CONSISTENT DYNAMICE CHOICE AND NON-EXPECTED UTILITY PREFERENCES

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Document de Travail
n°2007-20

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October 2007

1. Introduction

NonExpected Utility (NEU) models of choice under uncertainty have generated a growing interest over the last decades among decision theorists. In situations of uncertainty (*i.e*. where probability distributions on the outcomes are not given), these models allow to describe Ellsberg-type preferences by taking into account attitude toward uncertainty. In this paper, we focus on two approaches, namely Choquet Expected Utility (CEU) model and Multiple Priors (MP) model. This raises an important theoretical issue: how could NEU models be used in multi-stage decision problems? To preserve rationality in such situations, several principles can be imposed.

Sarin and Wakker (1998) show that under CEU, consequentialism, dynamic consistency and their sequential consistency property imply that deviations from Expected Utility (EU) are allowed in only one stage, the last one. Ghirardato (2002) shows that consequentialism and dynamic consistency together with standard assumptions imply that an EU representation exists in all stages. This paper first aims at explaining this paradox. We show that the only difference between these results is due to Ghirardato's (2002) assumption that the Decision Maker (DM) does not care about the timing of the resolution of uncertainty (Kreps and Porteus (1978)). In other words, in his set-up, the DM satisfies a subjective version of reduction of compound lotteries (RCL) axiom (Ghirardato (2002, p. 86)) of Von-Neumann and Morgenstern. In settings of 'objective uncertainty', or risk, Karni and Schmeidler (1991) show that if consequentialism and RCL hold together, then the dynamic consistency property is equivalent to the independence axiom of choice under risk in one-stage decision problems. In an identical set-up, Volij (1994) shows that given dynamic consistency and any of the two other concepts (consequentialism and RCL), the third is equivalent to the independence axiom.

Concerning MP model, Sarin and Wakker (1998) show that it can be applied to dynamic choice situations without restrictions. This result contrasts with the 'folk theorem' of decision theory, which enounces that consequentialism and dynamic consistency together imply Savage's postulate P2 (Sure Thing Principle). This

constitutes a second paradox, because MP model weakens Sure Thing Principle (STP). We show that these dynamic choice principles impose the same restrictions on MP and CEU.

In section 2, we present set-up and axioms, and prove a version of the folk theorem. In section 3, we present our result for CEU model. Section 4 reports results for MP model. For simplicity, section 3 and 4 only consider dynamic decision problems with two stages, two first stage events and two second stage events. Section 5 extends our results to cases with many stages and events. Section 6 discusses and concludes.

2. Set-up, axioms and definitions

S is a finite state space. A state in S is represented by s. Subsets of S are called events. $\forall A \subset S$, the event $S \setminus A$ is denoted A^c. X is an outcome space, *i.e.* a subset of $\mathbb{R}$, and we denote by $\mathbb{R}^S = \{f : S \to X\}$ the set of acts, or random variables. In a dynamic setting, S is endowed with the filtration $\{\mathcal{F}_t, t \in T\}$, which represents the information structure. We assume that time is discrete and that $T = \{0,1,...,T\}$ is finite. $\{\mathcal{F}_t, t \in T\}$ is given and fixed throughout. For each t in T, $\mathcal{E}_t$ is a finite partition which contains all events that occur at time t. $\{\mathcal{F}_t, t \in T\}$ can be rewritten as $\{\mathcal{E}_0,...,\mathcal{E}_T\}$. We denote by E_t an event which occurs at time t. Hence E_t is an element of $\mathcal{E}_t$. We only consider sequential choice, that is dynamic choice in which outcomes are obtained at time T. A decision maker (DM) is characterized by a preference relation, $\succcurlyeq_S$ (or $\succcurlyeq$), on $\mathbb{R}^S$. $\succcurlyeq$ is defined ex-ante, *i.e.* when no information is given to the DM. $\succcurlyeq_{E_t}$ compares acts conditionally to $E_t \in \mathcal{E}_t$, *i.e.* if the DM is informed that only $s \in E_t$ can obtain. $\forall t \in T$, we write $f =_{E_t} g$ if $\forall s \in E_t$, $f(s) = g(s)$. $\succ_{E_t}$ and $\sim_{E_t}$ are defined in the usual way. The class of binary relations

$\{\succcurlyeq_{E_t}\}_{t\in T}$ can satisfy several axioms. We first require that each conditional preference be a weak order:

Axiom 1 (Complete Weak Order). $\forall t \in T$, $\succcurlyeq_{E_t}$ is a weak order, *i.e.* it is complete, transitive and reflexive on $\mathbb{R}^S$.

An important axiom of the EU model (Savage (1954)) is the Sure-Thing Principle.

Axiom 2 (Sure-Thing Principle). $\forall f,g,f',g' \in \mathbb{R}^S$, $\forall \tau,t \in T, \tau < t,$, $\forall E_t \in \mathcal{E}_t$,

$$(f =_{E_t} f', g =_{E_t} g', f =_{E_t^c} g, f' =_{E_t^c} g') \Rightarrow (f \succcurlyeq_{E_\tau} g \Leftrightarrow f' \succcurlyeq_{E_\tau} g').$$

The next axiom states that each conditional preference is only dependent on the information received. We name this property "consequentialism" in reference to Machina (1989).

Axiom 3 (Consequentialism). $\forall t = 1,...,T-1$, $\forall E_t \in \mathcal{E}_t$, $\forall f,g \in \mathbb{R}^S$,

$$(f =_{E_t} g) \Rightarrow (f \sim_{E_t} g).$$

Such a definition can be found in Ghirardato (2002).

The following axiom imposes some dynamic restrictions:

Axiom 4 (Dynamic Consistency). $\forall f,g \in \mathbb{R}^S$, $\forall \tau,t = 0,...,T-1$ such that $\tau < t$, $\forall E_t \in \mathcal{E}_t$,

$$\left(f =_{E_t^c} g\right) \Rightarrow \left(f \succcurlyeq_{E_\tau} g \Leftrightarrow f \succcurlyeq_{E_t} g\right).$$

Roughly speaking, dynamic consistency property says that, given an information set, if the DM prefers f to g (or is indifferent between f and g), then he prefers f to g (or is indifferent between f and g) whatever new information obtains.

A well known result of decision theory is that consequentialism and dynamic consistency together imply STP. This result is proved in Ghirardato (2002). For him, this belongs to the "folk wisdom" of decision theory. However, his result does not hold

true if the information structure is given and fixed. In this case, STP does not necessary holds in all stages of $\{\mathcal{F}_t, t \in T\}$.

Proposition 2-1. Let $\{\mathcal{F}_t, t \in T\}$ be a filtration. Suppose that consequentialism holds on the complete weak order from $\{\succcurlyeq_{E_t}\}_{t=1,\ldots,T-1}$, and that dynamic consistency holds between $\succcurlyeq_{E_\tau}$ and $\succcurlyeq_{E_t}$ for all τ and t in T such that $\tau < t < T$. Then the Sure-Thing Principle (STP) holds on $\{\succcurlyeq_{E_t}\}_{t=0,\ldots,T-2}$.

