= s_i' - 2s_i, & w_i''(x) &= r_i', & w_i'''(x) &= 2r_i - r_i',\\
w_i(y) &= s_i' - 2s_i, & w_i'(y) &= -s_i', & w_i''(y) &= 2r_i - r_i', & w_i'''(y) &= r_i'.
\end{aligned}$$

Note that for all $i \in I$, we have $U_i(x) = U_i(y) = [-s_i', r_i']$ and $U_i(x\frac{1}{2}y) = [-s_i, r_i]$. Hence,

$$\begin{aligned}
\overline{G}(r,s) &= \max G(([-s_i, r_i])_{i\in I})\\
&= \max G((U_i(x\tfrac{1}{2}y))_{i\in I})\\
&= \max F((U_i)_{i\in I})(x\tfrac{1}{2}y)\\
&\leq \tfrac{1}{2}\max F((U_i)_{i\in I})(x) + \tfrac{1}{2}\max F((U_i)_{i\in I})(y)\\
&= \tfrac{1}{2}\max G((U_i(x))_{i\in I}) + \tfrac{1}{2}\max G((U_i(y))_{i\in I})\\
&= \tfrac{1}{2}\max G(([-s_i', r_i'])_{i\in I}) + \tfrac{1}{2}\max G(([-s_i', r_i'])_{i\in I})\\
&= \overline{G}(r',s'),
\end{aligned}$$

where the inequality follows from Lemma 1(c).

It only remains to prove that $\overline{G}(r',s') \geq \overline{G}(r,s)$ for all $(r,s),(r',s') \in T$ such that $(r',s') \geq (r,s)$ and $r+s=0$. So suppose there exist $(r,s),(r',s') \in T$ such that $(r',s') \geq (r,s)$, $r+s=0$, and $\overline{G}(r',s') < \overline{G}(r,s)$. Clearly, it must then be that $r'+s' > r+s$, so $(r',s') \in T'$. Since $\mathscr{D}$ is weakly regular, there exist $x,y \in X$, $x \neq y$, and $(U_i)_{i\in I} \in \mathscr{D}$ such that $U_i|_{\{x,y\}} = \text{conv}(\{w_i, w_i'\})$ for all $i \in I$, where $w_i, w_i' \in \mathbb{R}^{\{x,y\}}$ are defined by

$$\begin{aligned}
w_i(x) &= -s_i = r_i, & w_i'(x) &= -s_i = r_i,\\
w_i(y) &= -s_i', & w_i'(y) &= r_i'.
\end{aligned}$$

Note that for all $i \in I$ and all $\lambda \in [0,1]$, we have $U_i(x\lambda y) = [-\lambda s_i - (1-\lambda)s_i', \lambda r_i + (1-\lambda)r_i']$. Hence,

$$\begin{aligned}
\max F((U_i)_{i\in I})(x) &= \max G((U_i(x))_{i\in I})\\
&= \max G([-s_i, r_i])_{i\in I})\\
&= \overline{G}(r,s)\\
&> \overline{G}(r',s')
\end{aligned}$$

$$
\begin{aligned}
&= \max G([-s'_i, r'_i])_{i\in I}) \\
&= \max G((U_i(y))_{i\in I}) \\
&= \max F((U_i)_{i\in I})(y),
\end{aligned}
$$

so there exists $\lambda \in (0,1)$ such that $\lambda \max F((U_i)_{i\in I})(x) + (1-\lambda)\min F((U_i)_{i\in I})(y) > \max F((U_i)_{i\in I})(y)$. Moreover,

$$
\begin{aligned}
\max F((U_i)_{i\in I})(x\lambda y) &= \max G((U_i(x\lambda y))_{i\in I}) \\
&= \max G([-\lambda s_i - (1-\lambda)s'_i, \lambda r_i + (1-\lambda)r'_i])_{i\in I}) \\
&= \overline{G}(\lambda(r,s) + (1-\lambda)(r',s')) \\
&\leq \overline{G}(r',s') \\
&= \max F((U_i)_{i\in I})(y),
\end{aligned}
$$

where the inequality follows from the previous paragraph since $\lambda(r,s) + (1-\lambda)(r',s') \in T'$ and $\lambda(r,s) + (1-\lambda)(r',s') < (r,s)$. It follows that $\lambda \max F((U_i)_{i\in I})(x) + (1-\lambda)\min F((U_i)_{i\in I})(y) > \max F((U_i)_{i\in I})(x\lambda y)$, a contradiction by Lemma 1(d). □

Lemma 5. $\overline{G}$ and $\underline{G}$ are continuous.

Proof. We only state the proof for $\overline{G}$, the argument for $\underline{G}$ is similar. By Lemma 3, $\overline{G}$ is upper semi-continuous since T is a half-space (Rockafellar, 1970, Theorem 10.2), so it is sufficient to prove that $\overline{G}$ is lower semi-continuous, i.e. that $\operatorname{cl}\overline{G} = \overline{G}$.[17] By definition, $\operatorname{cl}\overline{G} \leq \overline{G}$. Conversely, let $(r,s) \in (\mathbb{R}^I)^2$. If $(r,s) \notin T$ then $\operatorname{cl}\overline{G}(r,s) = \overline{G}(r,s) = +\infty$ since T is closed. If $(r,s) \in T$ then let $(r',s') \in (\mathbb{R}^I)^2$ such that $(r',s') > (r,s)$. Then (r',s') belongs to the relative interior of T and, hence, $\operatorname{cl}\overline{G}(r,s) = \lim_{\lambda\to 0_+} \overline{G}((1-\lambda)(r,s) + \lambda(r',s'))$ (Rockafellar, 1970, Theorem 7.5). By Lemma 4, we have $\overline{G}(r,s) \leq \overline{G}((1-\lambda)(r,s) + \lambda(r',s'))$ for all $\lambda \in [0,1]$ and, hence, $\overline{G}(r,s) \leq \operatorname{cl}\overline{G}(r,s)$. □

Let $\overline{G}0^+, \underline{G}0^+ : (\mathbb{R}^I)^2 \to \mathbb{R} \cup \{-\infty, +\infty\}$ denote the recession functions of $\overline{G}$ and $\underline{G}$, respectively, i.e. (Rockafellar, 1970, Theorem 8.5) for all $(r,s) \in (\mathbb{R}^I)^2$,

$$
\overline{G}0^+(r,s) = \lim_{\mu\to+\infty} \frac{\overline{G}(\mu(r,s))}{\mu}, \qquad \underline{G}0^+(r,s) = \lim_{\mu\to+\infty} \frac{\underline{G}(\mu(r,s))}{\mu}.
$$

$\overline{G}0^+$ and $\underline{G}0^+$ are positively homogenous, proper convex functions by definition, and are closed since $\overline{G}$ and $\underline{G}$ are closed. Also note that $\overline{G}0^+$ and $\underline{G}0^+$ are non-decreasing by Lemma 4.

Lemma 6. $\operatorname{dom}(\overline{G}0^+) = T$ and $\underline{G}0^+ = \overline{G}0^+$.

[17]Given a function f, $\operatorname{cl} f$ denotes the closure of f.

