Ordering operations in square root extractions

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Analyzing some early medieval Sanskrit mathematical texts with the help of Speech Act Theory*

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Abstract

Procedures for extracting square roots written in Sanskrit in two treatises and their commentaries from the fifth to the twelfth centuries are explored with the help of Textology and Speech Act Theory. An analysis of the number and order of the steps presented in these texts is used to show that their aims were not limited to only describing how to carry out the algorithm. The intentions of authors of these Sanskrit mathematical texts are questioned by taking into account the expressivity of relationships established between the world and the text.

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1 Introduction

The Sanskrit scholarly tradition of composing compact procedural *sūtra*s with hairsplitting prose commentaries offers a fertile field for reflecting, as Speech Act Theory (SAT) does, on how prescriptive discourses relate to the real world[1]. Studying the construction and composition of cryptic statements of procedures and the way they are unraveled in

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[1]For a more general study on how Austins work could help contextualize Sanskrit scholarly knowledge see (Ganeri, 2008).

commentaries sheds light on just how diverse the relationships are between the texts that refer to algorithms and the actual physical execution of the algorithms. In other words the way the procedure is stated and the way the procedure is executed are two different realities whose relationships are studied here in the specific case of a connected set of two treatises and three commentaries.

Most available Sanskrit sources on mathematics provide procedures for extracting square and cube roots (*vargamūla*, *ghanamūla*). Such rules were part of the set of elementary operations (*parikarma*, *vidha*) that formed the basis of arithmetic and algebra. The square root procedure remained unchanged, except for small details in the inner workings, from the end of the fifth century to at least the beginning of the twelfth century and probably later[2].

Texts that hand down these rules are of two, tightly linked, kinds: treatises and commentaries. This study then will bring to light different ways in which a treatise and its commentary handled the tension of how a procedure is stated versus how procedure is executed. This analysis is part of a larger endeavor, with the aim of studying descriptive practices in Sanskrit mathematical texts, while focusing on how commentaries relate to their treatises[3].

In the following, the spotlight will be on how the different steps in square root extraction are presented in both the treatises and their commentaries. Attention will be paid to how different actions are stated and explained, in order to unravel the intentions with which these texts were composed.

1.1 Corpus

Five Sanskrit mathematical compositions serve as the basis for this study, as illustrated in Figure 1. The first, an astronomical *siddhānta* (that is, a theoretical text)[4] from the fifth century, the *Āryabhaṭīya* (Ab)[5], and two of its prose and rather prolix commentaries:

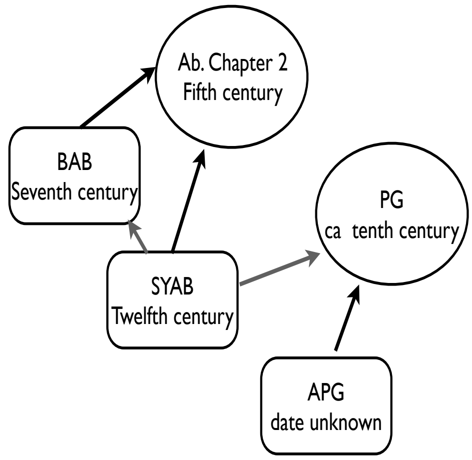
[2]Starting with the procedure given by Āryabhaṭa (499), remaining virtually unchanged in Bhāskarācāryas (fl. 1114) *Līlāvatī* and *Bījagaṇita*, and thus was still in use in later commentaries of these texts.

[3](Keller, 2010).

[4](Pingree, 1981, 13).

[5](
,
).

Figure 1: A connected set of five Sanskrit texts dealing with mathematics: 2 treatises, 3 commentaries



Bhāskaras *Āryabhaṭīyabhāṣya* [6] (BAB), from the seventh century, and Sūryadeva Yajvans twelfth century *Bhaṭaprakāśikā* (SYAB)[7]. Then, Śrīdharas "practical *Pāṭīgaṇita* (PG)(ca. five-twelve century) and its anonymous and undated commentary (APG)[8]. These texts belong to the early medieval period of Indian mathematics: after the ancient tradition of ritual geometry stated in the *Sulbasūtra*s and before Bhāskarācāryas (twelfth century) influential and synthetic works, such as the *Līlāvatī* and the *Bījagaṇita*.

The corpus consists of a set of connected texts, although they were composed at different times and in different places. As seen in Figure 1, commentaries are linked, naturally, to the text they comment on, here symbolized by black arrows. Furthermore, a commentator of the Ab, the author of SYAB, has read the PG, and quotes it. He also often paraphrases BAB. These relationships are symbolized by gray arrows. Considered together, these texts belong and testify to the cosmopolitan Sanskrit mathematics culture of the fifth to twelfth centuries[9]. However the two treatises examined here are different in nature: as stated previously the Ab is a theoretical astronomical text, with only one chapter devoted to mathematics (*gaṇita*), while the PG is solely a mathematical text,

[6](K. S. Shukla, 1976, 52-53). A translation of Bhāskaras commentary on Āryabhaṭas verse on root extraction can be found in (Keller, 2006, Volume 1, p. 20-21), and an explanation of the process in (Keller, 2006, Volume 2, p. 15-18).

[7](Sarma, 1976). A translation of his commentary on Āryabhaṭas verse for square root extractions is given in Appendix C.

[8](K. S. Shukla, 1959). A translation of the anonymous and undated commentary on the *Pāṭīgaṇita*s rule for extracting square roots is given in Appendix D). An explanation of this rule is given in Shuklas translation.

[9](Pollock, 2006, Part I).

devoted to wordly earthly (eg. everyday) practices (*lokavyavahāra*)[10]. Practices employed for stating procedures changed from one type of text to another. The aim of this study then is to forge tools to better describe and understand such differences.

The rules given by Āryabhaṭa and Śrīdhara are shown with their Sanskrit transliteration in Table 1. In the following, various analyses of these rules implicitly suppose that the reader has this table to hand, and can compare and analyze the graphics, the lists etc. with the texts presented here.

[10]Thus Śrīdhara starts his treatise with the following statement, (K. S. Shukla, 1959, Sanskrit: i, English: 1):

> *PG.1cd (aham) lokavyavahārārthaṃ gaṇitaṃ saṃkṣepato vakṣye*||
> I will briefly state mathematics aiming at wordly practices

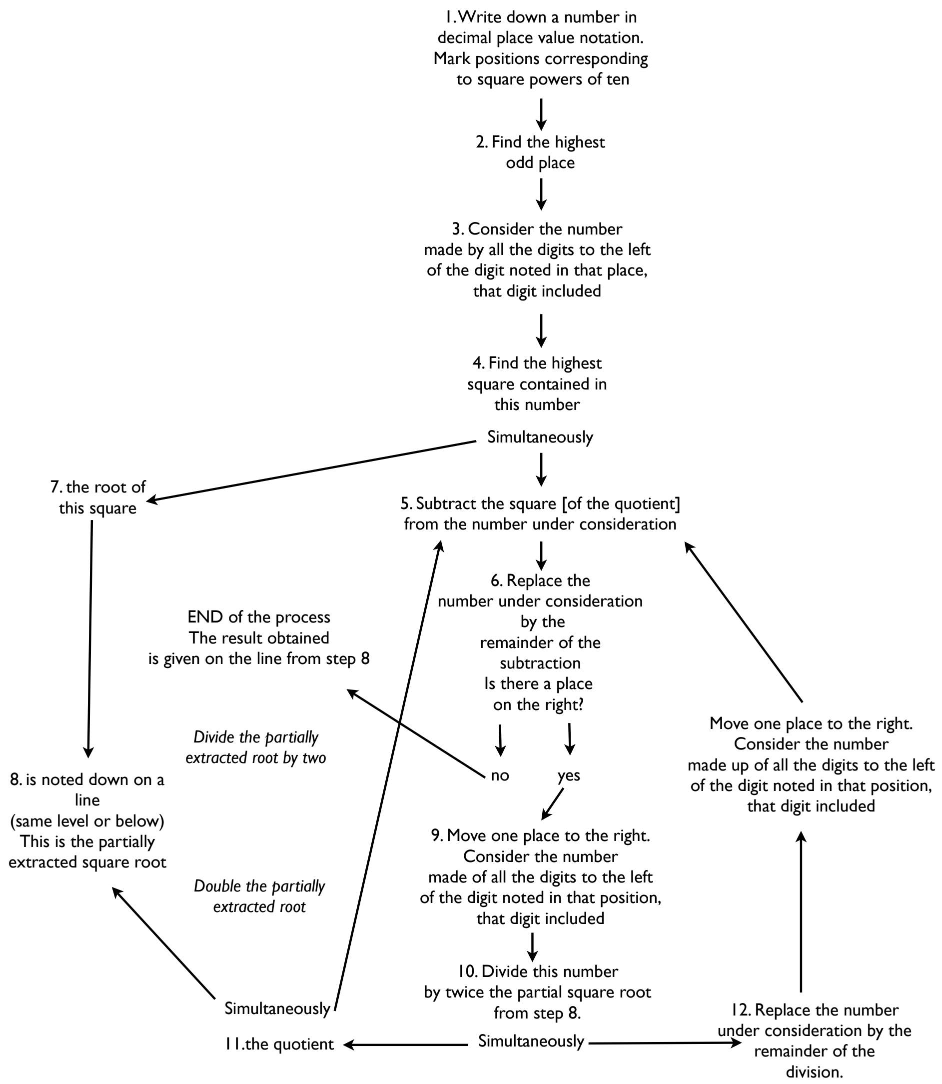
Table 1: Two Rules for extracting Square Roots

Treatise	Sanskrit Transliteration	English Translation[a]
Ab.2.4.ab[b]	*bhāgaṃ hared avargān nityaṃ dviguṇena vargamūlena*|	One should divide, repeatedly, the non-square [place] by twice the square-root|
Ab.2.4.cd	*vargād varge śuddhe labdhaṃ sthānāntare mūlam*||	When the square has been subtracted from the square [place], the result is a root in a different place||
PG.25.abcd[c]	*viṣamāt padas tyaktyvā vargaṃ sthānacyutena mūlena*| *dviguṇena bhajec cheṣaṃ labdhaṃ viniveśayet paṅktau* ||	Having removed the square from the odd term, one should divide the remainder by twice the root that has dropped down to a place [and] insert the quotient on a line||
PG.26.abcd	*tadvargaṃ saṃśodhya dviguṇaṃ kurvīt purvaval labdham*| *utsārya tato vibhajec śeṣaṃ dviguṇīkṛtaṃ dalayet*||	Having subtracted the square of that, having moved the previous result that has been doubled, then, one should divide the remainder. [Finally] one should halve what has been doubled.||

[a] [] indicate my own completions.
[b] (K. S. Shukla, 1976, 52).
[c] (K. S. Shukla, 1959, Sanskrit text, 18; English Translation, 9).

The procedure used for extracting square roots will not be discussed in what follows. Appendix A lists steps for extracting a square root, Appendix B illustrates this with the extraction of the square root of 186 624, an example carried out in the APG. A visualization of the process is given in Figure 2[11].

Figure 2: Different Steps in the Extraction of a Square Root



[11]This diagram should not be seen as an attempt to formalize the algorithm: it is only a heuristic illustration.

