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Value at Risk Computation in a Non-Stationary Setting

Dominique GUEGAN[1]

Abstract

This chapter recalls the main tools useful to compute Value at Risk associated with a m-dimensional portfolio. Then, the limitations of the use of these tools is explained, as soon as non-stationarities are observed in time series. Indeed, specific behaviours observed by financial assets, like volatility, jumps, explosions, and pseudo-seasonalities, provoke non-stationarities which affect the distribution function of the portfolio. Thus, a new way for computing VaR is proposed which allows the potential non-invariance of the m-dimensional portfolio distribution function to be avoided.

Keywords: Non-stationarity – Value-at-Risk – Dynamic copula –Meta-distribution – POT method.

JEL classification: C32, C51, G12

I – Introduction

Value at Risk (VaR) is now a major task of much financial analysis involved in risk management. It has become the standard market measure for portfolios holding various assets. Value at Risk is defined as the potential loss which is encountered in a specified period, for a given level of probability. Hence, VaR is essentially measured by quantiles.

The main objective of the 1988 Basel Accord is to develop a risk-based capital framework that strengthens and stabilises the banking system. In 1993, the group of thirty set out the following requirements: `Market risk is best measured as `Value at Risk' using probability analysis based upon a common confidence interval and time horizon', Gamrowski and Rachev

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(1996). Then, in 1996, following this recommendation, the Basel Committee decided to take into account the importance of market risks. The way to define capital requirements thus changed in order to be sufficiently sensitive to the risk profile of each institution. Until now, capital requirements are increasingly based on risk-sensitive measures, which are directly based on VaR for market risk. VaR is a common language to compare the risks of the different markets and can be translated directly into a minimum capital requirement, BCBS (1996). This measure has also permitted the financial institutions to develop their own internal model. On the other hand, this measure is based on some unrealistic assumptions that are specified latter, and the ES (Expected Shortfall) measure appears preferable, Artzner et al. (1997).

As VaR measures appear mainly as quantiles, the different ways to compute them are specified in univariate and multivariate settings. To do so, financial data sets are used which are characterized by structural behaviours like the volatility, jumps, explosions and seasonality that provoke non-stationarity. The question is how to model these features through the distribution function to obtain a robust VaR. New strategies have to be defined and some of them are proposed.

II – Risk Measures

2 – 1 Definition

Traditionally, the risk from an unexpected change in market prices (i.e. the market risk) was based on the mean of the deviation from the mean of the return distribution: the variance. In the case of a combination of assets, risk is computed via the covariance between the pairs of investments. Using this methodology makes it possible to describe the behaviour of the returns by the first two moments of the distributions and by the linear correlation coefficient $\rho(X,Y)$ between each pair of returns. This latter measure, which is a measure of dispersion,

can be adopted as a measure of risk only if the relevant distribution is symmetrical (and elliptical). On the other hand the correlation coefficient measures only the co-dependence between the linear components of two returns X and Y. This very intuitive method is the basis of the Markowitz (1959) portfolio theory, in which the returns on all assets, as well as their dependence structure are assumed to be Gaussian. This approach becomes incorrect as a measure of the dependence between returns, as soon as the cumulative distribution of returns is totally asymmetric, leptokurtic, and contains extreme values.

So far, since the 1996 Basel amendment, the official measure of market risk is the Value at Risk which is specified below.

Definition 2.1. For a given horizon and a probability level α, $0<\alpha<1$, VaR_α is the maximum loss that may be recorded in a specified period, with a level of confidence of $1-\alpha$. If X is a random return with distribution function F_X, then:

$$F_X(VaR_\alpha) = \Pr[X \le VaR_\alpha] = \alpha. \quad (2.1)$$

Thus, losses lower than VaR_α occur with probability α.

