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Lijue XIE, Michel GRABISCH


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# The core of games on $k$-regular set systems

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## 1 Introduction

In cooperative game theory, a central problem is to define a solution for the game, that is, assuming that all players decide to collaborate and form the grand coalition $N$, how to share among players the total benefit of the cooperation $v(N)$ (or, if it is a cost game, the total cost) so that every player is satisfied in some sense? Many kinds of solution concepts have been proposed, among them the Shapley value [21], the bargaining set of Aumann and Maschler [2], the kernel of Davis and Maschler [7, 8], the nucleolus of Schmeidler [20], etc., but the most popular one is the core [13]. Roughly speaking, the core is the set of solutions so that no coalition receives less than the amount it could have obtained by itself. It has been studied by many authors and has remarkable properties. In particular, when the game is convex, its structure is completely determined.

In the classical setting of cooperative game theory, it is assumed that any coalition can form, without any restriction. This strong assumption is rarely satisfied in practice, since the cooperation of players can be constrained in many respects (communication, personal incompatibilities, hierarchy, and so on). Therefore, many authors have proposed various ways to escape this strong assumption, mainly by imposing some convenient mathematical structure on the set of coalitions which can form, called *feasible* coalitions: distributive lattices (Faigle [11]), convex geometries (Bilbao and Edelman [4]), antimatroids (Algaba et al. [1]), augmenting systems (Bilbao and Ordóñez [3]), which have their respective merits and interpretations. Recently, Honda and Grabisch [15] (see also Lange and Grabisch [16]) have proposed *regular set systems*. Roughly speaking, a regular set system is a collection of coalitions containing the grand coalition and the empty set, such that any maximal sequence ordered by inclusion (called hereafter a maximal chain) from the empty coalition to $N$ has exactly $|N|$ nonempty coalitions, like for example, with $N = \{1, 2, 3\}$, the sequence $\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}$. Clearly, these maximal sequences correspond to permutations of $N$, that is, total orders on players. In the classical case, any permutation is possible (that is, the



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players can enter the game in any order), leading to the regular set system $2^N$, the set of all possible coalitions. However, for a regular set system, some permutations may be forbidden. Taking again the case of $N = \{1, 2, 3\}$, the collection $\{\emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ is a regular set system, where among the 6 possible permutations, only $1, 3, 2$ and $2, 3, 1$ are permitted. Another interesting interpretation of regular set systems is given by Lange and Grabisch [16]: if feasible coalitions correspond to connected components in a communication graph, as proposed by Myerson [17], then the set of feasible coalitions forms a regular set system, provided $N$ itself is connnected. The converse is however not true, which means that regular set systems are more general. Also, it has been proved in [16] that distributive lattices, convex geometries and antimatroids are particular cases of regular set systems.

The problem of studying the core of games defined on a set of feasible coalitions which is not $2^N$ is often difficult. The core has been studied for most of the above cited structures (see, e.g., Faigle [11], Derks and Gilles [9] for set systems closed under union and intersection, Grabisch and Xie [14] for distributive lattices,...) and is often unbounded. The existence of vertices and rays for the core on arbitrary set systems has been studied through the concept of core of a set system by Derks and Reijnierse [10]. We mention also Pulido and Sánchez-Soriano [19], who use the notion of core of a set system, but for axiomatization purpose. Our paper aims at studying the core for games on regular set systems. For the sake of generality, and to recover the case of set systems closed under union and intersection of Derks and Gilles, we slightly weaken the definition of regular set systems, saying that any maximal sequence from the empty set to $N$ should have a fixed number $k$ of nonempty coalitions, $k$ being at most $|N|$. We call such set systems *$k$-regular set systems*, and regular set systems are $|N|$-regular set systems. This generalization allows players to enter the game by groups, instead of being obliged to enter one by one.

The paper is organized as follows. Section 2 introduces the basic material for games on $k$-regular set systems. Section 3 recalls the main results known for the core for classical cooperative games on $2^N$. The study of the core begins in Section 4 by establishing general results (nonemptiness, general form of rays and vertices), some of them of more algorithmic nature are put in the appendix. Then Section 5 is more particularly devoted to the study of vertices, while Section 6 addresses the case of convex games. A final section makes comparison with results obtained by the authors for games on distributive lattices, and concludes the paper.

# 2 Basic concepts and notations

Throughout this paper, we consider a finite set of players $N = \{1, \ldots, n\}$. Subsets of $N$ are called *coalitions*. The *grand coalition* is $N$ itself. For simplicity, we often omit braces and commas for denoting subsets, e.g., $v(12)$ instead of $v(\{1, 2\})$.

Consider a coalition $S$. The *characteristic vector* $e_S \in \{0, 1\}^n$ of $S$ is defined by $(e_S)_i = 1$ if $i \in S$, and 0 otherwise. Following our convention, we denote by $e_i$ the characteristic vector of the singleton $i$.

Let $\mathcal{R}$ be a collection of subsets of $N$, called the set of *feasible coalitions*. If $\mathcal{R}$ embodies

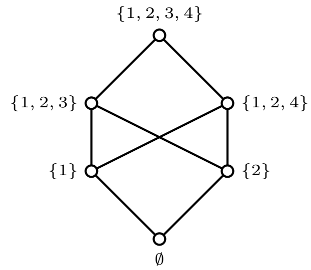


Figure 1: $\mathcal{R} = \{\emptyset, \{1\}, \{2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}$

$\emptyset$ and $N$, we call it a *set system* on $N$. A set system can be considered as a *partially ordered set (or poset)* with set-inclusion "$\subseteq$" as a partial order [6]. First, we recall some concepts about set systems.

Let $\mathcal{R}$ be a set system with a partial order $\subseteq$. Let $S, T \in \mathcal{R}$ and $S \subseteq T$. If there is no $R \in \mathcal{R}$, such that $S \subsetneq R \subsetneq T$, we say that $T$ *covers* $S$, denoted by $S \prec T$. If there exist $S_1, S_2, \ldots, S_k \in \mathcal{R}$, such that $S \prec S_1 \prec S_2 \prec \ldots \prec S_k = T$, we say that the sequence $S, S_1, S_2, \ldots, S_k$ is a *maximal chain from $S$ to $T$*, and the *length* of this maximal chain is $k$. Denote the set of all maximal chains from $S$ to $T$ of $\mathcal{R}$ by $\mathcal{C}_{S-T}(\mathcal{R})$. We define the supremum $S \vee T$ of $S, T$ as the least element in $\mathcal{R}$ which is greater than $S$ and $T$, and the infimum $S \wedge T$ of $S, T$ as the greatest element in $\mathcal{R}$ which is less than $S$ and $T$. Clearly, $\emptyset$ and $N$ are the least and greatest elements of $\mathcal{R}$. The collection of all maximal chains from $\emptyset$ to $N$ is denoted by $\mathcal{C}(\mathcal{R})$.

**Definition 1** *Let $1 \leq k \leq n$. A set system is* $k$-regular *if all maximal chains from $\emptyset$ to $N$ have the same length $k$.*

For example, $\{\emptyset, \{1\}, \{1,2\}, \{2\}, \{2,3,4\}, \{1,2,3,4\}\}$ is a 3-regular set system. In [15, 16], $n$-regular set systems are called *regular set systems*.