1.2 The mathematical ideas underlying square root extraction procedures

The process for extracting square roots relies on the decompositional nature of decimal place-value notation: the number, say 186 624, whose square root is to be extracted is considered to be the numerical square of another number. That is $186624 = b^2$. Extracting its square root means recovering the different elements of the developed square. In other words, if we take the numerical example from the APG, 186 624, the process uncovers different b_i values (that is both the values of b and i, i giving the powers of ten concerned) such that

$$186624 = (\sum_{i=0}^{p} b_i 10^i)^2 = \sum_{i=0}^{p} b_i^2 10^{2i} + \sum_{0\leq i,j\leq p}^{i+j\leq 2p} 2b_i b_j$$

To do so, the process takes decimal development of 186 624 as the sum of squares, $\sum_{i=0}^{p} b_i^2 10^{2i}$, and of double products of the type $2b_i b_j 10^{i+j}$, for $0 \leq i, j \leq p$ and $i + j \leq 2p$. Consequently, the process of extracting square roots, an iterative process, characterized as the repeated subtraction of squares, and division by doubled numbers. The repeated division by a doubled number explains the difference between the process provided by the Ab and that given in the PG: the PG arrives at a doubled root (useful during the process), while the Ab describes a process that enables one to obtain the square root immediately.

1.3 The procedure for extracting square roots in Sanskrit texts: the difficult question of description

Trying to determine the intentions and meanings of Sanskrit mathematical texts is made difficult by the fact that, as historians, we know little of the context in which mathematical texts were produced and used. Furthermore, I do not possess native knowledge of Sanskrit. To put it with Austins words, the accompaniments and circumstances of the utterance of *sūtras* are largely lost to us as readers today[12]. Or, to state the difficulties inherent to the historians trade according to Searles categories[13], and as described by Virbel in this volume, condition 1 of Searles "how to promise (e.g. in this case, being able to execute an algorithm) involves native knowledge of the language. Furthermore ignorance

[12](Austin, 1962, 76).
[13](Searle, 1969, 57-61).

of the context means that we cannot satisfy Searles conditions 4, 6 and 9. Indeed, we are not sure of the authors aim (6), nor that of his imagined reader or hearer (4) and can thus only be poor judges of how well, or not, the authors intentions are conveyed by the texts we read (9). As pointed out by Virbel then, certain conditions on the possibility for communication (1 and 9) and for making commitments (4 and 6) are not fulfilled. Nonetheless, in the following, treading carefully, the intentions of the Sanskrit authors of these statements of mathematical algorithms will be discussed. To do so, the light shed on the authors by their commentators will be used.

The procedure for extracting square roots has consistently attracted attention from historians of Indian mathematics. It testifies an early use of decimal place-value notation. Furthermore, the process found here is very similar to the one taught until the middle of the twentieth century in secondary schools in Europe, the United States and probably elsewhere in the world. However, how the procedures were originally carried out, practically, step by step, remains obscured by variations developed over time and the concision of the rules. Various reconstructions have been offered by secondary sources, from Singh[14] to, more recently, Plofker[15]. How such processes were executed in practice is, however, rarely discussed or justified by a direct quotation of sources.

Indeed, there are several layers of difficulties in such reconstructions.

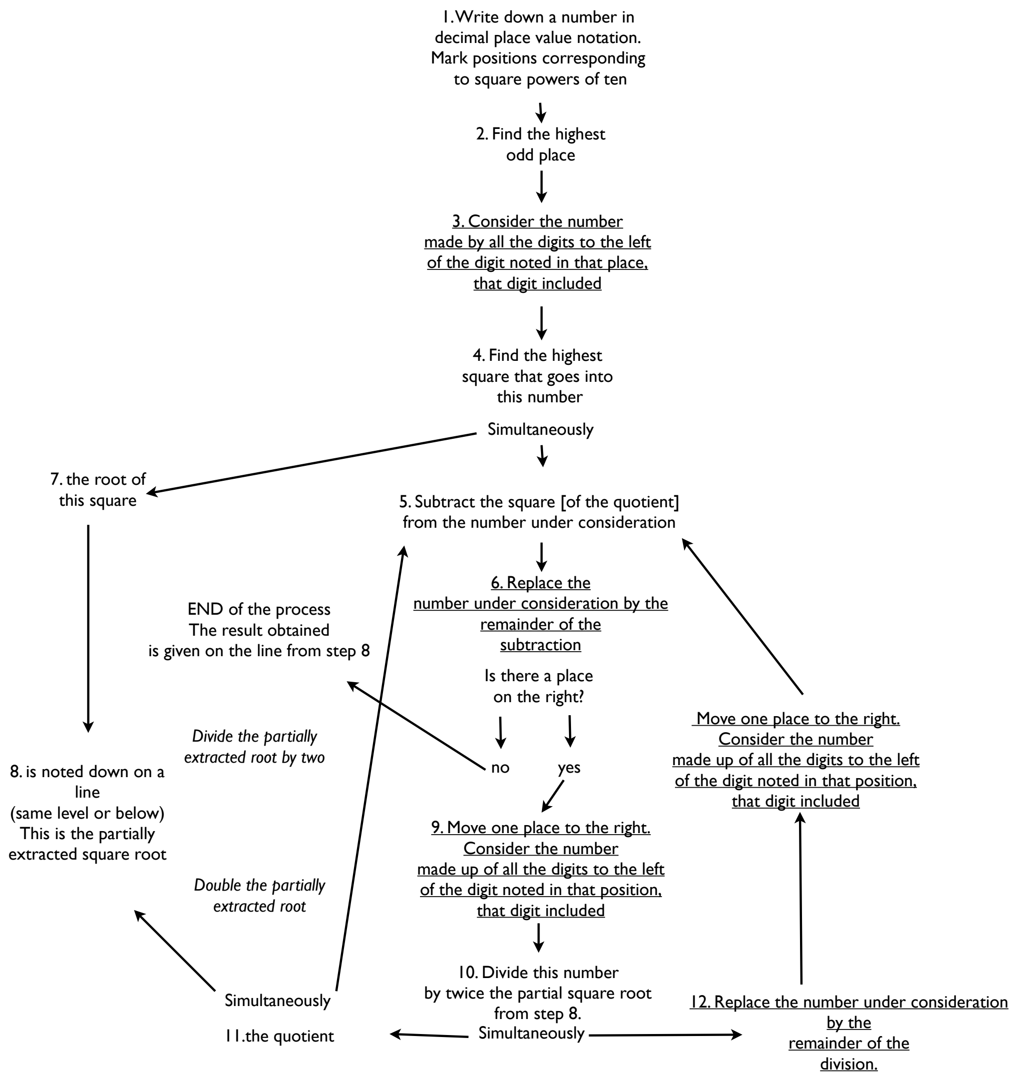
Even if we set aside the muddles inherent to the historians trade, the reconstruction of a procedural text is made arduous because of what one may term, following K. Chemla in this volume, the *granularity* of steps. This problem is certainly familiar to anyone who has had to describe an algorithm: what is stated as one step can often hide several others. For instance when “one should subtract the square from the square, given that the numbers and place where the subtraction should be carried out are detailed, this operation is considered as a single step, although a subtraction or a squaring might involve many steps. Thus “elementary operations in a more complex algorithm are stated without being described.

Of course, part of this granularity may have to do with “tacit knowledge. Thus some steps may have been considered so obvious that they did not need to be stated. For instance, none of the authors considered here specify that the remainder of the division should replace the initial dividend, or that the remainder of the subtraction replaces the minuend. Similarly, they do not state explicitly that after each arithmetical operation (division or subtraction) one needs to move one place to the right. Since all the texts

[14] (Singh, 1927).
[15] (Plofker, 2009, 123-125).

are silent on these steps that however are required, they may thus be considered tacit, as illustrated in Figure 3. Furthermore, commentators on theoretical *siddhāntas* may have considered, tacitly, that the“practical steps of the process were not to be specified. This may explain why so few layouts are indeed provided in the texts handed down to us. The following study will focus on the steps that are actually stated, leaving the tacit in the shade.

Figure 3: The Tacit Steps in a Square Root Extraction Underlined



However, concentrating on the steps “actually stated in the corpus only helps to bring out the difficulties in defining and specifying what “detailing an algorithmic step actually means. Indeed, obstacles in recovering “the algorithm may be inherent to the complexity of the relationships between what is stated about an algorithm, and the algorithms execution. A symptom of this difficulty has pervaded the writing of this article:

each new approach to (the texts on) square root extraction induced a new representation of the algorithm. Each new representation never exactly coincided with the others. Of course, I could try endlessly to coordinate such different representations: checking that they keep the same number of steps, respect the same identified actions and hierarchies between different steps. But I finally decided, on the contrary, to leave each description with its singular expressivity: none are wrong or faulty in respect to the text it illustrates, or the algorithm it refers to. But none coincide exactly either: each representation gives only part of the information. *No two representations coincide with each other.* Indeed these multiple representations demonstrate and illustrate how complex the relationships are between the executed algorithm and the way it can be referred to with words or figures. Each heuristic representation we forge to explain one or other aspect of the algorithm adds yet another layer to this complexity. In other words, there is no single, absolute way of describing the algorithms for extracting square roots and the different ways they are stated, whether it is to express the different ways it can be executed, or the different statements that can be made about it.

The analysis in this article will be restricted to three elements that are usually associated with algorithms. If a procedure is thought of as (1) an ordered (2) list of (3) actions to be carried out: the kinds of actions, the way the steps are listed and ordered will be discussed here. More specifically, in the following, first the kinds of statements Āryabhaṭas and Śrīdharas rules provide will be discussed, noting the paradox of *sūtra*s which both prescribe and are cryptic. What the different texts tell us of the algorithm will be studied, looking at how they detail and order actions and treat the procedures iteration. In the end, the relationships these texts weave with the real world will help provide a hypotheses on their different intentions.

In order to understand how and why an author "states" the steps in an algorithm, the focus needs to be on the kind of text that transmits the procedure. What kinds of statements on procedures are produced in Sanskrit mathematical texts: Descriptions? Incentives to actually carry out the procedure?

2 The prescriptive paradox of compact procedures

Procedures are transmitted through rules (*sūtra*s) and their commentaries. A *sūtra* as has been noted in some detail by Louis Renou, is a complex linguistic object used in a great diversity of communication acts[16]. In the following, the focus will be on how this

[16](Renou, 1963).

complexity is given an additional twist as mathematical algorithmic *sūtra*s are analyzed.[17] As *sūtra*s are often described as being cryptic, let us look closely at was this means in the case of *sūtra*s providing a procedure for extracting square roots.

2.1 Being cryptic

As seen in Table 1, read in isolation, the rules given by Āryabhaṭa and Śrīdhara are difficult to understand[18]: the Ab and the PG, do not specify what is produced if the rules are followed, nor do they specify what an odd term/square place is. None of the verses[19] indicate how to start the procedure, nor how to end it. Only some of the steps allowing the procedure to be carried out are given. This ellipse is illustrated in Figure 4 and in Figure 5.

In both Figures, the steps as given are contrasted with the actual steps required to carry out the process as analyzed in Figure 2. The underlined "tacit steps in Figure 3 are left out. The specificities of Śrīdharaś extraction of a double square root are not represented in Figure 4 on Āryabhaṭas rule. Āryabhaṭas *sūtra* gives only part of the process, its core: reduced to four steps, the algorithm is given in an unspecified order and seems restricted to a succession of divisions and subtractions around which other steps gravitate. Śrīdharas rule, although more detailed, also gives only part of the process: reduced to seven steps, unspecified in order (how does one go from step A to step B?), the emphasis is less on the heart of the iteration and more on the detail of what is done to the number "inserted on a line.

Thus, a first level of reading immediately reveals the ellipses of the rules when contrasted with the execution of the algorithm.

Another way to state the same fact consists in listing the detailed steps. With arbitrariness and limitations in mind, the steps in the procedure given in Āryabhaṭas verse can be listed as follows:

i. Divide the non-square place by twice the square root

[17]Incidentally, this study shows that these mathematical rules do not correspond either to Group A or Group B as defined by Renou in (?, ?, part C). Features of group A such as the use of the optative, are combined here with the prescriptive norms of group B.