It is now well-known, that the VaR number can provide an inadequate representation of risk because some assumptions are often unrealistic. The main problem is its incoherent property. Indeed, the VaR measure does not verify the sub-additivity property, meaning that the VaR of the sum of two positions X and Y is not less or equal to the sum of the VaR of the individual positions. This situation arises with nonlinear financial instruments as the options. Alternatively, the VaR can also indicate what the worst loss incurred in $(1-\alpha)$% of time is, but it says nothing about the loss on the remaining α %. This means that during turmoil periods, the VaR measure is unable to provide information about the largest losses. This could lead a risk manager to select the worst portfolio, thinking it to be the least risky. Finally existence of non-stationarities inside most financial data sets makes the computation of the

VaR often very irrelevant. Another measure of market risk is the Expected Shortfall (ES) (also called Conditional Value at Risk). This coherent measure represents the expectation of a loss, given that a threshold is exceeded, for instance VaR_α, and for a probability level α is equal to:

$$ES_\alpha = E[X \,|\, X \leq VaR_\alpha].$$

In that latter case, the ES measure is a lower bound of the VaR_α introduced in (2.1).

This chapter discusses the different problems encountered in computing a robust VaR from sample data sets, given non-stationarity. All these discussions can be extended without difficulty to the ES risk measure.

The Basel amendment has imposed several rules, the most important being the daily calculation of a capital charge to cover the market risk of a portfolio. This calculus is linked to the estimated VaR and has led the financial institutions to develop their internal models. The rule needs to develop methods to estimate the distribution function F_X every day in order to compute (2.1). Now, assuming the invariance of the distribution for any asset during the whole period under study is not always reasonable, since the basic properties of financial assets are not the same in stable periods and during crisis, Guégan (2008) provides a recent discussion of this problem, so that specific strategies need to be developed in the context discussed below. The next section specifies the tools used to compute a VaR.

2 – 2 Tools to compute a VaR

Assuming a portfolio is composed of a unique asset, its distribution function can be estimated analytically, using tests (a Kolmogorov test, a χ^2 test), graphical methods (a Q-Q Plot, etc.) or using a non-parametrical kernel method. When the portfolios are composed of more than one asset, the joint distribution of all assets making up the portfolio needs to be composed as well. In case of independent assets, this last distribution is the product of the assets' distribution. When the assets exhibit dependence between each other, the best way to compute the

distribution function of the portfolio is to use the notion of copula, if the aim is to obtain an analytical form of the distribution; if not, non-parametric techniques like the kernel method can be used, which are not studied here. A copula may be defined as follows.

2 – 2 – 1 Definition of a copula

Consider a general random vector $X = (X_1, ..., X_m)'$ which may represent m components of a portfolio measured at the same time. It is assumed that X has an m-dimensional joint distribution function $F(x_1, ..., x_m) = \Pr[X_1 \le x_1, ..., X_m \le x_m]$. It is further assumed that for all $i \in \{1, ..., m\}$, the random variables X_i have continuous margins F_i such that $F_i(x) = \Pr[X_i \le x]$. Accordingly, it has been shown by Sklar (1959) that:

Definition 2. 2 The joint distribution F of a vector $X = (X_1, ..., X_m)'$ with continuous margins $F_1, ..., F_m$ can be written as:

$$F(x_1, ...x_m) = C(F_1(x_1), ..., F_m(x_m)), (x_1, ..., x_m) \in R^m. \quad (2.2)$$

The unique function C in (2.2) is called the copula associated to the vector X.

The function C is a multivariate distribution function, generally depending on a parameter θ, with uniform margins on [0,1] and it provides a natural link between F and $F_1, ..., F_m$. From (2.2), it may be observed that the univariate margins and the dependence structure can be separated, and it makes sense to interpret the function C as the dependence structure of the random vector X.

2 – 2 – 2 Estimation of a Copula

To choose a copula associated to a portfolio, the bivariate case is restricted here. Let $X = (X_1, X_2)$ be a random vector with a bivariate distribution function F, continuous univariate marginal distribution functions F_1 and F_2, and the copula C:

$$F(x_1, x_2; \theta) = C(F_1(x_1), F_2(x_2); \theta).$$

Here, the copula is parameterized by the vector $\theta \in R^q$, with $q \in N$. $X = \{(X_{i1}, X_{i2}), i = 1,2,...,n\}$ denotes a sample of n observations and the procedure to determine the copula is the following:

1 - The marginal distribution functions F_j, $j = 1,2$ are estimated by the rescaled empirical distribution functions:

$$\hat{F}_{nj}(x) = \frac{1}{n+1}\sum_{i=1}^{n} I(X_{ij} \le x) \text{ , j=1,2.(2.3)}$$