A consequence of the definition is that in a $k$-regular set system $\mathcal{R}^k$, for any $S, T \in \mathcal{R}^k$, all maximal chains from $S$ to $T$ have the same length. Hence, it makes sense to define the *height function* $h$: for any $S \in \mathcal{R}^k$, $h(S)$ is the length of a maximal chain from $\emptyset$ to $S$. This induces levels in the set system (subsets of same height), and we say that the set system is *ranked* (by $h$).

A *set lattice* on $N$ is a set system on $N$ in which any two subsets have a supremum and an infimum. We can call set lattices simply *lattices* without confusion. Not all set systems are lattices, e.g, $\mathcal{R} = \{\emptyset, \{1\}, \{2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}$ (see Figure 1) is a set system but not a lattice. A lattice $\mathcal{R}$ is *distributive* if the operations $\vee, \wedge$ are mutually distributive. Any distributive lattice is ranked. If a set system $\mathcal{R}$ is closed under $\cup, \cap$ as in Derks and Gilles [9], then clearly $\mathcal{R}$ is a distributive lattice with $\vee = \cup$ and $\wedge = \cap$, but the converse is not true.

Let $P$ be a partially ordered set with partial order $\leq$. A *downset* of $P$ is a subset $S$ of $P$ such that, if $x \in S$, then $y \leq x$ implies $y \in S$. We denote the collection of downsets of $P$ by

$\mathcal{O}(P)$. It is a distributive set lattice on $P$ closed under union and intersection. Conversely, for any distributive lattice $\mathcal{R}$, there exists a poset $P$ with partial order $\leq$ such that $\mathcal{O}(P)$ is isomorphic to $\mathcal{R}$ (we say that $P$ *generates* $\mathcal{R}$). When $P = N$, we recover the definition of distributive games [14] (see Definition 3 (i)), and games under precedence constraints [11, 12]. The next lemma and Figure 2 summarize the above facts and clarify the relative situation of the main families of set systems used in the paper.

**Lemma 1** *Let $\mathcal{R}$ be a distributive set lattice on $N$ of height $k$.*

*(i) $\mathcal{R}$ is a $k$-regular set system, which is generated by a poset $P$ of $k$ elements.*

*(ii) $\mathcal{R}$ is closed under union and intersection if and only if $\mathcal{R}$ is isomorphic to $\mathcal{O}(P)$, where $P$ can be chosen as a partition of $N$.*

*(iii) $k = n$ if and only if $\mathcal{R}$ is isomorphic to $\mathcal{O}(N)$.*

**Proof** We only prove the "only if" part of (ii), since the rest is clear from definitions, and (iii) from (ii). Since $\mathcal{R}$ is closed under $\cup, \cap$, it is a distributive lattice with $k$ join-irreducible subsets $J_1, \ldots, J_k$ (see, e.g., [6]). Define the family of subsets

$$P_j := J_j \setminus \{J_\ell \mid J_\ell \subset J_j\}, \quad j = 1, \ldots, k.$$

Define the poset $(P, \leq)$ by $P = \{P_1, \ldots, P_k\}$ and $P_i \leq P_j$ iff $J_i \subseteq J_j$. Hence $(\{J_1, \ldots, J_k\}, \subseteq)$ and $(P, \leq)$ are isomorphic, and they generate the same downsets. Consequently $\mathcal{R}$ is isomorphic to $\mathcal{O}(P)$. Note also that for any $S \in \mathcal{R}$, we have $S = \bigcup_{i \mid J_i \subseteq S} J_i = \bigcup_{i \mid J_i \subseteq S} P_i$, so we can write (with slight abuse) $\mathcal{R} = \mathcal{O}(P)$. It remains to show that $P$ is a partition of $N$.

Since $N = \bigcup_{i=1}^{k} J_i = \bigcup_{i=1}^{k} P_i$, clearly $P$ must be a covering of $N$. It remains to show that the subsets $P_1, \ldots, P_k$ are pairwise disjoint. Assume on the contrary that $P_j$ and $P_{j'}$ intersect, say $i \in P_j \cap P_{j'}$. Then $i \in J_j$ and $i \notin J_\ell$, for all $J_\ell \subset J_j$, and similarly for $J_{j'}$. Remark that any $S \in \mathcal{R}$ such that $S \subset J_j$ does not contain $i$ (because such a $S$ is the union of some $J_\ell$'s included in $J_j$), and the same holds for $S \subset J_{j'}$. However, since $\mathcal{R}$ is closed under intersection, we must have $J_j \cap J_{j'} \in \mathcal{R}$, which is a subset of $J_j$ containing $i$, a contradiction. □

As explained in the proof, in cases (ii) and (iii), we may write that $\mathcal{R} = \mathcal{O}(P)$.

**Definition 2** *Let $\mathcal{R}^k$ be a $k$-regular set system. A* game *on $\mathcal{R}^k$ is a mapping $v : \mathcal{R}^k \to \mathbb{R}$ such that $v(\emptyset) = 0$. We denote the set of all these games on $\mathcal{R}^k$ by $\mathcal{G}(\mathcal{R}^k)$.*

Some particular cases are of interest.

**Definition 3** *Let $v$ be a game on a $k$-regular set system $\mathcal{R}^k$.*

*(i) If $\mathcal{R}^k$ is a distributive lattice $\mathcal{O}(N)$ induced by some order on $N$ (and in this case $k = n$), then $v$ is called a* distributive game *on $N$.*

*(ii) If $v(S) \leq v(T)$ for any $S, T \in \mathcal{R}^k$ such that $S \subseteq T$, we say that $v$ is* monotone.

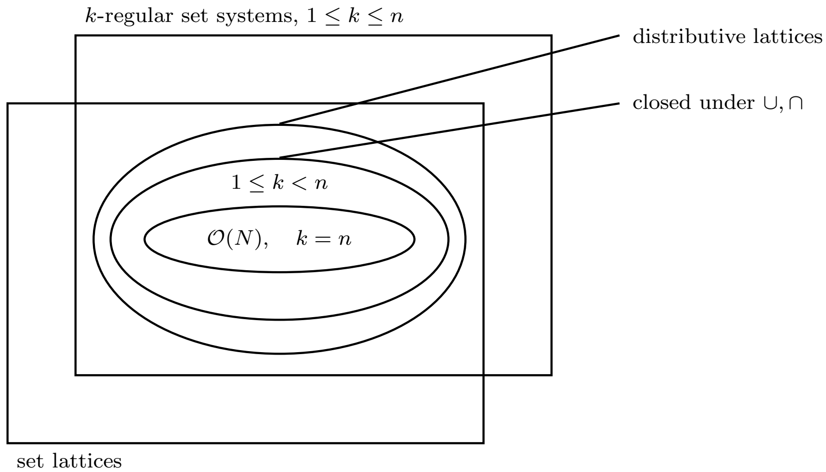



Figure 2: The relative position of set lattices and $k$-regular set systems on $N$

*(iii) Suppose in addition that $\mathcal{R}^k$ is a lattice. We say that the game is* convex*, if* $\forall S, T \in \mathcal{R}^k, v(S \vee T) + v(S \wedge T) \geq v(S) + v(T)$.

Let $x := (x_i)_{i \in N}$ be an $n$-dimensional vector on $\mathbb{R}^n$. $x$ is called a *payoff vector* when every $x_i$ for $i \in N$ represents what the player $i$ earns, like income, bonus, salary, etc. The core is defined as the set of payoff vectors such that no feasible coalition receives less than that it could achieve by itself.

