The perfect foresights' assumption revisited : (I) the existence of equilibrium with multiple price expectations

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The perfect foresights' assumption revisited: (I) the existence of equilibrium with multiple price expectations

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Abstract

Our earlier papers [2,3,4,5,6] had extended to asymmetric information the classical existence theorems of general equilibrium theory [1,7,10], under the standard assumption that agents had perfect foresights, that is, they knew, ex ante, which price would prevail on each spot market.

Common observation suggests, however, that agents more often trade with an unprecise knowledge of future prices. We now let agents anticipate, in each random state, a set of plausible prices, called expectations, endowed with a probability distribution. These expectations are assumed to define a so-called 'structure of beliefs', along which agents' expectations sets intersect on each spot market. We introduce a related concept of 'correct foresights equilibrium' (CFE), in which equilibrium prices belong to all agents expectations sets. We prove that the existence of a CFE is still characterized by the no-arbitrage condition of [2]. This result, which extends our earlier theorems [4,5,6], shows that private information or price uncertainty would not affect the existence but only the value of equilibrium prices and allocations.

Key words: general equilibrium, incomplete markets, asymmetric information, arbitrage, existence of equilibrium.

JEL Classification: D52

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1 Introduction

In earlier papers [2,3], we had introduced a notion sequential equilibrium, where agents could have asymmetric information, and learn from prices, without having rational expectations along Radner [11], that is, a model of how equilibrium prices are determined. We showed in [4,5,6] that the classical existence results of symmetric information [1,7,10] extended to that broader notion of equilibrium, namely, that existence was guaranteed by a no-arbitrage condition.

These earlier papers retained the standard assumption that agents had a perfect foresight of each future spot price. This assumption is intrinsic to the definition of sequential equilibrium in the current literature, but curiously at odds with the idea that economic agents might have no price model or precise anticipations. To be consistent with that idea, and with our dropping rational expectations in a first step, we now enlarge our notion of equilibrium by also dropping perfect foresights.

We introduce the simplest setting to deal with that problem, namely, a two-period pure exchange financial economy with nominal assets and spot markets for consumption goods, where each agent has a finite set of price anticipations in each state, and a joint probability distribution over expected prices (the case of infinite price expectations and continuous probability distributions is dealt with in a companion paper). These expectations are assumed to define a so-called '*structure of beliefs*', along which agents' price expectations sets intersect on each spot market.

We now replace the condition of perfect foresights at equilibrium by a milder one of '*correct foresights*', along which the '*true*' spot prices (each one of which may prevail tomorrow, conditionally on nature's play) are, ex ante, in all agents' expectations sets. Thus, a '*correct foresights equilibrium*' (C.F.E.) is a collection of prices, of correct expectations in the above sense, and of strategies which clear on all markets and are optimal within the budget set for every agent at both periods. This notion embeds and extends the equilibrium concept of [2].

Extending the result of [4], we show as our main Theorem that the existence of a C.F.E. is still characterized by the no-arbitrage condition of [2]. Thus, agent's private information or idiosyncratic anticipations would not affect the existence, but only the value, of equilibrium prices and allocations. The companion paper with infinitely many price expectations explains how private information, beliefs and uncertainty may account for phenomena such as speculation, bubbles or crashes on financial markets, within a general equilibrium framework.

Section 2 presents the basic model, where agents may have asymmetric information and finitely many price expectations in each future state. It presents the notions of structure of payoffs, information and beliefs and of correct foresights equilibrium, and states the existence Theorem. Section 3 proves the Theorem.

2 The basic model

We consider a pure-exchange financial economy with two periods ($t \in \{0,1\}$). The economy is finite, in the sense that the sets of agents, $I := \{1, ..., m\}$, of commodities, $\{1, ..., L\}$, of states of nature, S, and assets, $\{1, ..., J\}$, are all finite.

There is an a priori uncertainty at the first period ($t = 0$) about which state $s \in S$ will prevail at the second period ($t = 1$). Throughout, we shall denote by $s = 0$ the non-random state at $t = 0$ and let $\Sigma' := \{0\} \cup \Sigma$, for each subset Σ of S.

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2.1 Information, markets and expectations

2.1.1 Information

At $t = 0$, each agent $i \in I$ receives or infers a private information signal, or set, $S_i \subset S$, along which the true state will be in S_i. Henceforth, we let $(S_i) = (S_i)_{i=1}^m$ be given, and $\underline{\mathbf{S}} := \cap_{i=1}^m S_i$ be agents' pooled information set, which is assumed to contain the true state. Thus, along all information available at $t = 0$, every state $s \in \underline{\mathbf{S}}$ (and only those states) may prevail tomorrow.[2]

2.1.2 Markets

There are two markets: a commodity market and a financial market.

The commodity market consists in $\#\underline{\mathbf{S}}'$ spot markets for commodities, which agents consume, or exchange, at $t = 0$, on the spot market of the non-random state $s = 0$, and, at $t = 1$, on the spot market of the particular state $s \in \underline{\mathbf{S}}$, which will

[2] Throughout the paper, we denote by $\cdot$ and $\|.\|$, respectively, the scalar product and Euclidean norm. For every $\widetilde{\Sigma} \subset S' := \{0\} \cup S$ and every $\Sigma \subset \widetilde{\Sigma}$, for every $S \times J$-matrix $V := (v_j[s])_{(s,j) \in S \times J}$ and every $\Sigma \times J$-matrix A, for all set $X \subset \mathbb{R}^{L\widetilde{\Sigma}}$ and all set $X' \subset \mathbb{R}^{\widetilde{\Sigma}}$, for all collections $(q, s, l) \in \mathbb{R}^J \times \Sigma \times \{1, ..., L\}$ and $(x, x', (y, y'), (z, z')) \in \mathbb{R}^{L\widetilde{\Sigma}} \times \mathbb{R}^{\widetilde{\Sigma}} \times (\mathbb{R}^{\Sigma})^2 \times (\mathbb{R}^{L\Sigma})^2$, we denote by:
1) $x[\Sigma]$ and $x'[\Sigma]$, respectively, the truncations of x on $\mathbb{R}^{L\Sigma}$ and x' on $\mathbb{R}^{\Sigma}$;
2) $X[\Sigma] := \{x \in \mathbb{R}^{L\Sigma} : \exists \widetilde{x} \in X, x = \widetilde{x}[\Sigma]\}$;
3) $X'[\Sigma] := \{x' \in \mathbb{R}^{\Sigma} : \exists \widetilde{x} \in X', x = \widetilde{x}[\Sigma]\}$;
4) $A[s]$, $y[s]$, $z[s]$, resp., the row, scalar, vector, indexed by $s \in \Sigma$, of A, y, z;
5) $z^l[s]$ the l^{th} component of $z[s] \in \mathbb{R}^L$;
6) $y \leqslant y'$ and $z \leqslant z'$ (resp. $y << y'$ & $z << z'$) the relationships $y[s] \leqslant y'[s]$ and $z^l[s] \leqslant z'^l[s]$ (resp. $y[s] < y'[s]$ and $z^l[s] < z'^l[s]$) for every $(l, s) \in \{1, ..., L\} \times \Sigma$;
7) $y < y'$ (resp. $z < z'$) the relationships $y \leqslant y'$, $y \neq y'$ (resp. $z \leqslant z'$, $z \neq z'$);
8) $z_\Box z'$ the vector $(z[s] \cdot z'[s]) \in \mathbb{R}^{\Sigma}$, $y_\Box z$ the vector $(y[s]z[s]) \in \mathbb{R}^{L\Sigma}$;
9) $V(\Sigma)$ (when $0 \notin \Sigma$) the $\Sigma \times J-$matrix s.t. $V(\Sigma)[s] := V[s]$, for each $s \in \Sigma$;
10) $W(\Sigma, q)$ (when $0 \notin \Sigma$) the $\Sigma' \times J-$matrix such that $W(\Sigma, q)[0] := -q$ and, for each $s \in \Sigma$, $W(\Sigma, q)[s] := V[s]$ and we let $W(q) := W(S, q)$;
11) $\mathbb{R}_+^{L\Sigma} := \{x \in \mathbb{R}^{L\Sigma} : x \geqslant 0\}$, $\mathbb{R}_+^{\Sigma} := \{x \in \mathbb{R}^{\Sigma} : x \geqslant 0\}$,
$\mathbb{R}_{++}^{L\Sigma} := \{x \in \mathbb{R}^{L\Sigma} : x >> 0\}$, $\mathbb{R}_{++}^{\Sigma} := \{x \in \mathbb{R}^{\Sigma} : x >> 0\}$.