2 - The parameter θ of the copula C is estimated by a maximum log-likelihood method. It is assumed, in this case that the density c of the copula exists, and then $\hat{\theta}$ maximizes the following expression:

$$L(\theta, X) = \sum_{i=1}^{n} \log c(\hat{F}_{n1}(X_{i1}), \hat{F}_{n2}(X_{i2}); \theta) \text{ (2.4)}$$

where $\hat{F}_{nj}$, j=1,2 is introduced in (2.3) and:

$$c(u_1, u_2; \theta) = \frac{\partial C(u_1, u_2; \theta)}{\partial u_1 \partial u_2}, (u_1, u_2) \in [0,1]^2.$$

The estimator $\hat{\theta}$ is known to be consistent and asymptotically normally distributed under regular conditions.

3 - In order to apply the maximum likelihood method to estimate θ, we need to work with independent, identically random variables. X_{ij} is known for i=1,2,…, n and j = 1,2 to be not independent time series, so that each time series can start being filtered using an adequate filter (ARMA processes, related GARCH processes, Long Memory models, Markov Switching models, etc.). Then, the previous step is applied to obtain the copula C_θ on the residuals $(\varepsilon_{i1}, \varepsilon_{i2})$ for i=1,2, … ,n, associated with each time series. It should be noted that the copula which permits the dependence between (X_1, X_2) and $(\varepsilon_1, \varepsilon_2)$ to be measured will be the same.

4 - In order to choose the best copula C_θ, several criteria can be used:

- **The D^2 criteria**. the D^2 distance is associated to the vector (X_1, X_2):

$$D_C^2 = \sum_{x_1, x_2} \left| C_{\hat{\theta}}(\hat{F}_1(x_1), \hat{F}_2(x_2)) - \hat{F}(x_1, x_2) \right|^2 .$$

Then, the copula $C_{\hat{\theta}}$ for which the smallest D_C^2 is obtained will be chosen as the best copula. Here $\hat{F}(x_1, x_2)$ is the empirical joint distribution function associated to the vector (X_1, X_2).

- **AIC criteria**. When the parameter of the copula by maximizing the log-likelihood (2.4) is obtained, the maximisation provides a value of the AIC criteria. This value can be used to discriminate between different copulas. The copula for which this criterion is minimum, is retained.

- Graphical criteria. From the definition of a copula C, it is known that if U and V are two uniform random variables then the random variables $C(V/U) = \frac{\partial C}{\partial U}(U,V)$ and $C(U/V) = \frac{\partial C}{\partial V}(U,V)$ are also uniformly distributed. This property can be used to estimate the adjustment between the empirical joint distribution and the different copulas, by way of the classical QQ-plot method. For this, it is necessary to calculate the partial derivatives of the various copulas considered. In the case of Archimedean copulas (see below), only $C_\theta(U/V)$ are investigated, since they are symmetrical.

2 – 2 – 3 Classes of copulas

The previous methods can be adjusted on a lot of copulas. Two classes of copulas are mainly used: Elliptical copulas and Archimedean copulas.

1 - Elliptical Copulas. The most commonly used elliptical distribution - to model financial assets - are the Gaussian and the Student - t ones. Their expressions are:

$$C_\Phi(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \exp\left(-\frac{z_1^2 + z_2^2 - 2z_1 z_2}{2(1-\rho^2)}\right) dz_1 dz_2 ,$$

and

$$C_T(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \exp(1+\frac{z_1^2+z_2^2-2z_1z_2}{2(1-\rho^2)})^{-\frac{\nu+2}{2}} dz_1 dz_2 ,$$

where $\Phi^{-1}(u)$ is the inverted Gaussian probability distribution function and t_ν^{-1} is the inverted Student-t probability distribution function with ν degrees of freedom. These copulas are both symmetrical but they have different tail dependence behaviour.