Proof. Let $(r,s) \in (\mathbb{R}^I)^2$. If $(r,s) \notin T$ then, clearly, $\overline{G}0^+(r,s) = \underline{G}0^+(r,s) = +\infty$. Hence it is sufficient to show that if $(r,s) \in T$ then $\lim_{\mu\to+\infty} \frac{\overline{G}(\mu(r,s))}{\mu} = \lim_{\mu\to+\infty} \frac{\underline{G}(\mu(r,s))}{\mu} < +\infty$. So assume $r+s \geq 0$ and let $\mu > 1$. Since $\mathscr{D}$ is weakly regular, there exist $x, y \in X$, $x \neq y$, and $(U_i)_{i\in I} \in \mathscr{D}$ such that $U_i|_{\{x,y\}} = \operatorname{conv}(\{w_i, w_i'\})$ for all $i \in I$, where $w_i, w_i' \in \mathbb{R}^{\{x,y\}}$ are defined by

$$w_i(x) = -\mu s_i, \qquad w_i'(x) = -\mu r_i,$$
$$w_i(y) = \mu r_i, \qquad w_i'(y) = \mu s_i.$$

Note that for all $i \in I$, we have $U_i(x) = [-\mu s_i, \mu r_i]$, $U_i(y) = [-\mu r_i, \mu s_i]$, $U_i(x\frac{1}{2}y) = \{0\}$, and $U_i(x\frac{1}{\mu}(x\frac{1}{2}y)) = [-s_i, r_i]$. Hence,

$$\begin{aligned}\tfrac{1}{\mu}\overline{G}(\mu(r,s)) - (1-\tfrac{1}{\mu})\underline{G}(0,0) &= \tfrac{1}{\mu}\max F((U_i)_{i\in I})(x) + (1-\tfrac{1}{\mu})\min F((U_i)_{i\in I})(x\tfrac{1}{2}y)\\ &\leq \max F((U_i)_{i\in I})(x\tfrac{1}{\mu}(x\tfrac{1}{2}y))\\ &= \overline{G}(r,s),\end{aligned}$$

where the inequality follows from Lemma 1(d), so $\frac{1}{\mu}\overline{G}(\mu(r,s)) \leq \overline{G}(r,s) + (1-\frac{1}{\mu})\underline{G}(0,0)$. Since this must hold for all $\mu > 1$, we have $\lim_{\mu\to+\infty} \frac{1}{\mu}\overline{G}(\mu(r,s)) \leq \overline{G}(r,s) + \underline{G}(0,0) < +\infty$. Moreover, again by Lemma 1(d),

$$\begin{aligned}\tfrac{1}{2}\overline{G}(\mu(r,s)) - \tfrac{1}{2}\underline{G}(\mu(r,s)) &= \tfrac{1}{2}\max F((U_i)_{i\in I})(x) + \tfrac{1}{2}\min F((U_i)_{i\in I})(y)\\ &\in [\min F((U_i)_{i\in I})(x\tfrac{1}{2}y), \max F((U_i)_{i\in I})(x\tfrac{1}{2}y)]\\ &= [-\underline{G}(0,0), \overline{G}(0,0)],\end{aligned}$$

so $-\frac{2}{\mu}\underline{G}(0,0) \leq \frac{1}{\mu}\overline{G}(\mu(r,s)) - \frac{1}{\mu}\underline{G}(\mu(r,s)) \leq \frac{2}{\mu}\overline{G}(0,0)$. Since this must hold for all $\mu > 1$, we have $\lim_{\mu\to+\infty} \frac{\overline{G}(\mu(r,s))}{\mu} = \lim_{\mu\to+\infty} \frac{\underline{G}(\mu(r,s))}{\mu}$. □

Lemma 7. $\overline{G}$, $\underline{G}$, $\overline{G}0^+$, and $\underline{G}0^+$ are Lipschitzian.

Proof. We only state the proof for $\overline{G}$ and $\overline{G}0^+$, the argument for $\underline{G}$ and $\underline{G}0^+$ is similar. First, for all $(r,s) \in (\mathbb{R}^I)^2$, let $T(r,s) = ((-\min\{-r_i, s_i\}, \max\{-r_i, s_i\})_{i\in I})$. Note that $T(r,s) \in T$ for all $(r,s) \in (\mathbb{R}^I)^2$ and $T(r,s) = (r,s)$ for all $(r,s) \in T$. Define the function $\overline{g} : (\mathbb{R}^I)^2 \to \mathbb{R} \cup \{-\infty, +\infty\}$ by, for all $(r,s) \in (\mathbb{R}^I)^2$, $\overline{g}(r,s) = \overline{G}(T(r,s))$. Clearly, $\operatorname{dom}(\overline{g}) = (\mathbb{R}^I)^2$ and, for all $(r,s) \in T$, $\overline{g}(r,s) = \overline{G}(r,s)$. Moreover, $\overline{g}$ is convex since, for all $(r,s), (r',s') \in (\mathbb{R}^I)^2$ and all $\lambda \in [0,1]$,

$$\overline{g}(\lambda(r,s) + (1-\lambda)(r',s')) = \overline{G}\left(\begin{pmatrix} -\min\{\lambda(-r_i) + (1-\lambda)(-r_i'), \lambda s_i + (1-\lambda)s_i'\}, \\ \max\{\lambda(-r_i) + (1-\lambda)(-r_i'), \lambda s_i + (1-\lambda)s_i'\} \end{pmatrix}_{i\in I}\right)$$

$$
\begin{aligned}
&\leq \overline{G}\left(\begin{pmatrix} -(\lambda\min\{-r_i,s_i\}+(1-\lambda)\min\{-r'_i,s'_i\}), \\ \lambda\max\{-r_i,s_i\}+(1-\lambda)\max\{-r'_i,s'_i\}\end{pmatrix}_{i\in I}\right) \\
&= \overline{G}\begin{pmatrix} \lambda(-\min\{-r_i,s_i\},\max\{-r_i,s_i\})_{i\in I} \\ +(1-\lambda)(-\min\{-r'_i,s'_i\},\max\{-r'_i,s'_i\})_{i\in I}\end{pmatrix} \\
&\leq \begin{matrix} \lambda\overline{G}((-\min\{-r_i,s_i\},\max\{-r_i,s_i\})_{i\in I}) \\ +(1-\lambda)\overline{G}((-\min\{-r'_i,s'_i\},\max\{-r'_i,s'_i\})_{i\in I})\end{matrix} \\
&= \lambda\overline{g}(r,s)+(1-\lambda)\overline{g}(r',s'),
\end{aligned}
$$

where the first inequality follows from Lemma 4 and the second one from Lemma 3. Now, suppose $\overline{G}$ is not Lipschitzian. Then $\overline{g}$ is not Lipschitzian either. Hence, since $\overline{g}$ is finite and convex, there must exist $(r,s)\in(\mathbb{R}^I)^2$ such that $\overline{g}0^+(r,s)=+\infty$ (Rockafellar, 1970, Theorem 10.5), i.e. $\overline{G}0^+(T(r,s))=+\infty$, a contradiction by Lemma 6. Finally, note that for all $(r,s)\in(\mathbb{R}^I)^2$, $\overline{g}0^+(r,s)=\overline{G}0^+(T(r,s))<+\infty$ by Lemma 6. Hence $\overline{g}0^+$ is Lipschitzian since it is its own recession function (Rockafellar, 1970, Theorem 10.5), so $\overline{G}0^+$ is Lipschitzian. □

Lemma 8. For all $(r,s),(r',s')\in(\mathbb{R}^I)^2$, if $\overline{G}0^{+\prime}((r,s),(r',s'))=-\overline{G}0^{+\prime}((r,s),-(r',s'))$ then[18]

$$
\begin{aligned}
\overline{G}0^{+\prime}((r,s),(r',s')) &= \lim_{\mu\to+\infty}\overline{G}'(\mu(r,s),(r',s')) \\
&= \lim_{\mu\to+\infty}-\overline{G}'(\mu(r,s),-(r',s')) \\
&= \lim_{\mu\to+\infty}\underline{G}'(\mu(r,s),(r',s')) \\
&= \lim_{\mu\to+\infty}-\underline{G}'(\mu(r,s),-(r',s')).
\end{aligned}
$$

