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Brahmagupta's derivation of the area of a cyclic quadrilateral

Satyanad Kichenassamy[1]

Abstract. This paper shows that Propositions XII.21–27 of Brahmagupta's *Brāhmasphuṭasiddhānta* (628 A.D.) constitute a coherent mathematical discourse leading to the expression of the area of a cyclic quadrilateral in terms of its sides. The radius of the circumcircle is determined by considering two auxiliary quadrilaterals. Observing that a cyclic quadrilateral is split by a diagonal into two triangles with the same circumcenter and the same circumradius, the result follows, using the tools available to Brahmagupta. The expression for the diagonals (XII.28) is a consequence. The shortcomings of earlier attempts at reconstructing Brahmagupta's method are overcome by restoring the mathematical consistency of the text. This leads to a new interpretation of Brahmagupta's terminology for quadrilaterals of different types.

Résumé. On montre que les propositions XII.21–27 du *Brāhmasphuṭasiddhānta* (628 ap. J.-C.) forment un discours cohérent conduisant à l'expression de l'aire d'un quadrilatère cyclique en termes de ses côtés. Le rayon du cercle circonscrit est déterminé en considérant deux quadrilatères auxiliaires. Exprimant que le quadrilatère cyclique est partagé par une diagonale en deux triangles ayant en commun le centre et le rayon de leur cercle circonscrit, on obtient l'aire du quadrilatère, à l'aide des outils connus de Brahmagupta. L'expression des diagonales (XII.28) en découle. Les difficultés des tentatives antérieures en vue de retrouver la démarche de Brahmagupta sont résolues en restituant la cohérence mathématique du texte. On est ainsi conduit à une nouvelle interprétation des termes qu'utilise Brahmagupta pour désigner des quadrilatères de différentes classes.

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1. Introduction

Brahmagupta, an Indian mathematician of the seventh century A.D., authored two treatises of astronomy in Sanskrit, *Brāhmasphuṭasiddhānta* (*BSS*, 628 A.D. [Dvivedin, 1902]) and *Khaṇḍakhādyaka* (665 A.D. [Sengupta, 1934; Chatterjee, 1970]). The mathematical sections of the former have attracted the attention of historians of mathematics and mathematicians alike since Colebrooke's [1817] translation of Chapters XII and XVIII of *BSS*. Recall that Brahmagupta gave—for the first time, as far as we know—rules for handling negative numbers and zero, described the solution of linear equations of the form $ax-by=c$ in integers, and initiated the study of the equation $Nx^2+k=y^2$, also in integers.[2] Furthermore, he introduced a second-order interpolation method for the computation of sines. His expression for the area of a quadrilateral bounded by four chords of a circle (a cyclic, or chord quadrilateral), in Proposition XII.21 of *BSS*, is the focus of this paper.

The purpose of *BSS* was to establish a corrected form of an older astronomical system, and to offer a rebuttal of other systems current in India at the time, including that of *Āryabhaṭa I* (499 A.D.) [Clark, 1930; Shukla and Sarma, 1976].[3] Chapter XII presents mathematical results under nine sections and a supplement. Proposition XII.21 begins the fourth section, on closed figures (*kṣetra*[4]). XII.21–38[5] deal with quadrilaterals and trilaterals. Proposition XII.21 gives the following result: if a "triquadrilateral" (*tricaturbhuja*) has sides of lengths a, b, c and d, and $s=(a+b+c+d)/2$ is the half-perimeter, then its area is given by

$$(*) \qquad \text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

The significance of the word *tricaturbhuja*, apparently introduced by Brahmagupta and used only once more, in XII.27, will be discussed in due course. The purpose of this paper is to investigate whether Propositions XII.21–27 form a connected mathematical discourse suggesting the steps whereby this formula was derived.

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[2]The unknowns are, in both cases, x and y, the other numbers being given integers.

[3]This work is often referred to by the names of the Sanskrit metres it uses: "the ten *gīti*" and "the hundred and eight *ārya*". *BSS* was also composed in the *ārya* metre.

[4]A *kṣetra* is commonly an enclosed plot of ground and, by extension, a figure enclosed by lines. See e.g. Monier-Williams's Sanskrit-English Dictionary.

[5]Throughout the paper, a reference such as XII.21 stands for *BSS* XII.21, and the translations are from [Colebrooke, 1817] unless indicated otherwise; modifications have been put in italics. Colebrooke's translation formed the basis of most of the earlier contributions to this problem; it is generally adequate, and carefully worded. It would be misleading to indicate the copulae or other words he added for clarity since this would not reflect other untranslated minor features of the original, such as the use of different words for the basic operations, or the metre. A more recent translation of the verses we study, and of surrounding ones, may be found in [Plofker, 2007]. For the issue considered in this paper, there is no advantage in starting from any of the translations later than Colebrooke's, since later translations also fail to provide a rendering that is consistent from a mathematical point of view. The abbreviation *Ābh* refers to the Āryabhaṭīya. Thus, *Ābh* II.17 is verse 17 of section II of the Āryabhaṭīya. Translations from the French or German are mine.

1.1. **Basic issues and difficulties.** The formula (*) was first brought into the mainstream of mathematics by Colebrooke [1817], whose translation of chapters XII and XVIII of *BSS* was widely read by historians and mathematicians alike. As a result, Brahmagupta's formula is to be found in most textbooks on the history of mathematics.[6] The contemporary reaction to Colebrooke's translation is summarized by Chasles [1837, 429]: "This formula for the area of the triangle as a function of the sides, has been noticed in Brahmagupta's work, by the geometers who reported on it, and has been regarded as the most considerable proposition in it; and one has never quoted, I think, the formula for the area of the quadrilateral. The latter however deserved in all respects the preference; for, not only is it more general, more difficult to prove, presupposes a more advanced Geometry, and, in a nutshell, is of higher scientific value, it appears, so far to belong specifically to the Hindu author; for it is not found in any work of the Greeks, and such is not the case for the formula for the triangle..."[7]

Colebrooke does not discuss the derivation: the commentary he quotes[8] provides clarification and examples, but does not address the issue. Several attempts at finding a derivation of (*), summarized in Section 2.3, were made. In a nutshell, earlier derivations differ in the mathematical tools they take for granted, in the weight they give to Indian sources later than Brahmagupta, and in the interpretation of Brahmagupta's terminology.

Regarding Brahmagupta's mathematical tools, his derivation cannot be based on angles, parallels, or similar triangles, because Indian geometry does not use these tools: rather, it works with perpendiculars, arcs of a circle, and a restricted form of similarity for right triangles, or rather, half-oblongs.

It is unfortunately difficult to rely on commentaries or later sources to understand (*), because Brahmagupta's results were misunderstood by his immediate successors.[9] His result

[6]Surveys analyzing Brahmagupta's results in context include [Sarasvati Amma, 1999; Chasles, 1837; Hankel, 1874; Srinivasaiengar, 1988] and Colebrooke's work, including his "Dissertation" with its extensive "Notes" [Colebrooke, 1817, i–lxxxiv]. They all provide background information. We also mention general surveys dealing specifically with Indian mathematics [Datta and Singh, 1935, 1938, 1980; Bag, 1979; Joseph, 1991; Plofker, 2009] without aiming at completeness. Most general histories of mathematics mention Brahmagupta's results on the cyclic quadrilateral, see e.g. [Cajori, 1991]. The formula was extended to non-cyclic quadrilaterals, and to polygons with five sides or more, but these developments belong to modern mathematics. Their methods do not shed light on the problem at hand, and are therefore not discussed here.

[7]"Cette formule de l'aire du triangle en fonction des côtés, a été remarquée dans l'ouvrage de Brahmegupta, par les géomètres qui en ont rendu compte, et a été regardée comme en étant la proposition la plus considérable; et l'on n'a jamais cité, je crois, la formule de l'aire du quadrilatère. Celle-ci cependant méritait à tous égards la préférence; car, outre qu'elle est plus générale, plus difficile à démontrer, qu'elle suppose une Géométrie plus avancée, et, en un mot, qu'elle est d'une plus grande valeur scientifique, elle paraît, jusqu'ici appartenir en propre à l'auteur hindou; car on ne la trouve dans aucun ouvrage des Grecs, et il n'en est pas de même de la formule de l'aire du triangle..."

[8]It is called *Vāsanābhāṣya*, and is due to Caturveda Pṛthūdakasvāmi (ca. 860 A.D. [Colebrooke, 1817, v; Srinivasiengar, 1967, 58]) who, according to Pingree [1983], relied extensively on Balabhadra's eighth-century commentary, now lost.

[9]"More or less complete lack of understanding of Brahmagupta's theorems about the cyclic quadrilateral, more and more correct approximate formulae for the volume of a sphere, and various approximation

(*) was considered to be inexact in the twelfth century by the influential mathematician Bhāskara II.[10] Brahmagupta's results on the radius of the circumcircle, or circumradius (*hṛdayarajju* (XII.26–27)) were even omitted;[11] as a result, the nature of his "triquadrilaterals" was ignored.[12] Therefore, later sources do not necessarily represent Brahmagupta's terminology or methods.

This state of affairs made Brahmagupta's terminology difficult to understand. He deals in XII.21–28 with three types of quadrilaterals: the "triquadrilateral," the "not-equal" (*viṣama*) and the "not-not-equal" (*aviṣama*).[13] According to Colebrooke, commentaries assume that they correspond to "a trilateral and quadrilateral," "an isosceles trapezium," and a "scalene quadrilateral" respectively.[14] Some authors assume that the last must have perpendicular diagonals, because this is the case in XII.26. These assumptions lead to inconsistencies. Thus, if one accepts that Brahmagupta did not specify that his quadrilaterals must be cyclic, this would mean that Brahmagupta discovered a result without being able to state it. If his contribution had been limited to the discovery of (*) for quadrilaterals with perpendicular diagonals, then he would have realized that a much simpler formula follows from his results, as was observed by Weissenborn [1879] (see Section 2.3). It is nowadays generally accepted that Brahmagupta considered only cyclic quadrilaterals. Indeed, the radius of the circumcircle is determined in XII.27 for the same class of figures as in XII.21: the "triquadrilaterals." We suggest that "triquadrilateral" is Brahmagupta's term for quadrilaterals obtained by adding to a trilateral a point on its circumcircle, and that the two occurrences of this term indicate the beginning and conclusion of the derivation of (*).

1.2. **Notation and outline of reconstruction.** It appears that Brahmagupta's text should be viewed as a set of private notes, such as a modern lecturer might prepare for his own use, and bring to class as a reminder of the logical structure of his discourse. Such

formulae for the arc and area of a segment mark [the works of the period between Brahmagupta and Bhāskara II (b. 1114)]" [Sarasvati Amma, 1999, 10].

[10] *Līlāvatī* 167–170 [Colebrooke, 1817, 72–73].

[11] To the best of our knowledge, this term does not appear again until much later. On the circumcircle in later Indian literature, see the texts quoted in [Sarasvati Amma, 1999, 93–94, 108–109, 120–127].

[12] In fact, "[t]wo sections of Indian mathematicians have approached the study of the quadrilateral from two different angles. One section viewed it merely as a figure enclosed by four chords of a circle, whereas the other viewed it as a figure enclosed by any four lines, i. e. the general quadrilateral but, strangely enough, excluding the cyclic quadrilateral. The former includes the majority of Indian mathematicians, Brahmagupta, Śrīdhara, Mahāvīra, Śrīpati and the later Āryabhaṭa School; the latter Āryabhaṭa II and Bhāskara II. We do not know to which camp Āryabhaṭa I belonged, since his extant works accord the quadrilateral a doubtful passing notice only, but it is likely he belonged to the first camp." [ibid., 81]. Of these, all are later than Brahmagupta, except Āryabhaṭa I.

[13] Special quadrilaterals built from half-oblongs put together along their sides are described in XII.33–38.

[14] According to the Oxford English Dictionary, a trapezium is a quadrilateral in which two opposite sides are parallel, and the other two are not, and a trapezoid is one in which no two sides are parallel. In American English, it is the other way around: the trapezoid has parallel sides and the trapezium doesn't. The isosceles trapezium has two parallel, unequal sides, and two equal sides. The scalene quadrilateral has all its sides different, and is not necessarily cyclic.

notes would merely contain the key steps of the exposition, as well as hints about logical connections to point out, or results on which to elaborate. Indeed, Brahmagupta states, in XII.66, the closing verse of the chapter, "this (is) only the (general) direction."[15] This point of view has made it possible to make sense of another, earlier proposition in ancient Indian mathematics [Kichenassamy, 2006]. We therefore propose to take into account the mathematical relations between propositions together with the wording of each proposition, to reconstruct the steps of Brahmagupta's derivation of (*).

