BL-GARCH model with elliptical distributed innovations

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Exact Maximum Likelihood estimator for the BL-GARCH model under elliptical distributed innovations ⋆

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Abstract

We are interested in the parametric class of Bilinear GARCH (BL-GARCH) models which are capable of simultaneously capturing the well known properties of financial retrun series, volatility clustering and leverage effects. Specifically, as it is often observed that the distribution of many financial time series data has heavy tails, heavier than the Normal distribution, we examine, in this paper, the BL-GARCH model in a general setting under some non-normal distributions. We also propose and implement a maximum likelihood estimation (MLE) methodology for parameter estimation. To evaluate the small-sample performance of this method for various models, a Monte Carlo study is conducted. Finally, the capability of within-sample estimation, using the S&P 500 daily returns, is also studied.

Key words: BL-GARCH process, Leverage effects, Maximum likelihood, Monte Carlo method, Volatility clustering.
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1 Introduction

The stylized facts of time varying variances have been long recognized in the literature. A popular and prominent tool used to describe this phenomenon is the autoregressive conditional heteroskedasticity (ARCH) model. The ARCH model, developed by Engle (1982) and extended to the GARCH model by Bollerslev (1986), formulates the conditional variance of a random variable as a linear function of its past squared realizations. After the seminal work of Engle (1982), a number of applied works were published to illustrate the usefulness of the GARCH models in economic and financial areas, as in Gouriéroux (1997) and recently Giraitis *et al.* (2007). Moreover, it is well established in empirical finance that the volatility of many financial assets is asymmetric. Particularly, it has been observed that stock price changes are negatively correlated with changes in volatility which implies that volatility is higher after negative shocks than after positive shocks of the same magnitude: see Black (1976). In the light of this empirical finding, various models with asymmetry in volatility have been proposed. Among others, they include the exponential GARCH (EGARCH) of Nelson (1991), the APARCH model, proposed by Ding *et al.* (1993), and the BL-GARCH model recently introduced by Storti and Vitale (2003a). The BL-GARCH in Storti and Vitale (2003a) parallel the Gaussian bilinear (BL) model of means dealt with Granger and Andersen (1978) which is used in such fields as signal, environmental studies, demography, economics and finance: see Tong (1990) for a classical review.

The present work focuses on the bilinear model in volatility (BL-GARCH), which has been investigated recently by Storti and Vitale (2003a) for its statistical properties, in that they have derived conditions for the positivity of the conditional variance and for the second order stationarity of the general model. In particluar, for the BL-GARCH$(1,1)$, they have provided analytical expressions for the autocorrelation function of the squared process, the unconditional fourth moment and the kurtosis coefficient, together with conditions for the existence of the higher order moments of the process. Furthermore, in order to compare the ability of the model to estimate the volatility of financial time series data with other asymmetric GARCH models, they have applied the model to the continuously compounded returns S&P 500 index. However, parameter estimation of the BL-GARCH model remains problematic. Indeed, to deal with the parameter estimation for the BL-GARCH model, Storti and Vitale (2003b) used an indirect maximum likelihood procedure based on the EM algorithm. Despite the fact that the EM algorithm has become a very popular computational method in statistics, this approach presents some limitations: slow numerical convergence, the convergence to a maximum likelihood estimator depending on a judicious choice of the starting value, and the underlying assumption of normally distributed data. These would seem to limit its applicability and also there is the issue of the non-existence of a measure of the standard errors of the estimates. In addition, while, BL-GARCH models adequately capture volatility clustering and asymmetry, coupled with the auxiliary assumption of normally distributed errors, the model does not fully capture the very fat-tailed property of high frequency financial time series, often observed in exchange rates, stock returns and commodity returns, as in Baillie and Bollerslev (1991), McAleer (2005), Zivot and Wang (2005) and references therein. Actually, several alternative distributions have been proposed in the literature to model such excess kurtosis in the conditional distribution of returns better including the Student-t of Bollerslev (1987) and the generalized error distribution (GED) of Nelson (1991). Hansen (1994) introduced the use of an asymmetric Student-t type distribution to capture the skewness property well. Since then, other articles have studied different skew Student-t type distributions for financial and other applications: Azzalini and Capitanio (2003), Jones and Faddy (2003) and Patton (2004). All of these distributions try to account for substantial departure from the Normal distribution, specifically large kurtosis evident in the empirical distribution of the returns, more peakedness and with fatter tails than the Normal distribution. However, while large kurtosis of the returns is a well-established fact, the situation is much more obscure with regard to the symmetry of the distribution. Indeed, testing the symmetry of unconditional distributions of eight international stock market returns, Peiró (1999) concluded that, under the alternative of non-normal distributions, the symmetry of the returns cannot be rejected for most markets. Furthermore, investigating distribution-free methods, he finds that in most markets, daily financial returns are symmetric or, at least, do not present strong evidence of skewness.

The main purpose of the present study is first, following the empirical evidence of leptokurtosis in financial returns, to present an extension of the BL-GARCH with Normal distribution to other alternative non-Gaussian distributions. We therefore consider the elliptical distribution family, as it is rich enough to include the Normal, Student-t, GED and many others, Bingham and Kiesel (2002). Secondly, given that the maximum likelihood method is widely used when dealing with parameter estimation for GARCH models, Li *et al.* (2002) or McNeil *et al.* (2005), we also investigate the problem of estimating parameters of the BL-GARCH model using MLE and examine its finite sample properties in a series of Monte Carlo simulation experiments. This is completely new, in that Storti and Vitale (2003b) only give an empirical approach for the EM method.

The paper is organized as follows. In Section 2, the BL-GARCH model is presented with some important properties concerning conditions for the conditional variance to be finite, as well as for strictly stationarity and ergodicity solutions. In Section 3, we present the MLE method under elliptical distributions, such as the Normal, Student-t and GED. We also provide the score functions as well as the Hessian matrices for these models. Section 4 studies the performance of the estimation procedure through Monte Carlo simulations. Section 5 presents the data and contains the main empirical findings along with the goodness-of-fit tests, while Section 6 provides concluding remarks.

2 BL-GARCH model and its specification

Over the past two decades, enormous efforts have been devoted to modelling and forecasting the volatility of stock returns and other financial time series. Seminal work in this area of research can be attributed to Engle (1982), who introduced the standard autoregressive conditional heteroscedasticity (ARCH) model. Since then, further extensions have been investigated including the GARCH model proposed by Bollerslev (1986), and asymmetric GARCH models such as the EGARCH model of Nelson (1991), the APARCH model of Ding *et al.* (1993), and recently the BL-GARCH model, with which we deal in this paper, studied by Storti and Vitale (2003a).