[18]The cryptic character of this rule has been analyzed in (Keller, 2006, xvii), (Keller, 2010, 235-236) and is noted in (Plofker, 2009, 123-125). Some of its characteristics are described in (Singh, 1927).

[19]The Ab and the PG provide rules for extraction in a verse form that counts the number of syllabic units, the *āryā*. This is a very common verse form for prescriptive texts.

Figure 4: Steps provided by Āryabhaṭa (in Rectangles)

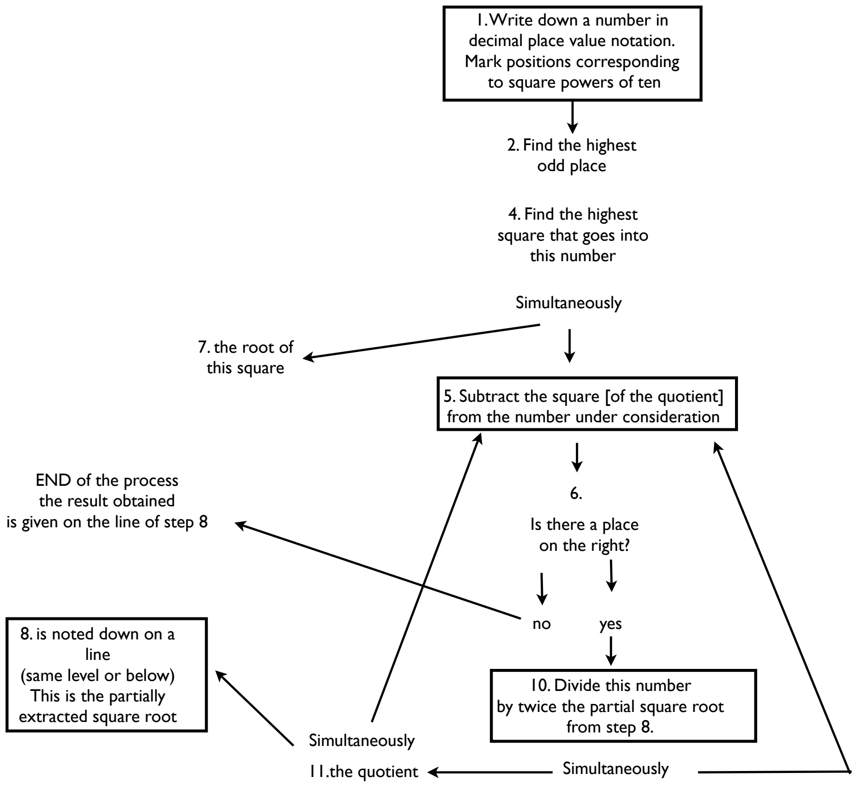
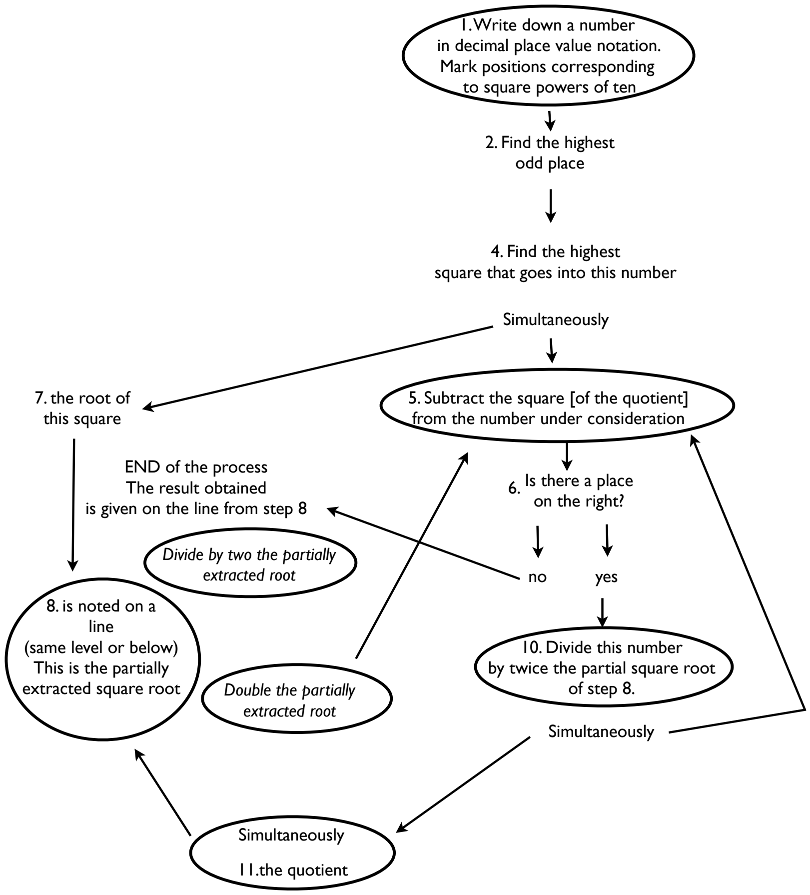


Figure 5: Steps provided by Śrīdhara (in Ovals)



ii. Iterate ("repeatedly)

iii. Subtract the square from the square place

iv. The result, a quotient, noted in a "different place is the (square) root

One could add an implicit step, the one which notes down the number to be extracted in a grid which identifies even powers of ten. This step can also be considered as included in the subtraction step. Similarly step i and step iii could also be interpreted as, in fact, including two steps each.[20]

Whatever the nuances we might want to add, this enumeration highlights how compact Āryabhaṭas verse is. Indeed the square root extraction as reconstructed in Appendix A in order to carry it out includes 16/17 steps, while Āryabhaṭa states between 3 and 8 steps.

Although, compared to the Ab, the PG is less concise- indeed Śrīdhara states the process in two verses while Āryabhaṭa uses only one- the process given in the *Pāṭīgaṇita* is also quite compact.

Śrīdharaś rule states the following steps:

i. Remove the square from an odd place

ii. Divide the remainder by twice the root

iii. The digits of the partial root are placed on a line below

iv. The square of the quotient is subtracted (from what is not specified)

v. Double the quotient and place it on a line

vi. Divide the remainder as in step 2, that is: Iterate

vii. The final result is divided by two

As can also be seen in Figure 6, doubling and dividing by two adds two steps to the process described in Āryabhaṭas compact verse. Furthermore Śrīdhara indicates more explicitly how the process ends.

[20] The difficulty of actually singling out the steps in Āryabhaṭas verse, addressed in the next section, can be seen when this enumeration is compared with Figure 4. With less contrast, the same can be seen for Śrīdharaś rule as well.

This first analysis of the different steps provided by the authors shows that the *sūtra*s considered here - whether overtly short as in the Ab, or more explicit as in the PG- are not sufficient to actually carry out the algorithm. If these rules aim to describe the process or prescribe actions, then some steps are missing. If these rules do not have such an aim, we can only remark that their initial intention is, at this stage, unknown to us. Thus, in both cases some information is lacking. These rules are so compact as be difficult to understand as they stand: they are cryptic. The difficulty of properly isolating the different steps stated in each rule shows that the tools necessary for further, rigorous description of the kind of compactness which characterizes different mathematical *sūtra*s elude us.

Elliptic formulations are often understood by Indologists as recalling the oral sphere. The enigma of cryptic *sūtra*s could have a mnemonic value unraveled through oral explanation. For instance, part of Ab.2.4s obscurity is rooted in wordplay on the word ‘square (*varga*). The Ab gives this name to both the square of a number and to the places in place-value notation having an even power of ten. Such places have the value of a square power of ten. It is also from such ‘square places that we find the ‘square numbers the process tries to bring out. Thus such wordplay recalls the main mathematical idea behind the procedure while simultaneously giving rhythm to the verse and making it confusing. Ab.2.4 can thus be understood as a mnemonic “chimera[21]. Other reasons commonly advanced for using short forms include secrecy, the desire to emphasize the difficulty of the given technical knowledge to add prestige for a profession living on patronage.

While these rules are compact to the point of being cryptic, they nonetheless prescribe an action to be taken. This prescription is voiced by an optative.

2.2 Using optatives

Sanskrit uses nominal forms extensively. Therefore, the use of conjugated forms is in itself an expressive statement. Conjugated verbs in mathematical *sūtra*s indicate a prescription. Indeed, most *sūtra*s of *jyotiṣa* texts (astral science including mathematics) use the optative[22]. Theoretically it is an equivalent to our conditional: it expresses doubt.

[21]To apply in this context the concept that (Severi, 2007) uses to denote pictorial mnemonic artifacts mostly used by North American Indians: the important idea is stamped into the artifact by relating two things that normally should not be connected. This association works like a knot in a handkerchief: something should be remembered here. In this case: where a digit is noted down and a square quantity are given a common name. They are associated in a confusing way in the verse, creating such a chimera.

[22]Note that in grammar (*vyakaraṇa*), according to an oral communication by Jan Houben, the optative belongs essentially to the commentary.

However, it should be understood here as expressing requirement. [23]

In Ab.2.4, “one should divide, is the translation of an expression which uses an optative: *bhāgaṃ hared*, ‘one should withdraw the share- the usual expression of a division. The verb to withdraw (*hṛ*) is in the optative voice. PG 25-26, is a succession of optatives: one should divide (*bhajed*), place (*viniveśayet*), make (*kurvāt*) the double, divide (*vibhajet*), then halve (*dalayet*).

The commentaries follow closely the use of optatives given in the treatises. Thus Bhāskara comments on Āryabhaṭas optative by providing a synonym (*gṛh-*), conjugated as an optative[24]: “One should remove the part, that is, one should divide. Sūryadeva (whose commentary is translated in Appendix C) does not comment upon Āryabhaṭas terms for division but repeats the verb in the optative form, while commenting on what a non-square is[25]: “One should divide by the (last) non-square place. The APG (a translation of the commentary is given in Appendix D) preserves Śrīdharaś optatives sometimes supplying a synonym for others. Thus it uses *bhāgam apaharet* for *bhājet* (one should divide). The APG provides optatives for a number of actions: the first subtraction of a square (*tyajet*), the placement of the root of this first square (*sthāpayed*), the subtraction of the square of the quotient (*śodhayet*), the fact that results should be considered as a unique quantity (*jñāyet*), etc.

Commentaries also use other moods to voice prescriptions: Imperatives when inviting one to solve a problem, obligational verbal adjectives when describing the steps to be taken. In the APG the optative is only used while commenting directly on Śrīdharas verse. When solving the problem, actions are given with absolutives (which give precise temporal orders), such as *śodhayitvā* (“having subtracted) used twice, and by verbal obligation adjectives (such as *kartavya*, “one should carry out).

Therefore, such algorithmic Sanskrit texts are prescriptive. Their prescription is first voiced in the treatises by an optative. These optatives are also taken up and declined in

[23](Renou, 1984, §292):

> Loptatif exprime les nuances variées dun optatif propre- souhait, hortatif, délibératif, éventualité, prescriptif, hypothétiques (...). La coexistence de ces divers emplois nest relevable que dans la poésie littéraire; dans les textes techniques prédomine la valeur prescriptive.

That is, in English (my translation):

> The optative expresses diverse nuances of a true optative: wish, hortative, deliberative, possibility, prescriptive, hypothetical (...) voices. The coexistence of these various uses are only found in literary poetry; in technical texts a prescriptive value prevails.