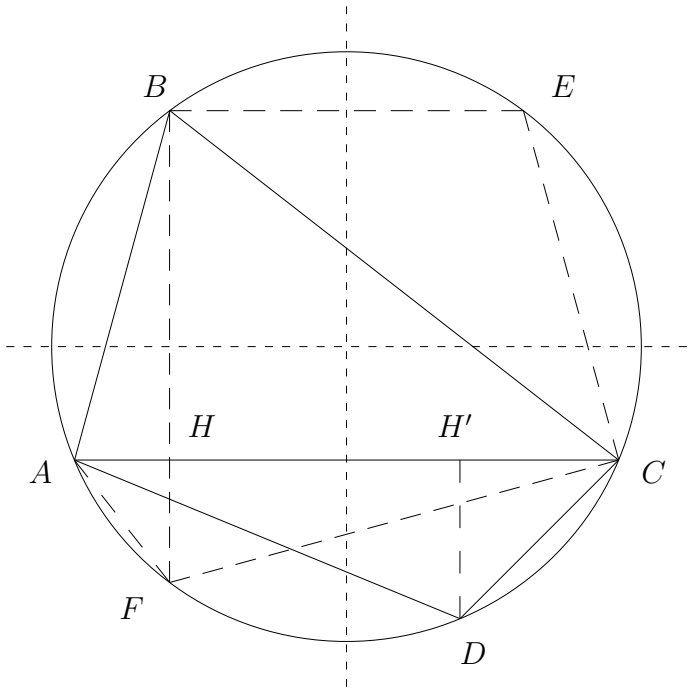


FIGURE 1. Cyclic quadrilateral $ABCD$, its segments, and associated symmetric and asymmetric quadrilaterals $ABEC$ and $ABCF$; the points E and F are obtained from B by symmetry with respect to the axes. Notation for side lengths is given in Section 1.2.

We now fix the notation that will be used in the analysis of all propositions. We consider a chord quadrilateral $ABCD$ (see Fig. 1), with sides a, b, c and d: $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Its diagonals are $AC = \gamma$ and $BD = \delta$. The perpendicular $BH = h$ dropped from B to AC determines segments $AH = \alpha$ and $HC = \beta$ on AC; see Fig. 2. We therefore have $\alpha + \beta = \gamma$. Similarly, the perpendicular $DH' = h'$ dropped from D to AC determines two segments $\alpha' = H'C$ and $\beta' = AH'$, with $\alpha' + \beta' = \gamma$. The circumcircle has radius r and diameter $2r$. All figures are inscribed in a circle having two perpendicular axes of symmetry: a vertical and a horizontal one.

A general, not necessarily cyclic, quadrilateral is not determined by its sides alone; it is determined if, in addition, a diagonal is given. The expression "triquadrilateral" suggests

[15]In Sanskrit: *diṅmātrametat*. Colebrooke translates as: "This is a portion only of the subject."

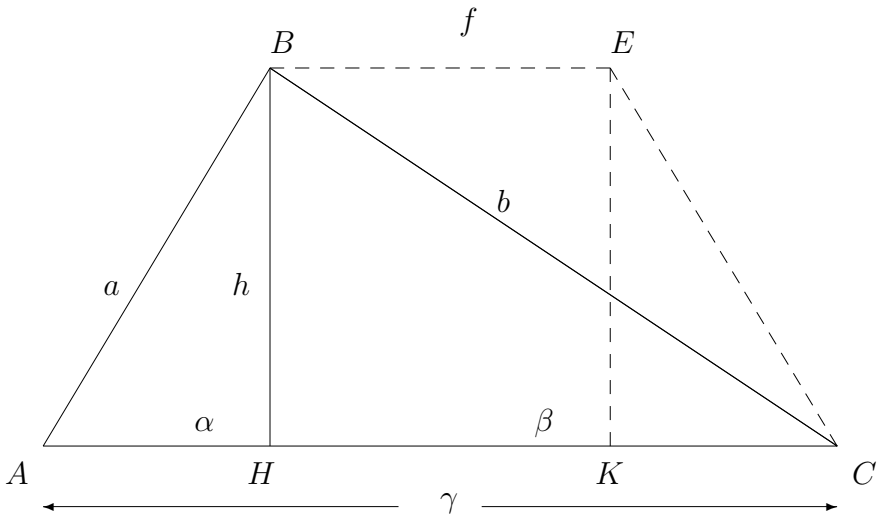


FIGURE 2. Triangle and associated symmetric quadrilateral (*aviṣama*).

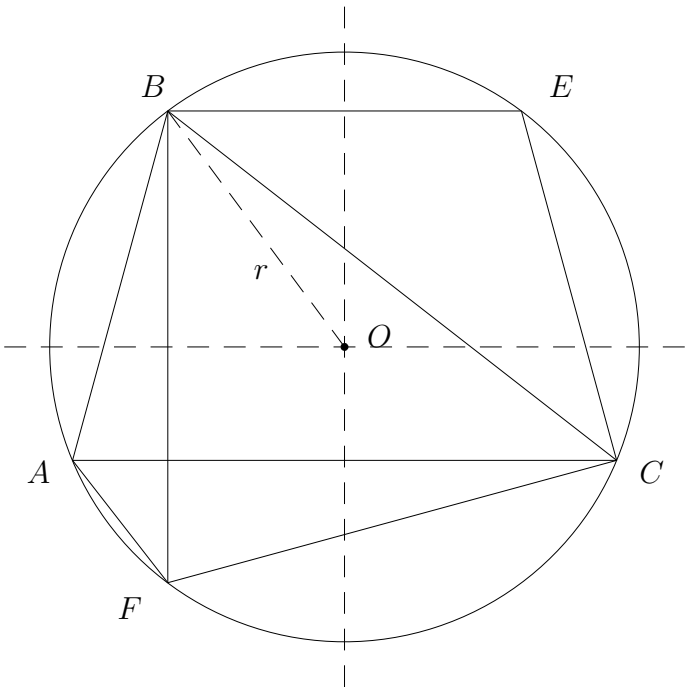


FIGURE 3. The symmetric and asymmetric quadrilaterals $ABCE$ and $ABCF$ generated by triangle ABC.

that Brahmagupta's quadrilaterals are obtained from a trilateral ABC by adding a fourth vertex D, chosen arbitrarily on the circumcircle of ABC; the basis AC of the trilateral is then a diagonal of the quadrilateral. All previous works have rendered "triquadrilateral" as "trilateral *and (unrelated)* quadrilateral" (see Section 2.3), leading to insuperable difficulties. The present paper suggests that "triquadrilateral" is a technical term meaning "trilateral *and (associated)* quadrilateral." It is not necessary to specify that one deals with cyclic figures because a trilateral is always inscribed in a circle. With this interpretation, Brahmagupta did specify the conditions of applicability of his result. The wording of XII.26–27 will confirm this interpretation. Brahmagupta gives a special place to two auxiliary quadrilaterals determined by ABC, $ABEC$ and $ABCF$, where B and E are symmetric with respect to the vertical axis, while B and F are symmetric with respect to the horizontal (see Fig. 3). The points A and C are also symmetric with respect to the vertical. In this sense, $ABEC$ is symmetric (*aviṣama*[16]). Since the pairs (A, C) and (B, F) are symmetric with respect to different axes, $ABCF$ is asymmetric (*viṣama*). The general cyclic quadrilateral $ABCD$ is also asymmetric, but is not determined by completion of a specific triangle by symmetry.

We now summarize the content of XII.22–28 and outline the strategy followed in this paper. We begin with XII.22–25, which deal with a triangle ABC and the symmetric quadrilateral $ABEC$; see Fig. 2. XII.22 shows how to compute from the sides a, b, and γ of ABC the segments α and β, and from them the length h of the perpendicular. XII.23 translates the results in terms of $ABEC$, and expresses the diagonal b of the latter in terms of its sides. XII.24 has been interpreted as giving three versions of "Pythagoras' theorem," but this is not consistent, for reasons that will be detailed in Section 4.4. We suggest that it expresses that $a^2 + b^2 = \gamma^2$, provided that AC is a diameter of the circumcircle. This seems consistent with the text. XII.25 shows how to determine, in $ABEC$, the portions of the diagonals and perpendiculars determined by their points of mutual intersection. XII.26 determines a point at equal distance to all the vertices of quadrilateral $ABEF$ or $ABCF$. In XII.27, the same is accomplished for a triangle, and twice this distance is identified as the diameter of the circumcircle of a *tricaturbhuja*. Observing that $ABCD$ is split by diagonal AC into two triangles with the same circumcenter and the same circumradius, and using the previous results, one is led to (*). The expressions for the diagonals in terms of the sides (XII.28) may then be derived as a consequence of (*).

1.3. **Organization of the paper.** Section 2.1 reviews the background of (*), and Section 2.2, its reception in India. Previous contributions on (*) are discussed in Section 2.3. Section 3 collects the mathematical tools that may be taken for granted in the reconstruction of Brahmagupta's derivation. Propositions XII.21–28 are analyzed in Section 4. Each of XII.21–27 is treated in a separate section (4.1–4.7), divided into four parts: text and translation; gloss, clarifying technical terms and drawing attention to the structure of the text; comments, explicating what comes out if one carries out the indications in the text;

[16]In this word, both prefixes *a-* and *vi-* have a negative meaning, so that *aviṣama* literally means "not-not-equal." Brahmagupta could not avoid the double negative and describe $ABEC$ as an "equal (quadrilateral)" (*sama(caturasra)*) because the latter is a classical name for the square.

and temporary conclusions drawn from the analysis of each verse. The derivation of (*) follows in Section 4.8. The formula for the diagonals (XII.28) is obtained as a consequence, in Section 4.9. The conclusion (Section 5) summarizes the results.

2. The cyclic quadrilateral before and after Brahmagupta

After reviewing briefly a few early texts, from which the independence of the area of a cyclic quadrilateral from the order of its sides may be concluded, the reception of Brahmagupta's results in India is discussed. Previous attempts at understanding Brahmagupta's text are then analyzed.

2.1. **Cyclic quadrilaterals in India before Brahmagupta.** The oblong and the isosceles trapezium occur as early as the *Śulva-sūtra*s,[17] the earliest treatises of geometry extant in India. The objective of these works is the construction of complex figures of prescribed area by tiling bricks having elementary shapes, such as squares and oblongs, and their subdivisions by diagonals.[18] Complex areas are computed by expressing them as sums or differences of other areas. General results on the transformation of figures into one another, without change of area, are given. Besides rectilinear figures, a figure in the shape of a wheel without spokes is constructed by adding to a solid square four segments of its circumcircle, called *pradhi*s. Thus, the area of the square is obtained from the area of the circle by subtracting the area of four segments of a circle (Baudhāyana, III.179–186[19] [Thibaut, 1875]). A consequence of the construction of the wheel without spokes that will be used below is the independence of the area of the cyclic quadrilateral from the order of the sides. Consider a cyclic quadrilateral $ABCD$, enclosed by chords of lengths a, b, c, and d determining a decomposition of the circle into four arcs $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$. The area of the quadrilateral is obtained from that of the circle by removing four segments of a circle, each determined by one chord. Forming a cyclic quadrilateral by subtracting segments of a circle shows that its area does not depend on the order of the chords that enclose it.

Among figures inscribable in a circle, the isosceles trapezium, which figures prominently in Brahmagupta's geometry, had been studied earlier in India (see, e.g., [Sarasvati Amma, 1999, Chap. 4; Datta and Singh, 1980]). In the *Śulvasūtra*s, it is called *ekato'ṇimat*, literally, "shorter on one side." *Ābh* II.8 expresses in particular its area as the product of the half sum of the (unequal) sides by the (perpendicular) distance separating them.

2.2. **Reception of Brahmagupta's results in India.** After Brahmagupta, one finds his formula (*), and many of his other propositions, repeated in later works without proper interpretation, to the extent that Bhāskara II (b. 1114 A.D.) considered it inexact. He gives an example of a noncyclic trapezium in which the formula does not give the correct

[17] "Aphorisms of the Cord," ca. 800-400 B.C. [Thibaut, 1875, 1882; Bürk, 1901/1902; Baudhāyana, 1979; Sen and Bag, 1983; van Gelder, 1963; Datta, 1932, Ch. 5; Kichenassamy, 2006].