**Definition 4** *Let $\mathcal{R}^k$ be a $k$-regular set system, $v \in \mathcal{G}(\mathcal{R}^k)$. The* core *of $v$ is defined by*

$$\mathcal{C}(v) := \{x := (x_i)_{i \in N} \in \mathbb{R}^n \mid x(S) := \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \mathcal{R}^k \text{ and } x(N) = v(N)\}.$$

Let $\mathcal{R}$ be a set system on $N$. The *core of* $\mathcal{R}$, introduced by Derks and Reijnierse [10], is defined by

$$\mathcal{C}(\mathcal{R}) := \{x := (x_i)_{i \in N} \in \mathbb{R}^n \mid x(S) := \sum_{i \in S} x_i \geq 0 \text{ for all } S \in \mathcal{R} \text{ and } x(N) = 0\}.$$

It is the set of side payments which are not unfavourable to any coalition in $\mathcal{R}$.

Since $\mathcal{C}(\mathcal{R})$ is always convex, then it is either a pointed cone or a linear subspace, or both, i.e., it reduces to the singleton $\{0\}$ (more on cones, rays and vertices in Section 4).

It is easy to see that, if the core of a given game on $\mathcal{R}$ is nonempty, then $\mathcal{C}(v) = \mathcal{C}(v) + \mathcal{C}(\mathcal{R})$, that is, $x + y \in \mathcal{C}(v)$ for any $x \in \mathcal{C}(v), y \in \mathcal{C}(\mathcal{R})$, and reciprocally, for any $z \in \mathcal{C}(v)$ there exist $x \in \mathcal{C}(v), y \in \mathcal{C}(\mathcal{R})$ such that $x + y = z$. Hence if the core of a given game $v$ on $\mathcal{R}$ is nonempty, we have

(i) $\mathcal{C}(\mathcal{R})$ is a pointed cone different from $\{0\}$ if and only if $\mathcal{C}(v)$ has rays. Then $\mathcal{C}(\mathcal{R})$ corresponds to the conic part of $\mathcal{C}(v)$.

(ii) $\mathcal{C}(\mathcal{R})$ is a linear subspace different from $\{0\}$, i.e., $\mathcal{C}(\mathcal{R})$ contains lines, if and only if $\mathcal{C}(v)$ has no vertices.

(iii) $\mathcal{C}(\mathcal{R}) = \{0\}$ if and only if $\mathcal{C}(v)$ is a polytope.

# 3 A review of classical results

The classical definition of games is recovered when the set system is $2^N$, which is $n$-regular. In this case, the condition of convexity simplifies into $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all $S, T \subseteq N$.

In the Boolean lattice $(2^N, \subseteq)$, a maximal chain between $\emptyset$ and $N$ is of the form $\{\emptyset \subsetneq \{i\} \subsetneq \{i,j\} \subsetneq \cdots \subsetneq N\}$, and hence is associated in a bijective way with a permutation $\pi$ on $N$, defining the order in which elements $i, j, \dots$ appear. For any permutation $\pi$ on $N$, we define the maximal chain

$$A_0^\pi := \emptyset \subsetneq A_1^\pi := \{\pi(1)\} \subsetneq A_2^\pi := \{\pi(1), \pi(2)\} \subsetneq \cdots \subsetneq A_n^\pi := N$$

with $A_i^\pi := \{\pi(1), \dots, \pi(i)\}$. Let us define the *marginal worth vector* $x^\pi(v)$ by:

$$x^\pi_{\pi(i)}(v) := v(A_i^\pi) - v(A_{i-1}^\pi), \quad i = 1, \dots, n.$$

The set of all marginal worth vectors is denoted by $\mathcal{M}(v)$. The $i$th coordinate represents the contribution of player $i$ in the chain.

Let us denote by $conv(A)$ the convex hull of the set $A$. For any convex set $A$, we denote by $Ext(A)$ the set of its extreme points.

A collection $\mathcal{B}$ of nonempty subsets of $N$ is *balanced* if there exist positive coefficients $\mu(S)$, $S \in \mathcal{B}$, such that

$$\sum_{S : S \ni i, S \in \mathcal{B}} \mu(S) = 1, \quad \forall i \in N.$$

Any partition $\{P_1, \dots, P_k\}$ of $N$ is a balanced collection, with coefficients $\mu(P_i) = 1$, $\forall i$. A game $v$ is *balanced* if for every balanced collection $\mathcal{B}$ with coefficients $\mu(S), S \in \mathcal{B}$, it holds

$$\sum_{S \in \mathcal{B}} \mu(S) v(S) \leq v(N).$$

The *core* of $v$ is defined as in Definition 4, with $\mathcal{R}^k = 2^N$.

The following proposition summarizes well-known results.

**Proposition 1** *[18] Let $v$ be a game on $N$. The following holds.*

*(i) $\mathcal{C}(v) \subseteq conv(\mathcal{M}(v))$.*

*(ii) $\mathcal{C}(v) \neq \emptyset$ if and only if $v$ is balanced.*

*(iii) $v$ is convex if and only if $\mathcal{C}(v) = conv(\mathcal{M}(v))$ (i.e., $\mathcal{M}(v)$ contains all vertices of the core).*

# 4 General facts on the core

We begin by deriving a necessary and sufficient condition for the nonemptiness of the core. A balanced collection of $\mathcal{R}^k \setminus \{\emptyset\}$ is a balanced collection in the classical sense (see Sec. 3) whose elements are all in $\mathcal{R}^k \setminus \{\emptyset\}$.

**Example 1** *Let* $\mathcal{R}_3 = \{\emptyset, \{1\}, \{1,3\}, \{2,3\}, \{2,3,4\}, \{1,2,3,4\}\}$. *The collection* $\mathcal{B}_1 = \{\{1\}, \{2,3,4\}, \{1,2,3,4\}\}$ *is balanced, because we can take the following coefficients* $\mu_1(\{1\}) = \mu_1(\{2,3,4\}) = \mu_1(\{1,2,3,4\}) = 0.5$.

**Definition 5** *A game $v$ on a $k$-regular set system $\mathcal{R}^k$ is* balanced *if for every balanced collection $\mathcal{B}$ of elements of $\mathcal{R}^k \setminus \{\emptyset\}$ with positive coefficients $\mu(S)$, $S \in \mathcal{B}$, it holds*

$$\sum_{S \in \mathcal{B}} \mu(S) v(S) \leq v(N).$$

By linear programming, we can show the following result (proof is omitted, since very close to the classical one).

**Proposition 2** *A game on a $k$-regular set system has a nonempty core if and only if it is balanced.*

Assuming that the core is nonempty, by its definition it is a closed convex polyhedron. We study its general structure in the rest of this section. We assume that the reader has some familiarity with polyhedra and linear programming (see, e.g., V. Chvátal [5]).

Let $v$ be a game on a $k$-regular set system $\mathcal{R}^k$, and put $r := |\mathcal{R}^k|$. The elements of the core of this game are all $n$-dimensional vectors $x$ satisfying $\sum_{i \in S} x_i \geq v(S)$ for all $S \in \mathcal{R}^k$ and $\sum_{i \in N} x_i = v(N)$. This can be written under the form of a system $Ax \geq b$ where $A$ is a $(r-1) \times n$ matrix, whose rows are characteristic vectors of the subsets $S$, $x$ is an $n$-column vector and $b = (v(S))_{S \in \mathcal{R}^k \setminus \{\emptyset\}}$ is a $(r-1)$-column vector:

$$A = \begin{pmatrix} 1 & \cdots & 0 \\ & \cdots & \\ 1 & \cdots & 1 \end{pmatrix}, b = \begin{pmatrix} \vdots \\ v(S) \\ \vdots \\ v(N) \end{pmatrix}, x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Recall that the last row ($S = N$) is in fact an equality, and that all variables $x_1, \ldots, x_n$ are free (i.e., unbounded). The core is the set of solutions of this system. We know that a vector $x$ is a vertex of the core if and only if $x$ is a basic feasible solution of the system $Ax \geq b$. By the fundamental theorem on polyhedra, the core of a game $v$ can be written as the Minkowski sum of a convex part $\mathcal{C}^F(v) := conv\big(Ext(\mathcal{C}(v))\big)$ and a conic part generated by $Ray\big(\mathcal{C}(v)\big)$, the set of rays (or basic feasible directions) of the core.