2.1 The BL-GARCH model

Let S_t denote the price of an asset at time t, $y_t = log\left(S_t/S_{t-1}\right)$ the continuously compounded return series and $\mu_t = E\left(y_t \mid \Psi_{t-1}\right)$ the conditional mean given an increasing sequence of σ-fields Ψ_{t-1} generated by $\left(y_{t-1}, y_{t-2}, \cdots\right)$. Assume that the series of interest, y_t, is given by, for $t = 1, \cdots, n$,

$$y_t = \mu_t + u_t, \tag{1a}$$

$$u_t = h_t \varepsilon_t, \tag{1b}$$

$$h_t^2 = a_0 + \sum_{i=1}^{p} a_i u_{t-i}^2 + \sum_{j=1}^{q} b_j h_{t-j}^2 + \sum_{k=1}^{r} c_k h_{t-k} u_{t-k}, \tag{1c}$$

where p, q, r are non-negative integers with $r = \min\left(p, q\right)$, h_t^2 the conditional variance of the process $\left(u_t\right)_t$ given the σ-fields Ψ_{t-1} and $\varepsilon_t \sim i.i.d.\, D\left(0, 1\right)$ with $D\left(.\right)$ a probability density function with mean 0 and unit variance (it will be defined in Section 3). The model (1b)-(1c) is more general than the standard GARCH model of Bollerslev (1986) in the sense that it allows innovations of different signs to have a different impact on volatility and allows larger shocks to have a larger impact on volatility than the standard GARCH model [2] .

[2] Changes in stock prices tend to be negatively correlated related to changes in volatility, Black (1976). Specifically, lagged unexpected declines in prices (bad news) increase current volatility more than to lagged unexpected increases in prices (good news). This asymmetry in stock returns volatility has been termed the "leverage-effect".

2.2.1 Positivity conditions

Sufficient conditions for the positivity of the conditional variance h_t^2 have been provided in the paper by Storti and Vitale (2003a). To this end, let

$$R = \begin{bmatrix} a_0 & 0 & \cdots & \cdots & 0 \\ 0 & R_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & R_r \end{bmatrix} \text{ and } S = \begin{bmatrix} s_1 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & s_{m-1} & 0 \\ 0 & \cdots & \cdots & 0 & s_m \end{bmatrix}, \qquad (2)$$

where, for $i = 1, \cdots, r$, $R_i = \begin{bmatrix} a_i & \frac{1}{2}c_i \\ \frac{1}{2}c_i & b_i \end{bmatrix}$ and, for $i = 1, \cdots, m$, with $m = \max(p,q) - r$, we set $s_i = a_{r+i}$, if $p > q$, or $s_i = b_{r+i}$, if $p < q$. Hence, if $a_0 > 0$, a sufficient condition for the positivity of h_t^2 is given by $s_i > 0$ ($i = 1, \cdots, m$) and all matrices R_i ($i = 1, \cdots, r$) being positive semi-definite. Furthermore, for the BL-GARCH$(1,1)$ model, the positive definiteness of the matrix R_1 is a necessary and sufficient condition for the positivity of h_t^2.

2.2.2 Covariance stationarity conditions

Storti and Vitale (2003a) show also that the stationarity condition of the BL-GARCH model is analogous to that stated for GARCH models by Bollerslev (1986). Hence, the shocks $h_t \varepsilon_t$ are covariance stationary and non-degenerate if

(i) $\text{var}(\varepsilon_t) < \infty$;
(ii) $a_0 > 0$; and
(iii) $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$.

Under the assumption that the innovations are elliptically distributed with zero mean and unit variance, the first condition is obviously satisfied by construction. The second and third conditions can either be enforced during estimation or verified afterwards.

3 Maximum likelihood approach

Estimation of conditional volatility models are typically performed by an MLE procedure, as in Bollerslev and Wooldridge (1992). The conditional likelihood function is given by

$$\mathscr{L}(\omega) = \mathscr{L}(y_1, \cdots, y_n \mid \omega) = \prod_{t=1}^{n} g(y_t, \mu_t(\alpha), h_t(\omega)), \tag{3}$$

where $g(y_t, \mu_t(\alpha), h_t(\omega))$ denotes the conditional density function for the random variables y_t with mean μ_t and standard deviation h_t, and $\omega = (\alpha, \theta)$ is the parameter vector to be estimated, and where α corresponds to the set of parameters in the condtional mean assumed, in what follows, to be an ARMA(k,l) model and $\theta = \left(a_0, a_1, \cdots, a_p, b_1, \cdots, b_q, c_1, \cdots, c_r\right)$. Following Storti and Vitale (2003a), we make the following theoretical assumption about the parameter space, Ω, and the true parameter vector, $\omega_0 = (\alpha_0, \theta_0)^{'}$:
Assumption 1: the parameter $\omega_0 \in \Omega \subseteq \mathbb{R}^{k+l+p+q+r+1}$ is in the interior of Ω, a compact parameter space. Specifically for any vector $\omega \in \Omega$, assume that

(1) the AR and MA polynomials have no common roots and that all their roots lie outside the unit circle;

(2) $a_0 > 0$, $a_1, \cdots, a_p \geq 0$, and $b_1, \cdots, b_q \geq 0$;

(3) $c_i^2 < 4a_i b_i$, for $i = 1, \cdots, r$; and

(4) $\sum_{i=1}^{p} a_i + \sum_{i=1}^{q} b_i < 1$.

Thus, estimation proceeds by maximising, under the Assumption 1, $L(\omega) = \log(\mathscr{L}(\omega))$, where pre-sample values of h_t^2 are set to the unconditional sample variance.

Since it may be expected that excess kurtosis and skewness displayed by the residuals of conditional heteroscedasticity models will be reduced when a more appropriate distribution is used, we consider in this study the three most typical elliptical normalized distributions that have been applied so far: the Normal, Student-t and GED distributions.

3.1 Normal distribution

The Normal distribution is the most widely used when estimating GARCH models. If we assume that the innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$ have a conditional Gaussian distribution then the conditional log-likelihood function associated to y_t, $u_t = y_t - \mu_t$, is given by

$$L(\omega) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left[\log\left(h_t^2\right) + \frac{u_t^2}{h_t^2}\right], \tag{4}$$

where n is the number of observations. To obtain an analytical or numerical solution for the MLE, we need the first-order derivative and to solve the equation $\partial L(\omega)/\partial\omega = 0$. Taking the differential of $L(\omega)$ in (4) with respect to the full set of parameter ω yields

$$\frac{\partial L(\omega)}{\partial \omega} = \sum_{t=1}^{n}\frac{u_t}{h_t^2}\frac{\partial \mu_t}{\partial \omega} + \frac{1}{2}\sum_{t=1}^{n}\frac{1}{h_t^2}\left(\frac{u_t^2}{h_t^2} - 1\right)\frac{\partial h_t^2}{\partial \omega}. \tag{5}$$