[18] Triangular bricks in the *Śulva-sūtras* are generally obtained by division of oblongs: "The division [of rectangular bricks] is always to be made following the diagonals in case there is no special direction given" (in Sanskrit: *nityamakṣṇayāpacchedanamanādeśe*, *Baudhāyana Śulvasūtra* III.65 [Thibaut, 1875]).

[19] This corresponds to 16.1–16.5 in the numbering of Sen and Bag [1983].

area, observes that the sides of a quadrilateral do not determine the diagonals, and calls a fiend (*piśāca*) one who would seek to determine the area by the sides alone without specifying a diagonal or a perpendicular.[20] He does not give Brahmagupta's formula for the circumradius; without knowledge that the quadrilateral is made of chords of a circle, the misunderstanding is perhaps logical. Most of the other results in Chapter XII of *BSS* are given by Bhāskara II, but in a different order. Thus, XII.24 appears, rearranged, as the second proposition of the chapter on closed figures in *Līlāvatī*, and is interpreted as stating the theorem on the square on the diagonal of an oblong in three equivalent ways, for the benefit of beginners. Quoting Brahmagupta's formula XII.28 for the diagonals of a cyclic quadrilateral, he gives a simpler formula, which, however, applies only in special cases, and wonders why earlier authors have given a more complicated result.[21]

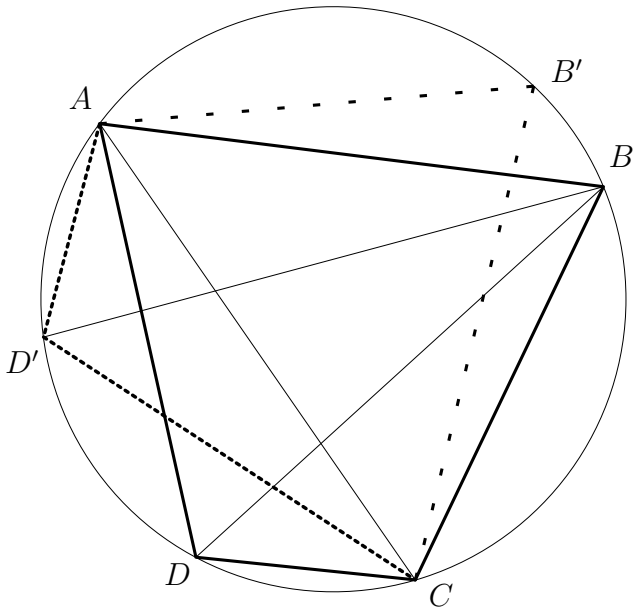


FIGURE 4. Quadrilaterals $ABCD$, $AB'CD$ and $ABCD'$ all have the same sides. Interchange of the arcs subtending AD and CD transforms $ABCD$ into $ABCD'$. Interchange of arc(AB) and arc(BC) yields $AB'CD$. The diagonals of $ABCD$ are AC and BD, and the "third diagonal" is $BD' = B'D$.

The study of the cyclic quadrilateral was taken up in the 14th century. These works, which led to the recognition that Brahmagupta's formula is correct for an arbitrary cyclic quadrilateral, make use of the "third diagonal," defined as follows (see Fig. 4). The basic operation is the interchange of two arcs of a circle[22] defined by the cyclic quadrilateral

[20] *Līlāvatī* 168–172 [Colebrooke, 1817, 72–73].

[21] *Līlāvatī* 190–192 [Colebrooke, 1817, 80–81].

[22] This operation is called *parivartana* (interchange) by Nārāyaṇa Paṇḍita. His *Gaṇita Kaumudī*, 1356 A.D. [Sarasvati Amma, 1999, 96 and 11], uses it to obtain properties of cyclic quadrilaterals. Similar

$ABCD$. Let $\tilde{a} = \text{arc}(AB)$, $\tilde{b} = \text{arc}(BC)$, $\tilde{c} = \text{arc}(CD)$ and $\tilde{d} = \text{arc}(DA)$. To interchange $\tilde{c}$ and $\tilde{d}$, define a new point D' on the circle such that $\text{arc}(CD) = \text{arc}(D'A)$ and $\text{arc}(DA) = \text{arc}(CD')$. Thus, $ABCD'$ defines a subdivision of the circle into arcs $\tilde{a}, \tilde{b}, \tilde{d}, \tilde{c}$, in this order. Quadrilaterals $ABCD$ and $ABCD'$ have the same set of sides, in a different order. They are obtained from the circle by removing four segments of the circle, determined by the same four chords. The enclosed area has therefore not been changed by the operation. The two quadrilaterals have diagonal AC in common. Similarly, to interchange $\tilde{a}$ and $\tilde{b}$, one introduces the point B' such that $\text{arc}(AB') = \text{arc}(BC)$. This generates a new quadrilateral $AB'CD$, with the same area and the same set of sides, in the order: b, a, c, d. Now, the lengths of the diagonals of all possible quadrilaterals obtained from $ABCD$ by repeated interchange of arcs may only take three values: the chords of arcs $\tilde{a}+\tilde{b}$, $\tilde{a}+\tilde{c}$, and $\tilde{a}+\tilde{d}$.[23] The chords subtending $\tilde{a}+\tilde{b} = \text{arc}(AC)$ and $\tilde{a}+\tilde{d} = \text{arc}(BD)$ are the diagonals AC and BD of quadrilateral $ABCD$. The chord subtending the third arc $\tilde{a}+\tilde{c} = \text{arc}(BD')$ is a diagonal of $ABCD'$; its length is called the "third diagonal" of the quadrilateral $ABCD$. The three diagonals of $ABCD$ are AC, BD and the "third diagonal" $BD' = B'D$.[24]

A proof of (*) is outlined in *Yuktibhāṣā*,[25] and summarized in [Sarasvati Amma, 1999, Chapter IV]. The *Yuktibhāṣā* does not refer to Brahmagupta for (*), but quotes Bhāskara II instead, including his comment that the result is unclear (*asphuṭa*) [Sarma, 2008, 122]. The *Yuktibhāṣā* also makes use of the third diagonal. Brahmagupta does not mention it, although its expression in terms of the sides is very similar to his Proposition XII.28. This suggests he did not follow the rationale described in *Yuktibhāṣā*.

Thus, (*) was finally recognized as correct, but Brahmagupta's derivation had been lost, and the result was rederived along other lines. It was left to the historians of mathematics to reconstruct Brahmagupta's derivation.

2.3. **Previous contributions to the understanding of XII.21.** Even though previous works have failed to provide a consistent explanation of how Brahmagupta could have arrived at (*) with the means at his disposal, the contributions discussed next helped contextualize Brahmagupta's propositions by bringing out which principles, procedures and results may be taken for granted, and which may not. Also, some of them have had an influence on modern mathematics. Contributions containing gross misrepresentations or factual errors are not reviewed.

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results are also given by Parameśvara [Gupta, 1977] and the mathematicians of the later Āryabhaṭa school of South India [Sarma, 2008].

[23]The chord of $\tilde{c}+\tilde{d}$ for instance, is equal to that of $\tilde{a}+\tilde{b}$.

[24]The diagonal $B'D$ of quadrilateral $AB'CD$ is equal to BD', because they subtend the same arcs; indeed, using the definition of B' and D', $\text{arc}(B'D) = \text{arc}(B'C) + \text{arc}(CD) = \text{arc}(D'A) + \text{arc}(AB) = \text{arc}(D'B)$.

[25]The *Gaṇita-yuktibhāṣā* by Jyeṣṭhadeva, ca. 1530 [Sarma, 2008], or *Yuktibhāṣā* for short, in Malayāḷam (a Dravidian language of South India), "purports to give an exposition of the techniques and theories employed in the computation of planetary motions" in the *Tantrasaṁgraha* (ca. 1500 A.D.). It often quotes the *Līlāvatī*, but, according to the index, it cites only two of Brahmagupta's propositions, on negative numbers and on the rule of three [Sarma, 2008, 74 and 596].

Chasles [1837, 420–447] sees in Brahmagupta's Propositions XII.21–38 an outline of a general theory of quadrilaterals in which all the lines considered by Brahmagupta are rational.[26] Indeed, XII.33–38 explain how to obtain, in terms of arbitrary quantities (*rāśi*), the sides of distinguished quadrilaterals.[27] Chasles considers that Brahmagupta's propositions constitute steps leading up to the construction of such rational quadrilaterals. This leads him to modify the order of the propositions.[28] He assumes that what Colebrooke translates by "trapezium" (the *viṣama*) must always have perpendicular diagonals. But since XII.21 does not use this word, he considers that Brahmagupta meant (*) to hold for all cyclic quadrilaterals. Chasles also points out that a given set of chords, a, b, c, and d in a circle in general do not determine one but three essentially different cyclic quadrilaterals, as explained in the previous section. Now, Chasles continues, if one starts with a quadrilateral with perpendicular diagonals, interchange leads in general to a quadrilateral with nonperpendicular diagonals. One should therefore include the consideration of general quadrilaterals when interpreting XII.21. He does not investigate a possible derivation of formula (*) because he considers the formula to be adequately understood in modern terms, and refers to Legendre's treatise [1817]. His detailed analysis of the history of (*) after the late sixteenth century will not be repeated here, since it is not germane to the question of Brahmagupta's derivation.[29] The modern history of (*), or its generalizations, are also not discussed.

Hankel [1874] also considers that XII.21 deals with cyclic quadrilaterals, because the radius of the circumcircle is given in XII.26, and all of XII.21–28 require it for their validity.[30] However, for him, Brahmagupta meant (*) to apply only to quadrilaterals with perpendicular diagonals, and XII.21–38 could not have constituted "a general theory of the

[26]This interpretation seems to have initiated the modern study of rational quadrilaterals, starting with Kummer's classic work [1848], in which he quotes Colebrooke's translation of XII.33–38.

[27]XII.38 uses the word *jātya*, literally, "(well) generated." Kusuba [1981] translates it by "well-born," and Datta and Singh [1980, 129] by "noble," "well-born," "genuine." Datta and Singh suggest that the term applies to right triangles only, and that this usage "seems to imply that all other triangles are derived from it."

[28]Chasles lists the propositions in the order: 24, 22, 21, 27; 35, 33, 34; 21, 28, 26, 23, 25, 29–32, 35–38. Here, XII.21 and XII.35 are repeated because the statements for triangles and quadrilaterals are separate in his argument.

[29]According to Chasles, the earliest proofs outside India appear to be by Naudé and Euler in the eighteenth century. According to Euler [1750, 57], Naudé's work was very complicated: "... non solum maxime est intricata et multitudine linearum in figura ductarum obruta, ut sine summa attentione ne capi quidem possit, sed etiam ubique nimis luculenta vestigia analytici calculi offendunt..." ("... not only exceedingly intricate with a multitude of lines covering the figure, so that indeed it cannot be understood without the greatest attention, but also the clear vestiges of analytical calculus everywhere create too much of a problem." Translation by A. Glover, under the supervision of H. White; it is part of the Euler Archive.) Euler therefore produced a different proof, and stated that a simpler one using "analysis" was available, but does not give it. He may have had in mind a proof similar to Legendre's [Legendre, 1817, Note V, Problème II, 297–298], based on the expression of the diagonals, i.e. Brahmagupta's XII.28. The desire to produce a proof along Euclidean lines, without using indeterminate analysis, seems to have been an important concern in works of this later period.

[30]See also [Sarasvati Amma, 1999, 92; Kusuba, 1981].

cyclic quadrilaterals, which would have far exceeded the geometric means at the Indians' disposal."[31] Those means, in his view, proceed from two principles: a "Principle of congruence," and a "Principle of similarity;" he takes it for granted that the concept of angle was known. The first principle expresses that "the same constructions lead to the same figure"[32] He considers that the principle of congruence implies a principle of symmetry.[33] About the second principle, he considers that it "finds frequent application in the restricted form of the proportionality of the sides of triangles that have one common angle and the sides opposite [the angle] parallel, at times however in a very general framework."[34] He does not give references for these points, apart from a quote from a commentary of XII.26, on the circumradius. However, this passage may just as well be taken to be a paraphrase in terms of proportions, rather than a hint at the reason why the proportion is true. He does not discuss the content of XII.21–38 in any further detail.