Proof. First consider a partition $\mathcal{E}_t$, with $t = 1,\ldots,T-1$, and an event $E_t \in \mathcal{E}_t$. $\forall \tau < t$, consider now a pair of acts f and g s.t. $f \succcurlyeq_{E_\tau} g$ and $\forall s \in E_t^c, f(s) = g(s)$. From dynamic consistency, $f \succcurlyeq_{E_\tau} g$ if and only if $f \succcurlyeq_{E_t} g$. From consequentialism and transitivity, $f \succcurlyeq_{E_t} g$ if and only if $f' \succcurlyeq_{E_t} g'$, where $\forall s \in E_t, f(s) = f'(s)$, $g(s) = g'(s)$, and $\forall s \in E_t^c, f'(s) = g'(s)$. By dynamic consistency, $f' \succcurlyeq_{E_t} g'$ if and only if $f' \succcurlyeq_{E_\tau} g'$. ■

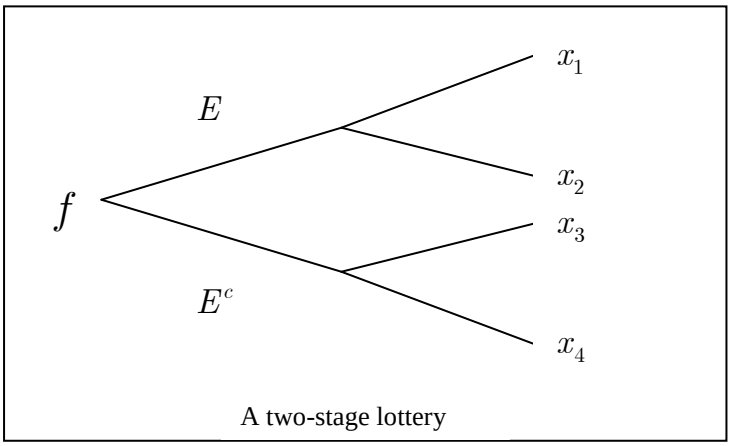
The class of preference relation $\{\succcurlyeq_{E_t}\}_{t \in T}$ on $\mathbb{R}^S$ induces a relation on X, also denoted by $\succcurlyeq$. Throughout the paper we assume the following hypothesis:

Hypothesis 1. The relation $\succcurlyeq$ on X is a weak order which satisfies continuity and strong monotonicity. Moreover, we avoid triviality: there are three distinguishable consequences in X. ■

If hypothesis 1 holds, then there exists a real-valued function $u : X \to \mathbb{R}$ which is continuous and strictly increasing, such that $\forall x, x' \in X, x > x'$ if and only if $u(x) > u(x')$.

For simplicity, we assume that $S = \{s_1, s_2, s_3, s_4\}$ and $T = \{0,1,2\}$ henceforth. $\{\mathcal{F}_t, t = 0,1,2\}$ is given and fixed throughout: $E = \{s_1, s_2\}$ and $E^c = \{s_3, s_4\}$ are first

stage events such that at time $t=1$ the DM is informed that only $s \in E$ or $s \in E^c$ can obtain. The elementary events $\{s_1\}_{i=1,...,4}$ occur in the second stage. In the following figure, f is a two-stage lottery, *i.e*. a compound lottery which yields sub-lotteries at the first stage.



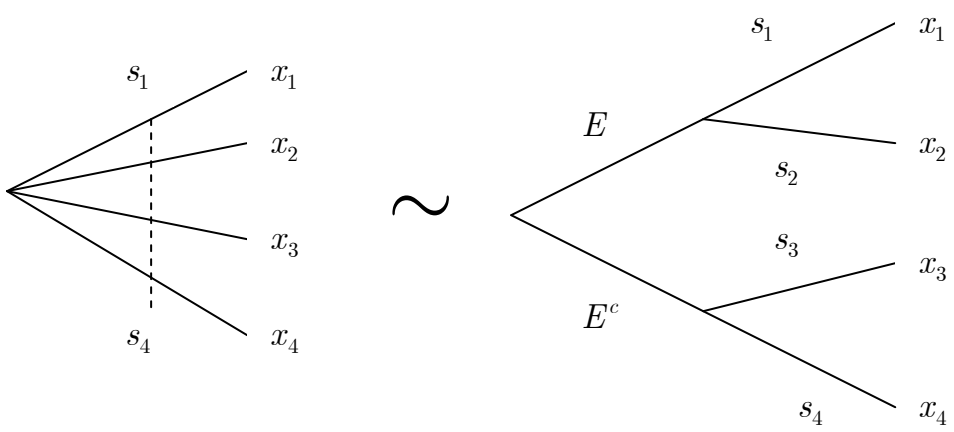
A two-stage lottery

Following Sarin and Wakker (1998), we suppose that the DM uses folding back procedure to value a compound act.

Axiom 5 (Folding back). $\forall x_1, x_2, x_3, x_4 \in X$,

$$W(x_1 ... x_4) = V(V_E(x_1, x_2), V_{E^c}(x_3, x_4)).$$

Once consequentialism and dynamic consistency assumed, folding back can be used without more restrictions. V_E, V_{E^c} are certainty equivalents of the sub-lotteries $(x_1 \text{ on } \{s_1\}, x_2 \text{ on } \{s_2\})$ and $(x_3 \text{ on } \{s_3\}, x_4 \text{ on } \{s_4\})$. V is the certainty equivalent of $(V_E(x_1, x_2), V_{E^c}(x_3, x_4))$, and W is used by the DM in the single stage evaluation $(x_1, ..., x_4)$. We assume that such functions exist and are well defined. For each NEU form (CEU or MP), we will specify which axiomatization is used. Note that folding back procedure implies that the DM is indifferent between the two following figures.



Single stage and two stages lotteries are equivalent.

In a dynamic setting, it seems natural to impose 'Model consistency'. Let $\mathbb{M}$ be a class of numerical representations, such that elements of $\mathbb{M}$ have the same axiomatic basis. Model consistency implies the use of the same numerical representation in the first stage, in the second and in the single stage evaluation: W, V, V_E, V_{E^c} are elements of $\mathbb{M}$.

We now present a general definition of model consistency:

Axiom 6 (Model consistency). Let $\mathbb{M}$ be a class of decision criteria defined by the same axiomatic basis. $\forall \mathcal{F}_T \subseteq 2^S, \forall t, \tau = 0, \ldots, T-1, t \neq \tau, \forall E_t \in \mathcal{E}_t, \forall E_\tau \in \mathcal{E}_\tau,$

$$W \in \mathbb{M} \Leftrightarrow V_{E_t} \in \mathbb{M} \Leftrightarrow V_{E_\tau} \in \mathbb{M}.$$

Our Model consistency condition is slightly different from the sequential consistency property of Sarin and Wakker (1998). Sequential consistency implies that

$$V_{E_\tau}, V_{E_t} \in \mathbb{M} \Rightarrow W \in \mathbb{M}.$$

Now we can present the following results:

Sarin and Wakker's (1998) theorem. Let $\{\mathcal{F}_t, t = 0,1,2\}$ be a filtration. Suppose that hypothesis 1 holds and that $\mathbb{M}$ is the family of CEU form. Then folding back and model consistency hold together if and only if there

are a utility function $u : X \to \mathbb{R}$ and a unique capacity $\nu : 2^S \to [0,1]$ necessary additive in the first stage of $\{\mathcal{F}_t, t = 0,1,2\}$.

Thus the DM is free to use a non-additive capacity in the second stage. This result is consistent with our proposition 2-1. However, this appears in contradiction with the following theorem:

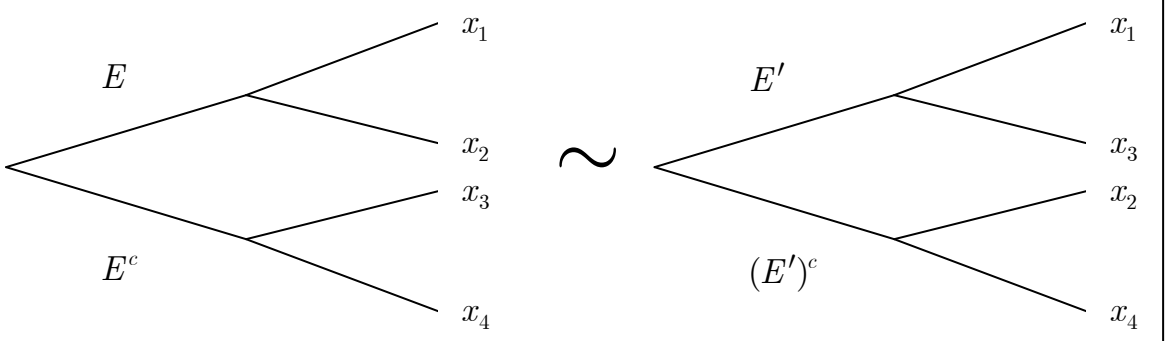
Ghirardato's (2002) theorem. $\forall A \in 2^S$, the class of binary relations $\{\succcurlyeq_A\}_{A \in 2^S}$ satisfies Savage postulates (except Sure Thing Principle), consequentialism and dynamic consistency if and only if there are a utility function $u : X \to \mathbb{R}$ and a unique additive measure $p : 2^S \to [0,1]$ s.t. all elements from $\mathbb{M}$ are expected utility representations.