The Hessian matrix is given by

$$\begin{aligned}\frac{\partial^2 L(\omega)}{\partial\omega\partial\omega'} = &-\sum_{t=1}^{n}\frac{1}{h_t^2}\frac{\partial \mu_t}{\partial \omega}\frac{\partial \mu_t}{\partial \omega'} - \sum_{t=1}^{n}\frac{u_t}{h_t^4}\frac{\partial \mu_t}{\partial \omega}\frac{\partial h_t^2}{\partial \omega'}\\ &-\sum_{t=1}^{n}\frac{u_t}{h_t^4}\frac{\partial \mu_t}{\partial \omega}\frac{\partial h_t^2}{\partial \omega'} + \sum_{t=1}^{n}\frac{1}{h_t^4}\left(\frac{1}{2} - \frac{u_t^2}{h_t^2}\right)\frac{\partial h_t^2}{\partial \omega}\frac{\partial h_t^2}{\partial \omega'}.\end{aligned} \tag{6}$$

3.2 Student-t distribution

Now, if we assume that the innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$ have a conditional Student-t distribution with ν degrees of freedom, then the MLE estimator $\hat{\omega}_n$ maximises the log-likelihood function $L(\omega)$ given by

$$\begin{aligned}L(\omega) = n&\left[\log\Gamma\left(\frac{\nu+1}{2}\right) - \log\Gamma\left(\frac{\nu}{2}\right) - \frac{1}{2}\log\pi(\nu-2)\right]\\ &-\frac{1}{2}\sum_{t=1}^{n}\left\{\log(h_t^2) + (\nu+1)\log\left[1 + \frac{u_t^2}{h_t^2(\nu-2)}\right]\right\},\end{aligned} \tag{7}$$

where $2 < \nu \leq \infty$ and Γ is the Euler gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. When $\nu \to \infty$, we have the Normal distribution, so that the smaller the value of ν the fatter the tails. The score function is given by

$$\begin{aligned}\frac{\partial L(\omega)}{\partial \omega} = &\sum_{t=1}^{n}\left[\frac{\nu+1}{\nu-2}\frac{u_t}{h_t^2}\left(1 + \frac{u_t^2}{h_t^2(\nu-2)}\right)^{-1}\right]\frac{\partial \mu_t}{\partial \omega}\\ &+\frac{1}{2}\sum_{t=1}^{n}\left[\frac{\nu+1}{\nu-2}\frac{u_t^2}{h_t^2}\left(1 + \frac{u_t^2}{h_t^2(\nu-2)}\right)^{-1} - 1\right]\frac{1}{h_t^2}\frac{\partial h_t^2}{\partial \omega},\end{aligned} \tag{8}$$

and the Hessian matrix is given by

$$\begin{aligned}\frac{\partial^2 L(\omega)}{\partial\omega\partial\omega'} &= \frac{\nu+1}{\nu-2}\sum_{t=1}^{n}\left(1+\frac{u_t^2}{(\nu-2)h_t^2}\right)^{-1}\frac{u_t}{h_t^2}\left[\frac{2}{\nu-2}\left(1+\frac{u_t^2}{(\nu-2)h_t^2}\right)^{-1}\frac{u_t}{h_t^2}-1\right]\frac{\partial\mu_t}{\partial\omega}\frac{\partial\mu_t}{\partial\omega'}\\
&+\frac{\nu+1}{\nu-2}\sum_{t=1}^{n}\left(1+\frac{u_t^2}{(\nu-2)h_t^2}\right)^{-1}\frac{u_t}{h_t^4}\left[\frac{1}{\nu-2}\left(1+\frac{u_t^2}{(\nu-2)h_t^2}\right)^{-1}\frac{u_t^2}{h_t^2}-1\right]\frac{\partial\mu_t}{\partial\omega}\frac{\partial h_t^2}{\partial\omega'}\\
&+\frac{\nu+1}{\nu-2}\sum_{t=1}^{n}\left(1+\frac{u_t^2}{(\nu-2)h_t^2}\right)^{-1}\frac{u_t}{h_t^4}\left[\frac{1}{\nu-2}\left(1+\frac{u_t^2}{(\nu-2)h_t^2}\right)^{-1}\frac{u_t^2}{h_t^2}-1\right]\frac{\partial h_t^2}{\partial\omega}\frac{\partial\mu_t}{\partial\omega'}\\
&+\frac{1}{2}\sum_{t=1}^{n}\frac{1}{h_t^4}\left[1+\frac{(\nu+1)u_t^2}{(\nu-2)h_t^2}\left(1+\frac{u_t^2}{(\nu-2)h_t^2}\right)^{-1}\right]\\
&\cdot\left[\frac{u_t^2}{(\nu-2)h_t^2}\left(1+\frac{u_t^2}{(\nu-2)h_t^2}\right)^{-1}-2\right]\frac{\partial h_t^2}{\partial\omega}\frac{\partial h_t^2}{\partial\omega'}.\end{aligned}\qquad(9)$$

3.3 GED distribution

Knowing that skewness and kurtosis are important in financial applications, Nelson (1991) suggested to consider the family of GEDs. The probability density function of a normalized GED random variable is given by

$$f(x)=\frac{\nu 2^{-\left(1+\frac{1}{\nu}\right)}}{\lambda_\nu\Gamma\left(\frac{1}{\nu}\right)}e^{-\frac{1}{2}\left|\frac{x}{\lambda_\nu}\right|^{\nu}},\ -\infty<x<\infty,\qquad(10)$$

with $\lambda_\nu=\sqrt{2^{-2/\nu}\Gamma(1/\nu)/\Gamma(3/\nu)}$ and $0<\nu<\infty$ is the tail-thickness parameter. The GED includes the Gaussian distribution ($\nu=2$) as a special case, along with many other distributions, some more fat-tailed than the Gaussian (e.g., the double exponential distribution corresponding to $\nu=1$) and some more thin-tailed (e.g., the Uniform distribution on the interval $\left[-\sqrt{3},\sqrt{3}\right]$ when $\nu\to\infty$). The GED log-likelihood function is given by

$$\begin{aligned}L(\omega)=n&\left[\log\left(\frac{\nu}{\lambda_\nu}\right)-\left(1+\frac{1}{\nu}\right)\log(2)-\log\Gamma\left(\frac{1}{\nu}\right)\right]\\
&-\frac{1}{2}\sum_{t=1}^{n}\left[\log\left(h_t^2\right)+h_t^{-\nu}\left|\frac{u_t}{\lambda_\nu}\right|^{\nu}\right].\end{aligned}\qquad(11)$$