A vertex of the core being determined by a nonsingular subsystem of $n$ equalities, no vertex exists if $rank(A) < n$. In particular, there is no vertex if $r - 1 < n$. We therefore assume in the rest of this section that $r - 1 \geq n$.

Adding slack variables $y_j \geq 0, j = 1, \ldots, r-2$ into this system, it turns to

$$A^* \cdot \begin{pmatrix} x \\ y \end{pmatrix} = b$$

with $y := (y_j)_j$ being a $(r-2)$-column vector, and $A^*$ being the $(r-1)\times(n+r-2)$ augmented matrix of $A$:

$$A^* \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{array}{ccc|ccccc} 1 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ & & & 0 & -1 & \cdots & 0 & 0 \\ & \cdots & & & & \cdots & & \\ & & & 0 & 0 & \cdots & 0 & -1 \\ \hline 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \end{array}\right) \times \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \hline y_1 \\ \vdots \\ y_{r-2} \end{pmatrix} = \begin{pmatrix} \vdots \\ v(S) \\ \vdots \\ v(N) \end{pmatrix}. \qquad (1)$$

We have $r-1$ basic variables and $n-1$ nonbasic variables.

**Proposition 3** *Let $v$ be a game on a $k$-regular set system $\mathcal{R}^k \subseteq 2^N$ with $N = \{1, 2, \ldots, n\}$. The components of the rays of the core of $v$ do not depend on $v$, but only on $\mathcal{R}^k$.*

**Proof** Any element $x$ of the core satisfies the system $A^*(x\ y)^T = b$. We know that the rays are all particular solutions of the system $A^*(x\ y)^T = 0$ with all nonbasic components all equal to 0 but one. The result holds because the system $A^*(x\ y)^T = 0$ depends only on $\mathcal{R}^k$, not on $v$. □

If $(x^*, y^*)$ is a vertex of system (1), since any variable $x_i$ is free, it must be basic. Because $r-1 \geq n$, we must choose $r-1-n$ variables $y_j$ as basic variables and let the $n-1$ nonbasic variables $y_j$ being equal to 0. We can suppose that $y^*_{r-n} = 0, \ldots, y^*_{r-2} = 0$. It leads to

$$\left(\begin{array}{ccc|cccc} 1 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ & \cdots & & 0 & -1 & \cdots & 0 \\ & \cdots & & \vdots & & \ddots & \\ & \cdots & & 0 & 0 & \cdots & -1 \\ \hline 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array}\right) \times \begin{pmatrix} x^*_1 \\ \vdots \\ x^*_n \\ \hline y^*_1 \\ \vdots \\ y^*_{r-1-n} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \vdots \\ v(S) \\ \vdots \\ v(N) \end{pmatrix}.$$

The basic $(r-1)\times(r-1)$ matrix of $A^*$ can be written as

$$\begin{array}{cc} & \begin{array}{cc} \quad n \quad & \quad r-1-n \end{array} \\ \begin{array}{c} r-1-n \\ \\ \\ n \end{array} & \left(\begin{array}{ccc|ccc} 1 & \dots & 0 & -1 & \cdots & 0 \\ & \vdots & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & -1 \\ \hline & \vdots & & & 0 & \\ 1 & \cdots & 1 & & & \end{array}\right) \end{array} =: \left(\begin{array}{c|c} A_1 & C_1 \\ \hline A_2 & 0 \end{array}\right),$$

that is,

$$\left(\begin{array}{ccc|ccc} & & & -1 & \cdots & 0 \\ & \vdots & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & -1 \\ \hline & \vdots & & & 0 & \\ 1 & \cdots & 1 & & & \end{array}\right) \times \left(\begin{array}{c} x_1^* \\ \vdots \\ x_n^* \\ \hline y_1^* \\ \vdots \\ y_{r-1-n}^* \end{array}\right) = \left(\begin{array}{c} v(1) \\ \vdots \\ v(S) \\ \vdots \\ v(N) \end{array}\right).$$

Because the basic matrix is nonsingular, then $A_2$ and $C_1$ are nonsingular too. Using well-known relations, we obtain

$$\left(\begin{array}{c} x_1^* \\ \vdots \\ x_n^* \\ \hline y_1^* \\ \vdots \\ y_{r-1-n}^* \end{array}\right) = \left(\begin{array}{cc} 0 & A_2^{-1} \\ C_1^{-1} & -C_1^{-1}A_1A_2^{-1} \end{array}\right) \times \left(\begin{array}{c} \vdots \\ v(S) \\ \vdots \\ v(N) \end{array}\right).$$

Hence a basic feasible solution of this system can be written as $(x_1, \cdots, x_n, y_1, \cdots, y_{r-1-n}, 0, \cdots, 0, \cdots, 0)^T$. It implies that a vertex of the core can be written as

$$\left(\begin{array}{cc} 0 & A_2^{-1} \end{array}\right) \times \left(\begin{array}{c} \vdots \\ v(S) \\ \vdots \\ v(N) \end{array}\right).$$

Let us now determine the rays of the core. An algorithmic procedure to determine all rays, which is a simplified version of the method given in Chvátal [5], is described in the Appendix. We remain in this section on a formal level. As above, let us suppose that the basic variables are $x_1, \cdots, x_n, y_1, \cdots, y_{r-1-n}$. Hence a basic feasible direction has the form

$(x_1, \cdots, x_n, y_1, \cdots, y_{r-1-n}, 0, \cdots, \overbrace{1}^{y_j}, \cdots, 0)^T$ for some $j \in \{r-n, \ldots, r-2\}$. We have

$$\left(\begin{array}{ccc|cccc} 1 & & 0 & -1 & 0 & \cdots & 0 \\ & & & 0 & -1 & \cdots & 0 \\ & \vdots & & \vdots & & \ddots & \\ & & & 0 & 0 & \cdots & -1 \\ \hline 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array}\right) \times \left(\begin{array}{c} x_1 \\ \vdots \\ x_n \\ \hline y_1^* \\ \vdots \\ y_{r-1-n} \\ \hline 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{array}\right) = 0,$$

i.e.,

$$\begin{pmatrix} A_1 & C_1 \\ A_2 & 0 \end{pmatrix} \times \left(\begin{array}{c} x_1 \\ \vdots \\ x_n \\ \hline y_1 \\ \vdots \\ y_{r-1-n} \end{array}\right) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{c} \\ \\ \leftarrow j \\ \\ \\ \end{array},$$

that is,

$$\left(\begin{array}{c} x_1 \\ \vdots \\ x_n \\ \hline y_1 \\ \vdots \\ y_{r-1-n} \end{array}\right) = \begin{pmatrix} 0 & A_2^{-1} \\ C_1^{-1} & -C_1^{-1}A_1A_2^{-1} \end{pmatrix} \times \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{c} \\ \\ \leftarrow j \\ \\ \\ \end{array},$$

with $r - n \leq j \leq r - 2$. Then

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & A_2^{-1} \end{pmatrix} \times \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_{r-1-n} \end{pmatrix} = \begin{pmatrix} C_1^{-1} & -C_1^{-1}A_1A_2^{-1} \end{pmatrix} \times \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