Sarin and Wakker (1998) preserve the dynamic structure of the decision problem: the exact sequence of decisions and events is relevant to the DM, hence they do not assume reduction of compound lotteries (p. 93). On the other hand, Ghirardato (2002, p. 86) applies his axioms on 2^S, and not only for a given and fixed filtration. He notes that this implies a subjective version of the reduction of compound lotteries axiom.

Axiom 7 (Reduction of Compound Acts). $\forall f, g \in \mathbb{R}^S$,

$$\forall s \in S, f(s) = g(s) \Rightarrow f \sim g\,.$$

This axiom is so called "neutrality assumption" or invariance. An important consequence of this assumption is that the DM is indifferent about the timing of the resolution of uncertainty.



The neutrality assumption

We take up the figure 5 of Sarin and Wakker (1998), with $E' = \{s_1, s_3\}$ and $E'^c = \{s_2, s_4\}$. Under folding back, Reduction of Compound Acts implies that

$$W(x_1,...,x_4) = V(V_{E'}(x_1, x_3), V_{E'^c}(x_2, x_4)).$$

Our purpose is to emphasize the implication of the RCA axiom. Given the filtration $\{\mathcal{F}_t, t = 0,1,2\}$ and a family $\mathbb{M}$ of CEU representations, we show that if RCA is assumed with folding back and model consistency, then all elements of $\mathbb{M}$ have an expected utility form.

An other theoretical paradox is linked with the use of the MP model in sequential choice situations. Sarin and Wakker (1998) show that MP model can be consistently used in dynamic choice without restriction. This result contrasts with the logic implication of the folk theorem, because MP model is obtained from EU model by weakening STP axiom (Gilboa and Schmeidler (1989), Casadesus-Masanell and al. (2000)). We show that MP model can be consistently used in dynamic choice if and only if the set of priors is reduced to a singleton in the first stage of $\{\mathcal{F}_t, t = 0,1,2\}$ ($V(.)$ is an expected utility form). Moreover, once RCA axiom is assumed, all elements of the family $\mathbb{M}$ of multiple priors forms use a unique additive measure.

3. Choquet Expected Utility

An important class of NEU models is the CEU one. In this model, the beliefs are represented by a Choquet capacity, *i.e.* a set function $\nu : 2^S \to [0,1]$ s.t. :

$\nu(S) = 1, \nu(\varnothing) = 0$ and $\forall A, B \in 2^S, B \subseteq A \Rightarrow \nu(A) \geq \nu(B)$.

Remark that if ν is convex, then CEU model is reduced to MP model (Gilboa and Schmeidler (1989), Denneberg (1994)).

The single stage evaluation W is a Choquet Expected Utility representation if and only if

$$W : (x_1, ..., x_4) \mapsto \int_S u(x(s)) d\nu(s).$$

If model consistency holds with respect to a CEU form, then the conditional evaluations V_E, V_{E^c} use the same utility $u : X \to \mathbb{R}$ and the update from ν. $\forall t = 1, ..., T,$ we denote by $\nu(.|E_t)$ the conditional set function for ν given $E_t \in \mathcal{E}_t$. Several rules can be used by the DM to update her capacities. In order to simplify notations, we only define these rules for $\{\mathcal{F}_t, t = 0,1,2\}$.

Definition 1. *Let ν be a capacity on S. The Full Bayes Updating Rule of ν conditional on $B \in \{E, E^c\}$ is given by :*

$$\forall C \subset B, \nu(C|B) = \frac{\nu(B \cap C)}{1 + \nu(B \cap C) - \nu(C \cup B^c)}. \quad \text{(FUBU).}$$

Definition 2. *Let ν be a capacity on S. The Bayes update of ν conditional on $B \in \{E, E^c\}$ is given by :*

$$\forall B \subset A, \nu(C|B) = \frac{\nu(B \cap C)}{\nu(B)}. \quad \text{(B)}$$

Definition 3. *Let ν be a capacity on S. The Dempster-Shafer update of ν conditional on $B \in \{E, E^c\}$ is given by :*

$$\forall C \subset B, \nu(C|B) = \frac{\nu((B \cap C) \cup B^c) - \nu(B^c)}{1 - \nu(B^c)}. \qquad \text{(DS)}$$

If the DM maximizes CEU in all stages of the filtration, then we impose the following hypothesis:

Hypothesis 2. $\forall \mathcal{F}_T \subseteq 2^S, \forall t = 1, \ldots, T-1, \forall E_t \in \mathcal{E}_t$, we suppose that the DM possibly uses FUBU, DS or B to calculate $\nu(.|E_t)$. ■

We suppose that each form from $\mathbb{M}$ is constructed with the axioms from Gilboa (1987), who gives an axiomatization of CEU with Savage acts. Therefore, all forms from $\mathbb{M}$ satisfy Sure Thing Principle on comonotonic acts and other axioms from Gilboa (1987).

Theorem 3-1. Let $\{\mathcal{F}_t, t = 0,1,2\}$ be a filtration. We suppose that hypothesis 1, 2, consequentialism, dynamic consistency and folding back hold. Then the following two statements are equivalent:

(i) Model consistency holds with respect to $\mathbb{M}$, the family of CEU forms, and reduction of compound acts (RCA) axiom holds.

(ii) There exist a utility function $u : X \to \mathbb{R}$ and a unique additive measure $p : 2^S \to [0,1]$ such that $\forall B \in \{E, E^c\}, W, V, V_B$ are expected utility forms, and V_B uses conditional probabilities $p(.|B) : \mathcal{E}_2 \to [0,1]$ calculated with Bayes rule.

Proof. The implication from (ii) to (i) is straightforward, because the expected utility representation verifies RCA. Moreover, it is clear that the statement (ii) implies folding back, consequentialism and dynamic consistency. Now we prove the implication from (i) to (ii).

If consequentialism holds on $\succcurlyeq_B$, for $B \in \{E, E^c\}$, and dynamic consistency holds between $\succcurlyeq$ and $\succcurlyeq_B$, then $\succcurlyeq$ verifies Sure Thing Principle (proposition 2-1). From Sarin and Wakker (1998) theorem 3-1, we can state the following equality:

$$\nu(C) + \nu(D) = \nu(C \cup D) \tag{E1}$$

for either $C \subseteq \{s_1, s_2\}$ and $D \subseteq \{s_3, s_4\}$ or $C \subseteq \{s_3, s_4\}$ and $D \subseteq \{s_1, s_2\}$.

Consider now a filtration $\{\mathcal{F}'_t, t = 0,1,2\}$ with first stage events $E' = \{s_1, s_3\}$ and $E'^c = \{s_2, s_4\}$. By RCA and folding back, we have

$$W(x_1, ..., x_4) = V(V_{E'}(x_1, x_3), V_{E'^c}(x_2, x_4)).$$

It's easy to see that consequentialism and dynamic consistency are satisfied and so STP axiom holds on $\succcurlyeq$. By model consistency, $W, V, V_{E'}, V_{E'^c}$ are elements of $\mathbb{M}$, *i.e.* they are all CEU forms. We have to show that

$$\nu(H) + \nu(J) = \nu(H \cup J) \tag{E2}$$

for either $H \subseteq \{s_1, s_3\}$ and $J \subseteq \{s_2, s_4\}$ or $H \subseteq \{s_2, s_4\}$ and $J \subseteq \{s_1, s_3\}$.