In this case, the score function and the Hessian matrix, respectively, are given by

$$\frac{\partial L(\omega)}{\partial\omega}=\frac{1}{2}\sum_{t=1}^{n}\frac{\nu}{|\lambda_\nu|}\left(\frac{u_t}{h_t}\right)^{\nu}\frac{1}{u_t}\frac{\partial\mu_t}{\partial\omega}+\frac{1}{2}\sum_{t=1}^{n}\frac{1}{h_t^2}\left[\frac{1}{2}\frac{l}{|\lambda_\nu|^{\nu}}\left(\frac{u_t}{h_t}\right)^{\nu}-1\right]\frac{\partial h_t^2}{\partial\omega},\qquad(12)$$

and

$$\frac{\partial^2 L(\omega)}{\partial\omega\partial\omega'} = -\frac{\nu}{|\lambda_\nu|^\nu}\left(\frac{\nu-3}{2}\right)\sum_{t=1}^{n}\frac{1}{h_t^2}\left(\frac{u_t^2}{h_t^2}\right)^{\frac{\nu}{2}-1}\frac{\partial\mu_t}{\partial\omega}\frac{\partial\mu_t}{\partial\omega'}$$
$$-\frac{1}{2}\frac{\nu}{|\lambda_\nu|^\nu}\sum_{t=1}^{n}\frac{u_t}{h_t^4}\left(\frac{u_t^2}{h_t^2}\right)^{\frac{\nu}{2}-1}\left[1+\left(\frac{\nu}{2}-1\right)\frac{u_t^2}{h_t^2}\right]$$
$$-\frac{1}{2}\frac{\nu}{|\lambda_\nu|^\nu}\sum_{t=1}^{n}\frac{u_t}{h_t^4}\left(\frac{u_t^2}{h_t^2}\right)^{\frac{\nu}{2}-1}\left[1+\left(\frac{\nu}{2}-1\right)\frac{u_t^2}{h_t^2}\right]$$
$$-\frac{1}{2}\sum_{t=1}^{n}\left[\frac{1}{4}\frac{\nu(\nu+2)}{|\lambda_\nu|^\nu}\left(\frac{u_t}{h_t}\right)^\nu-1\right]\frac{1}{h_t^4}\frac{\partial h_t^2}{\partial\omega}\frac{\partial h_t^2}{\partial\omega'}. \tag{13}$$

Equations (5) through (13) require the computation of $\partial h_t^2/\partial\omega$. This is given by

$$\frac{\partial h_t^2}{\partial\omega} = \left(1, u_{t-1}^2, \cdots, u_{t-p}^2, h_{t-1}^2, \cdots, h_{t-q}^2, h_{t-1}u_{t-1}, \cdots, h_{t-r}u_{t-r}\right)$$
$$+2\sum_{i=1}^{p}a_i u_{t-1}\frac{\partial u_{t-i}}{\partial\omega} + \sum_{j=1}^{q}b_j\frac{\partial h_{t-j}^2}{\partial\omega}$$
$$+\sum_{k=1}^{r}c_k\left(h_{t-k}\frac{\partial u_{t-k}}{\partial\omega} + \frac{1}{2}\frac{u_{t-k}}{h_{t-k}}\frac{\partial h_{t-k}^2}{\partial\omega}\right). \tag{14}$$

The use and analysis of the MLE method for the estimation problem is classical. A main attraction is the general feature that maximum likelihood estimators achieve optimal accuracy, in that they are asymptotically consistent (in data length n), and achieve the Cramér-Rao lower bound on estimate variability. Despite these advantages, an important obstacle to employing the method is the difficulty of computing a value $\hat{\omega}_{MLE}$ that satisfies Assumption 1. In the next section, the practical applicability and small sample performance of the MLE procedure for BL-GARCH processes are studied by Monte Carlo simulations.

4 Monte Carlo experiments

To our knowledge, no results exist on the properties of these estimators when we observe a finite segment of (1a)-(1c). Thus, we have designed and executed a Monte Carlo experiment using the different distributions described in the previous section as data generating processes, with the aim of analyzing the sampling properties of the exact MLE estimators of the parameter vector ω for the BL-GARCH model. Through the Monte Carlo experiment, the model considered for $u_t = y_t - \mu_t$ is a

BL-GARCH$(1,1)$ given by

$$u_t = h_t \varepsilon_t, \text{ for } t = 1, \cdots, n, \tag{15a}$$

$$h_t^2 = a_0 + a_1 u_{t-1}^2 + b_1 h_{t-1}^2 + c_1 h_{t-1} u_{t-1}, \tag{15b}$$

with ε_t is a standard Normal, Student-t or GED random variable and $n = 500$, 1000, 2000 and 3000. Two cases are studied in the simulation experiments. In the first case, the conditional mean, μ_t, is taken equal to zero while in the second one we assume that it follows an AR(1) model [3]. The data generating processes are summarized in Table 1 with the first three lines corresponding to the case $\mu_t = 0$ and the last line $\mu_t = \alpha_0 + \alpha_1 y_{t-1}$. The data generating processes, correponding to the case $\mu_t = 0$, are as in Storti and Vitale (2003a). Throughout the simulations, we consider a Student-t with five degrees of freedom, the same as used in Tsay (2002). Thus, the first four moments of the conditional density exist. For the GED distribution, we assume that the tail-thickness parameter is equal to three.

Table 1
Data generating processes (DGPs)

DGP	α_0	α_1	a_0	a_1	b_1	c_1
Model 1	0	0	0.01	0.09	0.9	0.15
Model 2	0	0	0.05	0.05	0.9	0.25
Model 3	0	0	0.2	0.05	0.75	0.35
Model 4	0.01	0.2	0.01	0.09	0.9	0.15

Tables 2-4 list the Monte Carlo mean, mean absolute error (MAE) and root mean square error (RMSE) for the parameter vector ω across $M = 1000$ Monte Carlo simulations. The simulation algorithm generates $n + 500$ observations for each series, saving only the last n. This operation is performed in order to avoid dependence on initial values. The calculations were carried out in Matlab on a Pentium IV CPU 3.00 GHz computer. Inspection of Table 2, corresponding to the Normal case, reveals that, for both sample sizes, the averages obtained from the exact MLE are close to the true parameter values. The corresponding MAE and RMSE are very small indicating that estimators are asymptotically unbiased and consistent, respectively. Tables 3 and 4 present the estimated results from a non-Gaussian BL-GARCH model. We read, from these tables, that the averages of the parameter estimates are close to the true values under each of the underlying non-Normal error terms. The MAE as well as the RMSE are quite small and decrease when the sample sizes increase. Finally, Table 5 summarizes the results from the AR(1)-BL-GARCH$(1,1)$. Results reveal that parameter estimates are satisfactory in that the

[3] Other simulation results are available upon request.

MAE and also RMSE are small. We can also remark that, in general, the estimators of the autoregressive models seem not to be affected by, the presence of the BL-GARCH errors. In addition, the method seems applicable, even if the sample size is less than 100, due to the fact that, in general, the true values are contained with ± 2 standard deviations of MLE's estimates: see Figures 1-4.