2 - Archimedean Copulas. A second class of copulas which is very attractive concerns the Archimedean copulas. To define these copulas, the following class of functions is introduced:

$$\Phi_\theta = \{\varphi_\theta : [0,1] \to [0,\infty], \varphi_\theta(1) = 0, \varphi_\theta^{'}(t) < 0, \varphi_\theta^{''}(t) > 0, \theta \in [-1,1]\} .$$

Classical functions for $\varphi_\theta \in \Phi_\theta$ are: $\varphi_\theta(t) = -\log t$, $\varphi_\theta(t) = (1-t)^\theta$, $\varphi_\theta(t) = t^{-\theta} - 1$ with $\theta > 1$. It is then easy to show that for all convex functions $\varphi_\theta \in \Phi_\theta$, a function C_θ exists such that

$$C_\theta(u,v) = \varphi_\theta^{-1}(\varphi_\theta(u) + \varphi_\theta(v)), if\ \varphi_\theta(u) + \varphi_\theta(v) \le \varphi_\theta(0) \quad (2.5)$$

and $C_\theta(u,v) = 0$ otherwise. The function $C_\theta(u,v)$ is a symmetric 2-dimensional distribution function whose margins are uniform in the interval [0,1]. This is called the Archimedean copula generated by Φ_θ. Amongst the Archimedean distributions, several laws exist: for instance the Frank law, the Cook and Johnson law, the Gumbel law, the Ali-Mikhail-Haq law, Joe (1997). The Archimedean property means that it is possible to construct a copula by way of a generator Φ_θ and that a formula exists which permits Kendall's tau to be computed from this operator, say:

$$\tau(C_\theta) = 1 + 4\int_0^1 \frac{\varphi_\theta(t)}{\varphi_\theta^{'}(t)} dt . \quad (2.6)$$

2 – 2 – 4 m-variate Archimedean Copulas

It is not easy to work in a multivariate setting using copula. Nevertheless a bivariate family of Archimedean copulas can be extended naturally enough to an m-variate family of

Archimedean copulas, m>2, under some constraints, Joe (1997). First of all, to get this extension, all the bivariate marginal copulas which make up the multivariate copulas have to belong to the given bivariate family. Secondly, all multivariate marginal copulas up 3 to m-1 may have the same multivariate form. This situation may be illustrated for a trivariate copula. It is assumed three markets denoted (X_1, X_2, X_3) may be observed, and for each there is an n-sample. It is assumed that each bivariate margin is characterized by a dependence parameter $\theta_{i,j}$, $(i \neq j \in \{1,2,3\})$. If $\theta_2 > \theta_1$ with $\theta_{1,2} = \theta_2$, and $\theta_{1,3} = \theta_{2,3} = \theta_1$, then a 3-variate Archimedean copula has the following form:

$$C_{\theta_1,\theta_2}(u_1,u_2,u_3) = \varphi_{\theta_1}^{-1}(\varphi_{\theta_1} o \varphi_{\theta_2}^{-1}(\varphi_{\theta_2}(u_1) + \varphi_{\theta_2}(u_2)) + \varphi_{\theta_1}(u_3)).$$

Empirically, for two random variables X_1 and X_2, $\theta(X_1, X_2)$ denotes the dependence parameter deduced from Kendall's tau, denoted by $\tau(X_1, X_2)$, by means of the formula (2.6). For a random vector $X = (X_1, X_2, X_3)'$ with joint distribution F and continuous marginal distribution functions F_1, F_2, F_3, expression (2.2) becomes for all $(x_1, x_2, x_3) \in R^3$:

$$F(x_1,x_2,x_3) = C_{\theta_1\theta_2}(F_1(x_1), F_2(x_2), F_3(x_3)) = C_{\theta_1}(C_{\theta_2}(F_1(x_1), F_2(x_2)), F_3(x_3)),$$

if $\theta_1 \leq \theta_2$ with $\theta_1 = \theta(X_1, X_2) = \theta(X_2, X_3)$ and $\theta_2 = \theta(X_1, X_2)$. When the copulas are retained, in order to choose the trivariate copula $C_{\hat{\theta}_1\hat{\theta}_2}$ that best models the empirical joint distribution $\hat{F}$ of the series (X_1, X_2, X_3), an extension of the numerical criterion D_C^2 can be derived:

$$D_C^3 = \sum_{x_1,x_2,x_3} \left| C_{\hat{\theta}_1}(C_{\hat{\theta}_2}(\hat{F}_1(x_1), \hat{F}_2(x_2)), \hat{F}_3(x_3)) - \hat{F}(x_1,x_2,x_3) \right|^2.$$

Then, the copula $C_{\hat{\theta}_1\hat{\theta}_2}$ which yields the lowest D_C^3 value, is retained as the best copula.