Suppose that (E2) holds. Then,

$$\nu(\{s_1\}) + \nu(\{s_2\}) = \nu(\{s_1\} \cup \{s_2\})$$

and

$$\nu(\{s_3\}) + \nu(\{s_4\}) = \nu(\{s_3\} \cup \{s_4\}).$$

Moreover, ν is additive on $\{E', E'^c\}$. Adding up these equalities with (E1) gives $\sum_{i=1}^{4} \nu(\{s_i\}) = 1$. This implies that the single-stage evaluation W is an expected utility form which uses an additive measure p on $\{s_i\}_{i=1,...,4}$ and a utility u. Now we derive (E2) from our axioms.

Case 1. $H = \{s_1\}$, $J \subseteq \{s_2, s_4\}, I = \{s_2, s_4\} \setminus J$.

We denote by x_I, x_J the outcomes on I, J. We suppose the followings rank-ordering on X. $x_J \geq x_1 \geq x_3 \geq x_I$ and $x_J \geq x'_1 \geq x'_3 \geq x_I$:

$$(x_1, x_3, x_J, x_I) \sim (x'_1, x'_3, x_J, x_I) \tag{E3}$$

Note that the utility $u : X \to \mathbb{R}$ keeps the rank-ordering because it's strictly increasing. We replace x_J by x'_J which is s.t. $x_1 \geq x'_J \geq x_3 \geq x_I$ and $x'_J \geq x'_1 \geq x'_3 \geq x_I$. By STP and by folding back, (E3) holds if and only if:

$$(x_1, x_3, x'_J, x_I) \sim (x'_1, x'_3, x'_J, x_I) \qquad \text{(E4)}$$

In the left outcomes, the decision weight associated to $u(x_1)$ is affected if x_J is replaced by x'_J. But in the right outcomes, the decision weight of $\{s_1\}$ is not affected in the CEU form. This implies that

$$\nu(\{s_1\}) + \nu(J) = \nu(\{s_1\} \cup J). \qquad \text{(E5)}$$

Case 2. $H = \{s_1, s_3\} = E', J \subseteq \{s_2, s_4\}, I = \{s_2, s_4\} \setminus J$.

Now we suppose $\hat{x}_J$ which is such that $x_1 \geq \hat{x}_J \geq x_3 \geq x_I$ and $x'_1 \geq x'_3 \geq \hat{x}_J \geq x_I$. By STP and folding back, (E4) holds if and only if

$$(x_1, x_3, \hat{x}_J, x_I) \sim (x'_1, x'_3, \hat{x}_J, x_I) \qquad \text{(E6)}$$

In the CEU form, the decision weight of $\{s_3\}$ must be affected in the right outcomes However, in the left outcomes the decision weight of $\{s_3\}$ is not modified. Together with the equality (E5), the indifferences (E4) and (E6) implies that

$$\nu(\{s_1, s_3\}) + \nu(J) = \nu(\{s_1, s_3\} \cup J).$$

Case 3. $H = \{s_3\}$, $J \subseteq \{s_2, s_4\}, I = \{s_2, s_4\} \setminus J$.

This case is similar to case 1.

Case 4. $J \subseteq \{s_1, s_3\}, H \subseteq \{s_2, s_4\}$. This case is straightforward.

(E2) has now been proved. (E1) and (E2) imply that $\forall A \in \{E, E^c, E', (E')^c\}$, the first stage evaluation V uses a unique additive measure $p : \{A, A^c\} \to [0,1]$ and $\forall s, s' \in S$, the single stage evaluation uses the same measure $p : \{s\} \mapsto p(\{s\})$.

Now we show that the DM must use Bayes rule to update her beliefs. By hypothesis 2, the DM is free to use FUBU, DS, or Bayes to update her capacities. However, the equalities (E1) and (E2) imply that:

FUBU is reduced to Bayes rule:

$$s \in B \Rightarrow p(\{s\}|B) = \frac{\nu(\{s\} \cap B)}{1 + \nu(\{s\} \cap \mathrm{B}) - \nu(\{s\} \cup \mathrm{B}^c)}$$
$$= \frac{p(\{s\})}{1 + p(\{s\}) - p(\{s\}) - p(B^c)} = \frac{p(\{s\})}{p(B)}.$$

DS is reduced to Bayes rule:

$$s \in B \Rightarrow p(\{s\}|B) = \frac{\nu((\{s\} \cap B) \cup B^c) - \nu(B^c)}{1 - \nu(B^c)}$$
$$= \frac{p(\{s\}) + p(B^c) - p(B^c)}{1 - p(B^c)} = \frac{p(\{s\})}{p(B)}.$$

If the DM uses Bayes rule to update her capacities, then

$$\sum_{s \in B} \nu(\{s\}|B) = \sum_{s \in B} \frac{p(\{s\})}{p(B)} = 1$$

and

$$\sum_{s \in B^c} \nu(\{s\}|B^c) = \sum_{s \in B^c} \frac{p(\{s\})}{p(B^c)} = 1.$$

It implies that $\forall B \in \{E, E^c\}$, the conditional capacities $\nu(\{s\}|B), \nu(\{s\}|B^c)$ are additive. Moreover, the capacity is additive on $\{\mathcal{F}'_t, t = 0,1,2\}$, hence p is well defined on 2^S. To conclude the demonstration, it is sufficient to remark that model consistency implies that W, V, V_E, V_{E^c} use the same utility $u : X \to \mathbb{R}$, s.t. they are all expected utility representations. ■

4. Multiple Priors

In this section, we suppose that the DM considers a set $\mathcal{C} = \left\{\pi \middle| \pi \text{ additive on } 2^S\right\}$ of priors, and maximizes minimal expected utility. $\mathcal{C}$ is assumed compact and convex. Maxmin Expected utility over Savage acts has been axiomatized by Casadesus-Masanell and al. (2000). Model consistency with respect to a multiple priors form means that all forms from $\mathbb{M}$ use elements of $\mathcal{C}$ and satisfy axioms from Casadesus-Masanell and al. (2000). We define the MP representation:

Definition 4. *$W : \mathbb{R}^S \to \mathbb{R}$ is a multiple priors representation if and only if*

$$W : (x_1,...,x_4) \mapsto \min_{\pi\in\mathcal{C}} \int_S u\big(x(s)\big)\, d\pi(s) = \sum_{i=1}^{4} \pi_i^j.u(x_i),$$

where $\pi^j = \arg\min W(x_1,...,x_4)$ and $\sum_{i=1}^{4} \pi_i^j = 1$.

The DM is pessimistic and uses the probability measure which minimizes expected utility. Note that the measure which minimizes expected utility overweights the minimal utility. In other words, the value of the expected utility of an act f is rank-dependent: $\pi^j = \arg\min W(x_1,...,x_4)$ is valid only for a given rank-ordering.

If $x_1 \geq x_2 \geq x_3 \geq x_4$, then $V(.)$ is a multiple priors representation if

$$V\left(V_E(x_1,x_2), V_{E^c}(x_3,x_4)\right) = \pi^j(E)V_E(x_1,x_2) + \pi^j(E^c)V_{E^c}(x_3,x_4),$$

hence such that $\pi^j = \arg\min V\left(V_E(x_1,x_2), V_{E^c}(x_3,x_4)\right)$. If u is strictly increasing on X, then $V_E(x_1,x_2) > V_{E^c}(x_3,x_4)$ and so $\pi^j = \min_{\pi\in\mathcal{C}} \pi(E)$.

If $x_1 \geq x_2 \geq x_3 \geq x_4$, then the conditional valuations are:

$$V_E(x_1,x_2) = \min_{\pi\in\mathcal{C}} \pi(\{s_1\}|E)u(x_1) + (1 - \min_{\pi\in\mathcal{C}} \pi(\{s_1\}|E))u(x_2),$$
$$V_{E^c}(x_3,x_4) = \min_{\pi\in\mathcal{C}} \pi(\{s_3\}|E^c)u(x_3) + (1 - \min_{\pi\in\mathcal{C}} \pi(\{s_3\}|E^c)u(x_4).$$

Proposition 4-1. Let $\{\mathcal{F}_t, t=0,1,2\}$ be a filtration. If hypothesis 1 holds and if all elements of $\mathbb{M}$ maximize minimal expected utility by using a non-unique prior, then model consistency and folding back cannot be simultaneously satisfied.