prevail. There is also a fictitious spot market in each state $s \in S/\underline{\mathbf{S}}$, which may only open in the eyes of the uninformed agents $i \in I$, such that $s \in S_i$. A commodity price $p \in \mathbb{R}_+^{L\underline{\mathbf{S}}'}$, embeds the spot price $p[0] \in \mathbb{R}_+^L$, at $t=0$, and the $\#\underline{\mathbf{S}}$ spot prices $p[s] \in \mathbb{R}_+^L$ (for $s \in \underline{\mathbf{S}}$), at which commodities would be traded if state $s \in \underline{\mathbf{S}}$ prevailed at $t=1$.

As is standard, agents' consumptions are bounded below, namely, they are non-negative bundles of the L goods in every state. Given her information set S_i, each agent $i \in I$, has, ex ante, an endowment, $e_i \in X_i := \mathbb{R}_+^{LS_i'}$, which provides her the bundle of commodities $e_i[0]$ at $t=0$ and $e_i[s] \in \mathbb{R}^L$ in each state $s \in S_i$ if it prevails.

The financial market permits limited transfers across periods and states, via J nominal assets $j \in \{1,...,J\}$, whose contingent payoffs, in each state $s \in S$, are denoted by $v_j[s]$ and yield a $S \times J-$matrix $V = (v_j[s])$, henceforth fixed, such that $J = rankV$. With no loss of generality, agents are not endowed with assets, but may exchange portfolios unrestrictively. A portofolio $z := (z^j) \in \mathbb{R}^J$ specifies the quantity z^j of each asset $j \in \{1,...,J\}$, positive, if purchased, or negative, if sold. Given an asset price $q \in \mathbb{R}^J$, an agent $i \in I$ may thus purchase a portofolio $z := (z^j) \in \mathbb{R}^J$ for $q \cdot z$ units of account at $t=0$, against the expected flow $V(S_i)z$ of payoffs at $t=1$.

We refer to the pair $[V,(S_i)]$ as the payoff and information structure. One particular class of structures $[V,(S_i)]$ is of interest: those for which there exists a price $q \in \mathbb{R}^J$ and a collection of weights $(\lambda_i) \in \Pi_{i=1}^m \mathbb{R}_{++}^{S_i}$, such that $q = {}^t\lambda_i V(S_i)$, for each $i \in I$. Then, by a standard separation argument, the financial market grants no agent an arbitrage at price q, that is, the possibility of a positive money transfer, in one state, at no cost in any other. When that condition holds, the structure $[V,(S_i)]$ is said to be (q-)arbitrage-free (see Cornet-De Boisdeffre, 2002 [2]).

Market prices may be bounded to one in each state with no loss of generality.

Henceforth, we restrict admissible prices to the following sets:

$\Lambda := \{p \in \mathbb{R}_+^{L\underline{\mathbf{S}}'} : \|p[s]\| \leqslant 1, \forall s \in \underline{\mathbf{S}}'\}$ and

$\mathcal{M} := \{(p,q) \in \Lambda \times \mathbb{R}^J : \|q\| \leqslant 1\}$.

Given $(p,q) \in \mathcal{M}$, we henceforth denote by, respectively, $p_0 := (p[0],q) \in \mathbb{R}_+^L \times \mathbb{R}^J$ and $p_1 := p[\underline{\mathbf{S}}] \in \mathbb{R}_+^{L\underline{\mathbf{S}}}$, the related market prices at $t=0$ and $t=1$.

2.1.3 Expectations

At $t=0$, only a fictitious observer would normally know, from the characteristics of all agents and of the economy, what the true price $p_1 \in \mathbb{R}_+^{L\underline{\mathbf{S}}}$ is at $t=1$. Yet, this price must exist in any state of equilibrium.

In this model, agents need not know, or anticipate with certainty, the true price p_1 at $t=1$. Hence, they will not necessarily have a unique price anticipation in each given state, as was the case so far in general equilibrium models, whether Grandmont's of temporary equilibrium, Radner's of rational expectations equilibrium with asymmetric information, or the classical models of perfect foresights.

Instead, we assume that agents have (at $t=0$) a set of plausible prices for the second period ($t=1$), called '*price expectations*', and an idiosyncratic probability distribution defining their uncertainty about future prices. To simplify presentation, we limit ourselves, in this introductory paper, to the case of nominal asset markets and discrete probability distributions over expectations. We also assume that, for each pair $(i,s) \in I \times S_i$, the number of price expectations in state s by the i^{th} agent is fixed (when the market price varies). In this setting, the standard fixed-point-like arguments of Euclidean spaces apply, which will simplify our proof in Section 3. In a companion paper, which builds on the results of this finite model, we extend our existence theorem to the case of infinite price sets and probability measures.

At any sequential equilibrium, all agents' expectations sets should include the actual spot prices, any of which may randomly prevail tomorrow (otherwise they might have to revise their anticipations and decisions ex post). Thus, given a market price $(p, q) \in \mathcal{M}$, agents, who observe $p_0 := (p[0], q) \in \mathbb{R}^{L+J}$ only, will not consider the possibility of another market price than p_0 at $t = 0$. However, they may be led to agree on p_1 (e.g., by a shared view, or by chance), that is, to agree that $p[s] \in \mathbb{R}^L_+$ (for each $s \in \underline{\mathbf{S}}$) is one possibility for the spot price in state s tomorrow.