Proof. We only state the proof for $\overline{G}$, the argument for $\underline{G}$ is similar, given Lemma 6. Let $(r,s),(r',s')\in(\mathbb{R}^I)^2$. If $(r,s)\notin T$ then the lemma is trivially true (all directional derivatives are equal to $+\infty$), so assume $(r,s)\in T$. Then, by definition of $\overline{G}0^{+\prime}$, for all $\varepsilon>0$, there exists $\delta_0>0$ such that

$$
\frac{\overline{G}0^+((r,s)+\delta_0(r',s'))-\overline{G}0^+(r,s)}{\delta_0}\leq\overline{G}0^{+\prime}(r,s)+\varepsilon
$$

and, hence,

$$
\lim_{\mu\to+\infty}\frac{\overline{G}(\mu(r,s)+\mu\delta_0(r',s'))-\overline{G}(\mu(r,s))}{\mu\delta_0}\leq\overline{G}0^{+\prime}(r,s)+\varepsilon.
$$

[18]Given a function f and a point $x\in\mathrm{dom}(f)$, $f'(x,y)$ denotes the (one-sided) directional derivative of f at x in direction y, i.e. $f'(x,y)=\lim_{\delta\to0_+}\frac{f(x+\delta y)-f(x)}{\delta}$ if $x\in\mathrm{dom}(f)$ and $f'(x,y)=+\infty$ otherwise.

Hence, for all $\varepsilon > 0$, there exist $\delta_0 > 0$ and $\mu_0 > 1$ such that, for all $\mu \geq \mu_0$,

$$\frac{\overline{G}(\mu(r,s) + \mu\delta_0(r',s')) - \overline{G}(\mu(r,s))}{\mu\delta_0} \leq \overline{G}0^{+'}(r,s) + \varepsilon.$$

Moreover, since $\mu(r,s) + \delta_0(r',s') = (1 - \frac{1}{\mu})(\mu(r,s)) + \frac{1}{\mu}(\mu(rs) + \mu\delta_0(r',s'))$, we have

$$\overline{G}(\mu(r,s) + \delta_0(r',s')) \leq (1 - \tfrac{1}{\mu})\overline{G}(\mu(r,s)) + \tfrac{1}{\mu}\overline{G}(\mu(rs) + \mu\delta_0(r',s'))$$

by Lemma 3 and, hence,

$$\frac{\overline{G}(\mu(r,s) + \delta_0(r',s')) - \overline{G}(\mu(r,s))}{\delta_0} \leq \frac{\overline{G}(\mu(r,s) + \mu\delta_0(r',s')) - \overline{G}(\mu(r,s))}{\mu\delta_0}.$$

Hence, for all $\varepsilon > 0$, there exist $\delta_0 > 0$ and $\mu_0 > 1$ such that, for all $\mu \geq \mu_0$,

$$\frac{\overline{G}(\mu(r,s) + \delta_0(r',s')) - \overline{G}(\mu(r,s))}{\delta_0} \leq \overline{G}0^{+'}(r,s) + \varepsilon$$

and, hence, for all $\delta \in (0, \delta_0)$,

$$\frac{\overline{G}(\mu(r,s) + \delta(r',s')) - \overline{G}(\mu(r,s))}{\delta} \leq \overline{G}0^{+'}(r,s) + \varepsilon$$

by Lemma 3 (Rockafellar, 1970, Theorem 23.1). Hence, by definition of $\overline{G}'$, for all $\varepsilon > 0$, there exists $\mu_0 > 1$ such that $\overline{G}'(\mu(r,s),(r',s')) \leq \overline{G}0^{+'}(r,s) + \varepsilon$ for all $\mu \geq \mu_0$.

Now, assume $\overline{G}0^{+'}((r,s),(r',s')) = -\overline{G}0^{+'}((r,s),-(r',s'))$. Then, by the preceding paragraph, for all $\varepsilon > 0$, there exists $\mu_0 > 1$ such that, for all $\mu \geq \mu_0$,

$$\begin{aligned}
\overline{G}'(\mu(r,s),(r',s')) - \varepsilon &\leq \overline{G}0^{+'}(r,s) \\
&= -\overline{G}0^{+'}((r,s),-(r',s')) \\
&\leq -\overline{G}'(\mu(r,s),-(r',s')) + \varepsilon \\
&\leq \overline{G}(\mu(r,s),(r',s')) + \varepsilon
\end{aligned}$$

(where the last inequality follows from Rockafellar, 1970, Theorem 23.1), so we obtain $\lim_{\mu\to+\infty} \overline{G}'(\mu(r,s),(r',s')) = \lim_{\mu\to+\infty} -\overline{G}'(\mu(r,s),-(r',s')) = \overline{G}0^{+'}((r,s),(r',s'))$ by passing to the limit as $\varepsilon \to 0_+$. $\square$

Let $\overline{G}^*, \underline{G}^* : (\mathbb{R}^I)^2 \to \mathbb{R} \cup \{-\infty, +\infty\}$ denote the conjugate functions of $\overline{G}$ and $\underline{G}$, respectively, i.e. for all $(\alpha,\beta) \in (\mathbb{R}^I)^2$,

$$\overline{G}^*(\alpha,\beta) = \sup_{(r,s)\in T} (\alpha r + \beta s - \overline{G}(r,s)), \qquad \underline{G}^*(\alpha,\beta) = \sup_{(r,s)\in T} (\alpha r + \beta s - \underline{G}(r,s)).$$

Since $\overline{G}$ and $\underline{G}$ are closed proper convex functions, so are $\overline{G}^*$ and $\underline{G}^*$ (Rockafellar, 1970, Theorem 12.2). Moreover, $\overline{G}^*$ and $\underline{G}^*$ are clearly bounded below since, for all $(\alpha, \beta) \in (\mathbb{R}^I)^2$, $\overline{G}^*(\alpha, \beta) \geq -\overline{G}(0, 0)$ and $\underline{G}^*(\alpha, \beta) \geq -\underline{G}(0, 0)$.

Let $C = \{(-\eta, -\eta) : \eta \in \mathbb{R}^I_+\} \subset (\mathbb{R}^I_+)^2$. Clearly, C is a non-empty, closed, and convex cone containing no line and, moreover, we have $C = T^\circ$.[19] Let $L \subset (\mathbb{R}^I)^2$ be the set of points where $\overline{G}0^+$ is differentiable. Clearly, $\mathrm{cl}(L) = T$ (Rockafellar, 1970, Theorem 25.5), so L is non-empty.[20] Let $E = \{\nabla\overline{G}0^+(r, s) : (r, s) \in L\}$ and $M = \mathrm{cl}(\mathrm{conv}(E))$. By Lemmas 4 and 7, E is a non-empty and bounded subset of $(\mathbb{R}^I_+)^2$ and, hence, M is a non-empty, compact, and convex subset of $(\mathbb{R}^I_+)^2$.

Lemma 9. $\mathrm{dom}(\overline{G}^*) = \mathrm{dom}(\underline{G}^*) = M + C$ and, for all $(\alpha, \beta) \in M + C$, $\overline{G}^*(\alpha, \beta) + \underline{G}^*(\alpha, \beta) \leq 0$.