If $y_1, \ldots, y_{r-1-n} \geq 0$, the vector $(x, y)^T$ is a ray; otherwise denote the set of indices of all $y_i < 0$ by $I$. Let $z_i := \dfrac{y_i^*}{-y_i}$ and $k$ be the index of $z_k$ such that $z_k = \min_{i \in I}\{z_i\}$. Then the variable $y_k$ leaves the basis and the variable $y_j$ enters the basis. It leads to another basic feasible solution $(x^{**}, y^{**})^T$ s.t.

$$(x^{**}, y^{**}) = (x_1^{**}, \cdots, x_n^{**}, y_1^{**}, \cdots, y_{k-1}^{**}, 0, y_{k+1}^{**}, \ldots, y_{r-1-n}^{**}, 0, \cdots, y_j^{**}, \cdots, 0)^T.$$

By choosing the indices of $r-1-n$ basic variables $y_i$ and the position of $y_j$ which is supposed to be equal to 1, we can find all basic feasible solutions with all free variables being basic, and all rays. Denote all these basic feasible solutions by $x^1, \ldots, x^p$ and all rays by $r^1, \ldots, r^q$. The following theorem, which is a direct consequence of the fundamental theorem on polyhedra, summarizes previous results.

**Theorem 1** *Let $Ax \geq b$ represent the core of a given balanced game $v$. Let*

$$\begin{pmatrix} x_1^i \\ \vdots \\ x_n^i \end{pmatrix} = \begin{pmatrix} 0 & A_2^{-1} \end{pmatrix} \times \begin{pmatrix} \vdots \\ v(S) \\ \vdots \\ v(N) \end{pmatrix}, \begin{pmatrix} r_1^i \\ \vdots \\ r_n^i \end{pmatrix} = \begin{pmatrix} 0 & A_2^{-1} \end{pmatrix} \times \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

*with $A_2$ being any nonsingular submatrix of $A$, which contains the row vector $(1, \ldots, 1)$. A vector $x \in \mathbb{R}^n$ belongs to the core if and only if $x = \sum_{i=1}^p \alpha_i x^i + \sum_{i=1}^q \beta_i r^i$ with $\sum_{i=1}^p \alpha_i = 1$ and $\alpha_i, \beta_j \geq 0$ for all $i = 1, \ldots, p, j = 1, \ldots, q$.*

The above theorem and the algorithm given in the Appendix permit to find all rays and vertices of the core, when nonempty. It does not give however an explicit expression of them. Derks and Gilles have given an explicit expressions of rays of the core in the particular case where $\mathcal{R}$ is closed under union and intersection [9]. We introduce the notation $D_i^{\mathcal{R}} := \bigcap\{S \in \mathcal{R} \mid i \in S\}$, for any $i \in N$. Recall that rays of $\mathcal{C}(\mathcal{R})$ correspond to rays of $\mathcal{C}(v)$.

**Theorem 2** *[9] Let $\mathcal{R}$ be a set system closed under $\cup$ and $\cap$. Then $\mathcal{C}(\mathcal{R})$ is the cone generated by $e_j - e_i$ with $i \in N$, $j \in D_i^{\mathcal{R}}$, where $e_i$ is the characteristic vector of $i$.*

Derks and Gilles have proved that such vectors $e_j - e_i$ are always rays of the core, for any set system.

# 5 Vertices of the core

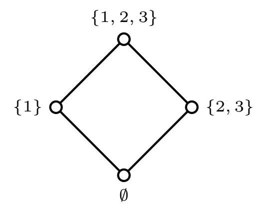
Assuming that the core is nonempty, we want to find conditions of existence of vertices of the core, and analyze them.

## 5.1 Existence of vertices

First, we consider the following example.

**Example 2** *Let $v$ be a game with three players on $\mathcal{R} = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\}$. The conditions for $x := (x_i)_{i=1,2,3}$ to be an element of the core are:*

$$x_1 \geq v(\{1\})$$
$$x_2 + x_3 \geq v(\{2,3\})$$
$$x_1 + x_2 + x_3 = v(\{1,2,3\}).$$




*Let $v(\emptyset) = 0, v(\{1\}) = 1, v(\{2,3\}) = 1.5, v(\{1,2,3\}) = 3$. The vector $x = (1,2,0) \in \mathcal{C}(v)$ as well as $x + l(0,-1,1)$ for any $l \in \mathbb{R}$. That is, $(0,-1,1)$ is a line of $\mathcal{C}(v)$, i.e., the core has no vertices.*

This example shows that the core may be without vertices.

The next proposition shows that in case of distributive lattices, the core has always vertices.

**Proposition 4** *Let $v$ be a distributive game on $\mathcal{R} = \mathcal{O}(N)$. Then, if the core of $v$ is nonempty, it has always some vertices.*

**Proof** Let $\{\emptyset, S_1 := \{i_1\}, \cdots, S_n := N\}$ be a maximal chain of $\mathcal{R}$ with $S_j := \{i_1, i_2, \cdots, i_j\}$. We know that the core of $v$ is the set of solutions of the system $Ax \geq b$. Then, after changing the order of variables of $x$, the submatrix of $A$ corresponding to $S_1, \cdots, S_n$ can be written as

$$\begin{array}{c c} & \begin{array}{ccc} i_1 & \cdots & i_n \end{array} \\ \begin{array}{c} S_1 \\ \vdots \\ S_n \end{array} & \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} \end{array}.$$

Its rank is $n$, then the rank of $A$ is $n$, and therefore the core has at least one vertex. □

The following result, due to Derks and Reijnierse, gives a necessary and sufficient condition for having vertices. Let $\mathcal{R}$ be a set system. The *span* of $\mathcal{R}$ is defined as follows

$$\text{span}(\mathcal{R}) := \Big\{S \subseteq N \mid e_S = \sum_{T \in \mathcal{R}} \lambda_T e_T \text{ for some } \lambda_T \in \mathbb{R}\Big\}$$

with $e_T = \sum_{i \in T} e_i$ for all $T \subseteq N$.

**Definition 6** *A set system $\mathcal{R}$ is called* non-degenerate *if the characteristic vectors $e_T, T \in \mathcal{R}$, span the whole allocation space $\mathbb{R}^N$, i.e.,* $\mathrm{span}(\mathcal{R}) = 2^N$*.*

**Theorem 3** *[10] The core of a set system $\mathcal{R}$ is a pointed cone if and only if $\mathcal{R}$ is non-degenerate.*

Hence, if a game $v$ has a nonempty core, it has vertices if and only if $\mathcal{R}$ is non-degenerate. For example, if $\mathcal{R}^k$ includes at least $n-1$ singletons, the core has at least one vertex.