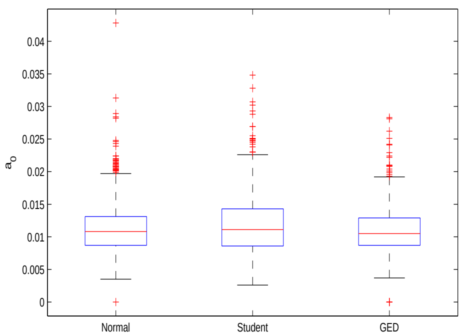

Fig. 1. Boxplot of the parameter a_0 of Model 1 under Normal, Student-t and GED distributions ($n = 1000$)

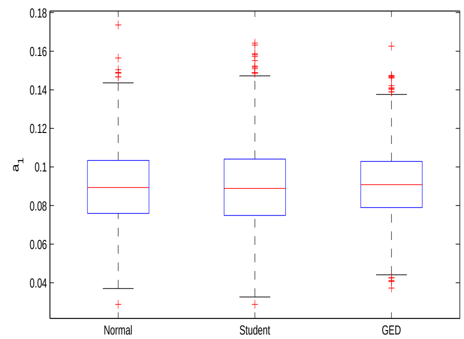

Fig. 2. Boxplot of the parameter a_1 of Model 1 under Normal, Student-t and GED distributions ($n = 1000$)

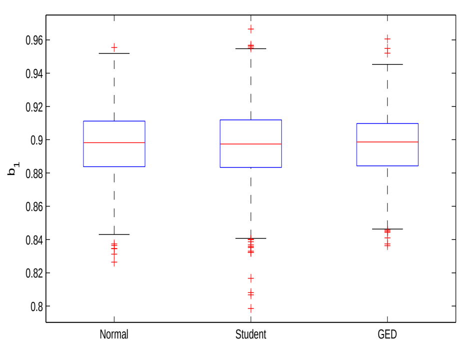

Fig. 3. Boxplot of the parameter b_1 of Model 1 under Normal, Student-t and GED distributions ($n = 1000$)

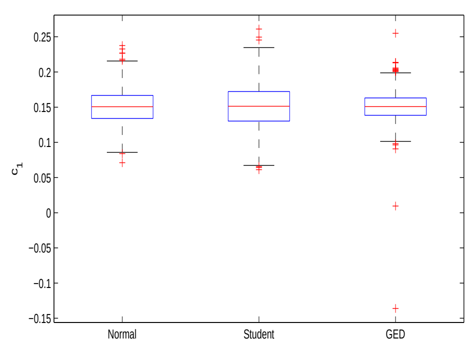

Fig. 4. Boxplot of the parameter c_1 of Model 1 under Normal, Student-t and GED distributions ($n = 1000$)

Table 2. Estimator parameters for the centered Gaussian BL-GARCH model defined by (15a)-(15b)

		$\hat{a}_0$			$\hat{a}_1$			$\hat{b}_1$			$\hat{c}_1$		
Model	n	Mean	MAE	RMSE	Mean	MAE	RMSE	Mean	MAE	RMSE	Mean	MAE	RMSE
1	100	0.09565	0.08864	0.26478	0.09236	0.07241	0.09654	0.76343	0.18036	0.29031	0.15901	0.09549	0.13342
	300	0.01770	0.01032	0.02273	0.08398	0.03705	0.04610	0.89119	0.04473	0.07083	0.15306	0.03807	0.04939
	1000	0.01150	0.00308	0.00422	0.08943	0.01529	0.01953	0.89718	0.01557	0.02021	0.15252	0.01951	0.03658
	3000	0.01047	0.00156	0.00205	0.09022	0.00875	0.01090	0.89883	0.00874	0.01089	0.15129	0.01124	0.01463
2	100	0.09694	0.06658	0.12746	0.06368	0.05110	0.08043	0.82400	0.12195	0.21326	0.27502	0.09176	0.12345
	300	0.05385	0.01643	0.02661	0.04573	0.02613	0.03235	0.89923	0.03529	0.05210	0.25481	0.04424	0.05712
	1000	0.05141	0.00741	0.00972	0.04756	0.01312	0.01637	0.90081	0.01644	0.02073	0.25072	0.02318	0.02904
	3000	0.05010	0.00369	0.00469	0.04905	0.00719	0.00884	0.90071	0.00870	0.01073	0.24942	0.01220	0.01540
3	100	0.22124	0.09400	0.12953	0.07752	0.04822	0.07592	0.70142	0.12867	0.18151	0.36293	0.10454	0.13423
	300	0.20409	0.04860	0.06385	0.05699	0.02454	0.03419	0.73837	0.06370	0.08437	0.35189	0.06176	0.07852
	1000	0.20094	0.02428	0.03056	0.05201	0.01420	0.01823	0.74715	0.03134	0.03988	0.35259	0.03335	0.04184
	3000	0.20013	0.01390	0.01751	0.05023	0.00884	0.01096	0.74947	0.01779	0.02252	0.35015	0.01918	0.02407

This table summarizes the estimates coefficients from the BL-GARCH$(1,1)$ model with the true value set of parameters $\{a_0, a_1, b_1, c_1\} = \{0.01, 0.09, 0.9, 0.15\}$, $\{0.05, 0.05, 0.9, 0.25\}$ and $\{0.2, 0.05, 0.75, 0.35\}$. MAE - Mean Absolute Errors, RMSE - root mean square error for the Gaussian BL-GARCH$(1,1)$. Monte Carlo simulations are computed with 1000 replications. Each replication gives a sample size $n = 100$, 200, 1000 and 3000 of observations.

Table 3. Estimator parameters for the centered Student-t BL-GARCH model defined by (15a)-(15b)