2 – 2 – 5 Copula's tail Behaviour

The copulas are also characterized by their tail behaviour, through their upper tail and lower tail coefficients. These coefficients are important for the computation of the VaR measure.

Indeed, if we retain a copula whose lower tail behaviour is null although there are co-movements inside the markets following negative shocks for instance, then the computation of the VaR will be biased. The tail dependence concept indicates the amount of dependence in the upper-right-quadrant tail or in the lower-left-quadrant tail of a bivariate distribution. The upper and lower tail dependence parameters of a random vector (X_1, X_2) with copula C can be defined as:

Definition 2.3. If a bivariate copula C is such that

$$\lim_{u\uparrow 1} \bar{C}(u,u) \Big/ (1-u) = \lambda_U$$

exists with $\bar{C}(u,u) = 1-2u+C(u,u)$, then the copula C has an upper tail dependence if $\lambda_U \in (0,1]$, and no upper tail dependence if $\lambda_U = 0$. Moreover if a bivariate copula C is such that:

$$\lim_{u\downarrow 0} C(u,u) \Big/ u = \lambda_L$$

exists, it may be said that the copula C has lower tail dependence if $\lambda_L \in (0,1]$, and no lower tail dependence if $\lambda_L = 0$. These tail coefficients can be computed in different ways with respect to the classes of copulas considered here.

1 – Student-t copula. For this copula, the lower and upper tail dependence coefficients are equal to $\lambda_U = \lambda_L = 2\bar{\tau}_{\nu+1}(\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}})$, where $\bar{\tau}_{\nu+1}(x) = 1-t_{\nu+1}(x)$, $t_{\nu+1}(x)$ is the Student distribution function with $\nu+1$ degrees of freedom, and ρ the linear correlation coefficient. Thus, λ_U is an increasing function of ρ. We can also observe that when ν tends to infinity, λ_U tends to 0.

2 - Archimedean copulas. If the generator function is such that $\varphi'(0)$ is finite, the copula C_θ does not have upper tail dependence. If C_θ has upper tail dependence, then $\varphi'(0) = -\infty$, the

upper tail dependence parameter is $\lambda_U = 2 - 2\lim_{t\downarrow 0} {\varphi_\theta^{-1'}(2t)}\big/{\varphi_\theta^{-1'}(t)}$, and the lower tail dependence parameter is $\lambda_L = 2\lim_{t\to\infty}({\varphi_\theta^{-1'}(2t)}\big/{\varphi_\theta^{-1'}(t)})$. If $\varphi^{-1'}$ is known, then the tail dependence of any Archimedean copula can be estimated.

3 – Survival copulas. λ_U, λ_L of a survival copula can also be derived from its associated copula (the survival copula of C is given by: C(S)(u,v)=u+v-1+C(1-u,1-v)). Thus, $\lambda_U^{C(S)} = \lambda_L^C$ and $\lambda_L^{C(S)} = \lambda_U^C$. This means that if a copula has an upper tail dependence then the associated survival copula has a lower tail dependence and vice-versa. Moreover, a survival copula and its associated copula have the same Kendall's tau.

4 – **Linear combinations of copulas**. In order to obtain copulas which have upper and lower tail dependences without being symmetrical, new copulas are constructed as convex linear combinations of two copulas. Hence, for $\omega \in [0,1]$ and two Archimedean copulas C_{θ_1} and C_{θ_2} a new copula C is obtained, which is defined as:

$$C(u,v) = \omega C_{\theta_1}(u,v) + (1-\omega)C_{\theta_2}(u,v).$$

The properties of these copulas can be derived from those of C_{θ_1} and C_{θ_2}. Suppose that C_{θ_1} and C_{θ_2} have respectively an upper and a lower tail dependence, then $\lambda_U^C = \omega\lambda_U^{C_{\theta_1}}$ and $\lambda_L^C = (1-\omega)\lambda_L^{C_{\theta_2}}$.