When this agreement takes place, and when $p_1 \in \mathbb{R}^{L\underline{\mathbf{S}}}_+$ turns out to be the actual price at $t = 1$ (i.e., if, for each $s \in \underline{\mathbf{S}}$, $p[s]$ will be observed on the spot market if state $s \in \underline{\mathbf{S}}$ prevails), agents are said to have '*correct price foresights*'. Indeed, correct foresights coincide with perfect foresights whenever agents have exactly one price expectation in each state of their information set, as was the case in [4].

This yields the following Definition of a '*structure of beliefs*'.

Definition 1 *For each pair* $(i, s) \in I{\times}S'_i$*, let* $\pi_{(i,s)} : p \in \Lambda \mapsto \pi_{(i,p,s)} \in \Pi$ *be a given mapping, from the price set* $\Lambda := \{p \in \mathbb{R}^{L\underline{\mathbf{S}}'}_+ : \|p[s]\| \leqslant 1, \forall s \in \underline{\mathbf{S}}'\}$ *to the set, denoted* Π*, of discrete probabilities over* $\mathbb{R}^L_+$*, and let* $P_{(i,p,s)} := \{f \in \mathbb{R}^L_+ : \pi_{(i,p,s)}(f) > 0\}$*, for every* $p \in \Lambda$*. The collection of mappings* $\pi := (\pi_{(i,s)}) := (\pi_{(i,s)})_{(i,s) \in I \times S'_i}$ *is said to be a structure of beliefs if the following Conditions hold:*

(a) $\forall (i, s) \in I{\times}S'_i$, $\exists n_{(i,s)} \in \mathbb{N}^* : \forall p \in \Lambda$, $\# P_{(i,p,s)} = n_{(i,s)}$*, so we let* $P_{(i,p,s)} := \{f^1_{(i,p,s)}, ..., f^{n_{(i,s)}}_{(i,p,s)}\}$*;*

(b) $\forall (i, p) \in I{\times}\Lambda$, $\pi_{(i,p,0)}(p[0]) = 1$*, so we let* $P_{(i,p,0)} := \{f^1_{(i,p,0)}\} := \{p[0]\}$*;*

(c) $\forall (i, p, s) \in I{\times}\Lambda{\times}S_i$, $\pi_{(i,p,s)}(p[s]) > 0$*, so we let* $f^1_{(i,p,s)} := p[s]$*;*

(d) $\forall (i, p, s, k, \varepsilon) \in I{\times}\Lambda{\times}S_i{\times}\{1, ..., n_{(i,s)}\}{\times}\mathbb{R}_{++}$, $\exists \eta \in \mathbb{R}_{++}$:

$(p' \in \Lambda$ *and* $\|p - p'\| < \eta) \Rightarrow (\|f^k_{(i,p,s)} - f^k_{(i,p',s)}\| + |\ \pi_{(i,p,s)}(f^k_{(i,p,s)}) - \pi_{(i,p',s)}(f^k_{(i,p',s)})\ | < \varepsilon)$.

For every $(i, p, s) \in I{\times}\Lambda{\times}S_i$*, we refer to the set* $P_{(i,p,s)}$ *as the* i^{th} *agent's price expectations set (in state* s*) and to* $P_{(i,p)} := \Pi_{s \in S'_i}\ P_{(i,p,s)}$ *as her price set.*

Remark 1 In the absence of public forecasts, information, or exchanged views regarding tomorrow's prices, the generic agent $i \in I$ should have an expectation rule, which maps $P_{(i,p)}$ into $p_0 \in \mathbb{R}^{L+J}$, that is, deduce her price set (only) from the prices (or other indicators) she observes at $t = 0$. But such a rule would be inconsistent (in a pure exchange economy) with the condition that all markets cleared ex post. Mathematically, this is due to the fact that the space of realizable consumptions tomorrow, namely, $\mathbb{R}^{L\underline{\mathbf{S}}}$, is (normally) of higher dimension than $L + J$. To insure market clearance ex post at agents' ex ante decisions and correct price foresights, expectations need not only depend on the price $p_0 \in \mathbb{R}_+^{L+J}$ agents observe, but also on the prices $p_1 \in \mathbb{R}_+^{L\underline{\mathbf{S}}}$, they cannot observe, but upon which they share a belief.

Though $p_1 \in \mathbb{R}_+^{L\underline{\mathbf{S}}}$, which reflects the characteristics of the economy (including agents' beliefs), should only be known by a fictitious observer, agents are assumed to have some glimpse of it, along Condition (c). When the characteristics of the economy change (which yield $(p, q) \in \mathcal{M}$), so do agents' overlapping expectations of spot prices. The mechanisms at stake may involve individual rationality, a shared view of the future, public forecasts, or whatever anticipation process, leaving room for insights and uncertainty to all agents.

In our companion paper, dealing with infinite expectations sets, the above assumption that agents' anticipations depend on unobserved prices is dropped. Agents' price expectations sets (and related probability distributions) may be set as given at the first period, with the only condition that they intersect on each spot market.

Henceforth, we assume that agents are endowed with a given structure of beliefs, $\pi := (\pi_{(i,s)})$, which is fixed and always referred to throughout the paper, jointly with the other notations of Definition 1.

2.1.4 Consumption plans and preferences

For each $i \in I$, the i^{th} agent's set of consumption plans is defined as follows:

$Y_i := \Pi_{s \in S_i'} (\mathbb{R}_+^L)^{n_{(i,s)}}$.

Its economic interpretation is the following: given a market price $p \in \Lambda$, a consumption plan $y \in Y_i$ embeds a non-random consumption decision $y(p[0]) \in \mathbb{R}_+^L$ at $t = 0$, and relates each expected state $s \in S_i$ and price $p_i[s] \in P_{(i,p,s)}$ at $t = 1$ to a conditional consumption decision $y(p_i[s])$ in state s. That consumption $y(p_i[s])$ is conditional on the joint conditions that state $s \in S_i$ and price $p_i[s] \in P_{(i,p,s)}$ prevailed.

Henceforth, for every tuple $(i, p, y, p_i) \in I \times \Lambda \times Y_i \times P_{(i,p)}$, we denote by $y(p_i) := (y(p_i[s])) \in \mathbb{R}_+^{LS_i'}$ the collection of consumption bundles $y(p_i[s]) \in \mathbb{R}_+^L$, across states $s \in S_i'$, as defined above. By construction, this vector $y(p_i)$ is independent of the market price and assigned a fixed place in the product set Y_i when $p \in \Lambda$ varies.

The above notion of consumption plan embeds the classical one, in which the future market price $p_1 \in \mathbb{R}_+^{L\mathbf{S}}$ is known with certainty by all agents and individual consumption decisions in each state are unique, that is, $Y_i := \mathbb{R}_+^{LS_i'}$, for each $i \in I$.