		$\hat{a}_0$			$\hat{a}_1$			$\hat{b}_1$			$\hat{c}_1$		
Model	n	Mean	MAE	RMSE	Mean	MAE	RMSE	Mean	MAE	RMSE	Mean	MAE	RMSE
1	100	0.10610	0.09896	0.27096	0.10400	0.08644	0.012400	0.73671	0.20345	0.31462	0.14972	0.13008	0.18310
	300	0.02153	0.01403	0.04723	0.08986	0.04244	0.05539	0.88264	0.05100	0.09447	0.15937	0.04995	0.06570
	1000	0.0119	0.00347	0.00473	0.09018	0.0189	0.0242	0.8961	0.0185	0.0239	0.1522	0.0256	0.0321
	3000	0.0104	0.00162	0.00214	0.08951	0.0103	0.0129	0.8994	0.0095	0.0119	0.1493	0.0145	0.0184
2	100	0.14528	0.11232	0.18852	0.07323	0.05744	0.09183	0.76433	0.17098	0.27763	0.28221	0.13470	0.18218
	300	0.07061	0.03569	0.08397	0.05241	0.03002	0.04431	0.87763	0.06095	0.12948	0.26215	0.06112	0.07734
	1000	0.05164	0.00949	0.01263	0.04824	0.01267	0.01606	0.90045	0.01684	0.02198	0.25186	0.03032	0.03909
	3000	0.05058	0.00525	0.00624	0.04958	0.00727	0.00906	0.90005	0.00959	0.01209	0.25087	0.01801	0.02241
3	100	0.24646	0.13090	0.25177	0.08733	0.06070	0.10274	0.67189	0.16486	0.23289	0.35433	0.13832	0.17808
	300	0.21268	0.07507	0.09401	0.06466	0.03185	0.04799	0.71045	0.10450	0.13967	0.35592	0.07987	0.10375
	1000	0.2018	0.03793	0.04975	0.05065	0.01584	0.02003	0.74725	0.04575	0.05793	0.34400	0.04590	0.05716
	3000	0.20206	0.02003	0.02479	0.04994	0.00940	0.01191	0.74910	0.02282	0.02897	0.35035	0.02673	0.03202

This table summarizes the estimates coefficients from the BL-GARCH$(1,1)$ model with the true value set of parameters $\{a_0, a_1, b_1, c_1\} = \{0.01, 0.09, 0.9, 0.15\}$, $\{0.05, 0.05, 0.9, 0.25\}$ and $\{0.2, 0.05, 0.75, 0.35\}$. MAE - Mean Absolute Errors, RMSE - root mean square error for the Student-t BL-GARCH$(1,1)$. Monte Carlo simulations are computed with 1000 replications. Each replication gives a sample size $n = 100$, 200, 1000 and 3000 of observations.

Table 4. Estimator parameters for the centered GED BL-GARCH model defined by (15a)-(15b)

Model	n	$\hat{a}_0$ Mean	$\hat{a}_0$ MAE	$\hat{a}_0$ RMSE	$\hat{a}_1$ Mean	$\hat{a}_1$ MAE	$\hat{a}_1$ RMSE	$\hat{b}_1$ Mean	$\hat{b}_1$ MAE	$\hat{b}_1$ RMSE	$\hat{c}_1$ Mean	$\hat{c}_1$ MAE	$\hat{c}_1$ RMSE
1	100	0.07398	0.06648	0.14326	0.10136	0.07136	0.08697	0.77613	0.15628	0.24907	0.16339	0.08021	0.10712
	300	0.01609	0.00806	0.01114	0.08845	0.02841	0.03609	0.89029	0.03428	0.04430	0.15484	0.03144	0.03870
	1000	0.01105	0.00261	0.00351	0.08917	0.01448	0.01832	0.89884	0.01432	0.01823	0.15094	0.01513	0.01919
	3000	0.01025	0.00132	0.00181	0.09067	0.00778	0.01027	0.89866	0.00778	0.01026	0.15070	0.00894	0.01183
2	100	0.08216	0.04731	0.09275	0.06423	0.05041	0.07831	0.84761	0.09293	0.16582	0.27555	0.07683	0.10431
	300	0.05405	0.01372	0.01935	0.04715	0.02526	0.03173	0.89845	0.03165	0.04357	0.25623	0.03677	0.04708
	1000	0.05077	0.00567	0.00738	0.04875	0.01225	0.01547	0.90018	0.01433	0.01819	0.25067	0.01906	0.02358
	3000	0.05037	0.00312	0.00393	0.04956	0.00684	0.00866	0.90008	0.00782	0.00976	0.25066	0.01022	0.01287
3	100	0.21931	0.08345	0.11359	0.07424	0.04509	0.07298	0.70846	0.11736	0.16225	0.36299	0.09512	0.12272
	300	0.21268	0.07507	0.09401	0.06466	0.03185	0.04799	0.71045	0.10450	0.13967	0.35592	0.07987	0.10375
	1000	0.20262	0.02017	0.02578	0.05130	0.01263	0.01631	0.74550	0.02690	0.03466	0.35097	0.02695	0.03392
	3000	0.20020	0.01124	0.01393	0.05082	0.00833	0.01040	0.74919	0.01494	0.01874	0.35114	0.01548	0.01958

This table summarizes the estimates coefficients from the BL-GARCH$(1,1)$ model with the true value set of parameters $\{a_0, a_1, b_1, c_1\} = \{0.01, 0.09, 0.9, 0.15\}$, $\{0.05, 0.05, 0.9, 0.25\}$ and $\{0.2, 0.05, 0.75, 0.35\}$. MAE - Mean Absolute Errors, RMSE - root mean square error for the GED BL-GARCH$(1,1)$. Monte Carlo simulations are computed with 1000 replications. Each replication gives a sample size $n = 500$, 1000, 2000 and 3000 of observations.

Table 5
Estimator parameters for the centered AR-BL-GARCH model defined by (15a)-(15b)