**Remark 1** *The following facts are noteworthy:*

*(i) The fact that the core of a game has vertices or no vertices, as well as the coordinates of rays (if any), depends solely on the set system, not on the game (see Theorems 2 and 3, and Proposition 3).*

*(ii) The fact that the core of a game is empty or not depends mainly on the game (see Proposition 2).*

## 5.2 Marginal worth vectors and the Weber set

Assuming that the core is nonempty and has vertices, let us try to characterize them. In the classical case, the marginal worth vectors play a key role for this. In our case, we will need something slightly more general. Let $\mathcal{R}^k$ be a $k$-regular set system on $N$, $v$ be a game on $\mathcal{R}^k$. Let $C := \{S_0 = \emptyset \prec S_1 \prec \cdots \prec S_k = N\} \in \mathcal{C}(\mathcal{R}^k)$. For any set $S_i \setminus S_{i-1} := \{s_{i_1}, \dots, s_{i_{j_i}}\}$, $\forall i = 1, 2, \dots, k$, we define an $n$-dimensional real-valued vector $\psi^{i,r} := (\psi^{i,r}(j))_{j\in N} \in \mathbb{R}^n$ with $i = 1, \dots, k,\ r \in S_i \setminus S_{i-1}$ by

$$\psi^{i,r} = [v(S_i) - v(S_{i-1})]e_r,$$

that is,

$$\psi^{i,r}(j) = \begin{cases} v(S_i) - v(S_{i-1}) & \text{if } j = r \\ 0 & \text{otherwise.} \end{cases}$$

Hence for any two sets $S_i, S_{i-1}$, there are $|S_i \setminus S_{i-1}|$ vectors $\psi^{i,r}$ to be defined. For any maximal chain $C$, we define some $n$-dimensional vectors $\psi^{C(r_1,\dots,r_k)} \in \mathbb{R}^n$ with $r_i \in S_i \setminus S_{i-1}$ $\forall i = 1, \dots, k$ by

$$\psi^{C(r_1,\dots,r_k)} := \psi^{1,r_1} + \psi^{2,r_2} + \cdots + \psi^{k,r_k} \text{ with } \psi^{i,r_i} = [v(S_i) - v(S_{i-1})]e_{r_i}.$$

There are $|S_1| \times |S_2 \setminus S_1| \times \cdots \times |S_k \setminus S_{k-1}|$ possibilities. We denote the set of all possibilities for $(r_1, \dots, r_k)$ corresponding to a maximal chain $C$ by $R^C$.

In Example 2, we consider the maximal chain $C := (S_0 := \emptyset \prec S_1 := \{2,3\} \prec S_2 := \{1,2,3\})$. We obtain:

$$\psi^{1,2} = (0, v(23), 0);$$

$$\psi^{1,3} = (0, 0, v(23));$$

$$\psi^{2,1} = (v(123) - v(23), 0, 0).$$

They lead to 2 possibilities for vectors $\psi^{C(r_1,\dots,r_k)}$:

$$\psi^{C(2,1)} = \psi^{1,2} + \psi^{2,1} = (v(123) - v(23), v(23), 0);$$

$$\psi^{C(3,1)} = \psi^{1,3} + \psi^{2,1} = (v(123) - v(23), 0, v(23)).$$

Let $C \in \mathcal{C}(\mathcal{R}^k)$, and let $\mathcal{V}^C(v, \mathcal{R}^k)$ denote the set of all possibilities for vectors $\psi^{C(r_1,\dots,r_k)}$. Let $\mathcal{V}(v, \mathcal{R}^k)$ denote the set of all vectors $\psi^{C(r_1,\dots,r_k)}$ for all maximal chains $C$ and all $r_1, \dots, r_k$, i.e., $\mathcal{V}(v, \mathcal{R}^k) := \cup_{C \in \mathcal{C}(\mathcal{R}^k)} \mathcal{V}^C(v, \mathcal{R}^k)$. By analogy with the classical case, we call *marginal worth vectors* the elements of $\mathcal{V}(v, \mathcal{R}^k)$, and we define the *Weber set* as the convex hull of all marginal vectors:

$$\mathcal{W}(v) := conv(\mathcal{V}(v, \mathcal{R}^k)).$$

We can show that

$$\psi^{C(r_1,\dots,r_k)}(N) = \sum_{i=1}^{k} \psi^{i,r_i}(N) = \sum_{i=1}^{k} \psi^{i,r_i}(r_i) = \sum_{i=1}^{k} [v(S_i) - v(S_{i-1})] = v(N)$$

and

$$\psi^{C(r_1,\dots,r_k)}(S_j) = \sum_{i=1}^{j} \psi^{i,r_i}(S_j) = \sum_{i=1}^{j} \psi^{i,r_i}(r_i) = \sum_{i=1}^{j} [v(S_i) - v(S_{i-1})] = v(S_j) \tag{2}$$

for all $S_j \in C$.

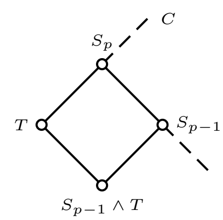
# 6 The core of convex games

In this section, we give some properties about the core of convex games.

**Theorem 4** *Let $\mathcal{R}^k$ be a $k$-regular lattice closed under union and intersection. If $v$ is convex, then $\mathcal{W}(v) \subseteq \mathcal{C}(v)$.*

**Proof** It suffices to prove that $\mathcal{V}(v, \mathcal{R}^k) \subseteq \mathcal{C}(v)$. Let $C := \{S_0 = \emptyset, S_1, \dots, S_k = N\}$ be a maximal chain of the $k$-regular lattice $\mathcal{R}^k$ closed under $\cup, \cap$. Let $\psi^{C(r_1,\dots,r_k)}$ be a vector in $\mathcal{V}^C(v, \mathcal{R}^k)$. We want to prove that $\psi^{C(r_1,\dots,r_k)} \in \mathcal{C}(v)$.

Let $T \notin C$ and $T \in \mathcal{R}^k$. Because $\emptyset \subsetneq T \subsetneq N$, then there exists a unique smallest set $S_p$, s.t. $T \subsetneq S_p \in C$, then $S_{p-1} \vee T = S_{p-1} \cup T = S_p$.



we have

$$v(S_{p-1} \cup T) + v(S_{p-1} \cap T) \geq v(S_{p-1}) + v(T).$$

we want to prove $\psi^{C(r_1,\ldots,r_k)}(T) \geq v(T)$ by induction on $p \geq 2$.

If $p = 2$, we have $v(S_1 \cup T) + v(\emptyset) \geq v(S_1) + v(T)$. By (1), it implies $\psi^{C(r_1,\ldots,r_k)}(T) = \psi^{C(r_1,\ldots,r_k)}(S_1 \cup T) - \psi^{C(r_1,\ldots,r_k)}(S_1) \geq v(T)$.

If $p > 2$, we deduce from (1) that

$$\psi^{C(r_1,\ldots,r_k)}(S_{p-1} \cup T) + v(S_{p-1} \cap T) \geq \psi^{C(r_1,\ldots,r_k)}(S_r) + v(T).$$

By induction

$$\psi^{C(r_1,\ldots,r_k)}(S_{p-1} \cap T) \geq v(S_{p-1} \cap T).$$

It indicates

$$\psi^{C(r_1,\ldots,r_k)}(S_{p-1} \cup T) + \psi^{C(r_1,\ldots,r_k)}(S_{p-1} \cap T) \geq \psi^{C(r_1,\ldots,r_k)}(S_{p-1}) + v(T).$$

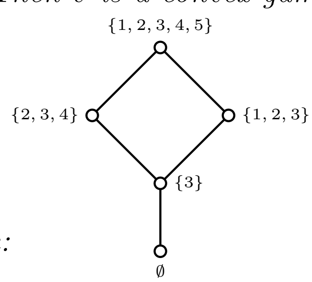
That is,

$$\psi^{C(r_1,\ldots,r_k)}(T) \geq v(T).$$

Consequently we have $\psi^{C(r_1,\ldots,r_k)} \in \mathcal{C}(v)$. $\square$

We cannot get the same result for any $k$-regular lattice, as the following example shows.

