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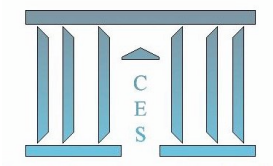
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The core of games on distributive lattices: how to share benefits in a hierarchy

Michel GRABISCH and Lijue XIE
University of Paris I – Panthéon-Sorbonne
Centre d'Economie de la Sorbonne
106-112, Bd de l'Hôpital, 75013 Paris, France
Email: michel.grabisch@univ-paris1.fr

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Abstract

Finding a solution concept is one of the central problems in cooperative game theory, and the notion of core is the most popular solution concept since it is based on some rationality condition. In many real situations, not all possible coalitions can form, so that classical TU-games cannot be used. An interesting case is when possible coalitions are defined through a partial ordering of the players (or hierarchy). Then feasible coalitions correspond to teams of players, that is, one or several players with all their subordinates. In these situations, it is not obvious to define a suitable notion of core, reflecting the team structure, and previous attempts are not satisfactory in this respect. We propose a new notion of core, which imposes efficiency of the allocation at each level of the hierarchy, and answers the problem of sharing benefits in a hierarchy. We show that the core we defined has properties very close to the classical case, with respect to marginal vectors, the Weber set, and balancedness.

Keywords: cooperative game, feasible coalition, core, hierarchy
JEL Classification: C71

1 Introduction

In cooperative game theory, a central topic is to define a rational way for distributing the total outcome among players (solution concept of this game). For transferable utility (TU) games, there exist two well-known solutions: the Shapley value [20], and the core [13]. The first one is defined by a set of rationality axioms: linearity, null player axiom, symmetry, and efficiency, and it is applicable to any game. The second one avoids the formation of subcoalitions of the grand coalition, in the sense that any subcoalition will receive at least the amount it can achieve by itself[1]. It may happen that no such solution

[1]In this paper, we consider games as *profit games*, hence the core is seen as a rational way to share benefits. We may consider *cost games* as well, reversing inequalities accordingly.

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exists. Classical results show under which conditions the core is nonempty, and give the structure of the core when the game is convex [21, 18].

In the classical setting of TU-games, any coalition $S \subseteq N$ can form, and each player can participate or not participate to the game. Mathematically speaking, this amounts to define the characteristic function of a game as a real-valued function v on the Boolean lattice 2^N, and vanishing at the empty set. More general definitions allowing a better modelling of reality have been proposed. We may distinguish between games having a restricted set of feasible coalitions (which may induce in some cases a hierarchy among players), and games permitting a more complex mechanism of participation. In the first category, we find games with precedence constraints [11], games on matroids, convex geometries and other combinatorial structures [2, 4], games on regular set systems [24], games on augmenting systems [3], games on communication graphs [19, 22, 23] (see a comparative survey of all these structures in [14]) ; in the second category, we find multichoice games of Hsiao and Raghavan [17], fuzzy games [6], and games on product of distributive lattices [15]. In many cases, the characteristic function of such general games can be considered as a real-valued function defined over a (often distributive) lattice.

In this paper, we propose a definition for the core of TU-games whose characteristic function is $v : L \to \mathbb{R}$, where L is a distributive lattice. There are two main reasons for focusing on this kind of game. The first one appears clearly from the previous discussion, since many of the above examples are related to lattices, or even their internal structure are exactly distributive lattices. The second reason is that a distributive lattice, by Birkhoff's theorem, is generated by defining a partial order on the set of players. This is in fact exactly the framework considered by Faigle and Kern [11], since precedence constraints among players are nothing else than that. A partial order on players can be interpreted in several ways, according to the application context, but there is one which is self-evident: it defines a hierarchy on players, in the sense that $j \leq i$ means that j is a subordinate of i. Moreover, the lattice generated by this order is composed by all possible downsets on N, where a downset is a subset of N where all subordinates of players in the downset are present. Therefore, the lattice can be interpreted as the set of all possible teams compatible with the hierarchy. This clearly applies for example to companies, or any other structured entity producing some benefit. In this context, defining the core of games on such structures amounts to define a way of sharing the total benefit $v(N)$ achieved among the members, in a way that fully respects the achievement of each team.