Proof. Given a rank-ordering $x_1 \geq ... \geq x_4$ on X, we assume a measure π^a which minimizes expected utility of $V_E(x_1, x_2)$, a measure π^b which minimizes expected utility of $V_{E^c}(x_3, x_4)$, and a measure π^c which minimizes expected utility of $V(.)$. Note that $\pi^c(E) = \min_{\pi \in \mathcal{C}} \pi(E)$ because $V_E(x_1, x_2) \geq V_{E^c}(x_3, x_4)$. Folding back holds if and only if $W(.)$ uses a measure π' s.t.

$$\forall s \in E, \pi'(\{s\}) = \pi^c(E) \times \pi^a(\{s\}|E)$$

$$\forall s \in E^c, \pi'(\{s\}) = \pi^c(E^c) \times \pi^b(\{s\}|E^c)$$

hence

$$\min_{\pi \in \mathcal{C}} \pi_1 = \pi'_1 = \pi^c(E) \times \pi^a(\{s_1\}|E).$$

It implies that

$$\min_{\pi \in \mathcal{C}} \pi(\{s_1\}|E) = \frac{\pi'_1}{\pi^c(E)}.$$

But in the Multiple Priors form, we have $\pi^c(E) = \min_{\pi \in \mathcal{C}} \pi(E)$. This last equality implies a contradiction. ■

This result leads us to establish the following theorem:

Theorem 4-2. Let $\{\mathcal{F}_t, t=0,1,2\}$ be a filtration. We suppose that hypothesis 1-1 holds, and that consequentialism, dynamic consistency and folding back hold. Then the following two statements are equivalent:

(i) Model Consistency holds with respect to $\mathbb{M}$, the family of multiple priors forms, such that W, V, V_E, V_{E^c} are elements of $\mathbb{M}$.

(ii) There exist a utility $u : X \to \mathbb{R}$ and a unique additive measure $\pi : \mathcal{F}_1 \to [0,1]$ such that $\succcurlyeq$ can be represented by an expected utility form

$$V : (x_1,...,x_4) \mapsto \pi(E)V_E(x_1,x_2) + \pi(E^c)V_{E^c}(x_3,x_4),$$

where V_E, V_{E^c} uses minimal conditional probabilities calculated with Bayes rule.

Proof. The implication from (ii) to (i) is straightforward, because it's easy to see that W, V, V_E, V_{E^c} are multiple priors forms. Moreover, it is clear that the statement (ii) implies folding back, consequentialism and dynamic consistency. We concentrate our attention on the implication from (i) to (ii). $V(.)$ is reduced to an expected utility form if

$$\forall B \in \{E, E^c\}, \max_{\pi \in \mathcal{C}} \pi(B) = \min_{\pi \in \mathcal{C}} \pi(B) \qquad \text{(E1)}$$

If (E1) holds, then π is unique on $\mathcal{F}_1$. The statement (ii) follows from (E1) and from the model consistency property. Now we prove (E1).

Case 1. $B = E, D \subseteq \{s_3, s_4\}, D' \subset \{s_3, s_4\} \setminus D$.

First consider the followings rank-ordering on X: $x_1 \leq x_D \leq x_{D'} \leq x_2$ and $x_2' \leq x_1' \leq x_D \leq x_{D'}$, where x_D is the outcome associated to event D and $x_{D'}$ is the outcome associated to D'. The utility $u : X \to \mathbb{R}$ keeps this rank-ordering because it is strictly increasing. We also suppose the following indifference:

$$f = (x_1, x_2, x_D, x_{D'}) \sim (x_1', x_2', x_D, x_{D'}) = g \qquad \text{(E2)}$$

We note π^i the measure which minimizes $V(V_E(x_1,x_2), V_{E^c}(x_D, x_{D'}))$. By folding back, we have

$$W(f) = W(g) \Leftrightarrow V(V_E(x_1,x_2), V_{E^c}(x_D,x_{D'})) = V(V_E(x_1',x_2'), V_{E^c}(x_D,x_{D'})).$$

By dynamic consistency,

$$V_E(x_1,x_2) = V_E(x_1',x_2'),$$

hence

$$\pi^i = \arg\min V(V_E(x_1,x_2), V_{E^c}(x_D,x_{D'})) \Leftrightarrow \pi^i = \arg\min V(V_E(x_1',x_2'), V_{E^c}(x_D,x_{D'})).$$

If $u : X \to \mathbb{R}$ is continuous and strictly increasing, then

$$\left(x'_2 \leq x'_1 \leq x_D \leq x_{D'}\right) \Rightarrow V_E(x'_1, x'_2) \leq V_{E^c}(x_D, x_{D'}).$$

In the Multiple Priors form, this implies that

$$\pi^i(E) = \max_{\pi \in \mathcal{C}} \pi(E)$$

and

$$\pi^i(E^c) = \min_{\pi \in \mathcal{C}} \pi(E^c).$$

By STP (consequentialism and dynamic consistency), (E2) holds if and only if

$$f' = (x_1, x_2, x'_D, x'_{D'}) \sim (x'_1, x'_2, x'_D, x'_{D'}) = g' \qquad \text{(E3)}$$

where $x'_D, x'_{D'}$ are such that $x_1 \leq x'_D \leq x'_{D'} \leq x_2$ and $x'_D \leq x'_{D'} \leq x'_2 \leq x'_1$. f and f' give the same rank-ordering on X. It implies that the single stage evaluation $W(.)$ and the conditional evaluations $V_E(.), V_{E^c}(.)$ use the same probability from (E2) to (E3). Then folding back and model consistency implies

$$\pi^i = \arg\min V(V_E(x_1, x_2), V_{E^c}(x'_D, x'_{D'})),$$

s.t. all elements of $\mathbb{M}$ use the same probabilities to value f and f'. Again, if $u : X \to \mathbb{R}$ is continuous and strictly increasing, then

$$\left(x'_D \leq x'_{D'} \leq x'_2 \leq x'_1\right) \Rightarrow V_E(x'_1, x'_2) \geq V_{E^c}(x'_D, x'_{D'}).$$

In the Multiple Priors form, this implies that

$$\pi^i(E) = \min_{\pi \in \mathcal{C}} \pi(E)$$

and

$$\pi^i(E^c) = \max_{\pi \in \mathcal{C}} \pi(E^c).$$

Therefore,

$$\max_{\pi \in \mathcal{C}} \pi(E) = \min_{\pi \in \mathcal{C}} \pi(E).$$

Case 2. $B = E^c, D \subseteq \{s_1, s_2\}, D' \subset \{s_1, s_2\} \setminus D$. This case is straightforward.

We have shown that $\forall B \in \{E, E^c\}, \max_{\pi \in \mathcal{C}} \pi(B) = \min_{\pi \in \mathcal{C}} \pi(B)$. Therefore π is unique on $\mathcal{F}_1$ and the first stage evaluation $V(.)$ uses a unique additive measure π. The minimal probabilities used by V_E, V_{E^c} are calculated with Bayes rule :

$$\forall s \in E, \min_{\pi \in \mathcal{C}} \pi(\{s\}|E) = \frac{\min_{\pi \in \mathcal{C}} \pi(\{s\})}{\pi(E)}$$

$$\forall s \in E^c, \min_{\pi \in \mathcal{C}} \pi(\{s\}|E^c) = \frac{\min_{\pi \in \mathcal{C}} \pi(\{s\})}{\pi(E^c)}.$$

Under folding back, it implies that W, V_E, V_{E^c} can used a non-unique prior. Moreover, W, V, V_E, V_{E^c} use the same utility $u : X \to \mathbb{R}$ and V is an expected utility form. ■

Corollary 4-3. Let $\{\mathcal{F}_t, t = 0,1,2\}$ be a filtration. We suppose that folding back and Model Consistency hold with respect to a family $\mathbb{M}$ of Multiple Priors forms. Then the DM must maximize expected utility in the first stage but she's free to use a non-unique prior in the second stage.