Each agent $i \in I$ is endowed with a continuous utility index $u_i : \mathbb{R}_+^{LS_i'} \to \mathbb{R}_+$ over consumptions bundles across states, and a related (continuous) utility function:

$U_i : (p, y) \in \Lambda \times Y_i \mapsto U_i(p, y) := \sum_{p_i \in P_{(i,p)}} \odot_{s \in S_i'} \pi_{(i,p,s)}(p_i[s]) u_i(y(p_i))$,

where $\odot_{s \in S_i'} \pi_{(i,p,s)}$ (for each $p \in \Lambda$) stands for the product probability of the independent probabilities $\pi_{(i,p,s)}$ across states $s \in S_i'$. This utility function yields a preference correspondence $\mathcal{R}_i$ for the generic agent $i \in I$, defined, for every pair $(p, y) \in \Lambda \times Y_i$, by the open set $\mathcal{R}_i(p, y)$ of consumption plans, which are strictly preferred to y when the market price is p, namely: $\mathcal{R}_i(p, y) := \{y' \in Y_i : U_i(p, y') > U_i(p, y)\}$.

We now present the equilibrium concept of the model and its existence property.

2.2 Definition and existence of the correct foresights equilibrium

2.2.1 The notion of correct foresights equilibrium

Agents make trade and consumption plans at $t = 0$, facing budget constraints in all prices they expect. Thus, using notations in footnote 2, for every $(i,(p,q)) \in I \times \mathcal{M}$, the i^{th} agent's budget set is the following set:

$$B_i(p,q) := \{(y,z) \in Y_i \times \mathbb{R}^J : p_i \,\square\, (y(p_i) - e_i) \leqslant W(S_i,q)z, \ \forall p_i \in P_{(i,p)}\}.$$

An allocation is a collection $y := (y_i) \in Y := \Pi_{i=1}^m Y_i$ of consumption plans for all agents. From above, it is defined indepently of the market price $(p,q) \in \mathcal{M}$.

For every triple $(i,p,y_i) \in I \times \Lambda \times Y_i$, we recall that $p \in P_{(i,p)}[\underline{\mathbf{S}}']$ and henceforth let $y_i^1 := (y_i(p[s]))_{s \in \underline{\mathbf{S}}'} \in \mathbb{R}_+^{L\underline{\mathbf{S}}'}$, which does not depend on p. An allocation $(y_i) \in Y$ is said to be attainable if the consumption decisions $y_i(f^1_{(i,p,s)}) := y_i(p[s])$ of all consumers $i \in I$ clear markets in each state $s \in \underline{\mathbf{S}}'$, i.e., $\sum_{i=1}^m (y_i^1 - e_i[\underline{\mathbf{S}}']) = 0$. Given $(p,q) \in \mathcal{M}$, we define the following sets of attainable allocations, portfolios and strategies:

$$\mathcal{A} := \{y := (y_i) \in Y : \textstyle\sum_{i=1}^m (y_i^1 - e_i[\underline{\mathbf{S}}']) = 0\};$$

$$\mathcal{Z} := \{(z_i) \in (\mathbb{R}^J)^m : \textstyle\sum_{i=1}^m z_i = 0\};$$

$$\mathcal{Y}(p,q) := \{[(y_i,z_i)] \in \Pi_{i=1}^m B_i(p,q) : (y_i) \in \mathcal{A}, \ (z_i) \in \mathcal{Z}\}.$$

The economy described above for given structures, $[V,(S_i)]$, of payoffs and information, π, of beliefs, and $(\mathcal{R}_i)$, of preferences is denoted by $\mathcal{E}_{[V,(S_i),\pi,(\mathcal{R}_i)]}$. At a sequential equilibrium of this economy, all consumptions should clear and optimise agents' welfare ex post, at their ex ante decisions.

Thus, a '*correct foresights equilibrium*', along Definition 2 below, is a collection of market prices $(p,q) \in \mathcal{M}$ and attainable strategies, which are optimal for each agent in the budget set. Under perfect foresights, it coincides with the equilibrium notion introduced in [4], hence, with the classical financial equilibrium when agents have symmetric information and perfect foresights.

Definition 2 *A price and strategy collection,* $((p,q),[(y_i,z_i)]) \in \mathcal{M} \times \Pi_{i=1}^{m} B_i(p,q)$*, is an equilibrium of the economy* $\mathcal{E}_{[V,(S_i),\pi,(\mathcal{R}_i)]}$*, or correct foresights equilibrium (CFE), if:*

(a) $\forall i \in I,\ B_i(p,q) \cap \mathcal{R}_i(p,y_i) \times \mathbb{R}^J = \varnothing$*;*

(b) $(y_i) \in \mathcal{A}$*;*

(c) $(z_i) \in \mathcal{Z}$*.*

Along Definition 2, at equilibrium, agents never need revise their anticipations or consumption decisions, so as to improve their welfare or let markets clear ex post. Indeed, for every $s \in \underline{\mathbf{S}}$, the true spot price belongs to every agent's expectations set from Condition (c) of Definition 1 and all agents' consumption decisions are optimal and clear market, ex ante and ex post. This is the main difference with the notion of temporary equilibrium, where agents' decisions and expectations need not be optimal or correct, or clear markets, ex post.

In our companion paper with infinite expectations sets, we argue that a temporary equilibrium is also well defined in this model (embedding the classical notion) and exists under weaker conditions than the CFE, e.g., Condition (c) of correct foresights may be dropped in Definition 1, and, for each $i \in I$, it is also possible to set as fixed and given, at $t=0$, the i^{th}agent's price expectations (and joint probabilities).

2.2.2 The existence Theorem

Extending the result of [4], the following Theorem characterizes the existence of equilibrium by the no-arbitrage condition of [2], under mild conditions. The economy $\mathcal{E}_{[V,(S_i),\pi,(\mathcal{R}_i)]}$ will be said to be standard if it meets Assumptions *A1* to *A4*:

- ***Assumption A1*** *(strong survival):* $\forall i \in I, e_i >> 0$;
- ***Assumption A2*** *(bounded beliefs):* $\exists \varepsilon \in \mathbb{R}_{++} : \varepsilon \leqslant p_i^l[s],\ \forall i \in I,\ \forall \overline{p} \in \Lambda$, $\forall (p_i, s, l) \in \{(p, s, l) \in P_{(i,\overline{p})} \times S_i \times \{1, ..., L\} : s \notin \underline{\mathbf{S}}$ *or* $(s \in \underline{\mathbf{S}}$ *and* $p[s] \neq \overline{p}[s])\}$;
- ***Assumption A3*** *(strictly increasing preferences):* $\forall i \in I, \forall (y, y') \in (\mathbb{R}_+^{LS'_i})^2,\ (y' > y) \Rightarrow (u_i(y') > u_i(y))$;
- ***Assumption A4*** *(quasi-concavity): for every* $(i, p) \in I \times \Lambda$*, the continuous mapping* $y \in Y_i \mapsto U_i(p, y)$ *is such that* $U_i(p, y + \alpha(y' \text{-} y)) \geqslant \min(U_i(p, y), U_i(p, y'))$*, for every* $(\alpha, (y, y')) \in]0, 1[\times Y_i^2$*, with strict inequality whenever* $y' \neq y$*.*