Now Hankel's insight into the underlying principles at work, if one removes from them the reference to angles and parallels, will be confirmed in this article in two respects. First, proportion[35] is applied to geometric problems; we shall see in the discussion of XII.25 that it is possible to derive proportions on right triangles without reference to the equality of angles. Second, mirror symmetry, or symmetry with respect to an axis, seems to be invoked in early geometric texts: the *Śulva-sūtra*s begin all their constructions with the definition of an "East(-West) axis" or *prācī*, and the construction of its perpendicular. For figures symmetric with respect to an axis, only half of the figure is described; it is considered as clear that the other half is then determined.[36] Problems involving reflected light occur in Chapter XIX of *BSS*, and lead to proportions between right triangles [Sarasvati Amma, 1999, 254–256].

Zeuthen [1876][37] accepts Hankel's hypothesis that Brahmagupta only considered quadrilaterals with perpendicular diagonals, and seeks to derive Brahmagupta's results without using the concept of angle. He also points out that the addition theorem for the sine follows easily from the consideration of trapezia, and suggests this was Brahmagupta's motivation. Weissenborn [1879] observes that the area of a cyclic quadrilateral with perpendicular diagonals admits a much simpler expression than (*): it is half the sum of the products of

[31] "Es handelte sich für den indischen Mathematiker hier keineswegs, wie Chasles meint, darum, eine allgemeine Theorie der Kreisvierecke zu geben, welche die geometrische Hilfsmittel der Inder bei weitem überstiegen hätte, als vielmehr Vierecke überhaupt berechnen zu können" [Hankel, 1874, 213].

[32] "... gleiche Constructionen zu derselben Figure führen" [Hankel, 1874, 206].

[33] He illustrates it on the figure obtained by completing a right triangle ABC inscribed in a circle of center O into a quadrilateral so that it is symmetric with respect to two perpendicular axes. Using this symmetry, he shows that all its angles are right: the quadrilateral is an oblong.

[34] "Dasselbe findet in der beschränkten Form der Proportionalität der Seiten von Dreiecken, welche einen Winkel gemein und die gegenüberliegenden Seiten parallel haben, häufig Anwendung; zuweilen aber in sehr allgemeiner Fassung." [Hankel, 1874, 207].

[35] Proportion is called *anupāta* in XII.32.

[36] See e.g. *Baudhāyana Śulvasūtra* 1.6, 2.8, 8.11 [Sen and Bag, 1983].

[37] The following account of [Zeuthen, 1876] and [Weissenborn, 1879] is based on the corresponding reports JFM 08.0001.02 and 11.0040.04 in the *Jahrbuch über die Fortschritte der Mathematik*.

opposite sides. Indeed, if we consider the quadrilateral $ABCF$ in Fig. 3, which has perpendicular diagonals, its area is $\frac{1}{2}AC \times BF$: half the product of the diagonals. It follows from XII.28 that the product of the diagonals is the sum of the products of opposite sides. The result follows.[38]

Pottage [1974] also assumes the validity of XII.21 to be restricted in Brahmagupta's mind, and favors an induction based on examples in which the area may be computed by other means. He shows in detail how the interchange operation leads to the refutation of any area formula that would not be completely symmetrical in the sides.[39] He notes, following Peet [1923, 94], that the gross formula in XII.21 is an approximation in excess, found in other traditions. He assumes that terminology and examples found in later texts are in the continuity of Brahmagupta's, and that (*) was derived by educated guessing. He also feels that (*) was first obtained for triangles, identified with quadrilaterals with one side of zero length, and then generalized to quadrilaterals. This generalization does not seem to be documented in any ancient or medieval tradition. On the contrary, (*) was rejected as inexact by Bhāskara II, who was aware that its analogue was correct for triangles.

Kusuba [1981] analyzes XII.33–38 and discusses the possible generation of distinguished quadrilaterals on the basis of "such figures that illustrate clearly the figures [Brahmagupta] has discussed." Regarding (*), he shows that it is a consequence of the expression of the diagonals in terms of the sides (XII.28). Kusuba does not discuss the derivation of XII.28.

Many texts on the history of mathematics summarize, or adopt one or the other of the above points of view, but do not add new elements. Proofs using modern methods, or generalizations of (*), are not reviewed here.

Thus, the earlier contributions establish that Brahmagupta meant his formula (*) to apply to a class of cyclic quadrilaterals, and that he was aware that the area depends on the lengths of the sides, but not on their order. He used two auxiliary types of quadrilaterals, called *viṣama* and *aviṣama*, the latter being an isosceles trapezium. The place of quadrilaterals with perpendicular diagonals in Brahmagupta's propositions remained problematic. The term *tricaturbhuja* was not investigated. Regarding mathematical tools at his disposal, it became gradually clear that he did not use angles, but may have used restricted symmetry and similarity arguments. We now list the tools that may be allowed in the analysis of Brahmagupta's propositions.

3. Mathematical tools known to Brahmagupta

The following tools or notions will be used freely in the reconstruction. They are found in *BSS*, or in Indian mathematical texts prior to *BSS*.

- The square of the diagonal of an oblong is the sum of the squares of the perpendicular sides (cf. *Baudhāyana Śulvasūtra* (I.48) [Thibaut, 1875] for the first statement

[38]This point was also made by R.C. Gupta in his review (MR 84d:01010) of [Kusuba, 1981] for the *Mathematical Reviews*. He quotes [Gupta, 1974], which was not available to me.

[39]A function is completely symmetrical if it remains unchanged when its arguments are permuted in any way whatsoever.

of this result; see also *Ābh* II.17). The area of an oblong is the product of its sides, and the area of a triangle is the product of the half-base and the perpendicular dropped on the base.

- Mirror symmetry with respect to the vertical or horizontal axis: its properties are used to solve the problems on mirror images and shadows in Chapters XII and XIX of *BSS* [Sarasvati Amma, 1999, Chap. X]. They involve the similarity of right triangles, see the discussion of XII.22 and XII.25 below.
- Identities: the difference of squares is the product of sum and difference ($x^2 - y^2 = (x+y)(x-y)$); the square of a sum is the sum of squares to which the double product is added; the square of a difference is computed similarly ($(x \pm y)^2 = x^2 + y^2 \pm 2xy$); two quantities x and y can be computed from their sum and difference.

The mathematical portions of *BSS* (including Chapters XII, XVIII, XIX and XXI) document the conceptual advances that Brahmagupta or his predecessors had achieved. Thus, the type of computation that could be expected of Brahmagupta may be illustrated by "Brahmagupta's identity," XVIII.65–66: if $Na^2 + k = b^2$ and $Nc^2 + l = d^2$, then

$$N(ad + bc)^2 + kl = (Nac + bd)^2.$$

This identity, and its application to the solution of quadratic equations in integers, are among his most celebrated contributions to mathematics; see, e.g., [Varadarajan, 17–31].

4. Analysis of XII.21–28

Brahmagupta's XII.21–28 is a list of propositions, without illustrations. All figures are described in words, using different names for lines in different positions. The mathematical operations on their lengths are formulated in such as way as to apply to several figures in one blow, an example being XII.23, which applies to a symmetric quadrilateral $ABEC$, whether its lower side is longer than its upper side or not. Thus, a single proposition may encompass what the modern reader might see as several closely related statements. The perpendiculars in the text are vertical.

4.1. **Proposition XII.21.**

4.1.1. *Text and translation.*

sthūlaphalaṁ tricaturbhujabāhupratibāhuyogadalaghātaḥ
bhujayogārdhacatuṣṭayabhujonaghātāt padaṁ sūkṣmam (XII.21)

The product of half the sides and countersides is the gross area of a *tri-quadrilateral*.[40] Half the sum of the sides set down four times, and severally lessened by the sides, being multiplied together, the square-root of the product is the exact area.

[40] Colebrooke has "triangle and tetragon."

4.1.2. *Gloss.* The first part of XII.21 states that the area enclosed by $ABCD$ is approximately

$$\frac{a+c}{2} \times \frac{b+d}{2}. \tag{1}$$

This gross area (1) gives the correct result for oblongs (and squares). The practice of giving results at different levels of accuracy side by side is widespread in Indian mathematics. It has two advantages. First, there is no point in giving a minutely accurate result in one part of a computation if other parts are affected by approximations. Second, the hierarchy of better and better results may reflect the gradual development of the subject. For an example that has been analyzed recently, see Kichenassamy [2006].

The commentary, equating "triquadrilateral" with "triangle and quadrilateral," considers that (*) is also meant to apply to triangles, and suggests the last term should then be s rather than $s - d$.

4.1.3. *Comment.* The gross formula refers to opposite sides. In contrast, the exact formula does not specify the order in which the four sides should be taken; this is consistent with the independence of the area on the order of the sides.

We may transform (*) using the identities $(x + y)(x - y) = x^2 - y^2$ and $(x \pm y)^2 = x^2 + y^2 \pm 2xy$. Since

$$s - d = \frac{1}{2}(a + b + c + d) - d = \frac{a+c}{2} + \frac{b-d}{2},$$

and

$$s - b = \frac{1}{2}(a + b + c + d) - b = \frac{a+c}{2} - \frac{b-d}{2},$$

we have

$$(s - b)(s - d) = \left[\frac{a+c}{2}\right]^2 - \left[\frac{b-d}{2}\right]^2. \tag{2}$$

In particular, unless $b = d$,

$$\sqrt{(s - b)(s - d)} < \frac{a+c}{2}.$$

Arguing similarly with $s - a$ and $s - c$, we obtain

$$\sqrt{(s - a)(s - c)} < \frac{b+d}{2}$$

unless $a = c$. Combining the two inequalities,

$$\sqrt{(s - a)(s - b)(s - c)(s - d)} < \frac{a+c}{2} \times \frac{b+d}{2}$$

unless $a = c$ and $b = d$. In particular, the gross area can never be less than the exact area given by (*).

4.1.4. *Conclusion.* The square of the exact area may be written

$$\left\{\left[\frac{a+c}{2}\right]^2-\left[\frac{b-d}{2}\right]^2\right\}\times\left\{\left[\frac{b+d}{2}\right]^2-\left[\frac{a-c}{2}\right]^2\right\}$$

This enables a comparison with the gross area. The expression in XII.21, however, shows that the sides play a symmetric rôle in the formula. For later reference, we note that, by exchanging b and c, and multiplying by 16, we obtain

$$16(\text{Area})^2=[(a+b)^2-(c-d)^2]\times[(c+d)^2-(a-b)^2].$$

4.2. **Proposition XII.22.**

4.2.1. *Text and translation.*

> *bhujakṛtyantarabhūhṛtahīnayutā bhūrdvibhājitā''vādhe*
> *svāvādhāvargonād bhujavargān mūlam avalambaḥ* (XII.22)
>
> The difference of the squares of the sides being divided by the base, the quotient is added to and subtracted from the base: the sum and the remainder, divided by two, are the segments. The square-root, extracted from the difference of the square of the side and the square of its corresponding segment of the base, is the perpendicular.

4.2.2. *Gloss.* We are dealing with a figure with a base γ and two other sides a and b, called arms;[41] there are only two of these arms since Brahmagupta speaks of the difference of their squares. He therefore suggests a triangle ABC with its base, perpendicular and segments. These lines are represented in Fig. 2, in the case when the perpendicular falls between A and C. Brahmagupta's formulae are also valid in the other case; see below. The sum of the segments is the base:

$$\alpha+\beta=\gamma. \tag{3}$$

The proposition states that

$$\alpha=\frac{1}{2}\left[\gamma+\frac{a^2-b^2}{\gamma}\right],\qquad \beta=\frac{1}{2}\left[\gamma-\frac{a^2-b^2}{\gamma}\right],$$

and

$$h=\sqrt{a^2-\alpha^2}=\sqrt{b^2-\beta^2}.$$

[41] Compare with the "legs" of a triangle in English.

4.2.3. *Comment.* Since BH is a perpendicular, $a^2 = \alpha^2 + h^2$ and $b^2 = \beta^2 + h^2$. Subtracting, we obtain

$$a^2 - b^2 = \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta).$$

Let us call f the difference $\beta - \alpha$; it is represented by line $BE = HK$ in Fig. 2, and will be used again in XII.23. We then have

$$f = \frac{b^2 - a^2}{\gamma}.$$

Since we know the sum and difference of α and β, we may recover their values. The expressions for α, β and h follow.

Instead of subtracting, we could have added a^2 and b^2. This yields

$$a^2 + b^2 = \alpha^2 + \beta^2 + 2h^2.$$

Since $(\alpha \pm \beta)^2 = \alpha^2 + \beta^2 \pm 2\alpha\beta$, $\alpha + \beta = \gamma$ and $\beta - \alpha = f$, we obtain relations we record for later reference:

$$a^2 + b^2 = \gamma^2 + 2(h^2 - \alpha\beta) = f^2 + 2(h^2 + \alpha\beta). \tag{4}$$

If the foot H of the perpendicular falls outside segment AC, we have $\gamma = \beta - \alpha$ and $f = \alpha + \beta$; the difference $\beta^2 - \alpha^2$ is equal to γf as before. As a consequence, Brahmagupta's statement, as he worded it, is valid in both cases.