III - Computation of the VaR

As soon as the distribution function of a portfolio is known, the VaR is directly computed from this multivariate distribution function or from the associated copula. The VaR measure corresponds to a quantile of a distribution function associated with a small probability. Several strategies have therefore been formulated to compute it:

- the quantile can be estimated directly from the multivariate distribution function using the previous approach.

- the behaviour of the distribution function above may be considered at a certain threshold to focus on the tail behaviour of the joint distribution function.

3 -1 VaR as a Quantile of the Whole Distribution Function

Computing the VaR from the whole sample is not simple because of non-stationarities which exist inside the financial data sets. Indeed, most of the financial data sets cover a reasonably long time period, so economic factors may induce some changes in the dependence structure. The basic properties of financial products may change in different periods (stable periods and crisis periods). Therefore, it seems important to detect changes in the dependence structure in order to adapt all the previous tools inside a non-stationary setting. Different strategies can be considered: one is based on the notion of dynamic copula, the other one on the notion of meta-distribution.

3 – 1 – 1 Dynamic Copulas

Dynamic copulas have recently been studied in risk management by Dias and Embrechts (2004) investigating the dynamic evolution of copulas' parameters. A change in a copula's family may also be examined, Caillault and Guégan (2005, 2009), and Guégan and Zhang (2008, 2009). Using dynamics inside a copula permits some time–varying evolutions inside the data sets to be modelled. Other non-stationary features can be modelled when a copula's family is changed, change-point techniques can be used to find the change times both for the parameters and the distribution functions. These changes can also be detected using moving windows along the data sets observed. This method makes all types of copula changes observable and makes the change trend clearer. However, how to decide the width of the moving window and the length of the time interval of movement is important and influences the accuracy of the result for the copula change. The following may be carried out:

1 - testing the changes inside the parameters when the copula family remains static, and,

2 - testing the changes inside the copulas.

In order to understand clearly copula changes, a series of nested tests based on the conditional pseudo copula can be used, Fermanian (2005). The different steps are:

A - A test is first carried out to see whether the copula does indeed change during a specified time period.

B - If the copula seems changeless, the result in the static case continues to hold.

C - Whether the copula's family changes is then detected.

D - If the result of the test shows that the copula's family may not change, then only changes of copula parameters are dealt with.

E - Otherwise, if the result of the test tells us that the copula family may change, then the changes of copula family are examined.

3 - Change-point tests can be used to detect when there is change inside the parameters. Now, considering that change-point tests have less power in case of ``small'' changes, it may be assumed that the parameters change according to a time-varying function of predetermined variables.

4 - U-statistics can also be used to detect change point.

Finally, this sequence of steps permits a sequence of copulas to be obtained: it can be a particular copula with evolutionary parameters and/or sequences of different copulas. At each step the VaR measure is computed providing a sequence of VaR measures that evolve over time.

3 – 1 – 2 Meta-distribution

In the previous approach, the complete information set was used in order to try to adapt empirically the evolution of the changes that are observed all along the trajectory. Sometimes the changes are very important corresponding to specific events and need to be clearly

identified. Indeed, in finance, structural behaviours like volatility, jumps, explosions and seasonality provoking strong non-stationarity may be observed. Alternatively, aggregation or distortion may also be at the origin of non-stationarity. Thus, the assumptions of strong or weak stationarity fail definitively. Indeed, the existence of volatility means that the variance must depend on time. With seasonality, the covariance depends on time producing evidence of non-stationarity. Existence of jumps produces several regimes within data sets. These different regimes can characterize the level of the data or its volatility. Changes in mean or in variance affect the properties of the distribution function characterizing the underlying process. Thus, this distribution function cannot be invariant under time-shifts and thus a global stationarity cannot be assumed. Distortion effects correspond to explosions that cannot be removed from any transformation. This behaviour can also be viewed as a structural effect. Existence of explosions means that some higher order moments of the distribution function do not exist. Concatenated data sets used to produce specific behaviour cannot have the same probability distribution function for the whole period, as soon as there is a juxtaposition of several data sets. Aggregation of independent or weakly dependent random variables is a source of specific features. All of these behaviours may provoke the non existence of higher order moments and non-invariance of the distribution function. Using the dynamic copula concept does not make it always possible to detect correctly the time at which changes arise, because the change point method is not always applicable. Thus, it appears necessary to work in another way, in order to integrate correctly the non-stationarities in the computation of the VaR.