Proof. By definition of $\overline{G}^*$, $\overline{G}0^+$, $\underline{G}^*$, and $\underline{G}0^+$, we have

$$\mathrm{cl}(\mathrm{dom}(\overline{G}^*)) = \{(\alpha, \beta) \in (\mathbb{R}^I)^2 : \forall (r, s) \in (\mathbb{R}^I)^2, \alpha r + \beta s \leq \overline{G}0^+(r, s)\},$$
$$\mathrm{cl}(\mathrm{dom}(\underline{G}^*)) = \{(\alpha, \beta) \in (\mathbb{R}^I)^2 : \forall (r, s) \in (\mathbb{R}^I)^2, \alpha r + \beta s \leq \underline{G}0^+(r, s)\}$$

(Rockafellar, 1970, Corollary 13.2.1, Theorem 13.3), so $\mathrm{cl}(\mathrm{dom}(\overline{G}^*)) = \mathrm{cl}(\mathrm{dom}(\underline{G}^*))$ by Lemma 6. Moreover, since $\mathrm{dom}(\overline{G}0^+) = T$, we have $0^+ \mathrm{cl}(\mathrm{dom}(\overline{G}^*)) = \{(\alpha, \beta) \in (\mathbb{R}^I)^2 : \forall (r, s) \in T, \alpha r + \beta s \leq 0\} = T^\circ = C$.[21] Hence $\mathrm{cl}(\mathrm{dom}(\overline{G}^*))$ contains no line, so

$$\mathrm{cl}(\mathrm{dom}(\overline{G}^*)) = \mathrm{cl}(\mathrm{conv}(\exp(\mathrm{cl}(\mathrm{dom}(\overline{G}^*)))) + C)$$

(Rockafellar, 1970, Theorem 18.7).[22] Moreover, $\exp(\mathrm{cl}(\mathrm{dom}(\overline{G}^*))) = E$ (Rockafellar, 1970, Corollary 25.1.3) and, hence, $\mathrm{cl}(\mathrm{dom}(\overline{G}^*)) = M + C$ since E is bounded (Rockafellar, 1970, Corollary 9.1.1).

Now we claim that $(\alpha, \beta) \in \mathrm{dom}(\overline{G}^*) \cap \mathrm{dom}(\underline{G}^*)$ and $\overline{G}^*(\alpha, \beta) + \underline{G}^*(\alpha, \beta) \leq 0$ for all $(\alpha, \beta) \in E$. If the claim is correct then, since $\overline{G}^*$ and $\underline{G}^*$ are closed and bounded below, we have $(\alpha, \beta) \in \mathrm{dom}(\overline{G}^*) \cap \mathrm{dom}(\underline{G}^*)$ and $\overline{G}^*(\alpha, \beta) + \underline{G}^*(\alpha, \beta) \leq 0$ for all $(\alpha, \beta) \in M$ (Rockafellar, 1970, Theorem 17.2, Theorem 18.6). Moreover, for all $\eta \in \mathbb{R}^I_+$, we have $\overline{G}^*(\alpha - \eta, \beta - \eta) = \sup_{(r,s) \in T}(\alpha r + \beta s - \eta(r + s) - \overline{G}(r, s)) \leq \overline{G}^*(\alpha, \beta)$ and $\underline{G}^*(\alpha - \eta, \beta - \eta) = \sup_{(r,s) \in T}(\alpha r + \beta s - \eta(r + s) - \underline{G}(r, s)) \leq \underline{G}^*(\alpha, \beta)$ since $r + s \geq 0$ for all $(r, s) \in T$. Hence $(\alpha, \beta) \in \mathrm{dom}(\overline{G}^*) \cap \mathrm{dom}(\underline{G}^*)$ and $\overline{G}^*(\alpha, \beta) + \underline{G}^*(\alpha, \beta) \leq 0$ for all $(\alpha, \beta) \in M + C$, so the proof is complete.

To prove the claim, let $(\alpha, \beta) \in E$, i.e. $(\alpha, \beta) = \nabla\overline{G}0^+(r, s)$ for some $(r, s) \in L$. Clearly, (r, s) must then belong to the interior of T (Rockafellar, 1970, Corollary 25.1.1).

[19] Given a cone S, S° denotes the polar cone of S, i.e. $S^\circ = \{y : \forall s \in S, xy \leq 0\}$.
[20] Given a set S, $\mathrm{cl}(S)$ denotes the closure of S.
[21] Given a set S, 0^+S denotes the recession cone of S, i.e. $0^+S = \{y : \forall x \in S, \forall \mu > 0, x + \mu y \in S\}$.
[22] Given a set S, $\exp(S)$ denotes the set of exposed points of S.

Hence, for all $\mu \geq 1$, $\mu(r,s)$ also belongs to the interior of T, so $\partial\overline{G}(\mu(r,s))$ is non-empty and compact (Rockafellar, 1970, Theorem 23.4).[23] It follows that there exists $(\overline{\alpha}_\mu, \overline{\beta}_\mu) \in \partial\overline{G}(\mu(r,s))$ such that $\overline{G}'(\mu(r,s), -(r,s)) = -\overline{\alpha}_\mu r - \overline{\beta}_\mu s$ since $\overline{G}'(\mu(r,s), \cdot)$ is the support function of $\partial\overline{G}(\mu(r,s))$ (Rockafellar, 1970, Theorem 23.2, Theorem 23.4). Hence we have $\overline{G}(\mu(r,s)) + \overline{G}^*(\overline{\alpha}_\mu, \overline{\beta}_\mu) = \mu(\overline{\alpha}_\mu r + \overline{\beta}_\mu s)$ (Rockafellar, 1970, Theorem 23.5) and, hence, $-\overline{G}^*(\overline{\alpha}_\mu, \overline{\beta}_\mu) = \overline{G}(\mu(r,s)) + \mu\overline{G}'(\mu(r,s), -(r,s))$. Hence, by definition of $\overline{G}'$, for all $\kappa > -\overline{G}^*(\overline{\alpha}_\mu, \overline{\beta}_\mu)$, there exists $\lambda \in (0,1)$ such that $\lambda\overline{G}(\mu(r,s)) + (1-\lambda)\kappa > \overline{G}(\lambda\mu(r,s))$ (Rockafellar, 1970, Theorem 23.1). Similarly, there exists $(\underline{\alpha}_\mu, \underline{\beta}_\mu) \in \partial\underline{G}(\mu(r,s))$ such that, for all $\kappa > -\underline{G}^*(\underline{\alpha}_\mu, \underline{\beta}_\mu)$, there exists $\lambda \in (0,1)$ such that $\lambda\underline{G}(\mu(r,s)) + (1-\lambda)\kappa > \underline{G}(\lambda\mu(r,s))$. Equivalently, for all $\kappa < \underline{G}^*(\underline{\alpha}_\mu, \underline{\beta}_\mu)$, there exists $\lambda \in (0,1)$ such that $\lambda(-\underline{G}(\mu(r,s))) + (1-\lambda)\kappa < -\underline{G}(\lambda\mu(r,s))$. Consequently, if for all $i \in I$ we define u_i, v_i, and U_i as in the proof of Lemma 6, we obtain

$$-\overline{G}^*(\overline{\alpha}_\mu, \overline{\beta}_\mu) \geq \tfrac{1}{2}\overline{G}(\mu(r,s)) - \tfrac{1}{2}\underline{G}(\mu(r,s)) \geq \underline{G}^*(\underline{\alpha}_\mu, \underline{\beta}_\mu)$$