[24]*tam bhāgam haret gṛhṇīyāt.*

[25]*avargasthānād bhāgaṃ haret.*

other prescriptive forms in commentaries. A cryptic *sūtra* prescribing a procedure to be carried out is a paradox: Indeed, why elaborate short cryptic prescriptions, if the aim is to have them followed? In other words, if the aim is to have a procedure applied, the directive character of an algorithmic rule is contradicted here by its cryptic form. What then were the intentions of the authors of such rules? As this question cannot be replied to directly, how the commentators understood the authors intentions will be observed. But to do so requires further unraveling of the complexity of statements in mathematical *sūtra*s.

2.3 Stating a procedure

The commentators are quite explicit on how they understand the kinds of statements the treatises provide. All the commentators consider the rule primarily as a linguistic assertion: a text whose language is the primary subject of the commentary (which kind of verb(s) it uses, what it means and how it is constructed syntactically). In this respect, all three commentators refer to the text they explicate as a *sūtra*. They also sometimes refer to it as a verse, *kārikā*[26].

Furthermore, the commentaries use vocabulary that relates the verse to mathematical procedures. Thus, *ānayana*, "computation, derived from the verb *ā-Nī*, to lead towards, is used by our three commentators to refer to the mathematical content of the rule. Bhāskara writes as an introductory sentence[27]: "In order to compute (*ānayana*) square roots, he says: Later in the commentary he uses the word *gaṇitakarman* "mathematical process. Similarly Sūryadeva uses almost the same words, but different declensions to introduce the verse in this way[28]: "He states a square root computation with an *ārya*. The anonymous and undated commentary on the *Pāṭīgaṇita* starts by specifying[29]: "A two *ārya* algorithmic rule (*karaṇasūtra*) concerning square roots. He later refers to the process using the expression *ānayana*[30].

Thus the commentators understand the rules as primarily being about mathematical

[26]This cross-reference may refer to the merging of both forms as referred to by (Renou, 1963) who considers that real *sūtra*s are non-versified. The etymology of *kārikā*, derived from the verb *kṛ*-. , "to make, can maybe be understood is this context as "(verse) for action.

[27]*vargamūlānayanāyāha.*

[28]*vargamūlānayanam āryayāha.*

[29]*vargamūle karaṇasūtram āryadvayam.*

[30]Standard vocabulary is used throughout Sanskrit mathematical texts to refer to computations, methods and algorithms. We do not know if there was any difference in meaning between these different words, if their meaning changed over time, according to authors. We have adopted the following translations here: "computation for words derived from *ānī*; "method for *karman*; algorithm or procedure for *karaṇa*.

procedures. Because the rules are also prescriptive, they contain a “commitment, that of stating an algorithm that provides a correct answer to a given problem. Note that the commentarys first move is to provide the procedures intended result. Expecting commentaries, the authors of the *sūtra*s may not have felt it necessary to specify the result of the procedure in the rules they composed.

The commentators thus refer to the rules on both levels: as a statement (on whose language one may comment) and as a procedure (on whose steps one may comment). How do the commentators deal with the *sūtra*s on these two levels? In addition, if the rules for extracting square roots are thus understood as prescribing a process that should be executed, does this mean that they provide a list of steps to carry out?

3 Detailing steps for extracting a Square Root

As noted earlier, the speech act “stating an algorithmic step is complex. Two aspects of this act, the distinction between a certain number of steps and their subsequent ordering are studied now.

3.1 Expressing Actions and Enumerating Steps

Earlier in this article, in an attempt to show that rules provided both by Āryabhaṭa and Śrīdhara were compact, they were crudely restricted to a list of steps. Indeed, our contemporary representation of what a good prescription should be involves listing actions. But how then do the rules given here fair in this respect? Are they lists of actions? And if not, does this imply that they do not describe an algorithm?

Recall Śrīdharaś statement of the procedure, as given in Table 1. A certain number of steps are expressed by a succession of optatives. The essential ordered backbone of operations to be carried out is conveyed in this way: a division, the insertion of a quotient on a line, a doubling, another division and a halving[31]. If we understand this succession as

[31]The fact that the PGs process provides a doubled root that needs to be halved is highlighted (by mistake?) in SYAB. Indeed, this commentator on the *Āryabhaṭīya* notes (Sarma, 1976):

> *labdhe mūlarāśau dviguṇī kṛtaṃ dalayet*
> When the root quantity has been obtained, having multiplied it by two, it should be halved.

However, Āryabhaṭas rule does not provide a double root and therefore does not request a halving at the end.

being a list[32], thus we can take each action as being on the same level of co-enumerability.

The impression that Śrīdhara provides a list of actions is emphasized by the APGs way of taking each optative and following it through twice. Thus in the general commentary[33]:

> And one should divide (*bhāgam apaharet*) from above by twice this, just there. The result should be inserted on a line (*viniveśayet*), one should subtract (*śodhayet*) the square of that from above that, and this should be doubled (*dviguṇīkuryāt*). If when this is doubled an additional place is created (*jāyet*)[34], then it should be used as before (*yojayet*) when it is a result. (...) One should repeat (*utsārayet*) this, thus one should divide (*vibhajet*), one should insert (*viniveśayet*) the result on a line, etc. as before in as much as the serpentine progression is possible, when finished one should halve (*dalayet*) all the result.

The optative is used when commenting directly on Śrīdharaś verse. Moreover, the APG takes elements that Śrīdhara did not formulate with conjugated verbs, and transforms them into conjugated, optative forms. Thus the subtraction, expressed by Śrīdhara with an absolutive (*saṃśodhya*), becomes an optative in the commentary (*śodhayet*). The repetition, an absolutive in the verse (*utsārya*), is an optative in the APG (*utsārayet*). In the resolution of the problem, the APG uses a conjugated verb, to make a quantity slither onto a line (*sarpati*), where the *Pāṭīgaṇita* uses a non-conjugated form to describe a quantity that has been dropped down(*cyuta*). Both Śrīdhara and his commentator seem to consider the rule provided as a list of steps, identified by the use of verbal forms, conjugated or not. Consequently, unraveling here how the authors "detail steps seems fairly simple and straight-forward: conjugated verbs give us the clue.

Āryabhaṭa and his commentators Bhāskara and Sūryadeva provide a stark contrast to this attitude. Indeed, Āryabhaṭas verse itself cannot be reduced to a list of steps of actions to be carried out. It uses only one conjugated verb, referring to a division. Furthermore, the final assertion in the verse is a description of the fact that a result gains a new status by changing place: this declaration has a reflexive character. Such a statement

[32] Note that the Indian subcontinents diversity of manuscripts presents a great variety of material settings; its scholarly texts and a large number of lists. However, there seems to have been no specific typographical layout for lists in mathematical manuscripts in the Indian subcontinent.

[33] *dviguṇena ca tena tatraiva sthitena upariṣṭāt bhāgam apaharet labdhaṃ paṇktau viniveśayet tatas tad vargam upariṣṭāc chodhayet tac ca dviguṇīkuryāt tasmin dviguṇe kṛte yadi sthānam adhikaṃ jāyet tat prāglabdhe yojayet (...) tam utsārayet tato vibhajet labdhaṃ paṅktau viniveśayed ity ādi pūrvavat yāvat utsarpaṇasambhavaḥ / samāptau sarvaṃ labdhaṃ dalayed/.*

[34] In the corpus looked at here, this is the only use of an optative in a non-prescriptive form.

is partly what makes this rule neither a prescription nor an enumeration. Bhāskaras and Sūryadevaś readings of the actions in the rule emphasize division. Thus, the only conjugated verb referring to an action in the algorithm used by Bhāskara concerns this action.

In both cases, in the PG and in the Ab, the presence of conjugated verbs on one side, and of verbal non-conjugated forms on the other, constructs a hierarchy among the various steps of the stated procedure. In the PG the actions, to subtract/remove (*tyaktvā*, *saṃśodhya*), to drop down (*cyuta*), and to evoke the past doubling (*dviguṇīkṛta*) are stated with verbal but non-conjugated forms (absolutives and verbal adjectives). They also seem to provide a list of actions of lesser importance, thus creating a second level of co-enumerability. In other words, the two kinds of verbal form form a hierarchy in the actions to be carried out, as seen in Table 2. Verbs in italics represent non-conjugated forms. Bold verbs represent conjugated forms.

Table 2: Expressing Actions in Śrīdharaś verse and in its anonymous commentary (APG)

Śrīdhara	**APG**
Subtract	**Subtract**
Noting the number	*Noting the number*
Drop down	**Place (under)**
Divide	**Divide**
Insert on a line	**Insert on a line**
Subtract	**Subtract**
Move	**Lead, Slither**
Double	**Double**
Divide	**Repeat**
Halve	**Halve**

As the APG treats all actions on the level of execution, it does not reproduce Śrīdharas hierarchy of actions[35].

In Āryabhaṭas case, the voicing of steps to be carried out cannot be restricted to verbal forms. Aside from the division another action, a subtraction (*śuddha*), is stated with a non-conjugated verbal form, a verbal adjective. Other parts of the algorithm that could be expressed by actions, such as squaring and multiplying by two, are not described in that way: the square of the number (*varga*) is considered directly, as if it had already been computed. Multiplication by two is described with an adjective meaning "having two for multiplier (*dviguṇa*). Among all the actions to be carried out to extract a square root,

[35]Except for two ambiguous elements: the semi-tacit use of decimal place-value notation, and when APG considers the case of a two digit result.

two main actions emerge from Āryabhaṭas verse, those given in verbal form: the action of dividing (first in the verse, and by the fact that it is conjugated) and secondly, the action of subtracting, as illustrated in Table 3. Both commentators of the *Āryabhaṭīya* further respect Āryabhaṭas use of a verbal adjective to refer to subtraction.

Table 3: Expressing Actions

Āryabhaṭa	**Bhāskara**	**Sūryadeva**
		Noting the number
		Setting Aside
Divide	**Divide**	**Divide**
	Noting the number	
Subtraction	*Subtraction*	*Subtraction*
		Double
		Halve

Bhāskara does not introduce new intermediary steps with conjugated verbs of action. For instance, he does not comment upon the subtraction, nor on the multiplication, but underlines how (by contrast with square places), the numerical square in Āryabhaṭas verse, refers to an action[36]: "When subtracting the square, a computed square is the meaning. But the aim here is to accentuate Āryabhaṭas wordplay, while raising its ambiguities: a square operation (*vargagaṇita*) is not to be confused with a square place (*vargasthāna*).

Bhāskara, Sūryadeva and the APG take care to emphasize on the use of decimal place-value notation, especially when describing the grid that is used to carry it out[37]. The two later commentaries, SYAB and APG, express the use of the formal features of decimal place-value notation as an action. Thus, in the APG one should "make (*kṛ*) (marks for the abbreviations of) even and odd places before noting down the number. In Sūryadeva, numbers are set down, placed (*vinyās-*), and then noted down(*cihn-*). This is even more the case in Bhāskara who only refers indirectly to the notation: the settings of the two solved examples in the commentary involve writing numbers; decimal place-value notation also appears when a distinction between "odd and "even places is required. However, Bhāskara does not specify this as an action. In the description he makes of the process as the answer to a question, decimal place-value notation just seems to be the natural background[38]:

[36] *śuddhe varge vargagaṇita iti arthaḥ.*

[37] We studied this aspect of the process, and what it means for the concept of decimal place-value notation in (Keller, 2010), we will thus not dwell on this aspect here.

[38] *kasmāt sthānāt prabhṛtīty*

(One should divide) beginning with which place?
He says: 'From the non-square ⟨place⟩ (...). In this computation, the square is an odd place. Therefore a non-square (...) is an even place, because, indeed, a place is either odd or even.

However, this is not the central step of the process.