As we will show, the study of the core appears to be more complex than in the classical case, although similar results still hold. A first fact is that the core, defined as in the classical way, is still a polyhedron but possibly unbounded. This is not surprising, considering the general results obtained by Derks and Reijnierse [9], about the boundedness of the core for games defined on a set of feasible coalitions. However, this negative results prevents us to use the core in its original definition as a way of sharing benefits, for monetary amounts should remain bounded. The only way to get out of this situation is to impose further constraints on the core, that is, to add new inequalities or equalities in its definition, so as to get it bounded. An obvious way to do it is to impose the nonnegativity of the payoffs for players. Then we obtain what is generally called the positive core, introduced by Faigle [10]. Since we have no special reason to impose nonnegativity[2], we

[2]It could happen that a player may induce some loss when participating to certain teams. Such a player should be penalized when sharing the benefit, therefore payoffs could be negative.

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have to find another way to impose constraints, with should reflect some rationality. One of the main achievement of this paper is precisely to solve this issue, by adding equality constraints playing the rôle of efficiency, at each level of the hierarchy. The new definition of the core we obtain is called the restricted core, and it is always bounded. Moreover, it has a clear interpretation in our context of sharing benefits. The second achievement is that we prove that the restricted core has properties very similar to the core of classical games: in particular, the inclusion of the restricted core into the (restricted) Weber set always holds, and equality holds when the game is convex.

There are in the literature other works dealing with hierarchies, in particular by Demange [8], and van den Brink et al. [23]. The latter is more related to communication graphs and deals with the selectope, while the former consider a rather different definition of a team, where all subordinates need not be present. In particular, any singleton is a team, which we do not think meaningful in our context. We discuss these related works at the end of the paper.

The paper is organized as follows. We begin by introducing and recalling essential definitions about lattices and partially ordered sets (posets) in Section 2. Then Sections 3, 4 present the basic definitions for games on distributive lattices and the core. In the next sections 5, 6, 7, we study their properties. We indicate in Section 8 how to apply our results to the case of product lattices, encompassing the case of multichoice games. In Section 9, we give a brief account on related works in the literature.

2 Posets, distributive lattices and levels

(see, e.g., Davey and Priestley [7]) In all this section, sets are considered to be finite. A set P equipped with a binary relation $\leq$ is a *partially ordered set* (or *poset*) if the binary relation $\leq$ satisfies reflexivity, antisymmetry and transitivity (*partial order*). For any two elements $x, y \in P$, $x < y$ means $x \leq y$ and $x \neq y$. If neither $x \leq y$ nor $y \leq x$, we say that x and y are *incomparable*. If there exists no $y \in P$ such that $y < x$, we call x a *minimal element* of this poset; if there exists no $y \in P$ such that $y > x$, x is a *maximal element* of this poset. We say that x is a *greatest element* of P if $x \geq y$ for all $y \in P$ (and similarly for the *least element*). The greatest and least elements of P are unique whenever they exist, and are denoted by $\top$ and $\bot$ respectively.

Let $x, y \in P$ and $x < y$. If there is no $z \in P$, such that $x < z < y$, we say that y *covers* x, denoted by $x \prec y$. A subset $A \subseteq P$ is called an *antichain* if it is a singleton or if any two elements of A are incomparable. On the other hand, a subset $C \subseteq N$ is called a *chain* if it contains no pair of incomparable elements[3]. For $x, y \in P$ and $x < y$, a *chain* C *from* x *to* y can therefore be considered as a sequence of totally ordered elements from x to y, i.e., $C = \{x =: z_0 < z_1 < \cdots < z_{k-1} < z_k := y\}$. The chain is *maximal* if no other chain from x to y contains it (equivalently, if $z_0 \prec z_1 \prec z_2 \prec \ldots \prec z_k$). For convenience, a maximal chain from some minimal element to some maximal element of P is called simply a *maximal chain*. The set of all maximal chains of P is denoted by $\mathcal{C}(P)$. The *length* $\ell(C)$ of a chain C is $|C| - 1$. For any element $x \in P$, its *height* $h(x)$ is the length of a longest chain from some minimal element to x:

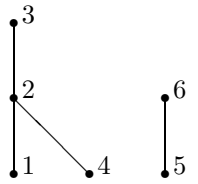
$$h(x) = \max\{\ell(C) \mid C = \{x_0, x_1, \ldots, x\}\}.$$

[3]Note that a singleton is both an antichain and a chain.

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The height function induces a natural partition $\{Q_1, \ldots, Q_q\}$ of P as follows: Q_i is the set of elements of height $i-1$, $i = 1, \ldots, q$. Evidently, Q_1 is the set of all minimal elements of P, and Q_q is a subset of its maximal elements. The set Q_i is called the *i-th level* of P.

Example 1. Let us consider the following poset.



$$P = \{1, 2, 3, 4, 5, 6\}$$

This poset has 3 levels: $Q_1 = \{1, 4, 5\}, Q_2 = \{2, 6\}$ and $Q_3 = \{3\} \subseteq \{3, 6\}$, the set of maximal elements.

Let P be a poset and its partition in levels $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$. Clearly, for any $x \in Q_i$, $y \in Q_j$ such that $x < y$, we have $i < j$. But the converse is not always true: even if $x \in Q_i$, $y \in Q_j$ and $i < j$, x and y may be incomparable.

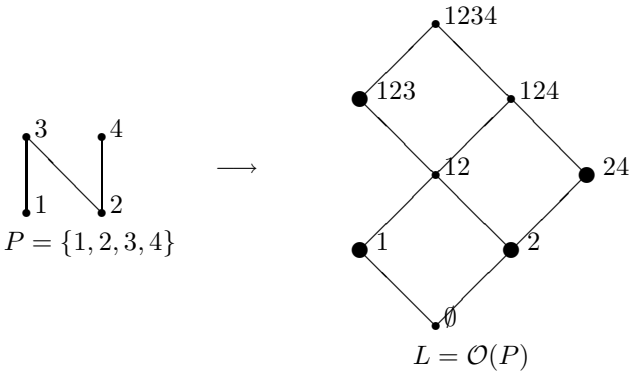
For any two elements $x, y \in P$, the *supremum* $x \vee y$ of x and y is the least element of all those greater than x and y (least upper bound), whenever it exists. Similarly, the *infimum* $x \wedge y$ of x and y is the greatest lower bound of x and y. A *lattice* L is a poset such that for any $x, y \in L$, $x \vee y$ and $x \wedge y$ exist. Clearly, in a finite lattice, $\top, \bot$ always exist. In addition, L is *distributive* if $\vee, \wedge$ satisfy the distributive law: for all $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{or equivalently} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Let L be a lattice and $x \in L$. If $x \neq \bot$ cannot be written as a supremum, i.e., $x = y \vee z$ implies $y = x$ or $z = x$, then x is said to be *join-irreducible*. Equivalently, a join-irreducible element covers only one element. Denote the set of all join-irreducible elements of L by $\mathcal{J}(L)$, and the set of join-irreducible elements less than or equal to an element $x \in L$ by $\eta(x)$. In a distributive lattice, any maximal chain has length $|\mathcal{J}(L)|$.