This research proposes to build homogeneity intervals on which the underlying distribution function is invariant almost up to the four first moments extending the works of Starica and Granger (2005), who propose a test based on the first two order moments of a distribution function. The principle of the test here is the following. It is assumed that a time

series $(Y_1,...,Y_n)$ is observed, and a subset $(Y_{m_1},...,Y_{m_2}), \forall m_1, m_2 \in N$ considered, on which the test is then applied, based on the four first moments. For this subset, the test provides a certain value and a confidence interval. Then, rolling windows are used, and another subset $(Y_{m_{2+1}},...Y_{m_{2+p}})$, for some $p \in N$, is considered, on which the test is again applied. This is extended in the same way, sequentially. For each interval, the value of the test is compared with the one obtained with the previous interval, using confidence intervals. Thus, a sequence of homogeneity intervals is constructed, for which invariance is known to exist for the fourth order moments using the following statistic:

$$\hat{T}(n,Y) = \sup_{\lambda \in [-\pi,\pi]} \left| \int_{[-\pi,\pi]^{k-1}} \left(\frac{I_{c_k,Y,n}(z)}{f_{c_k,Y}} - \frac{\hat{c}_k}{c_k} \right) dz \right|, \quad (3.1)$$

where $\hat{c}_k$ is an estimate of c_k, the cumulants of order k of the process $(Y_t)_t$, $f_{c_k,Y}$ denote the spectral density of cumulants of order k, and $I_{C_k,Y,n}$, its estimate using a sample $(Y_1,...,Y_n)$.

It may be shown – under the null hypothesis that the cumulants of order k are invariant in the subsamples – that statistic (3.1) converges in distribution to $\frac{(2\pi)^{k-1}}{c_k} B(\sum_{j=1}^{k-1} \lambda_j)$ where B(.) is the Brownian bridge, for k = 3, 4. The critical values associated with this test can be computed, and will permit confidence intervals to be built. Then, as soon as these homogeneity intervals have been identified, an invariant distribution function can be computed for each interval, and so a sequence of invariant distribution functions can be defined throughout the sample.

For a portfolio which is composed of m assets, this is calculated for each asset. Therefore, a copula linking these different assets using the margins can be estimated for a specific homogeneity interval, for instance on the last one. But other strategies can also be developed. This approach provides two kinds of results:

1 - Working with only one asset: this method associating this non-stationary time series with a distribution function obtained through a copula, permits to link a sequence of invariant distribution functions detected all along the trajectory. Indeed, as soon as the asset $(Y_t)_t$ is characterized by a sequence of r stationary subsamples $Y^{(1)},...,Y^{(r)}$, each characterized by an invariant distribution function $F_{Y^{(i)}}$, i= 1, …, r, the copula linking these margins permits the distribution function of $(Y_t)_t$ to be estimated, and provides an analytical expression of this distribution function that is called a meta-distribution function (this copula can also be characterized by sequence of parameters θ evolving over time):

$$F(Y^{(1)},...,Y^{(r)}) = C_\theta(F(Y^{(1)}),...,F(Y^{(r)})),$$

where $Y_t^{(i)}$ is the process defined in each subsample, i = 1, … , r.

2 – Working with m assets: if a portfolio which has m assets $(X_1, X_2,..., X_m)$ is considered next, the same procedures as used before may be used again. This means that for each asset, a sequence of invariant distribution functions $(F_{X_i}^{(1)},....,F_{X_i}^{(r)})$ is defined, for i = 1,…, m (assuming that r homogeneity intervals are detected for each asset). Then, in order to obtain a robust value of the VaR measure associated with this portfolio, the best copula C_θ is estimated which permits, for instance, the invariant distribution function associated to each market to be linked to the last homogeneity interval. This provides the following multivariate distribution function:

$$F(X_1,..., X_m \,|\, I_r) = C_\theta(F_{X_1}^{(r)}(X_1),...,F_{X_m}^{(r)}(X_m)), \quad (3.2)$$

where I_r is the r-th homogeneity interval. For a given α, the VaR_α is computed as the quantile of the expression (3.2) for this α .