The results take a simpler form if ABC is a half-oblong. We then have $a^2 + b^2 = \gamma^2$, and equation (4) yields

$$\alpha\beta = h^2 \tag{5}$$

This result also follows from *Ābh* II.17. As a consequence, $h/\alpha = \beta/h$: the two perpendicular sides of AHB and BHC are in the same ratio; their hypotenuses a and b are then also in the same ratio. We express this by saying that AHB and BHC are similar. On the other hand, the formulae for α and β give

$$\alpha = \frac{a^2}{\gamma}, \quad \beta = \frac{b^2}{\gamma},$$

hence the relations $\alpha/a = a/\gamma$ and $\beta/b = b/\gamma$. The relations $a/\gamma = h/b$ and $b/\gamma = h/a$ follow from the equality of the two expressions for the area of ABC: $\frac{1}{2}ab = \frac{1}{2}\gamma h$. Therefore, AHB and BHC are also similar to ABC.

4.2.4. *Conclusion.* Brahmagupta uses here the theorem on the square of the diagonal of an oblong ("Pythagoras's theorem") to determine, in terms of the sides of a triangle, the perpendicular and its segments on the base. In the special case of a half-oblong, this result implies that such a triangle is divided by a perpendicular into two triangles similar to itself.

4.3. **Proposition XII.23.**

4.3.1. *Text and translation.*

> *aviṣamacaturasrabhujapratibhujavadhayor yuteḥ padaṁ karṇaḥ*
> *karṇakṛtir bhūmukhayutidalavargonā padaṁ lambaḥ* (XII.23)
>
> *In a symmetric quadrilateral*,[42] the square-root of the sum of the products of the sides and countersides, is the diagonal. Subtracting from the square of the diagonal the square of half the sum of the base and *face*,[43] the square-root of the remainder is the perpendicular.

4.3.2. *Gloss.* This proposition refers to quadrilateral $ABEC$. Only one diagonal and one perpendicular are computed: $ABEC$ has equal diagonals and equal perpendiculars. The "face" $f = BE$ is the side opposite the base AC. The side BC of ABC is now the diagonal of $ABEC$; see Fig. 2. The proposition expresses that

$$b = \sqrt{a^2 + \gamma f} \tag{6}$$

and

$$h = \sqrt{b^2 - \left[\frac{1}{2}(\gamma + f)\right]^2}. \tag{7}$$

4.3.3. *Comment.* That it is preferable not to translate *aviṣama* by "isosceles trapezium" is confirmed by Propositions XII.36 and XII.37. The former gives the sides of a special isosceles trapezium, called "two-equal-quadrilateral" (*dvisamacaturasra*). This term would be the analogue of "isosceles quadrilateral." Similarly, XII.37 describes an isosceles trapezium with three equal sides (*trisama*, "three-equal"). XII.38 calls *caturviṣama* ("four-un-equal") a cyclic quadrilateral made by fitting two right triangles along their common hypotenuse. The terms *aviṣama* or *tricaturbhuja* are not used in these propositions. It might be wondered whether the phrase "two-equal" could refer to the equality of two diagonals rather than two sides. But if this were the case, the expression "four-un-equal" would not make sense. We conclude that the technical term *aviṣama* here specifically refers to the quadrilateral obtained by adding to ABC the point E symmetric to B with respect to the vertical, as in Fig. 2. Brahmagupta expresses in XII.23 that it has equal diagonals and perpendiculars. The symmetric quadrilateral $ABEC$ may also be obtained from ABE by adding the symmetric of A, namely C. From this perspective, f is the base of ABE, but its perpendicular is the same. However, Brahmagupta's formulae (6) and (7) remain unchanged because γ and f play symmetric rôles in them.

We saw in XII.22 that $b^2 - \beta^2 = a^2 - \alpha^2 = h^2$. If, as in Fig. 2, γ is larger than f, $\gamma = \alpha + \beta$. Since the perpendiculars BH and EK are equal and symmetrically placed with

[42] Colebrooke has "In any tetragon but a trapezium."

[43] Colebrooke translates *mukha* (literally, "face") by "summit." Chasles observes that "summit" suggests a point rather than a line, and proposes to translate *mukha* by "coraustus." All authors agree that the *mukha* is the side facing the base.

respect to the vertical, $BEKH$ is an oblong, and $f = BE = HK = \beta - \alpha$. Therefore, the sum and difference of β and α are γ and f respectively, and

$$b^2 = a^2 + (\beta^2 - \alpha^2) = a^2 + (\beta + \alpha)(\beta - \alpha) = a^2 + \gamma f,$$

hence (6). Since $\beta = \frac{1}{2}(\gamma + f)$, and $h^2 = b^2 - \beta^2$, (7) follows. If γ is shorter than f, we have $f = \alpha + \beta$ and $\gamma = \beta - \alpha$, and (6–7) follow as before.

Note that (4) should be replaced by

$$a^2 + b^2 = \gamma^2 + 2(h^2 + \alpha\beta) = f^2 + 2(h^2 - \alpha\beta) \tag{8}$$

if $f > \gamma$. The signs of the product $\alpha\beta$ in these formulae are due to our taking α and β positive.

4.3.4. *Conclusion.* Brahmagupta associates with any triangle a symmetric trapezium called *aviṣama*. The symmetry enables him to translate XII.22 in terms of the sides of this trapezium, and to obtain its diagonal and perpendicular in terms of them.

4.4. **Proposition XII.24.**

4.4.1. *Text and translation.*

> *karṇakṛteḥ koṭikṛtiṁ viśodhya mūlam bhujaḥ bhujasya kṛtim*
> *prohya padaṁ koṭiḥ koṭibāhukṛtiyutipadaṁ karṇaḥ* (XII.24)
>
> Subtracting the square of the upright from the square of the diagonal, the square-root of the remainder is the side; subtracting[44] the square of the side, the root of the remainder is the upright: the root of the sum of the squares of the upright and side is the diagonal.

4.4.2. *Gloss.* Colebrooke does not comment on the proposition, but his translation has a colon before the last sentence. He may therefore have felt that XII.24 was an argument in three parts, the third part being the conclusion. The terms *bhuja*, *koṭi* refer to two perpendicular lines; the *karṇa* is the diagonal of the oblong they define. The three sentences in XII.24 therefore refer to right triangles—but not necessarily to the same one. For it is difficult to interpret them as giving three obviously equivalent forms of the theorem on the square of the diagonal of an oblong. Given what has been omitted in the rest of the text, such redundant expressions would be odd on Brahmagupta's part. If his intention had been to present three different forms of the same result for teaching purposes, he would have put the last part of XII.24, on the sum of squares, in the first position, since this is the classical form of the result in earlier literature. In fact, this is what Bhāskara II did.[45] Also, this result has already been used several times in the preceding propositions.

[44] Colebrooke has "or subtracting."

[45] *Līlāvatī* 133–134 [Colebrooke, 1817, 58–59].

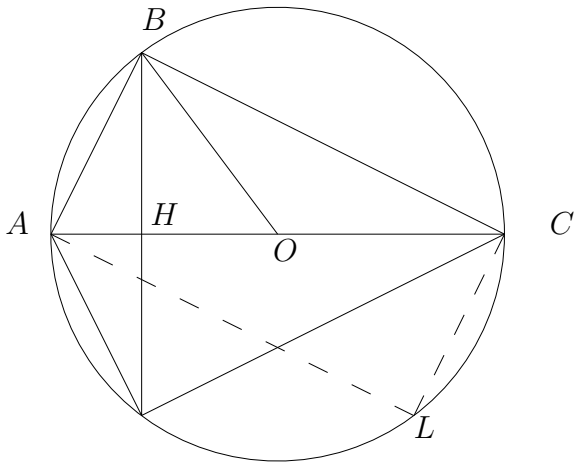


FIGURE 5. XII.24: Oblong $ABCL$ inscribed in a circle. Triangle ABC and its perpendicular BH are as in XII.22–23.

4.4.3. *Comment.* XII.22 led to the equation $h^2 = \alpha\beta$ (Eq. (5)) if ABC is half of oblong $ABCL$; see Fig. 5. This suggests the following question: What if we do not know that ABC is a half-oblong (i.e., a right triangle), but merely that A, B and C lie on the circle of radius r, centered at the midpoint O of AC? The first sentence of XII.24 suggests writing, using XII.22,

$$\alpha^2 = a^2 - h^2 \text{ and } \beta^2 = b^2 - h^2.$$

Now, apply the second sentence of XII.24 to triangle BHO: subtract in it the square of the arm from the square of the diagonal r^2. This yields a new expression for the perpendicular squared, namely

$$h^2 = r^2 - OH^2 = (r - OH)(r + OH) = \alpha\beta.$$

Combining with the previous equalities, we obtain

$$\begin{aligned} \gamma^2 &= (\alpha + \beta)^2 \\ &= \alpha^2 + \beta^2 + 2\alpha\beta \\ &= (a^2 - h^2) + (b^2 - h^2) + 2h^2 \\ &= a^2 + b^2. \end{aligned}$$

Therefore, the sum of the two arms squared equals the base squared. Thus, the last part of XII.24 is obtained. A confirmation of this interpretation is provided by *Ābh* II.17 [Clark, 1930]:

> *Ābh* II.17: The square of the *bhujā* plus the square of the *koṭī* is the square of the *karṇa*. In a circle the product of the two *śara*s [arrows] is the square of the half-chord of the two arcs.

With reference to Fig. 5, the arrows are AH and HC, and AC is the diameter through H. The second sentence is therefore the equation $h^2 = \alpha\beta$. This proposition suggests that the statement of the sum of squares was accompanied by the consideration of a half-oblong inscribed in a semicircle.

4.4.4. *Conclusion.* It is not cogent to assume that XII.24 states three times the theorem on the square of the diagonal. It is possible to interpret it as expressing that a triangle of which one side is a diameter of its circumcircle is a half-oblong.

4.5. **Proposition XII.25.**

4.5.1. *Text and translation.*

> *karṇayutāvūrdhvādharakhaṇḍe karṇāvalambayoge vā*
> *svāvādhe svayutihṛte dvidhā pṛthak karṇalambaguṇe* (XII.25)
>
> At the intersection of the diagonals, or the junction of diagonal and a perpendicular, the upper and lower portions of diagonal, or of the perpendicular and diagonal, are the quotients of these lines taken into the corresponding segment of the base and divided *by the corresponding connection.*[46]

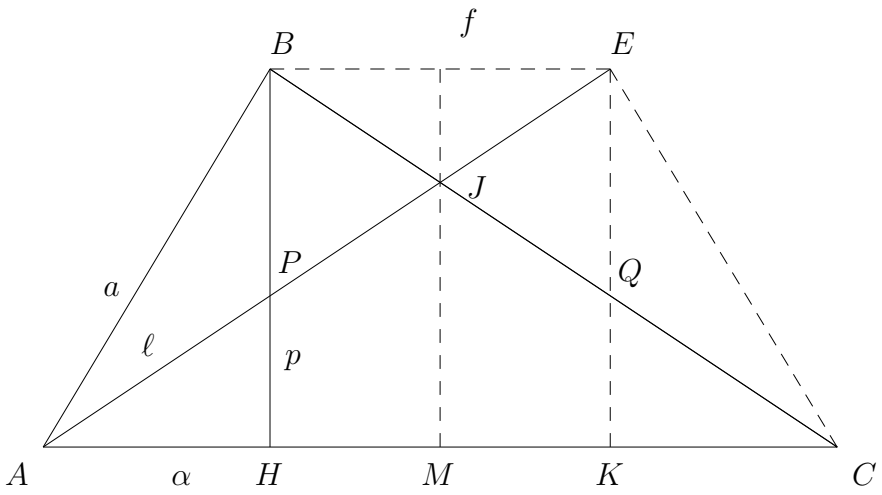


FIGURE 6. Symmetric quadrilateral with the intersections of diagonals and perpendiculars. Notation of Figure 2 applies.

[46]Colebrooke has "by the complement of the segments."