Because of the diverse ways of expressing algorithmic steps, rules do not appear directly in the form of a list - a format quite usual in Sanskrit technical texts (*śāstra*). However, loose enumerations of conjugated verbs, such as those given by Śrīdhara can quite easily be interpreted as a list of steps to be carried out[39]. When a rule only has one, unique conjugated verb, as in the case of the Āryabhaṭa then this interpretation although possible (as seen in our first section), distorts the statement of the rule itself.

Nonetheless, conjugated verbs by their contrast with the other verbs do tell us something of the hierarchization of the different steps of an algorithm. In the rule for extracting square roots studied here, the optative can be seen as first ordering the enumeration of steps contained in the algorithm. In this ordering, the optative provides the action around which the others are structured. This feeling may emerge from commentaries, which always carefully preserve the different ranges of voices: they do not transform the optatives or conjugate the nominal forms- except where the APG focuses on describing on an equal level each effective action of the process.

Therefore, while rules do not necessarily provide lists, they do transmit a hierarchy for the steps. The question now is, what order does this hierarchy reveal?

3.2 Ordering Steps

One of the difficulties of reconstructing algorithms concerns the temporal order in which the different steps of a procedure are to be carried out. Mathematical constraints might sometimes impose a temporal order, but not always. Thus in the procedure for extracting square roots, once the defined (largest) square has been found by trial and error, two actions then have to be carried out: the square has to be subtracted from the number under consideration and the root of the square noted down on a separate line. The order

āha avargāt (...) atra gaṇite viṣamaṃ sthānaṃ vargaḥ (...) avarga iti samaṃ sthānam, yato hi viṣamaṃ samaṃ ca sthānam/.

[39]Although, even in this case, there is still a great disparity between the representation of the action given in Figure 5 and in Table 2.

in which the actions are performed does not change the final result. This question of order can be seen as a consequence of having several implicit steps contained within one given step: when several actions are lumped together in a same step, the order in which they could be carried out, if there are no mathematical or exterior constraints, remains ambiguous. Does the hierarchy of steps observed in the previous section correspond to a temporal order for carrying out the algorithm?

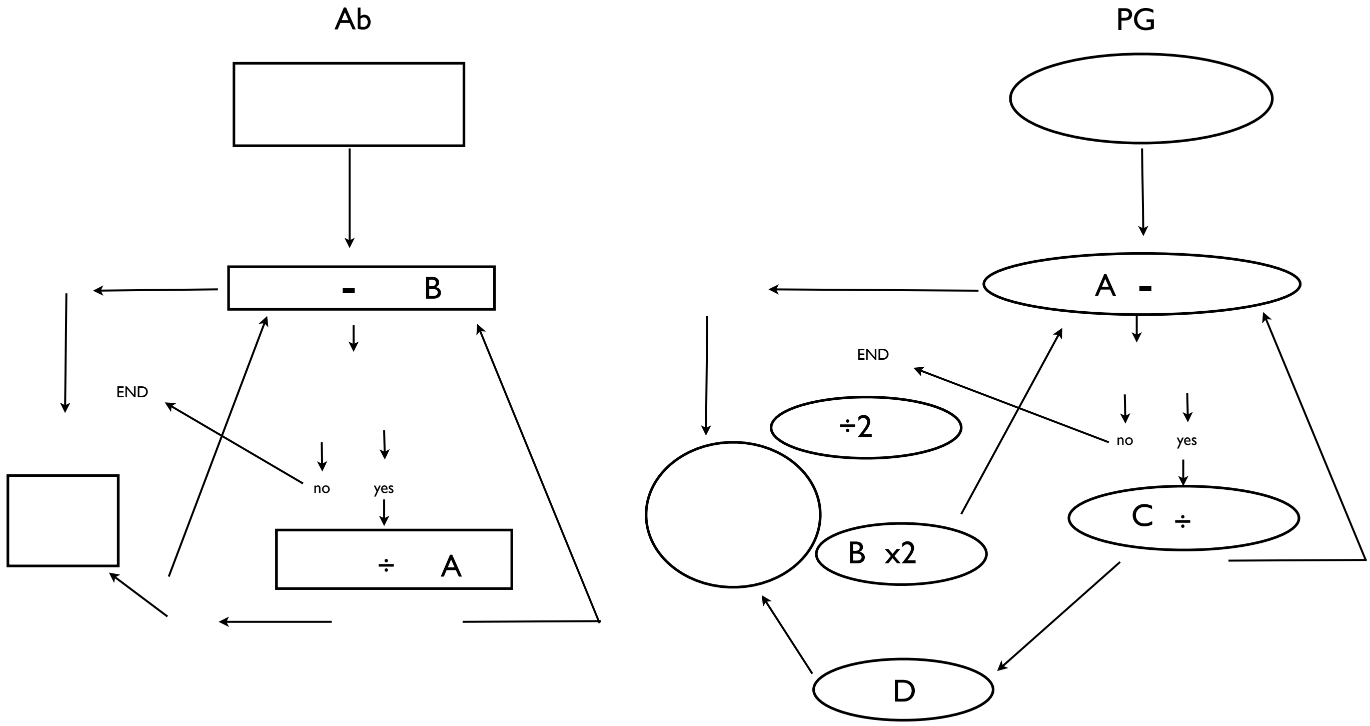
A specific verbal form is used in rules to order a set of actions in time. Absolutives are indeclinable. They are built on a verbal root and mean 'having carried out the action of the verb concerned. Absolutives thus indicate an action to be carried out before the main action indicated by a conjugated verb. Śrīdhara uses absolutives. In PG.2.25 one must subtract (*tyaktvā*, *saṃśodhya*), before dividing or doubling; one should move before dividing. This does not mean that the order of all different steps are elucidated in Śrīdharaś formulation. As illustrated in Figure 5, the placing of the quotient which has dropped down on a line is situated ambiguously in time during the process, as well as the doubling of its digits situated "after the subtraction".

The APG describes, in great detail, the part of Śrīdharaś process which seemed ambiguous: To do so, as noted previously, it does not use absolutives, but first the order in which conjugated verbs are enumerated, to which spatial modifiers (*upari* "above, *adhas* "below") are added. In its solved example the APG uses a wide variety of verbal forms: absolutives (*apāsya*, "having subtracted), verbal adjectives (*vyavasthita, sthita*, "placed), obligational adjectives (*neya*, "one should lead) . Thus not one device but many different types are used in this commentary to express with precision the temporal order of execution.

In certain specific parts of his commentary Sūryadeva uses absolutives such as *cihnayitvā* meaning "having noted, *dviguṇī kṛtya* (having multiplied by two), *vibhajya* (having divided), *apāsya* (having removed). In these sections, he spells out precisely the order in which different steps are to be carried out. Note that the first description concerns the use of decimal place-value notation, and the grid of even-odd places that is applied to it. The second part describes what happens when one computes the square root of fractions. In both cases then, the order does not concern Āryabhaṭas rule directly. The first case supplies actions that enable one to initialize Āryabhaṭas rule. In the second case, Āryabhaṭas rule provides the essential steps for another algorithm. The order expressed articulates steps additional to Āryabhaṭas procedure. In the general commentary[40] Sūryadeva more or less follows the appearance of a first, second and third digit of the square root being

[40]And if my interpretation of the use of "both and "three in this text is correct, as noted in the footnotes of Appendix C.

Figure 6: Comparing the Ab and the PG. Step order is indicated by capital letters.



extracted. This is how the order to perform the process is specified. In this part of his commentary a large diversity of verbal forms are used: verbal adjectives, absolutives and optatives. Although, this ordering of actions is certainly not the main part of his text, Sūryadeva can focus on one element of the process to detail its temporal order. Overall, Sūryadeva does not seem preoccupied by the order displayed in Āryabhaṭas rule.

Thus the ambiguous temporal order for different steps in the procedure is not always clarified by the commentators. They do not always use standard devices, such as the absolutive form of the verb. However there is no temporal ambiguity. To extract the square root you need to start with a subtraction. Āryabhaṭas verse starts with a division, the subtraction is stated at the end of the verse. That is, *in the* Āryabhaṭīya *steps to carry out the algorithm are given in reverse order.* This is underlined in Figure 4, by the letters A and B, which respectively denote the first and second steps to be carried out.

Note that as Sanskrit is a declensional language, a strict order for the words does not need be given. Even though a colloquial word order does exist, *sūtra*s often scramble them. The two actions stated in this rule are given in two successive verses: the action of division is emphasized by the fact that it is the first word, while Sanskrit usually positions the conjugated verb at the end of the sentence.

Neither Āryabhaṭa nor Bhāskara use absolutives. They thus show that the hierarchization implied by the use of conjugated and non-conjugated verbal forms, which gives emphasis of one action over another, may not concern the temporal order.

In BAB, the steps are spelled out in a succession of questions and answers:[41]

> One should take away [in other words] one should divide this [square].
> Beginning with which place?
> He says: 'From the non-square [place] (...) In this computation the square is the odd place.

41

> *taṃ bhāgam, haret gṛhṇīyāt/*
> *kasmāt sthānāt prabhṛtīty*
> *āha avargāt (...) atra gaṇite viṣamaṃ sthānaṃ vargaḥ*
> (...) *kena bhāgaṃ haret ity*
> *āha nityaṃ dviguṇena vargamūlena*
> (...)
> *kathaṃ punas tat vargamūlaṃ labhyate ity*
>
> *āha vargāt varge śuddhe labdhaṃ sthānāntare mūlam*

> (...)
> By what should one divide?
> He says:'Repeatedly, by twice the square root. (...)
> How, then, is this square root obtained?
> He says: 'When the square has been subtracted from the square [place], the result is a root in a different place.

Bhāskara reads Āryabhaṭas verse as being entirely structured around the division. The dialog argues that the order given by Āryabhaṭa is logical from this operative emphasis. Indeed, to carry out a division, a place to carry it out, a divisor and dividend are needed. By unraveling where the division is performed and what the divisor and dividend are, the steps are thus re-ordered and specified. The use of this staged dialog simultaneously emphasizes that the verses steps are disordered while at the same time making an argument for its coherence.

Different orders then can be layered in the statement of a single rule: a temporal order, a logical operative order, or even the order for different cases in which a rule could be applied. Furthermore, the statements of procedures do not necessarily list all the actions that are to be performed, and those listed are not necessarily in temporal order. But what then do they do?

4 Back to the Prescriptive Paradox of Procedural *sūtra*s

Bearing Austins "descriptive fallacy[42] in mind, it is (sometimes) difficult not to consider procedural *sūtra*s as descriptions of procedures. It is also tempting to understand the prescription they voice literally. However, Āryabhaṭas rule is a paradoxical act of communication: a cryptic scrambling of the algorithms steps. This is a clue, that Āryabhaṭas verse contains an indirectly stated intention (an illocutionary force), which may be evading us.

In the following, the initial questions will be raised again: What kinds of statements does a *sūtra* provide when it refers to a procedure? How *sūtra*s and commentaries deal here with the commitment contained in the incetive to perform the procedure will be examined. Afterwards, each authors relation to language- the meta-textual part of the

[42](Austin, 1962, 100).

rules- and each authors relation to the world where the algorithm is performed will be analyzed.

4.1 Commitments and Iteration

These *sūtra*s invite one to carry out a process (one or several operations): they thus contain a more or less implicit commitment, that of obtaining a result. The word used for "result is a substantivated verbal adjective, *labdha*, literally meaning "what has been obtained. It is sometimes translated, as in PG.2.25 as "quotient, being the result of a division.

As noted before, the rules examined here do not state explicitly what the procedure produces[43]. This is the heart of the paradox of prescriptive *sūtra*s : suggesting an action to be carried out, but being evanescent in the commitment the action will fulfill. The result literally shifts repeatedly. In Ab.2.4, *labdhaṃ sthānāntare mūlam*, the quotient/the result (a digit of the partial square-root) is the root in a different place. In PG.2.25, *purvaval labdham*, the previous result/quotient is doubled and moved. In both cases, we are implicitly in the midst of a process in which one result will produce another. The condition of success for the procedure, we understand, has less to do with "obtaining a particular result, than with repeating the process until it is completed[44].