Proof. It is sufficient to remark that the contradiction of proposition 3-1 is now removed. Then probabilities used by the single stage evaluation are obtained by multiplying probability used by the first stage evaluation $V(.)$ with probabilities used by V_E and V_{E^c}. Therefore, folding back and model consistency are simultaneously satisfied if and only if V uses a unique additive measure but W, V_E, V_{E^c} use a non-unique prior. ■

Similarly to CEU model (Sarin and Wakker (1998, corollary 3-3)) , MP model can be used in situations where first stage events involve no ambiguity but second stage events may involve ambiguity.

Example 4-4. Suppose that the utility u is the identity function. We also suppose a convex set $\mathcal{C} = \{\alpha\pi^1 + (1-\alpha)\pi^2 \,|\, \pi^1, \pi^2 \text{ additive on } 2^S, \alpha \in [0,1]\}$ of priors, where

$$\pi^1(E) = \pi^1(E^c) = \frac{1}{2}$$
$$\pi^2(E) = 0.6, \pi^2(E^c) = 0.4.$$

$\pi^1(.|E), \pi^1(.|E^c), \pi^2(.|E)$ and $\pi^2(.|E^c)$ are conditional probabilities calculated with Bayes rule:

$$\pi^1(\{s_1\}|E) = 0.8; \pi^1(\{s_2\}|E) = 0.2; \pi^1(\{s_3\}|E^c) = 0.2; \pi^1(\{s_4\}|E^c) = 0.8;$$
$$\pi^2(\{s_1\}|E) = 0.6; \pi^2(\{s_2\}|E) = 0.4; \pi^2(\{s_3\}|E^c) = 0.6; \pi^2(\{s_4\}|E^c) = 0.4.$$

If V_E, V_{E^c} are multiple priors forms, then the value of $(1,2,3,4)$ with the first stage evaluation V is given by

$$\left(1 - \min_{\pi \in \mathcal{C}} \pi(E^c)\right)\left[\min_{\pi \in \mathcal{C}} \pi(\{s_2\}|E) \times 2 + (1 - \min_{\pi \in \mathcal{C}} \pi(\{s_2\}|E)) \times 1\right]$$
$$+ \min_{\pi \in \mathcal{C}} \pi(E^c)\left[\min_{\pi \in \mathcal{C}} \pi(\{s_4\}|E^c)) \times 4 + (1 - \min_{\pi \in \mathcal{C}} \pi(\{s_4\}|E^c)) \times 3\right],$$

because $V_E(1,2) < V_{E^c}(3,4)$. Then,

$V(V_E(1,2), V_{E^c}(3,4)) = 0.6 \times (0.8 \times 1 + 0.2 \times 2) + 0.4 \times (0.6 \times 3 + 0.4 \times 4) = 2,08$.

Note that $V_E(.)$ uses the conditional measure $\pi^1(.|E)$ but $V_{E^c}(.)$ uses the conditional measures $\pi^2(.|E^c)$.

If the single stage evaluation W is also a multiple priors form, then

$$W(1,2,3,4) = 0.36 \times 1 + 0.24 \times 2 + 0.24 \times 3 + 0.16 \times 4 = 2,2$$

This implies that folding back does not hold for $\pi^1 \neq \pi^2$ on $\{E, E^c\}$, because we have $\pi^2 = \arg\min W(x_1, ..., x_4)$. By theorem 2, if folding back and model consistency hold together, then $\pi^1 = \pi^2$ on $\{E, E^c\}$. We suppose that $V(.)$ uses probabilities $\pi^1(E) = \pi^1(E^c) = \pi^2(E) = \pi^2(E^c)$. The value of $(1,2,3,4)$ with the first stage evaluation V is given by

$$\frac{1}{2} \times (0.8 \times 1 + 0.2 \times 2) + \frac{1}{2} \times (0.6 \times 3 + 0.4 \times 4) = 2,3.$$

Again, $V_E(.)$ uses the conditional measure $\pi^1(.|E)$ but $V_{E^c}(.)$ uses the conditional measures $\pi^2(.|E^c)$.

We can easily see that folding back holds, because if the single stage evaluation $W(.)$ is also a multiple priors representation, then

$$W(1,2,3,4) = 0.4 \times 1 + 0.1 \times 2 + 0.3 \times 3 + 0.2 \times 4 = 2,3\,.$$

Note that $W(.)$ uses the measure π^2 on $\{s_1, s_2\}$ and the measure π^1 on $\{s_3, s_4\}$, s.t. $(\pi^2, \pi^2, \pi^1, \pi^1) = \arg\min W(x_1, ..., x_4)$ under MP. ■

Moreover, MP is reduced to an expected utility form in all stages of $\{\mathcal{F}_t, t = 0,1,2\}$ if and only if RCA axiom holds.

Theorem 4-5. Let $\{\mathcal{F}_t, t = 0,1,2\}$ be a filtration. We suppose that hypothesis 1 holds. We also assume consequentialism, dynamic consistency and folding back. Then the following two statements are equivalent:

(i) Model consistency holds with respect to $\mathbb{M}$, the family of MP forms, and reduction of compound acts holds.

(ii) There exist a utility function $u : X \to \mathbb{R}$ and a unique additive measure $\pi : 2^S \to [0,1]$ such that $\forall B \in \{E, E^c\}$, W, V, V_B are expected utility forms, and V_B use conditional probabilities $\pi(.|B) : \mathcal{E}_2 \to [0,1]$ calculated with Bayes rule.

Proof. The implication from (ii) to (i) is straightforward, because the expected utility representation verify RCA. Now we prove the implication from (i) to (ii).
We assume that $x_1 \geq ... \geq x_4$. Note that the utility $u : X \to \mathbb{R}$ keeps this rank-ordering because it is strictly increasing. We note π^j the probability which minimizes the single stage evaluation $W(.)$ s.t. $\pi^j = \arg\min W(x_1, ..., x_4)$. If folding back and model consistency hold, we know from our theorem 4-1 that V is an expected utility form which uses a unique additive measure $\pi : \{E, E^c\} \to [0,1]$. Now we assume a

filtration $\{\mathcal{F}'_t, t=0,1,2\}$ with first stage events $E'=\{s_1,s_3\}$ and $E'^c=\{s_2,s_4\}$. By RCA and by folding back, we have

$$W(x_1,...,x_4)=V(V_{E'}(x_1,x_3),V_{E'^c}(x_2,x_4)).$$

If model consistency holds, then $V_{E'},V_{E'^c}$ are multiple priors forms and we can shown (similarly to the proof of the theorem 4-2) that V is an expected utility form which uses a unique additive measure $\pi:\{E',E'^c\}\to[0,1]$ and a utility $u:X\to\mathbb{R}$. Then,

$$\begin{aligned}
&W(x_1,...,x_4)=V(V_{E'}(x_1,x_3),V_{E'^c}(x_2,x_4))=V(V_E(x_1,x_2),V_{E^c}(x_3,x_4))\\
&\Leftrightarrow \pi_1^j u(x_1)+\pi_2^j u(x_2)+\pi_3^j u(x_3)+\pi_4^j u(x_4)\\
&=\pi(E)\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E)u(x_1)+\pi(E)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E))u(x_2)\\
&+\pi(E^c)\min_{\pi\in\mathcal{C}}\pi(\{s_3\}|E^c)u(x_3)+\pi(E^c)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_3\}|E^c))u(x_4)\\
&=\pi(E')\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E')u(x_1)+\pi(E')(1-\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E'))u(x_3)\\
&+\pi((E')^c)\min_{\pi\in\mathcal{C}}\pi(\{s_2\}|E'^c)u(x_2)+\pi((E')^c)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_2\}|E'^c))u(x_4)
\end{aligned}$$