Theorem 1 *A standard economy* $\mathcal{E}_{[V,(S_i),\pi,(\mathcal{R}_i)]}$ *admits an equilibrium if, and only if, the payoff and information structure,* $[V, (S_i)]$*, is arbitrage-free along* [2]*. Moreover, any equilibrium price,* $(p, q) \in \mathcal{M}$*, is such that* $p >> 0$ *and* $[V, (S_i)]$ *is* q*-arbitrage-free.*

Remark 2 From Theorem 1, in a standard economy, all spot prices are bounded apart from zero at equilibrium. So, a similar Condition as that of Assumption *A2* holds for all prices and price expectations. Assumption *A2* is asymmetrically restricted for the convenience of the proof.

Before proving that every standard economy $\mathcal{E}_{[V,(S_i),\pi,(\mathcal{R}_i)]}$ with an arbitrage-free structure $[V, (S_i)]$ admits an equilibrium, Claim 1, below, provides a converse result.

Claim 1 *Under Assumption A3, if a collection of prices* $(p, q) \in \mathcal{M}$ *and strategies* $[(y_i, z_i)] \in \Pi_{i=1}^m B_i(p, q)$ *satisfies Condition* (a) *of Definition 2 of equilibrium, then,* $p >> 0$ *and* $[V, (S_i)]$ *is* q*-arbitrage-free.*

Proof We set as given an economy $\mathcal{E}_{[V,(S_i),\pi,(\mathcal{R}_i)]}$, which meets Assumption *A3*, and a collection of prices $(p,q) \in \mathcal{M}$ and strategies $[(y_i,z_i)] \in \Pi_{i=1}^m B_i(p,q)$, which satisfy Condition (a) of Definition 2.

Assume, first, by contraposition that $p^l[s] = 0$, for some $(s,l) \in S_i' \times \{1,...,L\}$. Let $y \in Y_i$ be identical to y_i in any component but $y^l(p[s]) := 1 + y_i^l(p[s])$. Then, from Assumption *A3* and the definition of U_i and from Conditions (b)-(c) of Definition 1, $(y,z_i) \in B_i(p,q) \cap \mathcal{R}_i(p,y_i) \times \mathbb{R}^J$. This contradicts the fact that (y_i,z_i) meets Condition (a) of Definition 2. This contradiction proves: $p >> 0$.

Assume, next, that $[V,(S_i)]$ fails to be q-arbitrage-free, that is, $W(S_i,q)z > 0$, for some $(i,z) \in I \times \mathbb{R}^J$. Let $s \in S_i'$ be such that $\varepsilon := W(q)[s] \cdot z > 0$ and $y \in Y_i$ be identical to y_i in every component but $y^L(p[s]) := \frac{\varepsilon}{p^L[s]} + y_i^L(p[s])$ (since, from above, $p^L[s] > 0$). Then, from Assumption *A3* and from above, $(y, z_i + z) \in B_i(p,q) \cap \mathcal{R}_i(p,y_i) \times \mathbb{R}^J$, which contradicts the fact that (y_i,z_i) meets Condition (a) of Definition 2. $\square$

3 The existence proof

Henceforth, an arbitrage-free structure $[V,(S_i)]$ and a standard economy $\mathcal{E}_{[V,(S_i),\pi,(\mathcal{R}_i)]}$ are set as given. We define an auxiliary compact economy (sub-Section 3.1), which admits an equilibrium along a standard fixed-point-like argument, and show this equilibrium is also an equilibrium of the initial economy (sub-Section 3.2).

3.1 The auxiliary economy

As is in [4], we let first, for each $i \in I$:

$Z_i^o := \{z \in \mathbb{R}^J : V[s] \cdot z = 0, \forall s \in S_i\}$ and denote by $Z_i^{o\perp}$ its orthogonal;

$Z^o := \sum_{i=1}^m Z_i^o$ and denote by $Z^{o\perp} = \cap_{i\in I} Z_i^{o\perp}$ its orthogonal;

$\mathcal{M}^* := \{(p,q) \in \mathcal{M} : q \in Z^{o\perp}\}$.

And we state Lemma 1, serving to bound strategies in the equilibrium problem.

Lemma 1 *Let* $\mathcal{A}^- := \{(y_i) \in Y : (\sum_{i=1}^m (y_i^1 - e_i[\underline{\mathbf{S}}'])) \leqslant 0\}$. *The following Assertions hold:*
(i) $\exists r > 0 : ((y_i) \in \mathcal{A}^-) \Rightarrow (\sum_{i=1}^m \|y_i^1\| < r)$;
(ii) $\exists r > 0 : ((p,q) \in \mathcal{M}^*$ *and* $[(y_i, z_i)] \in \mathcal{A}(p,q)) \Rightarrow (\sum_{i=1}^m (\|y_i\| + \|z_i\|) < r)$.

Proof see ([4], p.p. 266-268): mutatis mutandis, Lemma 1 and its proof are the same as in [4], up to the required changes in the definitions of sets. □

We now set as given $r > 0$, which satisfies Lemma 1-(i)-(ii) and let, for each $i \in I$:

$Y_i^* := \{y \in Y_i : \|y(p_i[s])\| \leqslant r,\ \forall s \in S_i',\ \forall p \in \Lambda,\ \forall p_i \in P_{(i,p)}\ \}$ and $Y^* := \Pi_{i=1}^m Y_i^*$;
$Z_i^* := \{z \in Z_i^{o\perp} : \|z\| \leqslant r\}$;
$\mathcal{R}_i^* : \Lambda \times Y^* \to Y_i^*$ be defined by $\mathcal{R}_i^*(p,y) := \mathcal{R}_i(p,y_i) \cap Y_i^*$ for all $(p, y := (y_i)) \in \Lambda \times Y^*$.

3.2 The existence proof

Following [4,8], we define, for all $(i,(p,q)) \in I \times \mathcal{M}^*$, the modified budget sets:

$B_i'(p,q) := \{(y,z) \in Y_i^* \times Z_i^* : p_i \,\square\, (y(p_i) - e_i) \leqslant W(S_i,q)z + \gamma_{(p,q)}[S_i'], \forall p_i \in P_{(i,p)}\}$;
$B_i''(p,q) := \{(y,z) \in Y_i^* \times Z_i^* : p_i \,\square\, (y(p_i) - e_i) << W(S_i,q)z + \gamma_{(p,q)}[S_i'], \forall p_i \in P_{(i,p)}\}$,

where $\gamma_{(p,q)} \in \mathbb{R}_+^{S'}$ is defined by:
$\gamma_{(p,q)}[0] := 1 - \min(1, \|p[0]\| + \|q\|)$, $\gamma_{(p,q)}[s] := 1 - \|p[s]\|$, for each $s \in \underline{\mathbf{S}}$, and $\gamma_{(p,q)}[S \backslash \underline{\mathbf{S}}] := 0$.