by Lemma 1(d), so $\overline{G}^*(\overline{\alpha}_\mu, \overline{\beta}_\mu) + \underline{G}^*(\underline{\alpha}_\mu, \underline{\beta}_\mu) \leq 0$. Hence, since $\overline{G}^*$ and $\underline{G}^*$ are bounded below, the sequences $(\overline{G}^*(\overline{\alpha}_\mu, \overline{\beta}_\mu))_{\mu \geq 1}$ and $(\underline{G}^*(\underline{\alpha}_\mu, \underline{\beta}_\mu))_{\mu \geq 1}$ have subsequences converging as $\mu \to +\infty$ to limits $\overline{\kappa}$ and $\underline{\kappa}$, respectively, such that $\overline{\kappa} + \underline{\kappa} \leq 0$. Moreover, $\overline{G}0^{+'}((r,s), \cdot)$ is linear since $(r,s) \in L$ (Rockafellar, 1970, Theorem 25.2) and, hence, $(\overline{G}'(\mu(r,s), \cdot))_{\mu \geq 1}$ and $(\underline{G}'(\mu(r,s), \cdot))_{\mu \geq 1}$ converge pointwise to $\overline{G}0^{+'}((r,s), \cdot)$ as $\mu \to +\infty$ by Lemma 8. Since $\overline{G}0^{+'}((r,s), \cdot)$ is the support function of $\partial G0^+(r,s) = \{\nabla G0^+(r,s)\} = \{(\alpha,\beta)\}$, it follows that $\lim_{\mu \to +\infty}(\overline{\alpha}_\mu, \overline{\beta}_\mu) = \lim_{\mu \to +\infty}(\underline{\alpha}_\mu, \underline{\beta}_\mu) = (\alpha,\beta)$ (Schneider, 1993, Theorem 1.8.11, Theorem 1.8.12). Hence $\overline{G}^*(\alpha,\beta) \leq \overline{\kappa}$ and $\underline{G}^*(\alpha,\beta) \leq \underline{\kappa}$ since $\overline{G}^*$ and $\underline{G}^*$ are closed, so $(\alpha,\beta) \in \operatorname{dom}(\overline{G}^*) \cap \operatorname{dom}(\underline{G}^*)$ and $\overline{G}^*(\alpha,\beta) + \underline{G}^*(\alpha,\beta) \leq 0$. ☐

Let $C' = C \times \mathbb{R}_+ \subset (\mathbb{R}^I_+)^2 \times \mathbb{R}$. Clearly, C' is a non-empty, closed, and convex cone containing no line. Let $\overline{L}, \underline{L} \subset (\mathbb{R}^I)^2$ be the set of points where $\overline{G}$ and $\underline{G}$ are differentiable, respectively. Clearly, $\operatorname{cl}(\overline{L}) = \operatorname{cl}(\underline{L}) = T$ (Rockafellar, 1970, Theorem 25.5), so $\overline{L}$ and $\underline{L}$ are non-empty. Let $\overline{E} = \{\nabla\overline{G}(r,s) : (r,s) \in \overline{L}\}$ and $\underline{E} = \{\nabla\underline{G}(r,s) : (r,s) \in \underline{L}\}$. By Lemmas 4 and 7, $\overline{E}$ and $\underline{E}$ are non-empty and bounded subsets of $(\mathbb{R}^I_+)^2$. Moreover, we have $\overline{E} \subseteq \operatorname{dom}(\overline{G}^*)$ and $\underline{E} \subseteq \operatorname{dom}(\underline{G}^*)$ (Rockafellar, 1970, Theorem 23.5) and, hence, the sets $\overline{E}' = \{(\alpha,\beta,\overline{G}^*(\alpha,\beta)) : (\alpha,\beta) \in \overline{E}\}$ and $\underline{E}' = \{(\alpha,\beta,\underline{G}^*(\alpha,\beta)) : (\alpha,\beta) \in \underline{E}\}$ are non-empty and bounded subsets of $(\mathbb{R}^I_+)^2 \times \mathbb{R}$ since $\overline{G}^*$ are bounded below and above by Lemma 9. Hence the sets $\overline{M}' = \operatorname{cl}(\operatorname{conv}(\overline{E}'))$ and $\underline{M}' = \operatorname{cl}(\operatorname{conv}(\underline{E}'))$ are non-empty, compact, and convex subsets of $(\mathbb{R}^I_+)^2 \times \mathbb{R}$.

Lemma 10. For all non-empty, compact, and convex set $\overline{\Phi} \subset (\mathbb{R}^I_+)^2 \times \mathbb{R}$, $\overline{M}' \subseteq \overline{\Phi} \subseteq \overline{M}' + C'$ if and only if, for all $(r,s) \in T$, $\overline{G}(r,s) = \max_{(\alpha,\beta,\gamma) \in \overline{\Phi}}(\alpha r + \beta s - \gamma)$. Similarly,

[23]Given a function f and a point x, $\partial f(x)$ denotes the subdifferential of f at x, i.e. $\partial f(x) = \{x^* : \forall y, f(y) \geq f(x) + (y-x)x^*\}$.

for all non-empty, compact, and convex set $\underline{\Phi} \subset (\mathbb{R}^I_+)^2 \times \mathbb{R}$, $\underline{M}' \subseteq \underline{\Phi} \subseteq \underline{M}' + C'$ if and only if, for all $(r,s) \in T$, $\underline{G}(r,s) = \max_{(\alpha,\beta,\gamma)\in\underline{\Phi}}(\alpha r + \beta s - \gamma)$.

Proof. We only state the proof for $\overline{G}$, the argument for $\underline{G}$ is similar. By definition of $\overline{G}^*$, we have, for all $(r,s) \in T$,[24]

$$\overline{G}(r,s) = \sup_{(\alpha,\beta)\in\mathrm{dom}(\overline{G}^*)} (\alpha r + \beta s - G^*(\alpha,\beta)) = \sup_{(\alpha,\beta,\gamma)\in\mathrm{epi}(\overline{G}^*)} (\alpha r + \beta s - \gamma).$$

We now prove that $\mathrm{epi}(\overline{G}^*) = \overline{M}' + C'$. First, we clearly have $0^+ \mathrm{epi}(\overline{G}^*) \subseteq 0^+ \mathrm{dom}(\overline{G}^*) \times \mathbb{R} = C \times \mathbb{R}$. Moreover, since $\overline{G}^*$ is bounded below, we have in fact $0^+ \mathrm{epi}(\overline{G}^*) \subseteq C \times \mathbb{R}_+ = C'$. Conversely, since $\overline{G}^*(\alpha-\eta, \beta-\eta) \leq G^*(\alpha,\beta)$ for all $(\alpha,\beta) \in \mathrm{dom}(\overline{G}^*)$ and all $\eta \in \mathbb{R}^I_+$ (see the proof of Lemma 9), we have $C' \subseteq 0^+ \mathrm{epi}(\overline{G}^*)$. Hence $0^+ \mathrm{epi}(\overline{G}^*) = C'$ and, hence, $\mathrm{epi}(\overline{G}^*)$ contains no line, so we have $\mathrm{epi}(\overline{G}^*) = \mathrm{cl}(\mathrm{conv}(\exp(\mathrm{epi}(\overline{G}^*))) + C')$ since $\overline{G}^*$ is closed (Rockafellar, 1970, Theorem 18.7). Moreover, $\exp(\mathrm{epi}(\overline{G}^*)) = \overline{E}'$ (Rockafellar, 1970, Corollary 25.1.2) and, hence, $\mathrm{epi}(\overline{G}^*) = \overline{M}' + C'$ since $\overline{E}'$ is bounded (Rockafellar, 1970, Corollary 9.1.1).