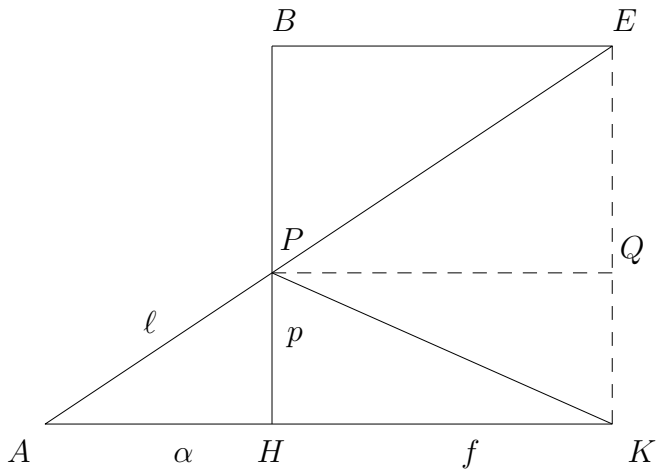


FIGURE 7. The proportions $\ell = b\alpha/\beta$ and $p = h\alpha/\beta$. Here, $b = AE$, $h = KE$ and $\beta = \alpha + f$.

4.5.2. *Gloss.* Consider again the symmetric quadrilateral $ABEC$ and its diagonals and perpendiculars; see Fig. 6. The diagonals meet at J; the diagonal AE and perpendicular BH meet at P; similarly, BC and EK meet at Q. The problem is to determine the segments of diagonals and perpendiculars thus defined, such as AP, PE or BP, PH. Brahmagupta gives a common method for all these problems: First, the segments of a diagonal are in the same proportion as the corresponding segments of the base, called here "own segment" and "own connection" (*svāvādhā* and *svayuti*). Second, the same proportion gives the segments of the perpendicular. For instance, regarding the lower segment $AP = \ell$ of the diagonal, and the lower segment $PH = p$ of the perpendicular, the proposition states that

$$\ell = b \times \frac{\alpha}{\beta} \text{ and } p = h \times \frac{\alpha}{\beta}.$$

The "own segment" is here the segment AH of AC determined by the foot H of the perpendicular dropped from the point of intersection, here P. The "own connection" is the length of the line AK connecting or joining A to the foot K of the perpendicular dropped from the end E of the diagonal. Colebrooke gives this interpretation, but translates *svayuti* by "complement of the segments," explaining that it is "the line which joins the extremities of the perpendicular and diagonal. It is the greater segment of the base or complement of the less." Since Brahmagupta introduces a new term, *svayuti*, rather than referring to the "greater segment," it may be that he had in mind a more general situation. Indeed, the formula for ℓ—unlike the result for the perpendicular—remains valid even if the quadrilateral is not symmetric, because ℓ does not depend on the position of B on the perpendicular

BH. We therefore suggested a more literal translation which would give a correct result in non-isosceles figures.[47]

4.5.3. *Comment.* The results reflect the similarity of half-oblongs AHP, AKE and EBP; see Fig. 7.[48] A possible derivation is as follows. With the notation of Fig. 7, the area $AKE = \frac{1}{2}\beta h$ is the sum of APK and KPE. The bases of the latter are respectively $\beta = AK$ and $h = KE$. Their perpendiculars are $PH = p$ and $PQ = f$. Therefore, $\frac{1}{2}\beta h = \frac{1}{2}(\beta p + fh)$. It follows that $\beta p = (\beta - f)h = \alpha h$, or

$$p = h\alpha/\beta,$$

as given in XII.25. Therefore, AHP and AKE are similar, and their hypotenuses ℓ and b are in the same relation as the other sides:

$$\ell = b\alpha/\beta.$$

By a similar argument, PQE and AKE are also similar. Since the oblong $BEQP$ is divided into equal parts PQE and EBP by its diagonal PE, EBP is also similar to AKE. Therefore, $BP/KE = BE/AK$, or $BP = h \times f/\beta$. This recovers the statement in XII.25 about the upper segment of the perpendicular. The upper segment of the diagonal may be computed similarly.

4.5.4. *Conclusion.* We saw in XII.22 that half-oblongs were divided by a perpendicular into similar triangles. Proposition XII.25 shows that a second form of similarity, the similarity of right triangles in the configuration of Fig. 7, may also be derived with the tools at hand. Part of the proposition also applies without assuming that the quadrilateral is symmetric, as suggested by the use of the word *svayuti*.

4.6. **Proposition XII.26.**

4.6.1. *Text and translation.*

> *aviṣamapārśvabhujaguṇaḥ karṇo dviguṇāvalambakavibhaktaḥ*
> *hṛdayaṁ viṣamasya bhujapratibhujakṛtiyogamūlārdham* (XII.26)
>
> The diagonal of *a symmetric quadrilateral*,[49] being multiplied by the flank, and divided by twice the perpendicular, is the central line; and so is, *in the (corresponding) asymmetric quadrilateral*,[50] half the square-root of the sum of the squares of opposite sides.

[47] Plofker [2009, 146] also observes that XII.25 partially applies to scalene quadrilaterals.

[48] Related results are given in the treatment of shadows (XII.53–54) and of problems combining shadows and mirror symmetry (XIX.17–20, analyzed in [Sarasvati Amma, 1999, 253–256]). The perpendicular JM, in the notation of Fig. 6, is also determined in *Ābh* II.8. Similarity relations involved in shadow problems are also found in *Ābh* II.15–16.

[49] Colebrooke has "a tetragon other than a trapezium."

[50] Colebrooke has "in a trapezium."

4.6.2. *Gloss.* This proposition locates the "heart" (*hṛdayaṁ*): a point O at equal distances from all vertices of a symmetric or asymmetric quadrilateral. We must determine which two quadrilaterals are meant. Propositions XII.26 contains two statements, but the word *hṛdayaṁ* is given only once, and is common to both sentences. Therefore, both quadrilaterals in XII.26 are inscribed in the same circle, the circumcircle of ABC, and O is its center. The first sentence refers to the symmetric quadrilateral $ABEC$; it states that

$$r = \frac{ab}{2h},$$

where r is the circumradius of $ABEC$, and a, b, and h have the same meaning as before, see Fig. 2. The second part of XII.26 refers to an asymmetric quadrilateral, which, like $ABEC$, should also be determined by ABC. Just as B and E were placed symmetrically with respect to the vertical, it is natural to introduce F, symmetric of B with respect to the horizontal axis of symmetry, see Fig. 3. The axes of symmetry are represented by dashed lines on the figure. Letting $c' = CF$ and $d' = FA$, the second part of XII.26 expresses that

$$r = \frac{1}{2}\sqrt{a^2 + c'^2} = \frac{1}{2}\sqrt{b^2 + d'^2}.$$

Since the text refers to opposite sides, without specifying which pair is meant, we conclude that the relation $a^2 + c'^2 = b^2 + d'^2$ is part of the proposition.

4.6.3. *Comment.* To derive the first line of XII.26, we have to locate the "heart" O of quadrilateral $ABEC$, see Figure 8. It is natural to consider the foot M of the perpendicular dropped from O to the base AC, and the foot N of the perpendicular dropped to the face BE. Let $h_1 = MO$, $h_2 = ON$, and recall the relations $\gamma = \alpha + \beta$, $f = \beta - \alpha$, $AM = \frac{1}{2}AC = \gamma/2$ and $BN = \frac{1}{2}BE = f/2$. We now express that the distances OA and OB are equal. We shall then have automatically $OA = OB = OE = OC$ by symmetry; the common value of these distances is r.

We distinguish three cases, depending on the position of O: in Case I (see Fig. 8), it lies between the base and face; in Cases II and III, it does not. In Case II, it is closer to the base than to the face; see Fig. 9. In Case III, it is closer to the face (see Fig. 10); this case may only occur if the face is longer than the base. The relations $r^2 = OA^2 = OB^2$ yield that

$$r^2 = h_1^2 + (\frac{1}{2}\gamma)^2 = (\frac{1}{2}f)^2 + h_2^2 \tag{9}$$

hold in all cases. They imply that

$$h_2^2 - h_1^2 = (\gamma/2)^2 - (f/2)^2. \tag{10}$$

One could reduce Case III to Case II by exchanging the rôles of base and face; however, the formulae in Case III will be useful in the derivation of (*).

Case I. Here, O lies between M and N (Fig. 8), and

$$h = h_1 + h_2.$$

If $\gamma > f$, $(\gamma/2)^2 - (f/2)^2 = \alpha\beta$, and (10) yields $h_2^2 - h_1^2 = \alpha\beta$. Since the sum and difference of h_1 and h_2 are known, we obtain

$$h_1 = \frac{1}{2}\left(h - \frac{\alpha\beta}{h}\right), \quad h_2 = \frac{1}{2}\left(h + \frac{\alpha\beta}{h}\right).$$

In particular,

$$2hh_1 = h^2 - \alpha\beta.$$

so that

$$(2hh_1)^2 = (h^2 - \alpha\beta)^2. \tag{11}$$

Since $r^2 = h_1^2 + (\gamma/2)^2$ and $\gamma = \alpha + \beta$,

$$\begin{aligned} 4r^2h^2 &= (2hh_1)^2 + (h\gamma)^2 \\ &= (h^2 - \alpha\beta)^2 + (h\alpha + h\beta)^2 \\ &= (h^2 + \alpha^2)(h^2 + \beta^2) = a^2b^2. \end{aligned}$$

Therefore,

$$r = \frac{ab}{2h}.$$

This is the first line of XII.26. If $\gamma < f$, the expressions for h_1 and h_2 should be exchanged,

$$2hh_1 = h^2 + \alpha\beta, \tag{12}$$

and $2hh_2 = h^2 - \alpha\beta$. Since $r^2 = h_2^2 + (f/2)^2$ by Eq. (9), and $f = \alpha + \beta$, the argument proceeds as before.

Case II. M lies between O and N (see Fig. 9), and

$$h = h_2 - h_1.$$

Since $\gamma > f$, $(\gamma/2)^2 - (f/2)^2 = \alpha\beta$, and $h_2^2 - h_1^2 = \alpha\beta$ as before. This leads to

$$h_1 = \frac{1}{2}\left(\frac{\alpha\beta}{h} - h\right);$$

the expression for h_2 is unchanged, and

$$2hh_1 = \alpha\beta - h^2.$$

Thus, Eq. (11) is still valid. The determination of r is therefore completed as in Case I.

Case III. N lies between O and M (see Fig. 10), and

$$h = h_1 - h_2.$$

Since now $\gamma < f$, $(f/2)^2 - (\gamma/2)^2 = \alpha\beta$, and $h_1^2 - h_2^2 = \alpha\beta$. This leads to

$$h_1 = \frac{1}{2}\left(\frac{\alpha\beta}{h} + h\right), \quad h_2 = \frac{1}{2}\left(\frac{\alpha\beta}{h} - h\right).$$

Thus,

$$2hh_1 = h^2 + \alpha\beta.$$

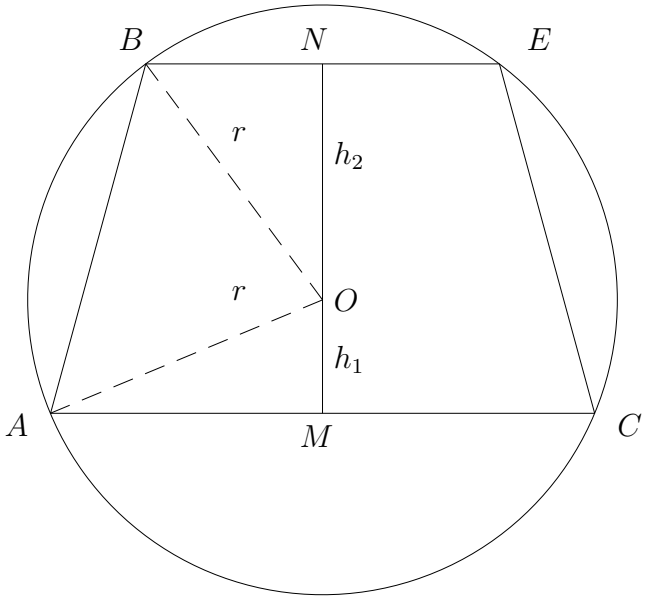


FIGURE 8. Proposition XII.26. Derivation of the circumradius of $ABCE$ using the circumcenter: Case I. The perpendicular $h = h_1 + h_2$, where $h_1 = OM$ and $h_2 = ON$.