Claim 2 *For every* $(i,(p,q)) \in I \times \mathcal{M}^*$, $B_i''(p,q) \neq \emptyset$.

Proof Let $(i,(p,q)) \in I \times \mathcal{M}^*$ be given.

From Assumption *A1*, we can choose $y \in Y_i^*$ such that $p_i[s] \cdot (y(p_i[s])\text{-}e_i[s]) \leqslant 0$, for every $(p_i, s) \in P_{(i,p)} \times S_i'$, with a strict inequality whenever $p_i[s] \neq 0$. Then, from

Assumption $A2$, $p_i[s]\cdot (y(p_i[s])\text{-}e_i[s]) < \gamma_{(p,q)}[s]$ for every $(p_i, s) \in P_{(i,p)} \times S_i$. Consequently, if $p[0] \neq 0$ or $(p[0], q) = 0$, the relation $(y, 0) \in B_i''(p, q)$ holds and, if $p[0] = 0$ and $q \neq 0$, the relation $(y, -\eta q) \in B_i''(p, q) \neq \varnothing$ holds for $\eta > 0$ small enough, since $q \in Z^{o\perp} \subset Z_i^{o\perp}$. □

Claim 3 *For every $i \in I$, B_i'' is convex-valued and lower semicontinuous.*

Proof Let $i \in I$ be given. The correspondence B_i'' is obviously convex-valued. From Definition 1, $(p, q) \in \mathcal{M}^* \mapsto P_{(i,p)}$ is continuous, hence, B_i'' has an open graph in a compact set, which implies, as a standard corollary, it is lower semicontinuous. □

Claim 4 *For every $i \in I$, B_i' is convex-valued and upper semicontinuous.*

Proof Given $i \in I$, B_i' is non-empty convex-valued. Since $(p, q) \in \mathcal{M}^* \mapsto P_{(i,p)}$ is continuous, B_i' has a closed graph in a compact set, and is, hence, upper semicontinuous. □

We now introduce an additional agent, $i = 0$, representing the market and, following Gale-Mas-Colell, 1975-1979 [9], a reaction correspondence for each agent, defined on the convex compact set $\Theta := \mathcal{M}^* \times (\Pi_{i=1}^m Y_i^* \times Z_i^*)$. Thus, we let, for each $i \in I$ and every $((p, q), (y, z) := [(y_i, z_i)]) \in \Theta$:

$$\Psi_i((p,q),(y,z)) := \left\{ \begin{array}{ll} B_i'(p,q) & \text{if } (y_i, z_i) \notin B_i'(p,q) \\ B_i''(p,q) \cap \mathcal{R}_i^*(p,y) \times Z_i^* & \text{if } (y_i, z_i) \in B_i'(p,q) \end{array} \right\}.$$

And we define the market reaction correspondence as follows:

$$\Psi_0((p,q),(y,z)) := \{(p', q') \in \mathcal{M}^* : (p'\text{-}p) \cdot \textstyle\sum_{i=1}^m (y_i^1\text{-}e_i[\underline{\mathbf{S}}']) + (q'\text{-}q) \cdot \sum_{i=1}^m z_i > 0\}.$$

Claim 5 *For each $i \in \{0, 1, ..., m\}$, Ψ_i is lower semicontinuous.*

Proof First, the correspondence Ψ_0 is lower semicontinuous for having an open graph. Second, we let $i \in I$ and $\omega := ((p, q), (y, z) := [(y_i, z_i)]) \in \Theta$ be given and consider separately the two alternatives $(y_i, z_i) \notin B_i'(p, q)$ and $(y_i, z_i) \in B_i'(p, q)$ and show that, in both cases, Ψ_i is lower semicontinuous at ω.

- Assume, first, that $(y_i, z_i) \notin B'_i(p,q)$. Then, $\Psi_i(\omega) = B'_i(p,q)$.

Let V be an open set in $Y_i^* \times Z_i^*$, such that $V \cap B'_i(p,q) \neq \emptyset$. It follows from the convexity of $B'_i(p,q)$ and the non-emptyness of the open set $B''_i(p,q)$ that $V \cap B''_i(p,q)$ is non-empty. From Claim 3, there exists a neighborhood U of (p,q) in $\mathcal{M}^*$, such that $V \cap B'_i(p',q') \supset V \cap B''_i(p',q') \neq \emptyset$, for every $(p',q') \in U$.

Since $B'_i(p,q)$ is non-empty, closed, convex in the compact set $Y_i^* \times Z_i^*$, there exist two open sets V_1 and V_2 in $Y_i^* \times Z_i^*$, such that $(y_i, z_i) \in V_1$, $B'_i(p,q) \subset V_2$ and $V_1 \cap V_2 = \emptyset$. From Claim 4, there exists a neighborhood $U_1 \subset U$ of (p,q), such that $B'_i(p',q') \subset V_2$, for every $(p',q') \in U_1$. Let $W = U_1 \times \Pi_{j=1}^m W_j$, where $W_i := V_1$ and $W_j := Y_j^* \times Z_j^*$, for every $j \in I \backslash \{i\}$. Then, W is a neighborhood of ω, such that $\Psi_i(\omega') = B'_i(p',q')$, and, from above, $V \cap \Psi_i(\omega') \neq \emptyset$, for every $\omega' := ((p',q'),(y',z')) \in W$. This proves the lower semicontinuity of Ψ_i at ω.

- Assume, now, that $(y_i, z_i) \in B'_i(p,q)$, i.e., $\Psi_i(\omega) = B''_i(p,q) \cap \mathcal{R}_i^*(p,y) \times Z_i^*$.

The lower semicontinuity of Ψ_i at ω is immediate if $\Psi_i(\omega) = \emptyset$, so, we assume $\Psi_i(\omega) \neq \emptyset$. From Definition 1 and Assumptions *A3-A4*, both correspondences B''_i and $(p',y') \in \mathcal{M}^* \times Y^* \mapsto \mathcal{R}_i^*(p',y')$ have open graphs. As a standard corollary, the correspondence $\Phi_i : ((p',q'),(y',z')) \in \Theta \mapsto B''_i(p',q') \cap \mathcal{R}_i^*(p',y') \times Z_i^*$ is lower semicontinuous. From the definition of Ψ_i and the inclusion $\Phi_i((p',q'),(y',z')) \subset B'_i(p',q')$ in a compact set, which holds for every $((p',q'),(y',z')) \in \Theta$, Ψ_i is lower semicontinuous at ω. □

Claim 6 *There exists* $(p^*, q^*), (y^*, z^*)) \in \Theta$*, such that:*

(i) $\forall (p,q) \in \mathcal{M}^*,\ (p^* - p) \cdot \sum_{i=1}^m (y_i^{*1} - e_i[\underline{\mathbf{S}}']) + (q^* - q) \cdot \sum_{i=1}^m z_i^* \geqslant 0$;

(ii) $\forall i \in I,\ (y_i^*, z_i^*) \in B'_i(p^*, q^*)$ *and* $B''_i(p^*,q^*) \cap \mathcal{R}_i^*(p^*, y^*) \times Z_i^* = \emptyset$.