Thus, for all $(r,s) \in T$, we have $\overline{G}(r,s) = \sup_{(\alpha,\beta,\gamma)\in\overline{M}'+C'}(\alpha r + \beta s - \gamma)$. Hence, since $\overline{G}$ is finite on T by definition and since $\overline{M}'$ is compact, it must be that $\overline{G}(r,s) = \sup_{(\alpha,\beta,\gamma)\in\overline{M}'}(\alpha r + \beta s - \gamma) = \max_{(\alpha,\beta,\gamma)\in\overline{M}'}(\alpha r + \beta s - \gamma)$. It follows that $\overline{G}(r,s) = \max_{(\alpha,\beta,\gamma)\in\overline{\Phi}}(\alpha r + \beta s - \gamma)$ for all non-empty, compact, and convex set $\overline{\Phi} \subset (\mathbb{R}^I_+)^2 \times \mathbb{R}$ such that $\overline{M}' \subseteq \overline{\Phi} \subseteq \overline{M}' + C'$. For the converse, first assume that $\overline{\Phi} \not\subseteq \overline{M}' + C'$, i.e. that there exists $(\alpha_0, \beta_0, \gamma_0) \in \overline{\Phi} \backslash (\overline{M}' + C')$. Then, since $\overline{M}' + C'$ is closed and convex and contains no line, there exists $(r,s,t) \in (\mathbb{R}^I)^2 \times \mathbb{R}$ with $t \neq 0$ such that $\alpha_0 r + \beta_0 s + \gamma_0 t > \alpha r + \beta s + \gamma t$ for all $(\alpha,\beta,\gamma) \in \overline{M}' + C'$. Clearly, it must then be that $(r,s,t) \in C'^\circ = T \times \mathbb{R}_-$. Hence, if we let $(r',s') = (\frac{r}{|t|}, \frac{s}{|t|}) \in T$, we have $\alpha_0 r' + \beta_0 s' - \gamma_0 > \alpha r' + \beta s' - \gamma$ for all $(\alpha,\beta,\gamma) \in \overline{M}' + C'$, so $\sup_{(\alpha,\beta,\gamma)\in\overline{\Phi}}(\alpha r' + \beta s' - \gamma) > \sup_{(\alpha,\beta,\gamma)\in\overline{M}'+C'}(\alpha r' + \beta s' - \gamma) = \overline{G}(r',s')$. Now, assume that $\overline{\Phi} \subseteq \overline{M}' + C'$ but $\overline{M}' \not\subseteq \overline{\Phi}$, i.e. that there exists $(\alpha_0,\beta_0,\gamma_0) \in \overline{M}' \setminus \overline{\Phi}$. Then there exists $(\alpha_0,\beta_0,\gamma_0) \in \overline{E}' \setminus \overline{\Phi}$ (Rockafellar, 1970, Theorem 18.6) and, hence, there exists $(r,s,t) \in (\mathbb{R}^I)^2 \times \mathbb{R}$ with $t \neq 0$ such that $\alpha_0 r + \beta_0 s + \gamma_0 t > \alpha r + \beta s + \gamma t$ for all $(\alpha,\beta,\gamma) \in \overline{\Phi} + C' \subseteq \overline{M}' + C'$ (Rockafellar, 1970, Corollary 25.1.2). Clearly, it must then be that $(r,s,t) \in C'^\circ = T \times \mathbb{R}_-$. Hence, if we let $(r',s') = (\frac{r}{|t|}, \frac{s}{|t|}) \in T$, we have $\alpha_0 r' + \beta_0 s' - \gamma_0 > \alpha r' + \beta s' - \gamma$ for all $(\alpha,\beta,\gamma) \in \overline{\Phi}$, so $\sup_{(\alpha,\beta,\gamma)\in\overline{\Phi}}(\alpha r' + \beta s' - \gamma) < \sup_{(\alpha,\beta,\gamma)\in\overline{M}'+C'}(\alpha r' + \beta s' - \gamma) = \overline{G}(r',s')$. □

Let $\overline{M}'' = \{(\alpha,\beta,-\gamma) : (\alpha,\beta,\gamma \in \overline{M}'\}$ and $C'' = C \times \mathbb{R}_-$. It follows from Lemma 10 that (9) holds if and only if $\mathrm{conv}(\underline{M}' \cup \overline{M}'') \subseteq \Phi \subseteq (\underline{M}' + C') \cap (\overline{M}'' + C'')$.

Lemma 11. $(\underline{M}' + C') \cap (\overline{M}'' + C'') = \mathrm{conv}(\underline{M}' \cup \overline{M}'') + (C \times \{0\})$.

[24]Given a real-valued function f, $\mathrm{epi}(f)$ denotes the epigraph of f, i.e. $\mathrm{epi}(f) = \{(x,\gamma) : \gamma \geq f(x)\}$.

Proof. Since $0 \in C' \cap C''$, we have $\underline{M}' \subseteq \underline{M}' + C'$ and $\overline{M}'' \subseteq \overline{M}'' + C''$. Moreover, $\underline{M}' \subseteq \overline{M}'' + C''$ and $\overline{M}'' \subseteq \underline{M}' + C'$ by Lemma 9. Hence $\mathrm{conv}(\underline{M}' \cup \overline{M}'') \subseteq (\underline{M}' + C') \cap (\overline{M}'' + C'')$ since the set on the right hand side is convex. Hence $\mathrm{conv}(\underline{M}' \cup \overline{M}'') + (C \times \{0\}) \subseteq (\underline{M}' + C') \cap (\overline{M}'' + C'')$ since $(C \times \{0\}) = C' \cap C''$. Conversely, let $\nu \in (\underline{M}' + C') \cap (\overline{M}'' + C'')$. By definition, there exist $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}) \in \underline{M}'$, $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) \in \overline{M}''$, and $\underline{\eta}, \overline{\eta}, \underline{\tau}, \overline{\tau} \in \mathbb{R}_+$ such that

$$\nu = (\underline{\alpha}, \underline{\beta}, \underline{\gamma}) + (-\underline{\eta}, -\underline{\eta}, \underline{\tau}) = (\overline{\alpha}, \overline{\beta}, \overline{\gamma}) + (-\overline{\eta}, -\overline{\eta}, -\overline{\tau}).$$

Hence, setting $\frac{\overline{\tau}}{\underline{\tau}+\overline{\tau}} = \frac{\underline{\tau}}{\underline{\tau}+\overline{\tau}} = \frac{1}{2}$ in case $\underline{\tau} = \overline{\tau} = 0$, we have

$$\begin{aligned}\nu &= \tfrac{\overline{\tau}}{\underline{\tau}+\overline{\tau}} \left((\underline{\alpha}, \underline{\beta}, \underline{\gamma}) + (-\underline{\eta}, -\underline{\eta}, \underline{\tau})\right) + \tfrac{\underline{\tau}}{\underline{\tau}+\overline{\tau}} \left((\overline{\alpha}, \overline{\beta}, \overline{\gamma}) + (-\overline{\eta}, -\overline{\eta}, -\overline{\tau})\right) \\ &= \tfrac{\overline{\tau}}{\underline{\tau}+\overline{\tau}} \left((\underline{\alpha}, \underline{\beta}, \underline{\gamma}) + (\overline{\alpha}, \overline{\beta}, \overline{\gamma})\right) + \left(-\left(\tfrac{\overline{\tau}}{\underline{\tau}+\overline{\tau}}\underline{\eta} + \tfrac{\underline{\tau}}{\underline{\tau}+\overline{\tau}}\overline{\eta}\right), -\left(\tfrac{\overline{\tau}}{\underline{\tau}+\overline{\tau}}\underline{\eta} + \tfrac{\underline{\tau}}{\underline{\tau}+\overline{\tau}}\overline{\eta}\right), 0\right),\end{aligned}$$

so $\nu \in \mathrm{conv}(\underline{M}' \cup \overline{M}'') + (C \times \{0\})$. □

Finally, let $\Phi = \mathrm{conv}(\underline{M}' \cup \overline{M}'')$. Then (9) holds, as well as the uniqueness result, by Lemma 11.