**Example 3** *Consider $\mathcal{R}$ defined in the figure below, and put $v(3) = 2$, $v(123) = 6$, $v(234) = 4$, and $v(12345) = 8$. Then $v$ is a convex game. The conditions for $x := (x_i)_{i=1,2,3}$ to be an element of the core are:*




$$x_3 \geq 2, \quad x_1 + x_2 + x_3 \geq 6, \quad x_2 + x_3 + x_4 \geq 4 \text{ and } x_1 + x_2 + x_3 + x_4 + x_5 = 8.$$

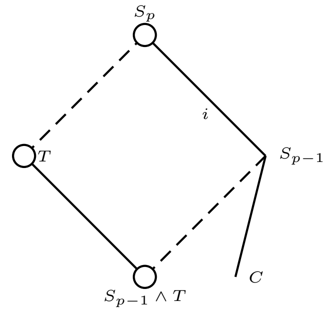
*Let $C = \{\emptyset, 3, 123, 12345\}$ be a maximal chain of $\mathcal{R}$. Then*

$$\psi^{C(3,1,5)} := (v(123) - v(3), 0, v(3), 0, v(12345) - v(123)) = (4, 0, 2, 0, 2)$$

*is not in the core of $v$.*

**Theorem 5** *Let $\mathcal{R}^n$ be a $n$-regular lattice. If $v$ is monotone and convex, then $\mathcal{W}(v) \subseteq \mathcal{C}(v)$.*

**Proof** The demonstration is similar to that of Th. 4. Let $C := \{S_0 = \emptyset, S_1, \ldots, S_n = N\}$ be a maximal chain of the $n$-regular set system $\mathcal{R}^n$ and $T \in \mathcal{R}^n, T \notin C$. Let $\psi^{C(r_1,\ldots,r_n)}$ be a vector in $\mathcal{V}^C(v, \mathcal{R}^n)$. It always exists a unique smallest set $S_p \in C$, s.t. $T \subseteq S_p$. Moreover since $\mathcal{R}^n$ is $n$-regular, then $\exists i \in N$ s.t. $S_{p-1} = S_p \setminus \{i\}$ and $T \vee S_{p-1} = S_p$.




By convexity,

$$v(S_{p-1} \vee T) + v(S_{p-1} \wedge T) \geq v(S_{p-1}) + v(T).$$

Let us prove $\psi^{C(r_1,\ldots,r_n)}(T) \geq v(T)$ by induction on $p \geq 2$.

If $p = 2$, then $T = \{i\}, S_1 = \{j\}, j \neq i$. We have $v(\{i,j\}) + v(\emptyset) \geq v(\{j\}) + v(T)$. By (1), it implies $\psi^{C(r_1,\ldots,r_n)}(\{i\}) = \psi^{C(r_1,\ldots,r_n)}(\{i,j\}) - \psi^{C(r_1,\ldots,r_n)}(\{j\}) \geq v(\{i\})$.

If $p > 2$, we deduce from (1) that

$$\psi^{C(r_1,\ldots,r_n)}(S_{p-1} \cup \{i\}) - \psi^{C(r_1,\ldots,r_n)}(S_{p-1}) + v(S_{p-1} \wedge T) \geq v(T).$$

i.e.,

$$\psi^{C(r_1,\ldots,r_n)}(\{i\}) + v(S_{p-1} \wedge T) \geq v(T).$$

By induction

$$\psi^{C(r_1,\ldots,r_n)}(S_{p-1} \wedge T) \geq v(S_{p-1} \wedge T).$$

We have

$$\psi^{C(r_1,\ldots,r_n)}(\{i\}) + \psi^{C(r_1,\ldots,r_n)}(S_{p-1} \wedge T) \geq v(T).$$

By definition of operation $\wedge$, $\exists S^- \subseteq S_{p-1} \cap T$, such that $S_{p-1} \wedge T = (S_{p-1} \cap T) \setminus S^-$. We have

$$\psi^{C(r_1,\ldots,r_n)}(\{i\}) + \psi^{C(r_1,\ldots,r_n)}(S_{p-1} \cap T) - \psi^{C(r_1,\ldots,r_n)}(S^-) \geq v(T).$$

Since $v$ is monotone, it implies $\psi^{C(r_1,\ldots,r_n)} \in \mathbb{R}^n_+$. Then,

$$\psi^{C(r_1,\ldots,r_n)}(T) \geq \psi^{C(r_1,\ldots,r_n)}(T) - \psi^{C(r_1,\ldots,r_n)}(S^-) \geq v(T).$$

Hence

$$\psi^{C(r_1,\ldots,r_n)}(T) \geq v(T)$$

Consequently we have $\psi^{C(r_1,\ldots,r_n)} \in \mathcal{C}(v)$. $\square$

# 7 Comparison with distributive games

Let $v$ be a distributive game on $\mathcal{R} := \mathcal{O}(N)$. In [14], the precore of $v$ is defined as follows.

**Definition 7** The precore of a distributive game $v$ on $\mathcal{O}(N)$ *is defined by*

$$\mathcal{PC}_d(v) := \{\psi \in \mathbb{R}^n \mid \psi(N) = v(N) \text{ and } \psi(S) \geq v(S)\, \forall S \in \mathcal{O}(N)\}.$$

Let $C := \{S_0 := \emptyset, S_1, \ldots, S_n := N\}$ be a maximal chain of $\mathcal{O}(N)$. It is associated in a bijective way with a permutation $\pi$ on $N$, defining the additional element between any 2 consecutive coalitions $S_i, S_{i-1}$ as $\pi(i)$. That is, we can write

$$S_i = \{\pi(1), \pi(2), \ldots, \pi(i)\}$$

The *pre-marginal worth vector* $\psi^\pi$ is defined by:

$$\psi^\pi(\pi(i)) := v(S_i) - v(S_{i-1}), \quad i = 1, \ldots, n.$$

The set of all pre-marginal worth vectors is denoted by $\mathcal{PM}(v)$.

Because there is a unique additional element between any two consecutive coalitions $S_i, S_{i-1}$, any vector $\psi^{C(r_1,\ldots,r_n)}$ in $\mathcal{V}^C(v, \mathcal{R})$ is uniquely written as

$$\psi^{C(\pi(1),\ldots,\pi(n))} = (v(S_1), v(S_2) - v(S_1), \ldots, v(S_n) - v(S_{n-1}))$$

Hence the set $\mathcal{PM}(v)$ is the same as $\mathcal{V}(v, \mathcal{R})$ in the case of $\mathcal{R} = \mathcal{O}(N)$.

In [14], we have proved the following results.

**Theorem 6** *For any distributive game $v$, the convex part of the precore is included in the convex hull of the set of all pre-marginal worth vectors, i.e, $\mathcal{PC}_d^F(v) \subseteq conv\big(\mathcal{PM}(v)\big)$.*

**Theorem 7** *For any distributive game $v$, it is convex if and only if $Ext(\mathcal{PC}_d(v)) = \mathcal{PM}(v)$.*

The first result does not hold in the general case of $k$-regular set systems.

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# Appendix A

We introduce a new method to find out all rays of a system. First, we recall a classical process to find all rays of a system $Ax = b, w \leq x \leq u$ in [5] where $A, b, w, u$ are defined as before.