Proof Quoting Gale-Mas-Colell, 1975-79 [9,10]: "Given $X = \Pi_{i=1}^m Xi$, where X_i is a non-empty compact convex subset of an Euclidean space, let $\varphi_i : X \to X_i$ be m

convex (possibly empty) valued lower semicontinuous correspondences. Then, there exists x in X such that for each i either $x_i \in \varphi_i(x)$ or $\varphi_i(x) = \emptyset$".

The correspondences $\Psi_0 : \Theta \to \mathcal{M}^*$ and $\Psi_i : \Theta \to X_i^* \times Z_i^*$ (for each $i \in I$) satisfy the conditions of the above theorem, hence, admit an element $((p^*, q^*), [(y_i^*, z_i^*)]) \in \Theta$, such that, either $(p^*, q^*) \in \Psi_0(\omega^*)$, or $\Psi_0(\omega^*) = \emptyset$, and, for each $i \in I$, either $(y_i^*, z_i^*) \in \Psi_i(\omega^*)$, or $\Psi_i(\omega^*) = \emptyset$. By construction, $(p^*, q^*) \notin \Psi_0(\omega^*)$ and, for each $i \in I$, $(y_i^*, z_i^*) \notin \Psi_i(\omega^*)$, since $y_i^* \notin \mathcal{R}_i^*(p^*, y^*)$. Hence, the relation $\Psi_0(\omega^*) = \emptyset$ holds, which yields Assertion (i), and the relations $\Psi_i(\omega^*) = \emptyset$ hold for each $i \in I$, and yield the above Assertion (ii). $\square$

Claim 7 $\sum_{i=1}^m z_i^* \in Z^o$.

Proof Assume, by contraposition, that $\sum_{i=1}^m z_i^* \notin Z^o$. Then, from Claim 6-$(i)$, the relation $q \cdot \sum_{i=1}^m z_i^* \leqslant q^* \cdot \sum_{i=1}^m z_i^*$ holds, for every $q \in Q := \{q \in Z^{o\perp} : \|q\| \leqslant 1\} \supset \{0\}$. This implies $q^* \cdot \sum_{i=1}^m z_i^* > 0$ and $\|q^*\| = 1$, hence, $\gamma_{(p^*, q^*)}[0] = 0$.

From Claim 6-(ii), $(y_i^*, z_i^*) \in B_i'(p^*, q^*)$, for each $i \in I$, whose budget constraint in state $s = 0$ is written: $p^*[0] \cdot (y_i^*(p^*[0]) - e_i[0]) \leqslant -q^* \cdot z_i^*$. Summing up the latter relations for $i \in I$, yields, from above, $p^*[0] \cdot \sum_{i=1}^m (y_i^*(p^*[0]) - e_i[0]) \leqslant -q^* \cdot \sum_{i=1}^m z_i^* < 0$. This contradicts Claim 6-(i), which obviously implies $p^*[0] \cdot \sum_{i=1}^m (y_i^*(p^*[0])\text{-}e_i[0]) \geqslant 0$. $\square$

Remark 3 Claim 7 implies, from Claim 6-(i) and the relation $q^* \in Z^{o\perp} = \cap_{i=1}^m Z_i^{o\perp}$:

(i) $\forall p \in \Lambda$, $(p^* - p) \cdot \sum_{i=1}^m (y_i^{*1} - e_i[\underline{\mathbf{S}}']) \geqslant 0$;

(ii) $\sum_{i=1}^m W(\underline{\mathbf{S}}, q^*) z_i^* = 0$.

Claim 7 also implies the existence of $(z_i') \in \Pi_{i=1}^m Z_i^o$ such that $\sum_{i=1}^m z_i^* = \sum_{i=1}^m z_i'$.

Henceforth, we let $z_i := (z_i^*\text{-}z_i')$, for each $i \in I$. These portfolios satisfy:

(iii) $z := (z_i) \in \mathcal{A}$, i.e.,$\sum_{i=1}^m z_i = 0$;

(iv) $\forall i \in I,\ W(S_i, q^*) z_i^* = W(S_i, q^*) z_i$ (since $(q^*, z_i') \in Z_i^{o\perp} \times Z_i^o$).

Claim 8 $\sum_{i=1}^{m}(y_i^{*1}\text{-}e_i[\underline{\mathbf{S}}']) \leqslant 0$, *hence,* $\sum_{i=1}^{m}\|y_i^{*1}\| < r$.

Proof Assume, by contraposition, that $\sum_{i=1}^{m}(y_i^*(p^*[s])\text{-}e_i[s]) \nleqslant 0$ for some $s \in \underline{\mathbf{S}}'$. Then, from Remark 3-($i$), the following relations hold:

$p^*[s] = \sup(0, \sum_{i=1}^{m}(y_i^*(p^*[s])\text{-}e_i[s]))/\|\sup(0, \sum_{i=1}^{m}(y_i^*(p^*[s])\text{-}e_i[s]))\|$,

$\gamma_{(p^*,q^*)}[s] = 0$ and $p^*[s] \cdot \sum_{i=1}^{m}(y_i^*(p^*[s])\text{-}e_i[s]) > 0$.

From Definition 1 $\left(p^* \in \cap_{i=1}^{m} P_{(i,p^*)}[\underline{\mathbf{S}}']\right)$, from Claim 6-($ii$) and above, the relations $p^*[s]\cdot(y_i^*(p^*[s])\text{-}e_i[s]) \leqslant W(q^*)[s]\cdot z_i^*$ hold, for each $i \in I$. Summing them up for $i = 1, ..., m$ yields, from Remark 4-(ii), $p^*[s]\cdot \sum_{i=1}^{m}(y_i^*(p^*[s])\text{-}e_i[s]) \leqslant 0$, which contradicts the opposite strict inequality above. This contradiction proves that $\sum_{i=1}^{m}(y_i^{*1}\text{-}e_i[\underline{\mathbf{S}}']) \leqslant 0$, hence, from Lemma 1 and the choice of r in sub-Section 3.1, $\sum_{i=1}^{m}\|y_i^{*1}\| < r$. □

Claim 9 *For each* $i \in I$, (y_i^*, z_i^*) *is optimal in* $B_i'(p^*, q^*)$.