Proof of Theorem 2

Assume X is a strongly regular mixture space and $\mathscr{D}$ is a strongly regular domain (see Section 6 for definitions). Clearly, if there exist a non-empty, compact, and convex set $\Theta \subset \mathbb{R}^I_+$ and a number $\gamma \in \mathbb{R}$ such that (6) holds then F satisfies Setwise Independence of Irrelevant Alternatives, Interval Pareto Weak Preference, and Weak Determinacy Preservation. Conversely, assume F satisfies these three axioms. Since Setwise Independence of Irrelevant Alternatives implies Interval Independence of Irrelevant Alternatives, there then exist a (unique) non-empty, compact, and convex set $\Theta \subset \mathbb{R}^I_+$ and a number $\gamma \in \mathbb{R}$ such that (5) holds. Hence, to complete the proof of the theorem (including the uniqueness part), it is sufficient to show that (5) can be strengthened to (6).

Since X is strongly non-degenerate, there exists a mixture preserving bijection from X into some convex subset of some linear space (Mongin, 2001), so we can assume without loss of generality that X itself is a convex subset of a linear space. Let Z be a maximal affinely independent subset of X, so that for all $x \in X$ there exists a unique $\zeta_x \in \mathbb{R}^Z_+$ with only finitely many non-zero components such that $\sum_{z \in Z} \zeta_x(z) = 1$ and $x = \sum_{z \in Z} \zeta_x(z) z$. Endow $\mathbb{R}^Z$ with the product topology, and recall that $\mathbb{R}^X$ is endowed with the product topology and $P \subseteq \mathbb{R}^X$ with the subspace topology, so that P and $\mathbb{R}^Z$ are linear topological spaces (Aliprantis and Border, 1999, Lemma 5.1, Theorem 5.2). Define the function $\varphi : P \to \mathbb{R}^Z$ by, for all $u \in P$, $\varphi(u) = u|_Z$.

Lemma 12. φ is a linear homeomorphism.

Proof. Clearly, φ is linear. We now show that φ is a bijection. First, let $w \in \mathbb{R}^Z$ and define the function $u \in \mathbb{R}^X$ by, for all $x \in X$, $u(x) = \sum_{z\in Z} \zeta_x(z)w(z)$. It is then easy to see that $u \in P$ and that $u|_Z = w$, so that φ is surjective. Moreover, for all $u' \in P$ such that $u'|_Z = w$, we have $u'(x) = \sum_{z\in Z} \zeta_x(z)u'(z) = u(x)$ for all $x \in X$ since u' is mixture preserving and, hence, $u' = u$, so that φ is injective. Hence φ is bijective and its inverse function $\varphi^{-1} : \mathbb{R}^Z \to P$ is defined by, for all $w \in \mathbb{R}^Z$ and all $x \in X$, $\varphi^{-1}(w)(x) = \sum_{z\in Z} \zeta_x(z)w(z)$. Hence it only remains to prove that φ and φ^{-1} are continuous.

To prove that φ is continuous, it is sufficient, by definition of the product topology on $\mathbb{R}^Z$, to prove that for all $z \in Z$ and all open subset O of $\mathbb{R}$, $\varphi^{-1}(O \times \mathbb{R}^{Z\setminus\{z\}})$ is an open subset of P. To this end, note that

$$\begin{aligned}\varphi^{-1}(O \times \mathbb{R}^{Z\setminus\{z\}}) &= \{u \in P : u(z) \in O\}\\ &= \{u \in \mathbb{R}^X : u(z) \in O\} \cap P\\ &= \left(O \cap \mathbb{R}^{X\setminus\{z\}}\right) \cap P.\end{aligned}$$

Moreover, $O \cap \mathbb{R}^{X\setminus\{z\}}$ is an open subset of $\mathbb{R}^X$ by definition of the product topology on $\mathbb{R}^X$, so that $\varphi^{-1}(O \times \mathbb{R}^{Z\setminus\{z\}})$ is an open subset of P by definition of the subspace topology on P.

To prove that φ^{-1} is continuous, it is sufficient, by definition of the product topology on $\mathbb{R}^X$ and of the subspace topology on P, to prove that for all $x \in X$ and all open subset O of $\mathbb{R}$, $\varphi((O \times \mathbb{R}^{X\setminus\{x\}}) \cap P)$ is an open subset of $\mathbb{R}^Z$. To this end, note that

$$\begin{aligned}\varphi((O \times \mathbb{R}^{X\setminus\{x\}}) \cap P) &= \left\{w \in \mathbb{R}^Z : \sum_{z\in Z} \zeta_x(z)w(z) \in O\right\}\\ &= \rho^{-1}(O),\end{aligned}$$

where $\rho : \mathbb{R}^Z \to \mathbb{R}$ is defined by, for all $w \in \mathbb{R}^Z$, $\rho(w) = \sum_{z\in Z} \zeta_x(z)w(z)$, so that it is sufficient to show that ρ is continuous. To this end, let $Z' = \{z \in Z : \zeta_x(z) \neq 0\}$, which is a finite subset of Z by definition, and note that $\rho = \rho_2 \circ \rho_1$, where $\rho_1 : \mathbb{R}^Z \to \mathbb{R}^{Z'}$ is defined by for all $w \in \mathbb{R}^Z$, $\rho_1(w) = w|_{Z'}$ and $\rho_2 : \mathbb{R}^{Z'} \to \mathbb{R}$ is defined by for all $w' \in \mathbb{R}^{Z'}$, $\rho_2(w') = \sum_{z\in Z'} \zeta_x(z)w'(z)$. Moreover, if $\mathbb{R}^{Z'}$ is endowed with the product topology then, since Z' is finite, ρ_1 is continuous by definition of the product topology and ρ_2 is continuous since it is linear, so that ρ is continuous. □

Let $\mathscr{W}$ denote the set of all non-empty, compact, and convex subsets of $\mathbb{R}^Z$. By Lemma 12, we then have $\mathscr{W} = \{\varphi(U) : U \in \mathscr{P}\}$ and $\mathscr{P} = \{\varphi^{-1}(W) : W \in \mathscr{W}\}$. Let $\mathscr{A} = \{(\varphi(U_i))_{i\in I} : (U_i)_{i\in I} \in \mathscr{D}\} \subseteq \mathscr{W}^I$, so that $\mathscr{D} = \{(\varphi^{-1}(W_i))_{i\in I} : (W_i)_{i\in I} \in \mathscr{A}\}$. Define the function $B : \mathscr{A} \to \mathscr{W}$ by, for all $(W_i)_{i\in I} \in \mathscr{A}$, $B((W_i)_{i\in I}) = \varphi(F((\varphi^{-1}(W_i)_{i\in I})))$, so that for all $(U_i)_{i\in I} \in \mathscr{D}$, $F((U_i)_{i\in I}) = \varphi^{-1}(B((\varphi(U_i))_{i\in I}))$. In order to establish (6), it is then sufficient to show that, for all $(W_i)_{i\in I} \in \mathscr{A}$,

$$B((W_i)_{i\in I}) = \left(\bigcup_{\theta\in\Theta}\sum_{i\in I}\theta_i W_i\right) + \gamma. \tag{10}$$

So fix such a $(W_i)_{i\in I}$.