Śrīdhara and Āryabhaṭa do not express the iteration in the same way. Āryabhaṭa states the procedure by beginning with the middle of the process. Moving the quantity from a line where it is a quotient, to a line where it is a digit of the square root, is what enables the procedure to be executed repeatedly. Śrīdhara repeats the process twice using different words: In the first verse it seems that he indicates how the process starts, while in the second verse, a second or final execution of the procedure is described. The rule ends with an evocation of the termination of the process. The authors of both these *sūtra*s use a literary device to explain how the procedure should be repeated: they do not so much describe the performance as offer a textual imitation of it.

Bhāskara and the APG do not reproduce this imitation of the process. Thus Bhāskara

[43]Thus Sūryadeva needs to explain:

> *taṃ saṅkhyāviśeṣaṃ mūlatvena gṛhnīyāt / tadatra vargamūlaphalam ity ucyate /*
> "This special number is referred to (in the rule) as a root. Consequently, here, the result which is a square root (*vargamūlaphala*) has been mentioned (*ucyate*).

[44]The difficulty of pinpointing exactly how the iteration is given in the verses explains why it appears and disappears in the previous illustrations we have given of the algorithm.

states plainly[45]: "This very rule is repeated again and again, until the mathematical process is completed (*parisamāptaṃ*). Bhāskara uses the expression *āvartate* (from *āvṛt-* to revolve, to turn). As seen previously, the APG takes up Śrīdharaś expression using the verb *utsṛj-* (to turn), and adds to it an ordered list of actions to be carried out. By contrast, by repeating the process for several digits, and then describing how it ends, Sūryadeva actually seems to use the *Pāṭīgaṇita*s device.

Turning a rule upside down and repeating it twice: textual devices are used by treatises to offer an imitation of what should be taking place on a working surface, where the algorithm is carried out. Thus for the iteration, the treatises observed here are intent on making their text and the world in which the algorithm is completed coincide not by describing what is going on *in words*, but by describing the repeated algorithm *with words*. Let us look more closely at how the treatises and their commentaries play with the world of the text and the world in which the algorithms are performed.

4.2 World and Text

Although Sanskrit mathematical texts may not list or give a temporal order for actions to be carried out, *sūtra*s and commentaries do nonetheless state something about the algorithm. The kind of adjustments between the world and the text included in the corpus will now be considered.

At times Śrīdhara gives a momentary description of what the computation should be at a given moment. Verbal adjectives seem, in this case, to indicate where the text and the algorithm should coincide. Thus Śrīdhara uses the expression *sthānacyuta*, "that has dropped to a place, *dviguṇīkṛtam*, "that has been doubled". An action is not spelled out but a description is made of the state of the working surface on which the process is performed. These descriptions enjoin the person performing the procedure to adjust the world to the statements in the rule.

Specifically because this is a part of the process which requires know-how that belongs to the world of algorithm execution, the APG is careful to describe, digit by digit, how numbers should be moved around on the working surface. Certain expressions are invitations to verify that at a given moment, the result obtained coincides with the text. Like for instance, in the purely descriptive[46]: "When twenty-four is subtracted by three

[45] *etat eva sūtraṃ punaḥ punararāvartaye yāvatparisamāptaṃ gaṇitakarmeti.*
[46] *tribhiḥ patanāt caturviṃśatau śuddhāyāṃ upari dvau śeṣaḥ.*

from below, above two remains. In other cases, the text describes the temporary state of the working surface, followed by a disposition[47]:

> Below, eighty-six is produced. This quantity slithers (*sarpati*) on a line. Below two, there is six, below seven, eight. Setting down: $\begin{array}{cccc} 1 & 7 & 2 & 4 \\ & 8 & 6 & \end{array}$

Thus the *Pāṭīgaṇita* and its anonymous undated commentary are intent on making the statement of the procedure and its performance- on a working surface using tabular dispositions- coincide.

Understanding conjugated verbs as an expressive device reveals how Bhāskaras commentary on Āryabhaṭas rule is mainly on the level of the language Āryabhaṭa uses. Bhāskara uses the expression “he says” (*āha*)[48] four times. His answers to the questions in the dialog always refer to Āryabhaṭas statements. Bhāskara then is not describing how the process is to be carried out independently from Āryabhaṭa. He is not adjusting Āryabhaṭas statements to how the process should be executed. Indeed, he is just modifying Āryabhaṭas statements in an attempt to show the internal coherence of their arrangement. He explains that this arrangement makes sense if the division is taken as the core from which all other steps in the process derive.

In three instances Bhāskara explores the limits of the mathematical reality expressed by Āryabhaṭa. First, when he specifies evenness as the opposite of oddness, then when the process ends because no other space to carry it out can be found, and finally as he gives an example concerned with fractions in which he then introduces his own rule. In other words, the world of Bhāskara is not like that of the APG, not a world of algorithm execution. His is one of mathematical objects.

Indeed, the iteration in the process is voiced by Āryabhaṭa as a change of status: as a quantity changes place, it becomes another quantity. Such a change needs to be properly identified. This is done by a name change. This way of expressing the iteration is repeated by Bhāskara as he explains that a result changes place, becomes a root, and re-enters the process. The end of the process appears when this change of status becomes

[47] *adhaḥ ṣāḍaṣītijayīte / eśa rāśiḥ sarpati, paṅktyāṃ dvayor adhaḥ (ṣāṭkaṃ) bhavati, aṣṭakaṃ saptādhāh / nyāsaḥ-* $\begin{array}{cccc} 1 & 7 & 2 & 4 \\ & 8 & 6 & \end{array}$.

[48] He also uses once each *labh* (to obtain), *bhū* (to be, have, produce), *vidyate* (to exist, discern). The three only other conjugated verbs of this part of his commentary are: (1) the optative used for division, and (2) verbs used while solving examples at the end of the commentary.

impossible. Bhāskara states it as follows[49]:

> The quotient here becomes, in a different place, what has the name root (*mūlasaṃjñā*). (...) In this different place, this quotient has the name root. However, precisely when a different place is not found, there, in that very place, that [result] has the name root.

The change of place which simultaneously is a change of status is acted out by a formal action: a name change. This action is very literally an attempt to adjust the statement of the process, to the mathematical world the quantity belongs to. In this case, Bhāskara emphasizes how what can seem a confusing change is actually what explains how the process works: each repeated division provides the digits of the square root. The centrality of the division is thus once again stated, even as the quotient disappears to leave space for the root. Finally, this name change is also what signals the end of the process.

All three commentaries link the movement of the quotient to a separate line to a status change. In the anonymous commentary on the *Pāṭīgaṇita* a digression discusses the status of the quantity that has been moved and noted down on a separate line. The double square root is called "the result (*labdha*). After having inserted the result/quotient of the division on the line, having subtracted its square and having doubled it, the APG examines the case where the doubling provides a number bigger than ten[50]: "If when this is doubled an additional place is created, then it should be used as before when it is a result (*praglabdhe*). Both have the quality of being a unique quantity (*rāśitā*). This quantity has the name "result" (*labdhasaṃjñā*). And when one arrives at the end of the process, the "result appears again, and has to be halved.

Sūryadeva starts by considering the first digit of the square root obtained by trial and error. He calls this quantity a "special number (*saṃkhyāviśeṣa*) and notes[51]: "this special number is referred to (in the rule) as a root. Then commenting on the last quarter of Āryabhaṭas verse, he adds[52]: "this ⟨quotient⟩, in the next square place, becomes (*bhavati*) the root. Although, the change of status is the same, here Sūryadeva does not change

[49] *yadatra labdhaṃ tat sthānāntare mūlasaṃjñaṃ bhavati/ (...) tasmin sthānāntare tasya labdhasya mūlasaṃjñā/ yatra worldplayaḥ sthānāntaram eva na vidyate, tatra tasya tatraiva mūlasaṃjñā.*

[50] *tasmin dviguṇe kṛte yadi sthānam adhikaṃ jāyet tat prāglabdhe yojayet, tayor ubhayor ekarāśitājñeyā / tasya rāśerlabdhasaṃjñā.*

[51] *taṃ saṃkhyāviśeṣaṃ mūlatvena gṛhnīyāt .*

[52] *tat purve vargasthāne mūlaṃ bhavati .*

the name. He does so earlier, when he establishes the equivalence between “square, non-square places and “odd, even ones[53]: “In places where numbers are set-down, the odd places have the name (*saṃjñā*) ‘square. Even places have the name ‘non-square.

Therefore, naming appears as a central commentarial activity: when a quantity is renamed by a commentator it reveals how the statements concerning an algorithm are adjusted to coincide with the world they refer to. This world can be on the level of performance (APG) of mathematical objects (Bhāskara), or a combination of both (Sūryadeva).

Therefore, the way the authors relate mathematical statements to the working surface on which a procedure is being carried out, provides us with a clue to what is important to them: the APG develops Śrīdharaś brief descriptions in order to give an algorithm to carry out that is as unambiguous as possible. Bhāskara highlights the fact that when the quotient is moved to a separate line, its name changes: what happens on the working surface is always coherent in the world of Āryabhaṭas statements. Finally, Sūryadeva, not surprisingly for a commentator on the Āryabhaṭa quoting Śrīdhara, seems to try to position himself between both approaches. Thus, he is intent on adjusting decimal place-value grid to Āryabhaṭas statements, and specifies how, in practice, the process should start by trial and error.

In other words, all the analyses carried out thus far shed light on the different intentions of the various authors of the corpus.

4.3 Intention

Specific tools for describing ways of making a text compact, expressing iteration, listing some actions and not others, and relating language and practice can help us infer authorial intention with greater assurance. For instance, by paying attention to which “essential elements of the algorithm a rule states and how different hierarchies of actions are imbedded in a *sūtra* provides us with each authors interpretation of the important points in his algorithm. In the following, the different kinds of statements on algorithms unraveled here will be re-examined focusing on the intentions of their authors.

We have thus seen the use of conjugated verbs (especially the optative) in the *sūtra*s as an expressive device. Śrīdhara singles out a certain number of actions (division, inserting the number on a line) over others (subtracting, doubling). The APG, on the other hand,

[53] *saṃkhyāvinyāsasthāneṣu viṣamasthānāni vargasaṃjñāni / samasthānānyavargasaṃjñāni* .

does not follow such a hierarchy. This is consistent with the aim of the commentary to treat each action in the execution of the algorithm on an equivalent level. Thus the hierarchy of steps in Śrīdharaś verse, not being included in the APG, sheds light on the aims of both. The APG describes how the algorithm is carried out on a working surface: all actions are equivalent from this point of view. Śrīdhara states (and maybe orders) the required actions. The APG with the dynamic image of a slithering snake enters into the detail of the process unraveling the ascending and descending operations, unwinding the intricate temporal order in which each step of the process should be carried out. In other words, the APG also sees the verse as evoking a dynamic process.

Śrīdhara and his anonymous commentator present a stark contrast to the intellectual couple formed by Āryabhaṭa and Bhāskara with their sparse number of conjugated verbs. Bhāskaras emphasis on the operative logic of Āryabhaṭas verse shows that his aim is to comment the coherence of Āryabhaṭas verse, not on how it should be carried out. Similarly, Sūryadevaś relative indifference to Āryabhaṭas scrambled order directs us to towards another aim. Indeed, as in the processes described by Karine Chemla in this volume, Sūryadeva integrates Āryabhaṭas verse in specific cases where the algorithm can be applied. The commentator describes how the rule is situated within other algorithms: root extractions which arrive at double roots and root extractions of fractions. Āryabhaṭas rule then is a general rule, whose temporality and logic is not in question. The hierarchy of operations to be carried out uses Āryabhaṭas rule as a central nod against which further operations are assessed. Sūryadevaś endeavor as a commentator is to make sure that the process covers all possible cases.