Proof Let $i \in I$ and $(y_i', z_i') \in B_i''(p^*, q^*) \subset B_i'(p^*, q^*)$, along Claim 2, be given. From Claim 6-(ii), $(y_i^*, z_i^*) \in B_i'(p^*, q^*)$. Assume, by contraposition, that there exists $(y_i, z_i) \in B_i'(p^*, q^*) \cap \mathcal{R}_i^*(p^*, y^*) \times Z_i^*$. Then, for every $k \in \mathbb{N}^*$, the convexity of $B_i'(p^*, q^*)$ insures $(y_i^k, z_i^k) := [\frac{1}{k}(y_i', z_i') + (1 - \frac{1}{k})(y_i, z_i)] \in B_i'(p^*, q^*)$, whereas $(y_i^k, z_i^k) \in B_i''(p^*, q^*)$ by construction. Since U_i is continuous, $\mathcal{R}_i^*(p^*, y^*)$ is open, hence, for $K > 0$ big enough, $y_i^K \in \mathcal{R}_i^*(p^*, y^*)$, i.e., $(y_i^K, z_i^K) \in B_i''(p^*, q^*) \cap \mathcal{R}_i^*(p^*, y^*) \times Z_i^*$, contradicting Claim 6-(ii). □

Claim 10 $\sum_{i=1}^{m}(y_i^{*1}\text{-}e_i[\underline{\mathbf{S}}']) = 0$.

Proof Assume, by contraposition, that $\sum_{i=1}^{m}(y_i^{*1}\text{-}e_i[\underline{\mathbf{S}}']) \neq 0$. From Claim 8, we may set as given some $(i, s, l) \in I \times \underline{\mathbf{S}}' \times \{1, ..., L\}$, such that $\sum_{j=1}^{m}(y_j^{*l}(p^*[s])\text{-}e_j^l[s]) < 0$ and $(y_i^{*l}(p^*[s])\text{-}e_i^l[s]) < 0$, hence, from Remark 3-($i$), $p^{*l}[s] = 0$. Then, from Claim 8, there exists $y_i \in Y_i^*$ identical to y_i^* in any consumption decision but $y_i^l(p^*[s]) > y_i^{*l}(p^*[s])$, which satisfies $(y_i, z_i^*) \in B_i'(p^*, q^*)$ and, from Assumption *A3* and Definition 1, $y_i \in$

$\mathcal{R}_i^*(p^*, y^*)$, that is, $(y_i, z_i^*) \in B_i'(p^*, q^*) \cap \mathcal{R}_i^*(p^*, y^*) \times Z_i^*$. This contradicts Claim 9. $\square$

Claim 11 $\gamma_{(p^*,q^*)} = 0$, *hence,* $\sum_{i=1}^m (\|y_i^*\| + \|z_i^*\|) \leqslant \sum_{i=1}^m (\|y_i^*\| + \|z_i\|) < r$.

Proof Given $(i, s) \in I \times \underline{\mathbf{S}}'$, the relation $p^*[s] \cdot (y_i^*(p^*[s])\text{-}e_i[s]) \leqslant W(q^*)[s] \cdot z_i^* + \gamma_{(p^*,q^*)})[\underline{\mathbf{S}}']$ holds, from Claim 6 and Definition 1. If $p^*[s] \cdot (y_i^*(p^*[s])\text{-}e_i[s]) < (W(q^*)z_i^* + \gamma_{(p^*,q^*)})[s]$, there exists $y_i \in Y_i^*$, identical to y_i^* in any consumption decision but $y_i[s] > y_i^*[s]$, and, from Claim 8 and Assumption *A4*, close enough to y_i^* so that $(y_i, z_i^*) \in B'(p^*, q^*)$. Then, from Assumption *A3* and Definition 1, $(y_i, z_i^*) \in B'(p^*, q^*) \cap \mathcal{R}_i^*(p^*, y^*) \times Z_i^*$, which contradicts Claim 9. Hence, $p^* \square (y_i^{*1}\text{-}e_i[\underline{\mathbf{S}}']) = W(\underline{\mathbf{S}}, q^*) z_i^* + \gamma_{(p^*,q^*)})[\underline{\mathbf{S}}']$, for each $i \in I$. Summing up these relations, for $i = 1, ..., m$, yields $\gamma_{(p^*,q^*)} = 0$, from Claim 10 and Remark 3-(ii). Then, from Claim 6-(ii), Claim 10 and Remark 3-(iii), the relation $[(y_i^*, z_i)] \in \mathcal{A}(p^*, q^*)$ holds and implies, from Lemma 1 and the fact that $z^* := (z_i^*)$ is an orthogonal projection of $z := (z_i)$: $\sum_{i=1}^m (\|y_i^*\| + \|z_i^*\|) \leqslant \sum_{i=1}^m (\|y_i^*\| + \|z_i\|) < r$. $\square$

Claim 12 *Let* $(z_i) \in \mathcal{Z}$ *be defined as in Remark 3. Then,* $((p^*, q^*), (y^*, z) := [(y_i^*, z_i)])$ *is a C.F.E., which satisfies the (price and no-arbitrage) Conditions of Theorem 1.*

Proof Let $\mathcal{C} := ((p^*, q^*), (y^*, z))$ be defined as in Claim 12. From Claim 1, if $\mathcal{C}$ is a C.F.E., $(p^*, q^*) \in \mathcal{M}$ meets all price Conditions of Theorem 1. From Claim 6-(ii), Claim 10, Claim 11 and Remark 3-(iii), the relation $(y^*, z) \in \mathcal{A}(p^*, q^*)$ holds, that is, $\mathcal{C}$ meets Conditions (b) and (c) of Definition 2 of equilibrium.

Assume, by contraposition, that $\mathcal{C}$ fails to meet Condition (a) of Definition 2, i.e., there exists $i \in I$ and $(\overline{y}_i, \overline{z}_i) \in B_i(p^*, q^*) \cap \mathcal{R}_i(p^*, y_i^*) \times \mathbb{R}^J$. Then, from Remark 3-$(iv)$, we may assume that $\overline{z}_i \in Z_i^{o\perp}$, that is, $(\overline{y}_i, \overline{z}_i) \in B_i(p^*, q^*) \cap Y_i \times Z_i^{o\perp}$. From Assumption *A4*, from Claim 11 and the convexity of $B_i(p^*, q^*) \cap Y_i \times Z_i^{o\perp}$, we may also assume that $(\overline{y}_i, \overline{z}_i)$ is so close to (y_i^*, z_i^*) that $(\overline{y}_i, \overline{z}_i) \in B_i'(p^*, q^*)$. Then, $(\overline{y}_i, \overline{z}_i) \in B_i'(p^*, q^*) \cap \mathcal{R}_i^*(y^*) \times \mathbb{R}^J$, which contradicts Claim 9. This contradiction proves $\mathcal{C}$ is a CFE and Theorem 1. $\square$

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