- 3 – 1 - 3 The Pot Method

To compute the VaR associated with a portfolio, it is also possible to consider an approach based on the behaviour of the tails of the empirical joint distribution of the assets, using the Peak Over Threshold method. The Peak Over Threshold method (POT) computes the associated distribution of excesses over a high threshold u, for a random variable X whose distribution function is F, as $F_u(y) = P[X - u \le y | X > u]$:

$$F_u(y) = \frac{F(y-u) - F(u)}{1 - F(u)}, \quad (3.3)$$

for $0 \le y < x_+ - u$, where $x_+ \le \infty$ is the upper endpoint of F. For a large class of distribution functions F (including all the common continuous distribution functions), the excess function F_u converges on a Generalized Pareto Distribution (GPD), denoted $G_{\xi\beta}$, as the threshold u rises. Furthermore, it may be assumed that the GPD models can approximate the unknown excess distribution function F_u. For a certain threshold u and for some ξ and β, (to be estimated):

$$F_u(y) = G_{\xi\beta}(y). \quad (3.4)$$

By setting x = u+y and combining expressions (3.3) and (3.4), the following is obtained:

$$F(x) = (1 - F(u))G_{\xi\beta}(x-u) + F(u), x > u,$$

which permits an approximation of the tail of the distribution F to be obtained. From an empirical point of view, the following steps are taken.

1 - If dealing with a time series with an unknown underlying distribution F, an estimate for F(u) may be constructed, using the N_u data exceeding the fixed threshold u and the parameters ξ and β of the GPD may be estimated. Then the following estimator for the tail distribution is obtained:

$$\hat{F}(x) = 1 - \frac{N_u}{N}(1 + \hat{\xi}\frac{x-u}{\hat{\beta}})^{-1/\hat{\xi}}, \quad (3.5)$$

which is only valid for x>u.

2 - Next, using the tail estimator (3.5) with the estimated values of $\hat{\xi}$ and $\hat{\beta}$, the tail of the empirical marginal distribution $\hat{F}_i$ may be computed for each market X_i for $x_i > u_i$, i= 1,2.

3 - In order to find the copula associated with these markets, the empirical values $\hat{\tau}$ of the Kendall's tau between the two markets X_i, i=1,2 may be computed. This $\hat{\tau}$ is computed in the tails (that are defined by the points on which the GPD is adjusted). The parameter $\hat{\theta}$ of the Archimedean copula is computed using the estimation $\hat{\tau}$.

4 - Using the empirical distribution $\hat{F}_i$ computed on the tails of each market X_i for $x_i > u_i$, i= 1,2 , the following relationship may be obtained for the market (X_1, X_2):

$$\hat{F}(x_i, x_j) = C_{\hat{\theta}}(\hat{F}(x_i), \hat{F}(x_j)), x_i > u_i, x_j > u_j$$

where $C_{\hat{\theta}}$ denotes a copula.

5 - Finally the diagnosis proposed in the Section 2.2.2 is employed to retain the best copula.

To use this method, the threshold u needs to be chosen, that can be a limitation on the method. One way to solve this problem is to use the approach developed in Section 3.1.2., Guégan and Ladoucette (2004).

IV - Concluding remarks

In this paper, we discuss extensively the influence of the presence of non-stationarity within data sets, in order to compute the VaR. To detect the existence of local or global stationarity on data sets, a new test based on the empirical moments above 2 is presented. Then, the concept of meta-distribution is introduced to characterize the joint distribution function of a non-stationary sample. This approach provides interesting solutions to some current, open questions. It is likely that more robust values for the VaR measure may be obtained using this approach, as well as for the ES.

Other points still need to be developed to improve the computations of the risk measures in a non-stationary setting.

- The use of the change point theory has to be developed to get the exact date at which the homogeneity intervals begin.

- The notion of "extreme" copulas need to be investigated in details, in order to build robust estimates for the VaR and the ES measures.

- The knowledge of the computation of VaR measures in an m-dimensional setting is still open. An approach has been proposed by Aas et al (2009) based on cascades method. Nevertheless the choice of the best copulas inside so many permutations is not clear and the computation of the VaR depends strongly of the choice of these permutations. Some new proposals have recently been put forward by Guégan and Maugis (2008), using vines.

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