Since γ is now $\beta - \alpha$, we obtain

$$\begin{aligned} 4r^2h^2 &= (2hh_1)^2 + (h\gamma)^2 \\ &= (h^2 + \alpha\beta)^2 + (h\beta - h\alpha)^2 \\ &= (h^2 + \alpha^2)(h^2 + \beta^2) = a^2b^2; \end{aligned}$$

hence $r = ab/(2h)$ as before. This completes the derivation of the first line of XII.26. The values of $2hh_1$ in each of the three cases are summarized in Table 1 for later reference.

(I)	$h = h_1 + h_2$,	$h_2^2 - h_1^2 = \alpha\beta$,	$2hh_1 = h^2 - \alpha\beta$;
(II)	$h = h_2 - h_1$,	$h_2^2 - h_1^2 = \alpha\beta$,	$2hh_1 = \alpha\beta - h^2$;
(III)	$h = h_1 - h_2$,	$h_1^2 - h_2^2 = \alpha\beta$,	$2hh_1 = h^2 + \alpha\beta$.

TABLE 1. XII.26: Basic results in each of the three cases: O between base and face, and the base is longer than the face (I); O below the base (II); O above the face, which is then larger than the base (III).

We now turn to the derivation of the second line of XII.26. Consider Fig. 11. Introducing G symmetric to F with respect to the vertical axis, we obtain a quadrilateral $BEGF$,

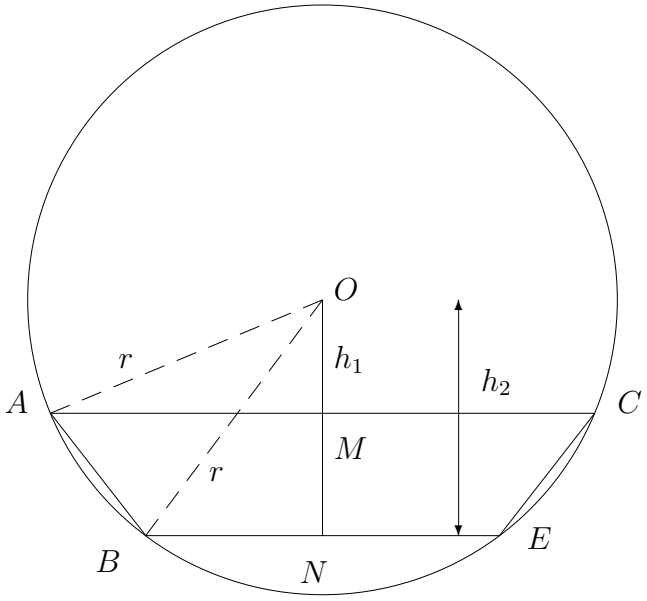


FIGURE 9. Proposition XII.26. Derivation of the circumradius of $ABCE$ using the circumcenter. Case II: $h = h_2 - h_1$. The base is larger than the face.

symmetric with respect to both axes. It is therefore an oblong, and its diagonals are diameters: EF and BG are diameters of the circumcircle. We may therefore apply XII.24 to any triangle having a diameter as a side, and any one of the other points on the circle as third vertex. Applying XII.24 to BAG, we obtain

$$(2r)^2 = AB^2 + AG^2.$$

By symmetry with respect to the vertical axis, $AG = FC = c'$. On the other hand, $AB = a$. Therefore,

$$(2r)^2 = a^2 + c'^2.$$

Similarly, applying XII.24 to EAF, we obtain

$$(2r)^2 = AE^2 + AF^2.$$

By symmetry again, $AE = BC = b$. On the other hand, $AF = d'$. Therefore,

$$(2r)^2 = b^2 + d'^2.$$

This establishes the second part of XII.26.

4.6.4. *Conclusion.* The circumradius of $ABEC$ is obtained first, by the most direct argument, using the center. Next, the circumradius of $ABCF$ is obtained using XII.24, by observing that EF—the line joining the two points that have been obtained from B by symmetry—is a diameter of the circumcircle.

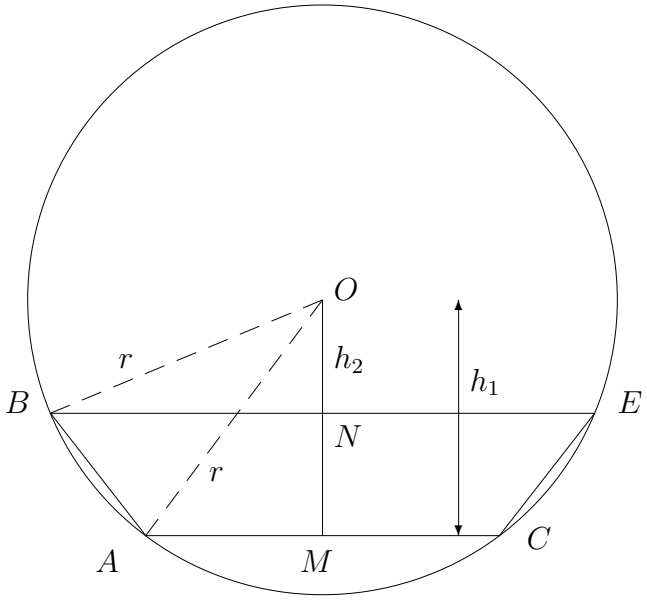


FIGURE 10. Proposition XII.26. Derivation of the circumradius of $ABCE$ using the circumcenter. Case III: the face is now longer than the base, and $h = h_1 - h_2$.

4.7. **Proposition XII.27.**

4.7.1. *Text and translation.*

tribhujasya vadho bhujayor dviguṇitalamboddhṛto hṛdayarajjuḥ
sā dviguṇā tricaturbhujakoṇaspṛgvṛttaviṣkambhaḥ (XII.27)

The product of the two sides of a triangle, divided by twice the perpendicular, is the central line: and the double of this is the diameter of the exterior circle[51] *of a triquadrilateral.*[52]

4.7.2. *Gloss.* This proposition is in two parts. The first line states that for a trilateral ABC, the central line (*hṛdayarajju*, literally, "heart-cord"), in other words, the radius r of the circumcircle is given by

$$r = \frac{ab}{2h},$$

in the notation of the derivation of XII.26. The association of a cord to a length could be a reference to the *Śulva*s, where lengths are prescribed by means of prepared cords; see Kichenassamy [2006]. The second line of XII.27 states that $2r$ is the diameter of the

[51] A literal translation would be "the corner-touching-circle."

[52] Colebrooke omits the last three words, which translate *tricaturbhuja.*

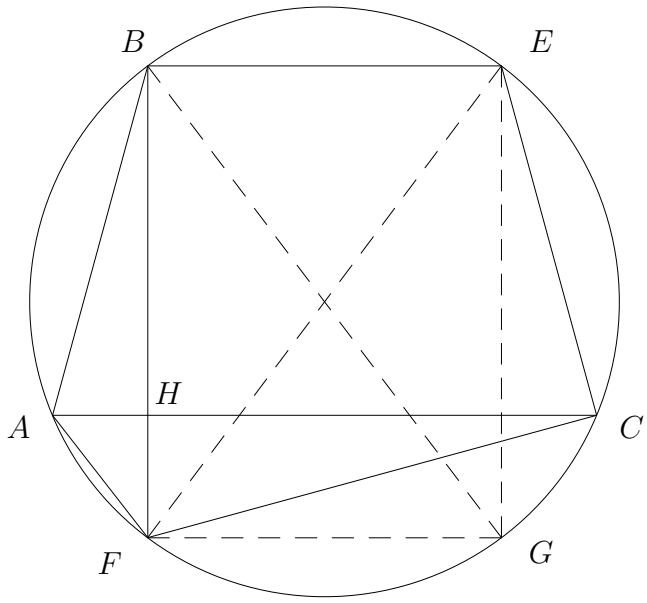


FIGURE 11. Proposition XII.26. Derivation of the formula for the circumradius of $ABCF$. Lines BG and EF are diameters of the circumcircle; $h = BH$ and $k = FH$.

circumcircle of a triquadrilateral. This is the first explicit mention of a circle in this chapter, and the only reference to the triquadrilateral since XII.21.

As in XII.24, we meet with an apparent redundancy: since the circumcircle of ABC is also the circumcircle of the symmetric quadrilateral $ABEC$, it would appear that the first line of XII.27 does not bring anything new compared to the first line of XII.26. Moreover, if "triquadrilateral" really meant "triangle and (unrelated) quadrilateral," this would mean that Brahmagupta stated the result regarding the triangle twice in the same proposition. We suggest below that the first line indicates a different way to determine r. Also, since the same length gives the circumradius of a trilateral and a triquadrilateral, it appears that the trilateral is not a special case of the triquadrilateral, and that the vertices of the triquadrilateral all lie on the circumcircle of a triangle. Thus, the triquadrilateral is determined by adding a vertex on the circumcircle of a triangle. Conversely, the circumcircle of a triquadrilateral is determined as the circumcircle of any of the associated triangles. Such a triangle is determined by the choice of two adjacent sides of the quadrilateral.

4.7.3. *Comment.* It is possible to derive the circumradius of triangle ABC from the second part of XII.26, using similarity of half-oblongs. This may be what Brahmagupta suggests in the first line of XII.27. Assume first that AC lies between BE and FG, see Fig. 11.

Applying XII.24 to EBF, we obtain

$$\begin{aligned}(2r)^2 &= EB^2 + BF^2 \\ &= f^2 + (h+k)^2 \\ &= (\beta-\alpha)^2 + (h+k)^2 \\ &= (h^2+\alpha^2) + (k^2+\beta^2) + 2(hk-\alpha\beta) \\ &= a^2 + c'^2 + 2(hk-\alpha\beta).\end{aligned}$$

Since $(2r)^2 = a^2 + c'^2$ (by XII.26), we obtain

$$hk = \alpha\beta.$$

Therefore, triangles AHB and FHC are similar: $\alpha/k = h/\beta = a/c'$, hence $c'h = a\beta$, and

$$\begin{aligned}(2rh)^2 &= (a^2+c'^2)h^2 \\ &= (ah)^2 + (c'h)^2 \\ &= (ah)^2 + (a\beta)^2 \\ &= a^2(h^2+\beta^2) = (ab)^2.\end{aligned}$$

This gives the first line of XII.27 in this case. If, on the contrary, AC does not lie between BE and FG, one has $(2r)^2 = (\alpha+\beta)^2 + (h-k)^2$, and the argument is completed along the same lines.

Up to now, no point on the circumcircle of ABC other than those that may be obtained by symmetry with respect to the axes has been used. Once the "heart" O of ABC has been determined, it is clear that any point D at distance r from O will have the property that O is the "heart" of $ABCD$. Since a circle is usually determined by its diameter, we conclude that Brahmagupta states, in the second line of XII.27, that $2r$ is the diameter of the circumcircle of any quadrilateral $ABCD$ with D on the circumcircle of ABC.

4.7.4. *Conclusion.* The circumradius of a triangle is obtained by a new application of XII.24. The second line of XII.27 applies the result to the triquadrilateral. Thus, triangles and triquadrilaterals form distinct classes of figures. This confirms that the generic term for a cyclic quadrilateral is *tricaturbhuja*, a quadrilateral defined by a trilateral and a point on its circumcircle.

4.8. **Derivation of formula (*).** XII.27 refers to the "triquadrilateral" (*tricaturbhuja*). This word was used in XII.21, and not in any of the intervening propositions. This suggests that the derivation of XII.21 may be completed on the basis of the information given so far. We therefore consider a general triquadrilateral $ABCD$ (see Fig. 12), obtained by completion of a triangle ABC. The base $AC = \gamma$ of the triangle is a diagonal of $ABCD$; it divides the quadrilateral into two triangles, with sides (a, b, γ) and (γ, c, d), and perpendiculars and segments h, α, β and h', α', β' respectively.

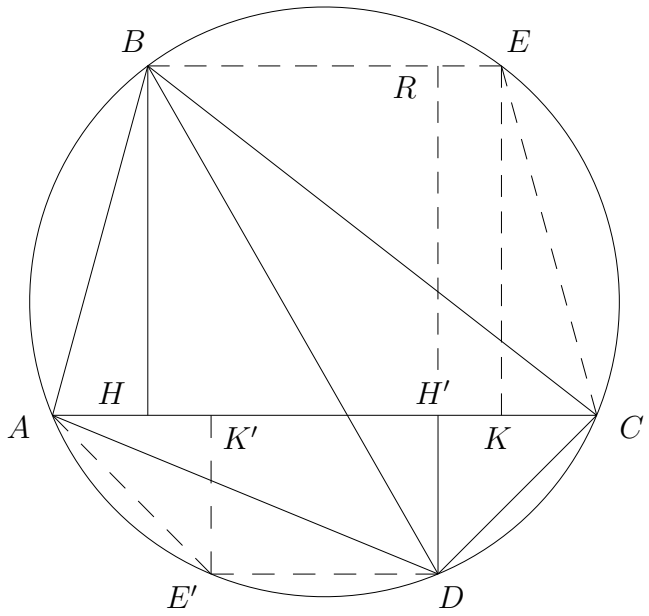


FIGURE 12. Derivation of (*) and XII.28. The perpendicular dropped from B is $BH = h$; that from D is $DH' = h'$; also, $AH = \alpha$, $H'C = \alpha'$, $HH' = \beta' - \alpha = \beta - \alpha'$. The diagonals are $AC = \gamma$ and $BD = \delta$.