If this last equality holds $\forall x_1,x_2,x_3,x_4\in X$ s.t. $x_1\geq...\geq x_4$, then W,V,V_E,V_{E^c} and $V_{E'},V_{E'^c}$ use the same probability $\pi^j\in\mathcal{C}$. Moreover, they use the same utility $u:X\to\mathbb{R}$. π^j is unique on 2^S if and only if

$$\left.\begin{array}{l}
\forall s\in A\in\{E,E^c,E',E'^c\},\sum_{s\in A}\min_{\pi\in\mathcal{C}}\pi(\{s\})=\pi(A)\\
\text{or}\\
\forall s\in A\in\{E,E^c,E',E'^c\},\sum_{s\in A}\max_{\pi\in\mathcal{C}}\pi(\{s\})=\pi(A)
\end{array}\right\}\qquad\text{(E1)}$$

If (E1) holds, then

$$\forall s\in A,\min_{\pi\in\mathcal{C}}\pi(\{s\})=\max_{\pi\in\mathcal{C}}\pi(\{s\})\Rightarrow\forall s\in S,\forall\pi^i,\pi^j\in\mathcal{C},\pi^i(\{s\})=\pi^j(\{s\}).$$

Together with the unicity of π on $(\mathcal{F}_1\cup\mathcal{F}'_1)$, this implies that there exists a unique additive measure π on 2^S. Now we prove (E1).

Case 1. $A=E$.

We have $\pi(E')=\pi_1^j+\pi_3^j$ and $\pi(E)=\pi_1^j+\pi_2^j$.

Remark that

$$\pi_1^j = \pi(E)\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E) = \pi(E')\min_{\pi\in\mathcal{C}}\pi(\{s_1\}\big|E')$$

and that

$$\pi_2^j = \pi(E)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E)) = \pi(E')^c\min_{\pi\in\mathcal{C}}\pi((\{s_2\}|E'^c)).$$

Moreover,

$$\min_{\pi\in\mathcal{C}}\pi(\{s_1\}) = \pi(E)\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E)$$
$$\min_{\pi\in\mathcal{C}}\pi(\{s_2\}) = \pi(E')^c\min_{\pi\in\mathcal{C}}\pi(\{s_2\}|E'^c).$$

This implies that

$$\pi(E)\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E) + \pi(E)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E))$$
$$= \pi(E')\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E') + \pi((E')^c)\min_{\pi\in\mathcal{C}}\pi(\{s_2\}|E'^c)$$
$$\Leftrightarrow \pi(E) = \min_{\pi\in\mathcal{C}}\pi(\{s_1\}) + \min_{\pi\in\mathcal{C}}\pi(\{s_2\}) \qquad \text{(E2)}$$

We have proved (E1) for $A = E$ s.t. $\forall s \in E, \forall \pi^i, \pi^j \in \mathcal{C}, \pi^i(\{s\}) = \pi^j(\{s\})$ and therefore π is unique on E.

Case 2. $A = E^c$.

We have $\pi(E') = \pi_1^j + \pi_3^j$ and $\pi(E^c) = \pi_3^j + \pi_4^j$.

Remark that

$$\pi_3^j = \pi(E^c)\min_{\pi\in\mathcal{C}}\pi(\{s_3\}|E^c) = \pi(E')(1-\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E'))$$

and that

$$\pi_4^j = \pi(E^c)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_3\}|E^c)) = \pi(E'^c)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_2\}|E'^c)).$$

Moreover,

$$\max_{\pi\in\mathcal{C}}\pi(\{s_3\}) = \pi(E')(1-\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E')),$$
$$\max_{\pi\in\mathcal{C}}\pi(\{s_4\}) = \pi(E'^c)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_2\}|E'^c)).$$

This implies that

$$\pi(E^c)\min_{\pi\in\mathcal{C}}\pi(\{s_3\}|E^c)+\pi(E^c)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_3\}|E^c))$$
$$=\pi(E')(1-\min_{\pi\in\mathcal{C}}\pi(\{s_1\}|E'))+\pi(E'^c)(1-\min_{\pi\in\mathcal{C}}\pi(\{s_2\}|E'^c))$$
$$\Leftrightarrow \pi(E^c)=\max_{\pi\in\mathcal{C}}\pi(\{s_3\})+\max_{\pi\in\mathcal{C}}\pi(\{s_4\}) \qquad \text{(E3)}$$

Therefore, $\forall s\in E^c, \forall\pi^i,\pi^j\in\mathcal{C}, \pi^i(\{s\})=\pi^j(\{s\})$ and so π is unique on E^c.

Case 3. $A=E'$ and $A=(E')^c$.

This case is similar to cases 1 and 2, so we can shown that

$$\pi(E')=\min_{\pi\in\mathcal{C}}\pi(\{s_1\})+\min_{\pi\in\mathcal{C}}\pi(\{s_3\}) \qquad \text{(E4)}$$

and that

$$\pi((E')^c)=\max_{\pi\in\mathcal{C}}\pi(\{s_2\})+\max_{\pi\in\mathcal{C}}\pi(\{s_4\}). \qquad \text{(E5)}$$

(E2) and (E3) imply that V_E, V_{E^c} use a unique additive measure updating from $\pi\in\mathcal{C}$. (E4) and (E5) imply that $V_{E'}, V_{E'^c}$ use a unique additive measure updating from $\pi\in\mathcal{C}$. Therefore, W uses a unique additive measure π. All elements of $\mathbb{M}$ are expected utility forms which use an unique additive measure π and an utility $u: X\to\mathbb{R}$. ■

5. Arbitrary finite numbers of events and stages

Now we consider the general case where S contains any finite number $S\geq 4$ of states, and T contains any finite number $T\geq 2$ of stages. $\mathcal{E}_{T-1}=\{E^1_{T-1},...,E^{N^{T-1}}_{T-1}\}$ is the finite partition which contains N^{T-1} events at time $T-1$. We note $f_{E_{T-1}}$ the restriction of an act f to the elements of event E_{T-1}.

Therefore, folding back can be rewritten as

$$W(f)=V\left(V_{E^1_1}(...(V_{E^i_{T-1}}(f_{E^i_{T-1}}))...),...,V_{E^{N^1}_1}(...(V_{E^j_{T-1}}(f_{E^j_{T-1}}))...)\right) \qquad \text{(I)}$$

where $i\in card\{E_{T-1}\in\mathcal{E}_{T-1}\,|\,E_{T-1}\subset E^1_1\}$ and $j\in card\{E_{T-1}\in\mathcal{E}_{T-1}\,|\,E_{T-1}\subset E^{N^1}_1\}$.

Theorem 5-1. Let $\mathcal{F}_T = \{\mathcal{E}_0,...,\mathcal{E}_T\}$ be a filtration with $T \geq 2$. Suppose that hypothesis 1 holds. $\forall t = 1,...,T-1, \forall E_t \in \mathcal{E}_t,$ we assume that consequentialism holds on $\{\succcurlyeq_{E_t}\}_{t=1,...,T-1}$ and that dynamic consistency holds between $\succcurlyeq$ and $\{\succcurlyeq_{E_t}\}_{t=1,...,T-1}$. If folding back and model consistency hold with respect to a family $\mathbb{M}$, then,

(i) If $\mathbb{M}$ is the Multiple Priors family, then $\forall \tau = 0,...,T-2,\ \forall E_\tau \in \mathcal{E}_\tau,\ V_{E_\tau}$ is an EU form which uses a unique additive measure $\pi : \mathcal{F}_{T-1} \to [0,1]$, and $\forall E_{T-1} \in \mathcal{E}_{T-1},\ W, V_{E_{T-1}}$ are all multiple priors forms which use a non-unique prior, and, $W, V_{E_\tau}, V_{E_{T-1}}$ use the same utility $u : X \to \mathbb{R}$.