Distribution	n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{a}_0$	$\hat{a}_1$	$\hat{b}_1$	$\hat{c}_1$
Normal	100	0.0257	0.18447	0.10767	0.09233	0.78056	0.16659
		(0.06246)	(0.08631)	(0.08060)	(0.07867)	(0.17161)	(0.09643)
		[0.09254]	[0.08737]	[0.24972]	[0.10633]	[0.28043]	[0.13219]
	300	0.01897	0.19377	0.01874	0.0838	0.88978	0.15629
		(0.03242)	(0.04678)	(0.01141)	(0.03729)	(0.04555)	(0.03919)
		[0.04338]	[0.05949]	[0.02728]	[0.04697]	[0.07751]	[0.05055]
	1000	0.01103	0.19871	0.01142	0.08847	0.89841	0.15103
		(0.01624)	(0.02544)	(0.00322)	(0.01555)	(0.01603)	(0.01932)
		[0.02069]	[0.03167]	[0.00441]	[0.01995]	[0.02064]	[0.02422]
	3000	0.01117	0.19905	0.01042	0.08998	0.89941	0.15086
		(0.00911)	(0.01477)	(0.00156)	(0.0088)	(0.00871)	(0.01129)
		[0.01189]	[0.0187]	[0.00209]	[0.01134]	[0.01117]	[0.01444]
Student-t	100	0.0209	0.18876	0.08852	0.09884	0.7457	0.15474
		(0.05193)	(0.08171)	(0.08201)	(0.08648)	(0.20145)	(0.13304)
		[0.07803]	[0.10338]	[0.23158]	[0.11957]	[0.32115]	[0.17796]
	300	0.01472	0.19351	0.02124	0.08836	0.87681	0.15205
		(0.02583)	(0.04575)	(0.01389)	(0.04023)	(0.05523)	(0.05247)
		[0.03494]	[0.05758]	[0.03351]	[0.05389]	[0.10593]	[0.06717]
	1000	0.01123	0.19838	0.01206	0.09095	0.89505	0.15105
		(0.01378)	(0.02474)	(0.0036)	(0.01713)	(0.01776)	(0.02524)
		[0.01737]	[0.031]	[0.00523]	[0.02215]	[0.0227]	[0.03156]
	3000	0.01053	0.19941	0.01054	0.09075	0.89861	0.15111
		(0.00794)	(0.01389)	(0.0018)	(0.00992)	(0.00957)	(0.0151)
		[0.01013]	[0.01748]	[0.00278]	[0.01286]	[0.01323]	[0.01999]
GED	100	0.01715	0.18247	0.09296	0.09692	0.77744	0.16364
		(0.06385)	(0.08462)	(0.08650)	(0.07818)	(0.17089)	(0.08186)
		[0.09838]	[0.10735]	[0.10735]	[0.10405]	[0.28519]	[0.11175]
	300	0.01680	0.19660	0.01619	0.08756	0.89222	0.15398
		(0.03025)	(0.04396)	(0.00882)	(0.03510)	(0.03925)	(0.0323)
		[0.04067]	[0.05591]	[0.02660]	[0.04534]	[0.06616]	[0.04133]
	1000	0.01253	0.19973	0.01098	0.0914	0.89718	0.15228
		(0.0163)	(0.02493)	(0.0028)	(0.01539)	(0.01545)	(0.01624)
		[0.021]	[0.03152]	[0.004]	[0.02045]	[0.02062]	[0.02077]
	3000	0.01059	0.19945	0.01026	0.09187	0.89782	0.15056
		(0.00949)	(0.01377)	(0.00159)	(0.00928)	(0.0914)	(0.01029)
		[0.01288]	[0.01753]	[0.00232]	[0.01533]	[0.01465]	[0.01379]

This table summarizes the estimates coefficients from the AR(1)-BL-GARCH$(1,1)$ model with the true value set of parameters $\{\alpha_0, \alpha_1, a_0, a_1, b_1, c_1\} = \{0.01, 0.2, 0.01, 0.09, 0.9, 0.15\}$. MAE - Mean Absolute Errors, RMSE - root mean square error for the AR(1)-BL-GARCH$(1,1)$. Monte Carlo simulations are computed with 1000 replications. Each replication gives a sample size $n = 500$, 1000, 2000 and 3000 of observations.

5 Empirical study

The daily continuously compounded returns of the S&P 500 stock market index are used for the empirical study in this paper to gauge the effectiveness of the BL-GARCH-type model with Normal, Student-t and GED innovations. In particular, we analyze the period from March 01, 1999 through January 31, 2001, which yields $n = 487$ daily observations, excluding public holidays. The sample closely corresponds to the data used by Storti and Vitale (2003b). Table 6 gives the summary statistics of the S&P 500 log returns for the full sample. The mean and the standard deviation are quite small, while the estimated measure of skewness is significantly positive, indicating that the S&P 500 has non-symmetric returns. The kurtosis is a little higher than that of a Normal distribution which is 3, suggesting that fat-tailed distributions could better describe the unconditional distribution of the data. The results of the non-Normality test agree with prior literature using financial data, that is, a leptokurtic distribution is found for the S&P 500 log return data. The Box-Pierce Q-tests of up to twenty-fourth order serial correlation for the levels and squares of the mean-corrected S&P 500 log returns were performed. $Q(24)$ and $Q^2(24)$ are significant for both the return and squared return series. The diagnostics suggest that a GARCH-class model would be appropriate, along with an error distribution that allows for greater kurtosis than the Normal distribution.

Table 6
Statistics of daily log returns of the S&P 500 stock market index.

Number of observations	487	Skewness	0.03708
Mean	0.0002055	Kurtosis	4.4643
Standard deviation	0.01283	Jarque Bera test	43.3524
Minimum	-0.06004	$Q(24)$	39.2169
Maximum	0.04888	$Q^2(24)$	45.7208

The Jarque Bera test critical value at significance level of 5% is 5.85423. Q and Q^2 are the Box Pierce statistics for the levels and squared of the S&P 500 log returns respectively, using 24 lags. The critical value at level of 5% is 36.4150.

Figure 5 gives, the time plot of the data while Figure 6 shows that the returns distribution also exhibits fat-tails confirming the results in Table 6.

Table 7 presents the results of maximum likelihood estimation of the parameters of the model under candidate the various assumptions on innovations, with the standard errors computed using the Hessian matrix [4]. The parameter estimates from

[4] The time for convergence of the algorithm under specific distribution is approximately 5 seconds.

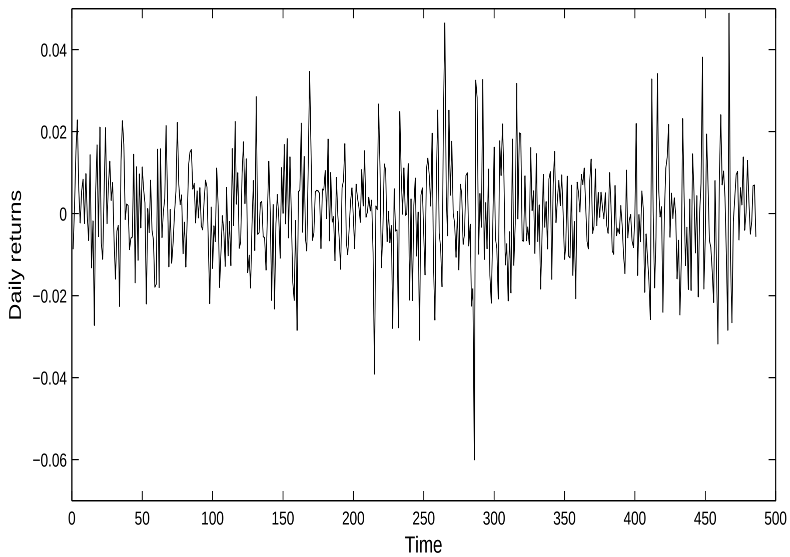

Fig. 5. S&P 500 daily returns 03/01/99-01/31/01

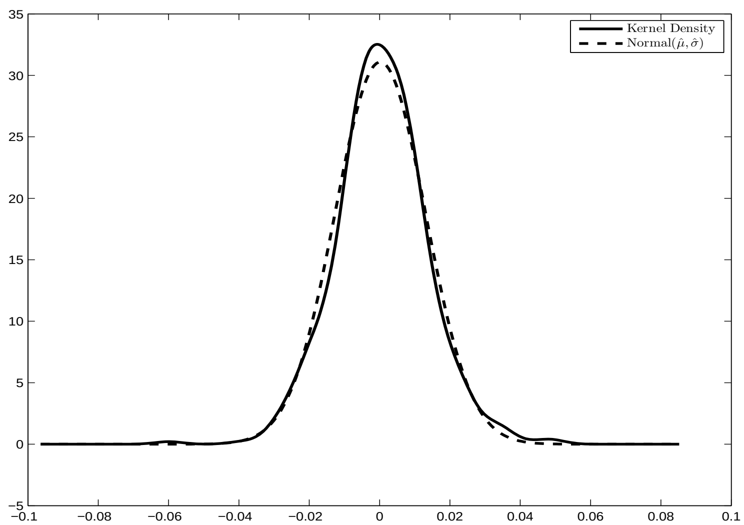

Fig. 6. Non-parametric density of S&P 500 daily returns and probability density function of the normal distribution, in dotted line