Lemma 13. Let W and W' be closed subsets of $\mathbb{R}^Z$. If $W|_{Z'} = W'|_{Z'}$ for all finite subset Z' of Z then $W = W'$.

Proof. Assume $W \neq W'$. Then (without loss of generality) there exists $w' \in W' \setminus W$. Hence $w' \in W^c$, where W^c denotes the complement of W in $\mathbb{R}^Z$ (which is an open set since W is closed). Hence, by definition of the product topology, there must exist a finite subset Z' of Z and a collection $(O_z)_{z\in Z'}$ of open subsets of $\mathbb{R}$ such that $w' \in O \subseteq W^c$, where $O = (\prod_{z\in Z'} O_z) \times \mathbb{R}^{Z\setminus Z'}$. By definition of O, it follows that $w \in O$ and, hence, $w \notin W$ for all $w \in \mathbb{R}^Z$ such that $w|_{Z'} = w'|_{Z'}$. Hence $w'|_{Z'} \notin W|_{Z'}$, so $W|_{Z'} \neq W'|_{Z'}$. □

By Lemma 13, in order to establish (10), it is sufficient to show that for all finite subset Z' of Z,

$$B((W_i)_{i\in I})|_{Z'} = \left.\left(\left(\bigcup_{\theta\in\Theta}\sum_{i\in I}\theta_i W_i\right) + \gamma\right)\right|_{Z'}. \tag{11}$$

So fix such a Z'. Since $\mathscr{D}$ is strongly regular, there exist a finite subset Y' of X, an alternative x' belonging to the relative interior of $\mathrm{conv}(X')$, where $X' = Y' \cup Z'$, and a profile $(U'_i)_{i\in I} \in \mathscr{D}$ such that, for all $i \in I$, $U'_i|_{Z'} = \varphi^{-1}(W_i)|_{Z'}$ and $U'_i(x') = 0$. Hence, by Setwise Independence of Irrelevant Alternatives, we have

$$\begin{aligned} F((U'_i)_{i\in I})|_{Z'} &= F((\varphi^{-1}(W_i))_{i\in I})\big|_{Z'} \\ &= F((\varphi^{-1}(W_i))_{i\in I})\big|_{Z}\big|_{Z'} \\ &= \varphi\left(F((\varphi^{-1}(W_i))_{i\in I})\right)\big|_{Z'} \\ &= B((W_i)_{i\in I})\big|_{Z'}. \end{aligned}$$

Moreover, by (5), we have

$$F((U'_i)_{i\in I})(x) = \left(\left(\bigcup_{\theta\in\Theta}\sum_{i\in I}\theta_i U'_i\right) + \gamma\right)(x)$$

for all $x \in \mathrm{conv}(X')$ and, in particular, $F((U'_i)_{i\in I})(x') = 0$.

Lemma 14. Let X' be a finite subset of X and let $U, U' \in \mathscr{P}$ such that $U(x) = U'(x)$ for all $x \in \mathrm{conv}(X')$ and $U(x')$ is a singleton for some x' belonging to the relative interior of $\mathrm{conv}(X')$. Then $U|_{X'} = U'|_{X'}$.

Proof. By assumption, we have $U(x') = U'(x') = \{\eta\}$ for some $\eta \in \mathbb{R}$. Let $V = U|_{X'} \subset \mathbb{R}^{X'}$ and $V' = U'|_{X'} \subset \mathbb{R}^{X'}$, and suppose $V \neq V'$. Then (without loss of generality) there exists $v' \in V' \setminus V$. Hence, since V is non-empty, closed and convex, there exists $t \in \mathbb{R}^{X'}$, $t \neq 0$, such that $\sum_{y \in X'} t(y)v'(y) > \sum_{y \in X'} t(y)v(y)$ for all $v \in V$. If $t > 0$ then the vector $x = \sum_{y \in X'} \frac{t(y)}{\sum_{y' \in X'} t(y')} y$ belongs to $\text{conv}(X')$ and we have $u'(x) = \sum_{y \in X'} x(y)v'(y) > \sum_{y \in X'} x(y)v(y) = u(x)$ for all $u \in U$ and, hence, $U'(x) \not\subseteq U(x)$, a contradiction. Otherwise, note that for all $\lambda \in (0,1)$ and all $v \in V$,

$$\begin{aligned}\sum_{y \in X'} (\lambda t + (1-\lambda)x')(y)v'(y) &= \lambda \sum_{y \in X'} t(y)v'(y) + (1-\lambda) \sum_{y \in X'} x'(y)v'(y) \\ &= \lambda \sum_{y \in X'} t(y)v'(y) + (1-\lambda)\eta \\ &> \lambda \sum_{y \in X'} t(y)v(y) + (1-\lambda)\eta \\ &= \lambda \sum_{y \in X'} t(y)v(y) + (1-\lambda) \sum_{y \in X'} x'(y)v(y) \\ &= \sum_{y \in X'} (\lambda t + (1-\lambda)x')(y)v(y).\end{aligned}$$

Moreover, $x'(y) > 0$ for all $y \in X'$ since x' belongs to the relative interior of $\text{conv}(X')$ and, hence, $\lambda t + (1-\lambda)x' > 0$ for all $\lambda \in (0,1)$ sufficiently close to 0. So pick such a λ and complete the argument as above by defining $x = \sum_{y \in X'} \frac{\lambda t(y) + (1-\lambda)x'(y)}{\sum_{y' \in X'} (\lambda t(y') + (1-\lambda)x'(y'))} y$. $\square$

By Lemma 14, we have

$$F((U'_i)_{i \in I})|_{X'} = \left(\left(\bigcup_{\theta \in \Theta} \sum_{i \in I} \theta_i U'_i \right) + \gamma \right) \Bigg|_{X'}$$

and, hence, since $Z' \subseteq X'$,

$$\begin{aligned}F((U'_i)_{i \in I})|_{Z'} &= F((U'_i)_{i \in I})|_{X'}|_{Z'} \\ &= \left(\left(\bigcup_{\theta \in \Theta} \sum_{i \in I} \theta_i U'_i \right) + \gamma \right) \Bigg|_{X'} \Bigg|_{Z'} \\ &= \left(\left(\bigcup_{\theta \in \Theta} \sum_{i \in I} \theta_i U'_i \right) + \gamma \right) \Bigg|_{Z'} \\ &= \left(\bigcup_{\theta \in \Theta} \sum_{i \in I} \theta_i U'_i|_{Z'} \right) + \gamma \\ &= \left(\bigcup_{\theta \in \Theta} \sum_{i \in I} \theta_i W_i|_{Z'} \right) + \gamma\end{aligned}$$

$$= \left(\left(\bigcup_{\theta \in \Theta} \sum_{i \in I} \theta_i W_i \right) + \gamma \right) \Bigg|_{Z'},$$

which yields (11) since $B((W_i)_{i \in I})|_{Z'} = F((U_i')_{i \in I})|_{Z'}$.

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