A *basic feasible partition* of $x$ is represented by the vector $e = [e_1, \ldots, e_n]$ such that

$$\begin{aligned} &e_i = 1 \text{ if } x_i \text{ is basic,} \\ &e_i = 0 \text{ if } x_i \text{ is nonbasic and } x_i^* = w_i, \\ &e_i = 2 \text{ if } x_i \text{ is nonbasic and } x_i^* = u_i. \end{aligned}$$

We say that two basic feasible partitions $e, e'$ are *neighbors* of each other if there one of two following cases happens $\begin{cases} \exists i \text{ s.t. } e_i \neq e'_i, e_i \neq 1, e'_i \neq 1 \text{ and } e_j = e'_j, \forall j \neq i, \\ \exists i, j \text{ s.t. } e_i = 1, e'_i \neq 1, e_j \neq 1, e'_j = 1 \text{ and } e_k = e'_k, \forall k \neq i, j. \end{cases}$

For a basic feasible partition $e$, it is very easy to find its neighbors in the first case. For the second case, in [5], an algorithm is proposed to produce all the neighbors of a basic feasible partition $e$ as follows: Let $x^*$ be the basic feasible solution corresponding to $e$. We consider some nonbasic variable $x_i$. In $Ax = b$, we replace the value $x_i^*$ by $x_i^* + t$ (if $x_i^* = w_i$) or by $x_i^* - t$ (if $x_i^* = u_i$) with $t \geq 0$ but keep the values of other nonbasic variables. If $t$ can be arbitrarily large, then we discover a ray associated with $e$; otherwise, it is a neighbor of $e$.

**Example 4** *Let*

$$A = \begin{pmatrix} 3 & 1 & 4 & 1 & 0 \\ 4 & 1 & 5 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 23 \\ 25 \end{pmatrix}, x = (x_1, x_2, x_3, x_4, x_5)^T,$$

$$w = (-2, 4, -\infty, 0, 0)^T, u = (3, +\infty, 5, +\infty, +\infty)^T.$$

*Let* $e^1 = (2, 0, 1, 1, 0)^T$ *be a basic feasible partition. It corresponds to the basic feasible solution* $x = (3, 4, 9/5, 14/5, 0)$.

*We begin from the nonbasic variable* $x_1$. *We use* $x_1 - t$ *instead of* $x_1$ *in* $Ax = b$. *It writes*

$$\begin{cases} 3x_1 - 3t + x_2 + 4x_3 + x_4 = 23 \\ 4x_1 - 4t + x_2 + 5x_3 + x_5 = 25. \end{cases}$$

*We hold* $x_1 = 3, x_2 = 4, x_5 = 0$. *We have*

$$\begin{cases} 4x_3 + x_4 = 10 + 3t \\ 5x_3 = 9 + 4t. \end{cases}$$

*Because $x_3 \leq 5, x_4 \geq 0$, it indicates that $t \leq 4$. Let $t = 4$, that is, $x_1$ enters the basis, but $x_3$ leaves it. We get one of neighbors of $e^1$: $e^2 = (1, 0, 2, 1, 0)^T$.*

*From the nonbasic variable $x_2$, we can get similarly another neighbor of $e^1$: $e^3 = (2, 1, 1, 0, 0)^T$.*

*Now we consider the nonbasic variable $x_5$. Replace $x_5$ by $x_5 + t$ with $t \geq 0$, but keeping $x_1 = 3$ and $x_2 = 4$. We have*

$$\begin{cases} 5x_3 = 9 - t \leq 25 \\ 5x_4 = 14 + 4t \geq 0. \end{cases}$$

*Then $t$ is not restricted and we have a ray. Let $y$ be the ray of $e^1$. It must satisfy $y_1 = y_2 = 0, y_5 = 1$ and $Ay = 0$. Then $y = (0, 0, -1/5, 4/5, 1)^T$.*

*By the same way, we consider the neighbors of $e^2, e^3$ and can find all rays of $Ax = b$.*

Because we can easily find all basic feasible partitions, we propose the following way to simplify the above process for finding all rays. Let $x^*$ be the basic feasible solution of a basic feasible partition $e$, $J$ be the set of indices of all basic variables. For some nonbasic variable $x_i$, $i \notin B$, we take $x'_i = 1$ (if $x^*_i = w_i$) or $x'_i = -1$ (if $x^*_i = u_i$) and $x'_j = 0\ \forall j : j \neq i, j \notin J$. We obtain the coordinates of $x'$ in $J$ by solving the system $Ax' = 0$. For any $k \in J$, if $x'_k \leq 0$ whenever $w_k = -\infty$ and $x'_k \geq 0$ whenever $u_k = +\infty$, then $x'$ is a ray (because $x^* + tx'$ is always feasible with $t \geq 0$, that is, $t$ can be arbitrarily large); otherwise, let $I$ be the set of all indices of variables not satisfying these two conditions and

$$z_k := \begin{cases} \dfrac{u_k - x^*_k}{x'_k} & \text{if } x'_k > 0, \\[2ex] \dfrac{x^*_k - w_k}{-x'_k} & \text{if } x'_k < 0. \end{cases}$$

Let $j$ be the index of the variable $z_j$ such that $z_j = \min_{k \in I} \{z_k\}$.

With $t$ increasing, the variable $x_j$ achieves first its upper bound (if $x'_j > 0$) or lower bound (if $x'_j < 0$). Hence it produces a neighbor $e^*$ of the basic feasible partition $e$. That is, $e^*_k = e_k, \forall k \neq j, k \neq i$, $e^*_i = 1$ and $e^*_j = 0$ (if $x'_j < 0$) or $e^*_j = 2$ (if $x'_j > 0$). By this method, we can find all neighbors of a basic feasible partition. Now we apply this method to find again all neighbors of $e^1$ of Example 4.

**Example 5** *We begin also from the nonbasic variable $x_1$. Let $x'_1 = -1$ (because $x_1 = 3 = u_1$) and $x'_2 = 0, x'_5 = 0$. The system $Ax' = 0$ can be written as*

$$\begin{cases} -3 + 4x'_3 + x'_4 = 0, \\ -4 + 5x'_3 = 0. \end{cases}$$

*We obtain $x'_3 = 4/5, x'_4 = -1/5$. Then $x'$ is not a ray but a neighbor of $e^1$ and $I = \{3, 4\}$. We put that*

$$z_3 := \frac{u_3 - x_3}{x'_3} = 4, z_4 := \frac{x_4 - w_4}{-x'_4} = 14.$$

*Then* $j = 3$. *A neighbor* $e^2$ *of* $e^1$ *is found, it is* $e^2 = (1, 0, 2, 1, 0)^T$.

*For the nonbasic variable* $x_2$, *similarly, let* $x'_2 = 1$ *and* $x'_1 = x'_5 = 0$. *We have* $x'_3 = x'_4 = -1/5$ *by* $Ax' = 0$. *Because* $x'_3 < 0$, *the set* $I$ *is a singleton* $\{4\}$, *i.e.,* $t = 4$. *Then another neighbor* $e^3$ *of* $e^1$ *is* $e^3 = (2, 1, 1, 0, 0)^T$.

*Now we consider the last nonbasic variable* $x_5$. *Let* $x'_5 = 1$ *and* $x'_1 = x'_2 = 0$. *By* $Ax' = 0$, *we have* $x'_3 = -1/5, x'_4 = 4/5$. *Then we obtain a ray* $x' = (0, 0, -1/5, 4/5, 1)^T$.