Finally, looking at how different authors treat the world of linguistic statements and the world in which a procedure is carried out on a working surface confirms these conjectured intentions. In places Śrīdhara seems to describe the working surface at specific tricky points that are detailed by his anonymous commentator. On the contrary, Bhāskara does not comment at this level, but rather on providing a name at the right time, for the right quantity: re-naming a quantity that has been moved assures us that the process is coherent with the stated rule. Sūryadeva does a bit of both, renaming the decimal place-value notation grid used in the process, and describing how it should be used.

For both BAB and SYAB, the *Āryabhaṭīya* gives the main mathematical ideas behind this procedure. Three hypotheses can be drawn on the intentions behind Āryabhaṭas way of describing the procedure: His first aim could be to establish the procedure (both by explaining it, and providing a way of being able to recall it easily), giving its gist. A second aim could be to transmit a reflection on what the procedure is about (how does one undo a squaring in decimal place-value notation) and what this tells us about numbers.

Most probably the aim was to add together all the above, eg prescribing a procedure, giving its gist, and hinting that this is less about doing than reflecting: an effort to be as general as possible. Since the square root process comes after the definition of a square, since Bhāskara contrasts operations of increase which include squaring with operations of decrease which includes seeking square roots[54], and since his general commentary on the *sūtra* is followed by the resolution of an example which calculates the roots of previously computed squares, we might conclude that, for Bhāskara the square root procedure was less a procedure to follow, and more a reflection on how one dismantles a squaring operation using decimal place-value notation. In a mathematical tradition where the correction of an algorithm was sometimes verified by inverting it, and finding the initial input, square root extraction may have been seen as inverting the squaring procedure. Āryabhaṭas rule then would seem to exhort one to carry out the process whose steps are described, but his real aim (as seen through Bhāskaras eyes at least) would be to transmit a reflection on the procedures mathematical grounding. He might actually be suggesting that the process itself is not only useful for extracting square roots but also as a reflection on what undoing a square operation using decimal place-value notation involves.

Conclusion

Part of the *sūtras* perlocutory or contextual effect is irremediably lost to us, as is the case for all historical texts, but even more so on the Indian subcontinent, where so little is known about the context in which mathematics was practiced. The cryptic algorithmic statements of mathematical *sūtra*s are, to put it in Austins words, neither unambiguous nor explicit. At first reading they can seem strangely vague and full of uncertain references. However this detailed study of rules for square root extraction gives us hope that we can uncover elements of how past milieus created, read and understood mathematical *sūtra*s.

Maybe the "descriptive fallacy of statements on algorithms is to consider that all such statements aim to describe the way algorithms should be carried out; and more specifically that all invitations to carry out an algorithm include a more or less explicit description of how to do so. Indeed, this study has showed firstly that an invitation to carry out an algorithm does not necessarily describe literally how to do so. Secondly, what can be classified as the description of an algorithm can be very diverse. Thirdly, that an invitation to execute an algorithm can also be a coded invitation to reflect on it.

[54](Keller, 2007).

Verses stating algorithms are not neutral descriptions of how to carry out an algorithm. They indicate what the authors wanted to transmit and to emphasize concerning these rules. Due to their expressivity, the procedural statements may also then be read using techniques usually ascribed to reading literature. The commentators readings of these rules show clearly that ambiguous expressions are doors opening onto several specific meanings; the obscure phrases are those that in the end highlight the meaning of the rule.

Conjugated verbs tell us here something of the emphasis, or not, which each text puts on action: Āryabhaṭas theoretical rule uses one conjugated verb, while Śrīdharas practical rule gives several. Bhāskara is intent on commenting on statements, and thus frequently conjugates the verb “to speak, state, while Sūryadeva who reflects on different forms of square-root extractions conjugates the verb *bhū*, “to produce, become, be. Finally the anonymous commentator of Śrīdharaś *sūtra*s, intent on reworking and specifying different steps, supplies many optatives. For different aims, different practices of algorithmic statements can be used. Practices of algorithmic statement appear to vary according to the type of text (theoretical, practical).

Thus, the authors did not necessarily list actions. The hierarchy of steps they do provide does not always represent a temporal order. The authors could specify actions, describe a working surface with a dynamic tabular layout, formulate relationships between *sūtra* statements and the world of mathematical objects, and elucidate different mathematical objects to which the procedure could be applied.

This study has used different tools to describe and understand how processes were made compact in *sūtra*s. Whether represented as a flow chart or as a list, Āryabhaṭas verse does not appear to be an arbitrary fragment of the algorithm. The study of how language related to the world of algorithm creation helped in understanding how the iteration of the process was expressed by the authors of the *sūtra*s. The iteration of the process, given by a repetition by Śrīdharais simply shown by Āryabhaṭa by reversing the usual order of the procedure. In both cases, the literary device of imitation is used to describe a complex reality. Such processes, like the play on the word *varga* used by Āryabhaṭa can be seen as striking stylistic idiosyncrasies- specific to each rule and to each author- which may have had the role of the “knots in ones handkerchief, if such rules were meant to be learned by heart.

The authors main aim then would not have been to describe an algorithm, but rather to comment on it, that is, to emphasize a point in the procedure: its mathematical grounding for Āryabhaṭa, its coherence for Bhāskara, the fact that it was worked out on several connected horizontal lines for Śrīdhara as understood by his anonymous commentator, and finally as a fundamental operation which can be carried out on both integers and fractions for Sūryadeva. The compression of the *sūtra*s then would have less to do with mnemonics and secrecy than with the expressive granularity of algorithm statement.

Finally, for Āryabhaṭa and Bhāskara, stating such rules seems to have had the aim of indicating how the algorithm was constructed, and the mathematical properties it was based on. In other words, they may have intended to highlight that such a procedure gave insights into the properties of numbers written in decimal place-value notation, and into what made them perfect squares or not. Maybe the procedure itself was thought of as a reflexive algorithm.

The definition of the *sūtra* as recalled by (Renou, 1963) is a paradox: a self-sufficient compact verse but also one of a series:

> "Le terme de *sūtra* ou "fil" désigne tantôt une règle énoncée sous la forme dune proposition (...) plus ou moins brêve, tantôt un ensemble de propositions concourant à constituer un même recueil. (...) Le genre du *sūtra* se définit par sa relation plutôt que par son contenu : un *sūtra* (au sens de "règle" ou "aphorisme") est dabord un élément dépendant du contexte, même sil est grammaticalement autonome; il est déterminé par le système et (...) corrélatif au groupe qui lenvironne."
>
> "The word *sūtra* or 'string refers sometimes to a rule stated as a more or less short proposition (...), sometimes as a set of propositions forming a collection. The *sūtra* genre is defined by its relationships rather than its content, a *sūtra* (understood as a 'rule or an 'aphorism) is first and foremost an element dependent on its context, even if it is autonomous grammatically; it is determined by the system and is correlated to the group that is around it

Indeed, this study has showed that *sūtra*s and commentaries are deeply intertwined. The (authors of the) *sūtra*s expected commentaries to provide the mathematical context, the procedures result and the detail required for the execution of the algorithm. Looking at the statements and the way they are formulated and interpreted has, no doubt, underlined the technical reading a *sūtra* requires. If there is expressivity in a *sūtra* we need its commentary to reveal it. There is a specific rhythm to reading a *sūtra* and its commentary: knowing the text of the *sūtra* by heart, understanding it, which means unfolding its meanings and understanding the texts expressivity. Possibly, neither the *sūtra*, nor the commentary were intended to be read just once in a linear way, but masticated over in the way Nietzsche defines aphoristic reflections... the way iterative algorithms are executed.

List of Abbreviations

Ab Āryabhaṭa I s *Āryabhaṭīya* (fifth century)

APG The Anonymous and undated commentary on the *Pāṭīgaṇita* of Śrīdhara

BAB Bhāskara I s commentary on the *Āryabhaṭīyabhāṣya* : *Āryabhaṭīya* (seventh century)

PG The *Pāṭīgaṇita* of Śrīdhara (tenth century)

SYAB Sūryadeva Yajvans commentary on the *Āryabhaṭīya* : *Bhaṭaprakāśikā* (Twelfth century)

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A Different steps in the algorithm for extracting square roots as spelled out in the corpus

Taking into account all the steps detailed by the authors considered here (with an arbitrary filter - the mesh of the net may at times seem too small and at others too large- that is underlined in paragraph 1.3), thirteen steps for extracting a square root can be listed. Step 3, 6 and 12 state common tacit steps. The algorithm may be more efficiently illustrated in Figure 2[55].

1. The number whose square root is to be extracted is noted down in decimal place-value notation. Places are categorized with a grid that enables one to identify square powers of ten. Either positions for square and non-square powers of ten are listed or the series enumerating positions starting with the place with the lowest power of ten is considered. This list categorizes places as even or odd places.
2. The highest odd/square place is identified.
3. Consider (tacitly) the number made by all the digits to the left of the digit noted down in that place, that digit included.
4. Find the largest square contained in the number noted down to the left in the last/highest odd place.

   From here, onwards, one could also start by considering step 8 and 9, before turning to step 5 to 7.
5. Subtract the square from the number under consideration.
6. Replace (tacitly) the minuend by the remainder of the subtraction.
7. The root of the subtracted square is the first digit of the square root being extracted.
8. The root of this square (Ab family)/ The double of the root of this square (PG family) is noted on the same line, to the left of the whole number/ on a line below the line of the number whose root is being extracted. In the PG family then, the doubling of the digit is a separate step in the process. The doubling does not necessarily need to take place immediately, one can note down the digit, and then double it just before it enters the division described in Step 10. This is what the APG recommends.

[55]The "reconstruction of these variants of the different steps of the process is not discussed here. Hopefully this issue will be tackled in a forthcoming article.

9. Consider the number whose highest digit is the previously noted remainder and the next digit to its right.

10. Divide this number by (twice) the partial square root from Step 8[56] In the following, Step 11 can be carried out after Step 12.

11. The quotient is the next digit in the partial square-root. It (Ab), or its double (PG), is thus noted down next to the previously found digit, as in Step 8. Its square is what will be subtracted as the process is iterated here from Step 5.

12. Replace the dividend with the remainder of the division. Then one should consider the next place on the right, which is a square/uneven place.

13. When there is no place on the right, the algorithm is finished. Examples only consider a process that extract a perfect square, consequently, either the square-root, or its double is obtained, according to the procedure followed. If we are in the latter case, the number obtained is halved.

B Extracting the square root of 186 624.

This is a numerical example addressed in APG. Footnotes and asterisks indicate non-attested forms.

1. The number whose square root is extracted is noted in decimal place-value notation. These decimal places are categorized using a grid: Square (*varga*), non-square (*avarga*) powers of ten (Ab), or even (*sama*, abr. *sa*) and odd (*viṣama*, abr. *vi*) place ranks - counted starting with the lowest power of ten- (BAB, PG, SYAB, APG).

avarga	*varga*	*avarga*	*varga*	*avarga*	*varga*
sa	*vi*	*sa*	*vi*	*sa*	*vi*
10^5	10^4	10^3	10^2	10^1	10^0
1	**8**	**6**	**6**	**2**	**4**

[56] Although, this is never mentioned in the ancient texts, the quotient obtained needs to be sufficiently small. This sometimes requires a subtraction by 1 or 2 (and a change in the remainder of the division) to find the adequate digit.

$$186624 = \mathbf{1}.10^5 + \mathbf{8}.10^4 + \mathbf{6}.10^3 + \mathbf{6}.10^2 + \mathbf{2}.10^1 + \mathbf{4}.10^0$$

2. Subtract the square from the highest odd place

   The highest "odd (*viama*) place or "square (*varga*) place is 10^4. The process starts by finding, by trial and error, the highest square number contained in the number noted to the left of this place. In this example, one looks for the highest square that will go into 18. And thus $18 - 4^2$ is the operation carried out.