Since ABC and ACD have areas $\gamma h/2$ and $\gamma h'/2$ respectively, the area of $ABCD$ is $\frac{1}{2}\gamma(h + h')$. Because triangles ABC and ACD have the same circumradius r, XII.27 yields

$$2r = \frac{ab}{h} = \frac{cd}{h'}.$$

Therefore,

$$h + h' = (ab + cd)/(2r),$$

and

$$\text{Area} = \frac{1}{2}\gamma(h + h') = \frac{\gamma}{4r}(ab + cd). \tag{13}$$

On the other hand, the diagonal AC is the common base of the two symmetric quadrilaterals $ABEC$ and $ACDE'$ associated with ABC and ACD respectively. There are two possibilities:[53] (a) AC larger than BE and $E'D$; (b) $BE > AC > E'D$. In terms of the discussion of XII.26, (a) means that one of the two quadrilaterals is in Case I, the other in Case II; (b) means that $ACDE'$ is in Case II, and $ABEC$ is in Case I or III.

[53]One may perform a symmetry with respect to the horizontal to reduce the problem to one of these cases if need be. An exceptional case such as $\gamma = BE$ may only occur if $ABEC$ is an oblong. The discussion is unchanged, with now $\alpha = 0$.

Consider the situation (a) (see Fig. 12). Both symmetric quadrilaterals have their base larger than their face: $AC > BE$ and $E'D$. $ABEC$ is in Case I, and $ACDE'$ in Case II. Since both quadrilaterals have the same circumcenter, $h_1 = OM$ is common to them. Therefore, by Table 1,

$$2hh_1 = h^2 - \alpha\beta; \quad 2h'h_1 = \alpha'^b eta a' - h'^2.$$

Using (4) and its analogue for ACD,

$$\begin{aligned} a^2 + b^2 &= \gamma^2 + 2(h^2 - \alpha\beta) = \gamma^2 + 4hh_1, \\ c^2 + d^2 &= \gamma^2 + 2(h'^2 - \alpha'\beta') = \gamma^2 - 4h'h_1. \end{aligned}$$

Subtracting, we obtain

$$4h_1(h + h') = a^2 + b^2 - c^2 - d^2. \tag{14}$$

Squaring (14), adding to it 16 times the square of (13), and using $h_1^2 + (\frac{1}{2}\gamma)^2 = r^2$ (Eq. (9)), we obtain

$$\begin{aligned} (a^2 + b^2 - c^2 - d^2)^2 + 16(\text{Area})^2 &= 16[h_1^2 + (\frac{1}{2}\gamma)^2](h + h')^2 \\ &= 16r^2(h + h')^2 \\ &= 4(ab + cd)^2. \end{aligned}$$

Thus, the square of the area is a difference of squares involving only the sides of $ABCD$:

$$\begin{aligned} 16(\text{Area})^2 &= 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 \\ &= [2ab + 2cd + a^2 + b^2 - c^2 - d^2] \times [2ab + 2cd - a^2 - b^2 + c^2 + d^2] \\ &= \left[(a + b)^2 - (c - d)^2\right] \times \left[(c + d)^2 - (a - b)^2\right]. \end{aligned} \tag{15}$$

From the conclusion of XII.21, (*) follows in this case.

The situation (b) is treated similarly. If $ABEC$ is in Case I, but, unlike the above, $BE > \gamma$, we need to use (8) for ACD rather than (4). Also, (12) gives $h^2 + \alpha\beta = 2hh_1$. The formulae for quadrilateral $ACDE'$ are unchanged. Therefore,

$$\begin{aligned} a^2 + b^2 &= \gamma^2 + 2(h^2 + \alpha\beta) = \gamma^2 + 4hh_1, \\ c^2 + d^2 &= \gamma^2 + 2(h'^2 - \alpha'\beta') = \gamma^2 - 4h'h_1. \end{aligned}$$

The rest of the derivation of (*) is as before. Finally, if quadrilateral $ABEC$ is in Case III, and $ACDE'$ in Case II, Table 1 now gives

$$2hh_1 = h^2 + \alpha\beta = \alpha'\beta' - h'^2.$$

Using (8) for ABC, and (4) for ADC, we obtain

$$\begin{aligned} a^2 + b^2 &= \gamma^2 + 2(h^2 + \alpha\beta) = \gamma^2 + 4hh_1, \\ c^2 + d^2 &= \gamma^2 + 2(h'^2 - \alpha'\beta') = \gamma^2 - 4h'h_1. \end{aligned}$$

Subtracting, we recover (14), and the derivation proceeds as before.

Since the area does not depend on the order of the sides, we have, taking them in the order a, c, b, d,

$$16(\text{Area})^2 = 4(ac+bd)^2 - [a^2+c^2-b^2-d^2]^2. \tag{16}$$

Thus, (15) and (16) give two expressions for the area: one involves the sum of products of adjacent sides $(ab+cd)$; the other the sum of products of opposite sides $(ac+bd)$.

4.9. **Proposition XII.28.**

4.9.1. *Text and translation.*

> *karṇāśritabhujaghātaikyam ubhayathā'nyonyabhājitaṁ guṇayet*
> *yogena bhujapratibhujavadhayoḥ karṇau pade viṣame* (XII.28)
>
> The sums of the products of the sides about both the diagonals being divided by each other, multiply the quotients by the sum of the products of opposite sides; the square-roots of the results are the diagonals *in an asymmetric quadrilateral*.[54]

4.9.2. *Gloss.* The equal diagonals of the symmetric quadrilateral (*aviṣama*) $ABCE$ have been determined in XII.23. Brahmagupta now determines the diagonals of the general, asymmetric (*viṣama*) quadrilateral $ABCD$. The proposition says that the diagonals are

$$\gamma = \sqrt{\frac{ad+bc}{ab+cd}(ac+bd)} \text{ and } \delta = \sqrt{\frac{ab+cd}{ad+bc}(ac+bd)}.$$

Multiplying the two equalities, it follows that $\gamma\delta = ac+bd$. This expression for the product of the diagonals is usually known as "Ptolemy's theorem."

It might be wondered whether the term *viṣama* (asymmetric) in this proposition could not refer to the special quadrilateral $ABCF$, with perpendicular diagonals, considered in XII.26 (see Fig. 11), rather than the general triquadrilateral mentioned in the immediately preceding proposition (XII.27). However, if Brahmagupta had only meant to refer to $ABCF$, he would have stated a much simpler formula. Indeed, the diagonals of $ABCF$ are $h+k$ and $\alpha+\beta$. Calling a, b, c', d' the sides of $ABCF$ as before, and applying the first line of XII.26, we obtain

$$h = \frac{ab}{2r} \text{ and } k = \frac{c'd'}{2r}.$$

By the second line of XII.26, $2r = \sqrt{a^2+c'^2} = \sqrt{b^2+d'^2}$. Therefore,

$$\delta = h+k = \frac{ab+c'd'}{\sqrt{a^2+c'^2}}.$$

A similar result holds for the other diagonal γ.

[54] Colebrooke has "in a trapezium."

4.9.3. *Comment.* From (13), we have

$$\text{Area} = \frac{1}{4r}\gamma(ab+cd).$$

Arguing similarly with the other diagonal,

$$\text{Area} = \frac{1}{4r}\delta(ad+bc).$$

Therefore,

$$\frac{\delta}{\gamma} = \frac{ab+cd}{ad+bc}. \tag{17}$$

Now, recall Eqs. (15) and (16),

$$\begin{aligned} 16(\text{Area})^2 &= 4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2 \\ &= 4(ac+bd)^2 - (a^2+c^2-b^2-d^2)^2. \end{aligned}$$

Let us compute $a^2+c^2-b^2-d^2$ in terms of the segments of the perpendiculars on AC (see Fig. 12, where $b>a$, $d>c$, and $HK>H'K'$). Now, as in the comment on XII.23, we have

$$\beta - \alpha = BE = HK.$$

Similarly, considering $ACDE'$,

$$\beta' - \alpha' = DE' = H'K'.$$

Also, $HK + H'K' = 2HH'$. Using XII.22, we obtain

$$\begin{aligned} b^2 - a^2 = \beta^2 - \alpha^2 = \gamma BE = \gamma HK, \\ d^2 - c^2 = \beta'^2 - \alpha'^2 = \gamma DE' = \gamma H'K'. \end{aligned}$$

Adding and squaring, we obtain

$$(a^2+c^2-b^2-d^2)^2 = 4\gamma^2 HH'^2.$$

Now, considering half-oblong BRD (see Fig. 12), we have

$$\delta^2 = HH'^2 + (h+h')^2.$$

Since $\frac{1}{2}\gamma(h+h') = \text{Area}$, we obtain

$$16(\text{Area})^2 + (a^2+c^2-b^2-d^2)^2 = 4\gamma^2(HH'^2 + (h+h')^2) = (2\gamma\delta)^2.$$

Comparing with (16), we obtain

$$(2\gamma\delta)^2 = 4(ac+bd)^2.$$

Taking into account (17), on the quotient of diagonals, XII.28 follows.

A similar argument applies in the other possible configurations. For instance, if $a>b$ and $d>c$, one should write $a^2-b^2 = \gamma HK$, $d^2-c^2 = \gamma H'K'$ and $4\gamma^2 HH'^2 = \gamma^2(HK-H'K')^2 = (a^2+c^2-b^2-d^2)^2$. The argument is completed as before.

4.9.4. *Conclusion.* The expression for the diagonals is a consequence of the determination of the area. Its counterpart for symmetric quadrilaterals has already been given in Proposition XII.23.

5. Conclusion

Our analysis of *BSS* XII.21–28, taking into account the structure of the propositions, their wording, and their mathematical consistency, establishes the following conclusions.

- The previous interpretations of XII.21–28, either following later Indian sources, or assuming that Brahmagupta referred only to a restricted class of quadrilaterals, lead to insuperable difficulties.
- The quadrilateral to which the area formula (*) applies is defined by a technical term, *tricaturbhuja* ("triquadrilateral"), that has been overlooked in previous works. It is formed by adding to a triangle ABC a fourth vertex D on the circumcircle of the triangle; see Fig. 1. Two special choices of the fourth vertex are determined by symmetry with respect to the vertical and horizontal axes of the circle, where the vertical is the diameter perpendicular to AC. One obtains in this way $ABEC$ and $ABCF$, where E and F are the mirror images of B with respect to the vertical and horizontal respectively. Of all these triquadrilaterals, only $ABEC$ is symmetric (*aviṣama*) with respect to the vertical; the others are asymmetric (*viṣama*).
- Proposition XII.24 need not be interpreted, as is generally done, as stating three times the same result. The analysis of XII.24, and its comparison with *Ābh* II.17 suggest that it expresses that a triangle in which one side is a diameter of its circumcircle is a half-oblong.
- A restricted form of similarity, for right triangles, underlies Brahmagupta's determination, in XII.25, of all the segments of the sides, perpendiculars and diagonals of $ABEC$, determined by their points of intersection. The analysis of the term *svayuti* shows that he was aware of more general situations than those involving a symmetric quadrilateral.
- The first two lines of XII.26 give two different methods for the determination of the circumradius of $ABEC$: one locating the center, the other involving the auxiliary quadrilateral $ABCF$.
- The first line of XII.27 derives the circumradius of a trilateral from the second line of XII.26. The second line of XII.27 applies the result to the triquadrilateral $ABCD$.
- The area formula (*) is obtained by expressing the area of $ABCD$ as the sum of the areas of ABC and ACD, and by eliminating from the result all quantities other than the sides, with the help of XII.22–27.
- The expressions for the diagonals of the triquadrilateral given in XII.28 are derived from the area formula, using the independence of the area on the order of the side-lengths.

Thus, we have established that Brahmagupta's text describes the conditions of applicability of his area formula, together with the elements needed for its derivation, using a precise and consistent terminology.

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