(ii) If $\mathbb{M}$ is the Choquet Expected Utility family and if hypothesis 2 and RCA axiom hold, then $\forall t = 0,...,T-1,\ \forall E_t \in \mathcal{E}_t,\ W, V_{E_t}$ are all expected utility forms which use a unique additive measure $p : 2^S \to [0,1]$ and the same utility $u : X \to \mathbb{R}$.

(iii) If $\mathbb{M}$ is the Multiple Priors family and if RCA axiom holds, then $\forall t = 0,...,T-1,\ \forall E_t \in \mathcal{E}_t,\ W, V_{E_t}$ are all expected utility forms which use a unique additive measure $\pi : 2^S \to [0,1]$ and the same utility $u : X \to \mathbb{R}$.

Proof. Throughout the proof, we assume folding back as depicted in (I). We first prove part (i).

(i) Note that the main argument of the proof of the theorem 2 is the Sure Thing-Principle, which allows to replace any result on an event E_τ, for $\tau = 0,...,T-1$, by another result and so another utility. By proposition 1, STP holds on $\{\succcurlyeq_{E_t}\}_{t=0,...,T-2}$. By verify all cases, we can show that $\forall \tau = 0,...,T-1,\ \forall E_\tau \in \mathcal{E}_\tau$, $\max_{\pi \in \mathcal{C}} \pi(E_\tau | E_{\tau-1}) = \min_{\pi \in \mathcal{C}} (E_\tau | E_{\tau-1})$. It implies that π is unique on $\mathcal{F}_{T-1}$ s.t. V_{E_τ} uses an unique additive measure. The rest of the proof requires no adaptation and $\forall E_{T-1} \in \mathcal{E}_{T-1},\ W, V_{E_{T-1}}$ use a non-unique prior because we do not impose STP on $\{\succcurlyeq_{E_{T-1}}\}_{E_{T-1} \in \mathcal{E}_{T-1}}$. Therefore, $V_{E_{T-1}}$ can use several conditional probabilities $\pi^i(.|E_{T-1}) : \mathcal{E}_T \to [0,1]$ s.t.

$$\forall E_T \in \mathcal{E}_T, \forall \pi^i \in \mathcal{C}, \pi^i(E_T \left| E_{T-1}\right.) = \frac{\pi^i(E_T)}{\pi(E_{T-1} \left| E_{T-2}\right.)}.$$

Moreover, model consistency implies that $\forall \tau = 0,...,T-2, \quad \forall E_\tau \in \mathcal{E}_\tau$, $\forall E_{T-1} \in \mathcal{E}_{T-1}$, $W, V_{E_\tau}, V_{E_{T-1}}$ use the same utility. Now we prove statement (ii).

(ii) For the CEU family, STP (consequentialism and dynamic consistency) implies the additivity of ν on $\mathcal{F}_\tau, \forall \tau = 0,...,T-1$. Therefore,

$$\forall E_\tau \in \mathcal{E}_\tau, \forall s \in E_\tau, \forall s' \in E_\tau^c, \nu(\{s\}) + \nu(\{s'\}) = \nu(\{s\} \cup \{s'\}).$$

The main implication of RCA is that it does not modify the preference ordering on $\mathbb{R}^S$ whatever partition fixed at time $\tau \in T$. It implies that STP holds on $T-2$ stages, whatever filtration faced to the DM. All elementary events from S are separable. Therefore, by verify all cases, we can show that $\forall s, s' \in S$, ν is additive such that $\nu(\{s\}) + \nu(\{s'\}) = \nu(\{s\} \cup \{s'\})$. Similarly to the proof of theorem 1, FUBU and DS are reduced to Bayes rules such that $\forall \tau = 1,...,T, \nu(E_\tau \left| E_{\tau-1}\right.) = \frac{\nu(E_\tau)}{\nu(E_{\tau-1} \left| E_{\tau-2}\right.)}$. Therefore, W uses an additive measure p and $V, V_{E_1},...,V_{E_{T-1}}$ use conditional probabilities calculated with Bayes rule. It implies that $\forall t = 0,...,T-1, \quad \forall E_t \in \mathcal{E}_t$, W, V_{E_t} are all expected utility forms which use an additive measure $p : 2^S \to [0,1]$ and the same utility $u : X \to \mathbb{R}$.

(iii) Again the main implication of RCA is that it implies STP on $T-2$ stages, whatever filtration faced to the DM. Therefore, $\forall \mathcal{F}_T \subseteq 2^S$, π is unique on $\mathcal{F}_{T-1}$. All probabilities used by W are obtained by probabilistic multiplication (Bayes' rule). Therefore, given a rank-ordering on X, folding back implies that the same probability is used by all elements of $\mathbb{M}$ on all filtrations. Extension of the proof of theorem 3 is straightforward, and we can shown that this probability is unique by verifying all cases. This implies that $\forall t = 0,...,T-1, \quad \forall E_t \in \mathcal{E}_t, \quad W, V_{E_t}$ are all expected utility

forms which use a unique additive measure $\pi : 2^S \to [0,1]$ and the same utility $u : X \to \mathbb{R}$. ■

6. Related literature and conclusion

Since Sarin and Wakker (1998), dynamic consistency of NonExpected Utility preferences has been studied in several papers. These papers give an axiomatic understanding of the links between NEU preferences and dynamic choice. It is no surprise that given NEU preferences, all of them have to relax a specific dynamic choice principle to preserve the other.

Regarding the CEU model, there are several ways to preserve consistency in dynamic choice situations. One is to impose some restrictions on behavior under uncertainty. Assuming Model Consistency with respect to a convex capacity updated with Bayes rule, Dempster-Shafer rule or FUBU (consequentialism) in all stages of the filtration, Eichberger and al. (2005) show that dynamic consistency holds if and only if the capacity is additive over the final stage. Therefore, to preserve dynamic consistency, they assume aversion to ambiguity on $T-1$ stages. Another way to preserve dynamic consistency is to relax consequentialism. Chateauneuf and al. (2001) relate conditioning and comonotony (or antimonotony) of information with the valued random variable. The DM minimizes the role of information (pessimism). She uses Bayes rule when information is comonotonic with the valued act and Dempster-Shafer rule when information is antimonotonic with it. As a consequence, counterfactuals outcomes do matter and hence consequentialism does not hold. This implies that dynamic consistency can be preserved when information is comonotone or antimonotone with the valued act. A third way to preserve dynamic consistency is to weaken model consistency. Nishimura and Ozaki (2003) preserve dynamic consistency of CEU preferences by weakening the axiom of comonotonic independence of Gilboa and Schmeidler (1989) from time 1 to time T on a given filtration. Therefore, they relax model consistency.

Similarly, several papers have extensively studied MP in dynamic choice situations. Assuming consequentialism and reduction of compound acts, Sinischalchi (2006) weakens dynamic consistency and shows that this allows the existence of a MP representation in all stages of the filtration. This also permits to take into account Ellsberg-type preferences while preserving a form of dynamic consistency.
It is also possible to preserve dynamic consistency of MP model by imposing some restrictions on the set of priors. Epstein and Schneider (2003) show that consequentialism, dynamic consistency and model consistency hold with respect to a MP representation if and only if the set of priors is rectangular. This assumption implies that the set of priors does not contain probability measures which do not ensure dynamic consistency of MP preferences. As a consequence, Ellsberg-type preferences cannot always be taken into account and Epstein and Schneider note that, in some settings, ambiguity may question dynamic consistency.

Pursuing the works of Sarin and Wakker (1998), we have studied how decision criteria that take attitude toward uncertainty into account could be consistently used in sequential choice situations. Our result imply that NonExpected Utility models cannot simultaneously satisfy consequentialism, dynamic consistency and model consistency. To be more precise, these axioms impose some restrictions on the information structure, which must contain unambiguous events on $T-1$ stages. Adding up the reduction of compound acts axiom implies that NonExpected Utility models collapse in Expected Utility in all stages of the filtration. The use of Multiple Priors and Choquet Expected Utility models in sequential choice situations involves the same restrictions. From a strictly technical point of view, this result is due to the fact that these models are based on a very similar axiomatic foundation.

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