3. Replace the minuend with (BAB) or place below (APG)

4. The remainder,

5. The root,4, of the subtracted square (16) is the first digit of the square root being extracted. The root of this square (Ab family-4)/ The double of the root of this square (PG family- 8) is noted down on the same line (BAB)/ or a separate line (PG).

   Bhāskara might have written[57]:

$$\begin{array}{cccccc} *10^5 & 10^4 & 10^3 & 10^2 & 10^1 & 10^0 \\ & \mathbf{4}/2 & 6 & 6 & 2 & 4 \end{array}$$

   While the APG writes:

$$\begin{array}{ccccc} 2 & 6 & 6 & 2 & 4 \\ & \mathbf{8} & & & \end{array}$$

   Because $1.10^5 + 8.10^4 = [4.10^2]^2 + 2.10^4$,

   $186624 = [\mathbf{4.10^2}]^2 + 2.10^4 + 6.10^3 + 6.10^2 + 2.10^1 + 4.10^0$.

6. Moving one place to the right, one should divide by twice the root

   In this example, 26 is divided by 8: $26 = 8 \times 3 + 2$. The quotient is 3, 2 is the remainder.

   This is then set down. Bhāskaras style

$$* \quad \mathbf{4} \quad \mathbf{3}/2 \quad 6 \quad 2 \quad 4$$

   APG style

$$\begin{array}{cccc} *2 & 6 & 2 & 4 \\ & \mathbf{8} & 3 & \end{array}$$

[57]$*$ mark non-attested forms.

In other words, because $2.10^4 + 6.10^3 = 8 \times 3.10^3 + 2.10^3$, $186624 = [\mathbf{4.10^2}]^2 + [2 \times (4.10^2)(3.10^1)] + 2.10^3 + 6.10^2 + 2.10^1 + 4.10^0$.

7. Moving one place to the right, iterate. That is "subtract the square again. This time the square of the quotient is subtracted. In this example 3^2 is subtracted from 26: $26 - 9 = 17$. The remainder is 17. This is noted down again:

   Bhāskara style:

$$* \quad \mathbf{4} \quad \mathbf{3}/1 \quad 7 \quad 2 \quad 4$$

   APG style:

$$\begin{array}{cccc} 1 & 7 & 2 & 4 \\ & \mathbf{8} & \mathbf{6} & \end{array}$$

   In other words, writing $26 - 9 = 17$ according to the corresponding powers of ten. $186624 = [\mathbf{4.10^2}]^2 + [2 \times (4.10^2)(3.10^1)] + [\mathbf{3.10^1}]^2 - [3.10^1]^2 + 2.10^3 + 6.10^2 + 2.10^1 + 4.10^0 = [\mathbf{4.10^2}]^2 + [2 \times (4.10^2)(3.10^1)] + [\mathbf{3.10^1}]^2 + 1.10^3 + 7.10^2 + 2.10^1 + 4.10^0$.

8. Moving one place to the right, divide by twice the root. In this example one should divide 172 by $2 \times 43 = 86$: $172 = 2 \times 86$. The quotient is 2 and there is no remainder. This is noted

   Bhāskara style:

$$* \quad \mathbf{4} \quad \mathbf{3} \quad \mathbf{2} \qquad 4$$

   APG style:

$$\begin{array}{cccc} * & & & 4 \\ & \mathbf{8} & \mathbf{6} & 2 \end{array}$$

   $186624 = [\mathbf{4.10^2}]^2 + [2 \times (4.10^2)(3.10^1)] + [\mathbf{3.10^1}]^2 + 2 \times (2.10^0)(4.10^2 + 3.10^1) + 4.10^0 = (\mathbf{4.10^2} + \mathbf{3.10^1} + \mathbf{2.10^0})^2$

9. The square root is 432. To end the procedure, moving one step to the right, one can "subtract the square of the quotient" (2^2):

   Bhāskara style:

$$\mathbf{4} \quad \mathbf{3} \quad \mathbf{2}$$

   APG style:

$$\mathbf{8} \quad \mathbf{6} \quad \mathbf{4}$$

   $186624 = (432)^2$

C SYAB.2.4.

[58] He states (*āha*) a square root computation (*vargamūlānayana*) with an *ārya*[59]:

One should divide, repeatedly, the non-square [place] by twice the square root|
When the square has been subtracted from the square [place], the result is a root in a different place||

In places where numbers are set-down (*vinyāsa*), the odd places have the technical name (*saṃjñā*) “square. Even places have the technical name “non-square. In this verse, when a square quantity is chosen (*uddiṣṭa*), having initially started by marking (*cihnayitvā*) the square and non-square places, when one is able to subtract (*śodhayituṃ śakyate*) the square of a special number- among those [squares of the digits] beginning with one and ending with nine- from the last square place, having subtracted (*apāsya*) that square ; this special number is referred to [in the rule] as a root (*mūlatvena gṛhṇīyāt*). Consequently, here, the result which is a square root (*vargamūlaphala*) has been mentionned (*ucyate*). One should divide (*bhāgaṃ haret*) the next adjacent non-square place by twice that [root]. In this verse, when the square of this quotient has been subtracted (*śuddhe*) from the next adjacent square place, that quotient from the non-square place, in a different place, in the next square place, that [quotient] becomes (*bhavati*) the root[60]. Also, when one has multiplied it (the quotient) by two (*dviguṇīkṛtya*), dividing (*bhāgahāraṇa*) in due order both [digits] from its adjacent non-square, as before, the computation of the third root [is accomplished]. Once again with three [digit numbers, the process is carried out][61]. In this way, one should perform (*kuryād*) [the process] until no square and non-square [place] remain (*bhavanti*). When the root quantity has been obtained (*labdhe*), having multiplied it by two (*dviguṇī kṛtaṃ*), it should be halved (*dalayet*). Concerning fractions also, having divided (*vibhajya*) the square root of the numerator by the square root of the denominator the quotient[62] becomes (*bhavanti*) the root. One states (*āha*) in this way:

[58]For a translation into English of BAB.2.4, see (Keller, 2006).

[59](Sarma, 1976, 36-37).

[60]This long sentence has an equivocal expression: is *sthānātare* (in a different place) glossed into *pūrve vargasthāne* (in the next square place), or should one understand that two actions are prescribed, first setting aside the quotient as a digit of the root on the one hand, and then that its square enters an operation in the next square place ?

[61]This is a mysterious cryptic expression, it is thus my interpretation that the three here, as the “both (*ubhaya*) used in the sentence before, refer to the number of digits of the square root being extracted.

[62]Reading *labdham* instead of the misprinted *ladhdham*.

> **When the square root of the numerator has been extracted, and the root born from the denominator [and also] the root [is obtained]**(PG 34)|

In order to obtain the roots of the square which were previously explained (in the commentary of verse 3 which is on squares), setting down: 15 625. The result is the square root 125. Setting down the second: $\frac{4}{9}$. The root of the numerator 2, the root of the denominator 3, having divided (*vibhajya*) the numerator by that, the result is the square root of the fraction: $\frac{2}{3}$. Thus the fourth rule [has been explained].

D APG

An algorithmic rule (*karaṇasūtra*) of two *āryas* for square roots[63]:

> **PG.25. Having removed the square from the odd term, one should divide the remainder by twice the root that has dropped down to a place [and] insert the quotient on a line**‖
> **PG.26. Having subtracted the square of that, having moved the previous result that has been doubled, then, one should divide the remainder. [Finally] one should halve what has been doubled.**‖

What is the root of a given quantity whose nature is a square? This is the aim of that procedure. One should subtract (*tyajet*) a possible/special square, from the *viṣama* ⟨place⟩ of the square quantity, ⟨in other words⟩ from what is called odd (*oja*), that is from the first, third, fifth, or seventh etc., ⟨place⟩; the places for one, one hundred, ten thousand, or one million, etc.; the *pada*, that is from the last among other places. This should provide (*syāt*) the root of that square which one should place (*sthāpayed*) beneath the place of decrease, ⟨under⟩ the place ⟨where⟩ the possible square is subtracted (*śodhita*) from that, ⟨the place for⟩ one, a hundred, ten or thousands, etc., the last among the other places. And one should divide (*bhāgam apaharet*) from above (*upariṣṭāt*) by twice that, just there. The result should be inserted (*viniveśayet*) on a line, one should subtract (*śodhayet*) the square of that from above that, and this should be doubled (*dviguṇīkūryāt*). If when this is doubled (*dviguṇe kṛte*) an additional place is created (*jāyet*), then it should be used (*yojayet*) as before when it is a result. Both have the quality of being a unique quantity

[63](K. S. Shukla, 1959, 18-19).

(*rāśitā*). This quantity has the name "result". One should repeat (*utsārayet*) this, thus one should divide (*vibhajet*), one should insert (*viniveśayet*) the result on a line, etc. as before in as much as the serpentine ⟨progression⟩ is possible (*sarpaṇasaṃbhava*), when finished (*samāpta*) one should halve (*dalayet*) all the result, thus obtaining the square root.

Thus for 186624, for which quantity is this a square?

In due order starting from the first place which consists of four, making (*karaṇa*) the names: "odd (*viṣama*), even (*sama*), odd (*viṣama*), even (*sama*)".

Setting down:

sa	vi	sa	vi	sa	vi
1	8	6	6	2	4

In this case, the odd terms which are the places for the ones, hundreds, and ten thousands, consist of four, six and eight. Therefore the last odd term is the ten thousand place which consists of eight. Then, the first quantity is eighteen. Having subtracted (*apāsya*) sixteen since it is a possible square for these quantities, the result is two. That last quantity is placed (*vyavatiṣṭhate*) separately above. Thus, where it is placed (*sthite sati*) the root of sixteen, four, with two for multiplier, eight, is to be led (*neyaḥ*) below the place where the square was subtracted (*vargaśuddhi*), which consists of six for the place of decrease. And then division (*bhāgāpahāraḥ*) of twenty six led above (*uparitanyā*). Setting down:

2	6	6	2	4
	8			

.

When twenty-four is subtracted by three from below, above two remains. Below, the quotient which is three should be inserted (*niveśya*) on a line, they (eg these three units) should be placed (*sthāpya*) under ⟨the place⟩ consisting of six. Its square is nine. Having subtracted (*śodhayitvā*) this from above, these (1724) ⟨are placed above⟩, three is multiplied by two, six is to be made (*kartavya*). Below, eighty six is produced. This quantity slithers (*sarpati*) on a line. Below two, there is six, below seven, eight. Setting down:

1	7	2	4
	8	6	

Division above of a hundred increased by seventy two by that eighty six. Decreasing from above the dividend without remainder by two, the result is two, having inserted (*niveśya*) that on a line, having placed (*sthāpyau*) the two ⟨units⟩ below four, its square is four; having subtracted (*śodhayitvā*) from above, those two multiplied by two should be made (*kartavyā*) four, therefore eight hundred increased by sixty four is produced (*jayite*).

Since above the quantity subtracted has no remainder, there is no sliding like a snake etc. method, remains just to halve the quantity obtained. Thus, when that is done (*kṛta*), the result is 432. Its square is 186624.