the GARCH model with Normal, Student-t and GED errors are also provided in this table. In order to compare objectively, various goodness-of-fit statistics are used. The diagnostics, summarized in Table 7, are the log-likelihood function at its maximum, the Akaike Information Criterion (AIC) and the Bayesian Information Crietrion (BIC). We further report the values of the Box-Pierce (Q) statistics for the standardized and squared standardized residuals with in parentheses, the corresponding p-values as a check of the empirical validity of the models. It is clear from Table 7 that estimate $\hat{a}_1$ and $\hat{b}_1$ in the GARCH$(1,1)$ are significant at the 5% level with the volatility coefficient greater in magnitude. Hence the hypothesis of constant variance can be rejected, at least within sample. Furthermore, the stationarity condition is satisfied for the three distributions, as $\hat{a}_1 + \hat{b}_1 < 1$ at the maximum of the respective log-likelihood functions.

From results obtained on estimation of the BL-GARCH$(1,1)$ model, we remark, once again, that the innovation and volatility spillovers are significantly different from zero. Further, the estimated asymmetric volatility response ($\hat{c}_1$) is negative and significant for all models confirming the usual expectation in stock markets where downward movements (falling returns) are followed by higher volatility than upward movements (increasing returns). In all cases, the tails parameter estimate is strongly significant far two for the GED distribution while it is very large through insignificant for the Student-t distribution but $1/\nu$ is also different to zero. However, the results confirm the empirical findings by Storti and Vitale (2003a), in that the kurtosis strongly depends to the leverage-effect response parameter.

The results for Q-statistics shown in Table 7 are not significant up to order 12 and also order 24, which indicates that the BL-GARCH$(1,1)$ as well as the GARCH$(1,1)$ process are appropriate to model the conditional variance of the S&P 500 log-returns. However, the goodness-of-fit statistics as well as the residuals diagnostics indicate that the BL-GARCH performs better in describing the conditional variance of the S&P 500 returns. Moreover, the possible usefulness of using fat-tailed innovations for the BL-GARCH model seems to be confirmed by the log-likelihood values and the AIC.

6 Conclusion

This study obtains exact maximum likelihood estimates of a BL-GARCH process with conditionally elliptical distributions. The small-sample properties indicate that the approach can yield asymptotically efficient estimates. In addition, these results strongly suggest that the maximum likelihood estimation inference procedure can be used to estimate the parameters of the BL-GARCH model, even in samples as small as 100 observations. Further, as the simulation experiments show, one advantage of the maximum likelihood estimator procedure, proposed in this paper, compared to the method used by Storti and Vitale (2003b), is that it could simultaneously estimate the parameters of the BL-GARCH when the conditional mean is assumed non-constant. Further, the empirical results reveal that the BL-GARCH-t$(1,1)$, i.e a BL-GARCH model with conditional errors that are t-distributed, fits the data best.

Table 7
Conditional variance model estimates for the S&P 500 return

	GARCH model			BL-GARCH model		
Parameters	Normal	Student-t	GED	Normal	Student-t	GED
$\hat{a}_0$	0.00000715	0.000006361	0.000006108	0.000011394	0.000009243	0.00001057
	(0.28104 10^{-10})	(0.20586 10^{-10})	(0.20144 10^{-10})	(0.14775 10^{-10})	(0.13496 10^{-10})	(0.15424 10^{-10})
$\hat{a}_1$	0.0568184	0.0504452	0.0499355	0.060119	0.0513906	0.0559965
	(0.0007625)	(0.0006168)	(0.0006111)	(0.0006226)	(0.0005249)	(0.0006235)
$\hat{b}_1$	0.900428	0.911250	0.913097	0.880531	0.900687	0.888784
	(0.0026979)	(0.0017480)	(0.0018101)	(0.0013771)	(0.0011259)	(0.0013873)
$\hat{c}_1$	-	-	-	-0.271323	-0.249673	-0.261943
	(-)	(-)	(-)	(0.0027800)	(0.0030085)	(0.0031051)
$\hat{v}$	-	8.676471	1.514841	-	14.943269	1.741412
	(-)	(9.929491)	(0.017712)	(-)	(67.446372)	(0.024844)
Goodness-of-fit statistics						
Log-lik	1435.91706	1441.94452	1441.04650	1456.47965	1458.63396	1457.65676
AIC	-2865.83441	-2875.88904	-2874.09300	-2904.95930	-2907.26792	-2905.31353
Diagnostics						
$Q(12)$	15.892571	16.018537	16.010225	15.876516	15.739415	15.833319
	(0.196206)	(0.190388)	(0.190768)	(0.196958)	(0.203465)	(0.198990)
$Q(24)$	32.082112	32.206641	32.159427	32.314595	32.501633	32.417392
	(0.124969)	(0.121951)	(0.123088)	(0.119381)	(0.115033)	(0.116975)
$Q^2(12)$	4.682633	4.589348	4.566436	4.477202	4.455119	4.485078
	(0.967748)	(0.970309)	(0.970918)	(0.973208)	(0.973756)	(0.973011)
$Q^2(24)$	19.326186	19.483667	19.377018	22.177684	22.542585	22.475412
	(0.734378)	(0.725728)	(0.731595)	(0.568662)	(0.546916)	(0.550914)

This table provides the estimated coefficients, standard errors for the conditional standard deviation equation for the S&P 500 log returns index market. $\hat{a}_0$ is the constant in the conditional standard deviation equation, $\hat{a}_1$ is the ARCH coefficient, $\hat{b}_1$ is the GARCH coefficient, $\hat{c}_1$ is the leverage effect, $\hat{v}$ is the degrees of freedom. Log-lik is the maximized log likelihood. AIC is the Akaike Information Citerion and BIC the Bayesian Information Criterion. Q and Q^2 are the Box Pierce statistics for the standardized and squared standardized residuals respectively, using 12 and 24 lags with p-values in square brackets. The critical values at significant level of 5% are 21.026069 and 36.415028 respectively.

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