Let P be a poset and consider $x \in P$. The *principal ideal of* x is defined by $\downarrow x := \{y \in P \mid y \leq x\}$. Similarly, the *principal filter of* x is $\uparrow x := \{y \in P \mid y \geq x\}$. Let $Q \subset P$. The subset Q is a *downset* of P if $x \in Q$, $y \leq x$ imply $y \in Q$. Any downset is a union of principal filters. We denote the set of all downsets of P by $\mathcal{O}(P)$. Birkhoff proved that, if L is a distributive lattice, L is isomorphic to $\mathcal{O}(\mathcal{J}(L))$ by the isomorphism η [5]. Put otherwise, any poset P generates a distributive lattice $\mathcal{O}(P)$, whose set of join-irreducible elements is isomorphic to P. This well-known result, fundamental in this paper, is illustrated in the next example.

Example 2. We consider the poset $(P, \leq)$ given in the left. The set $\mathcal{O}(P)$ of all its downsets is given in the right (for ease of notation,$\{i, j\}$ is denoted by ij and so on.). It is a distributive lattice, and its join-irreducible elements are 1, 2, 24 and 123 (figured with larger circles on the figure). Observe that the sub-poset $\mathcal{J}(\mathcal{O}(P))$ of its join-irreducible elements is isomorphic to P.



Let P be a poset. The partition of P into levels $Q_1, \ldots, Q_q$ induces a partition $\{S_1, \ldots, S_q\}$ of its corresponding distributive lattice $\mathcal{O}(P)$ in the following way:

$$S_1 := \mathcal{O}(Q_1), \quad S_2 := \mathcal{O}(Q_1 \cup Q_2) \setminus S_1, \ldots, \quad S_q := \mathcal{O}(P) \setminus (S_1 \cup \cdots \cup S_{q-1}).$$

The following proposition shows some properties of the partition $\{S_1, \ldots, S_q\}$.

Proposition 1. Let P be a poset and $\{Q_1, \ldots, Q_q\}$ be its partition into levels. Then the following holds.

(i) $\top_i := \cup_{j=1}^{i} Q_j$ is the greatest element of S_i for all $i = 1, \ldots, q$;

(ii) Denoting respectively by $\bot, \top$ the bottom and top elements of $\mathcal{O}(P)$, we have $\bot < \top_1 < \cdots < \top_q = \top$;

(iii) $S_1 = \downarrow \top_1$, and $S_i = (\downarrow \top_i) \setminus (\downarrow \top_{i-1})$ for $i = 2, \ldots, q$.

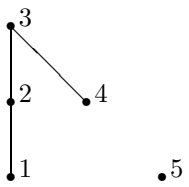
Proof. (i) We first show that $\cup_{j=1}^{i} Q_j \in \mathcal{O}(P)$. Consider $x \in \cup_{j=1}^{i} Q_j$ and $y < x$. Suppose $y \in Q_j$ for some $j > i$. Then there exists a longest maximal chain of length at least i from some minimal element y_0 to y. But this chain could be prolongated till x and would have a length greater than i, a contradiction with the definition of x. Hence $y \in \cup_{j=1}^{i} Q_j$, and $\cup_{j=1}^{i} Q_j$ is a downset of P.

Moreover, $\cup_{j=1}^{i} Q_j \in \mathcal{O}(\cup_{j=1}^{i} Q_j)$ and does not belong to $S_1, \ldots, S_{i-1}$, hence $\cup_{j=1}^{i} Q_j \in S_i$. Since any $x \in S_i$ is such that $x \subseteq \cup_{j=1}^{i} Q_j$, this proves that $\cup_{j=1}^{i} Q_j$ is the greatest element of S_i.

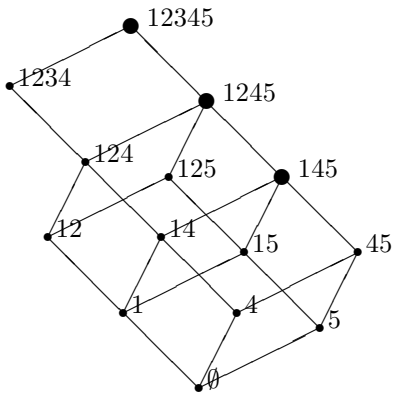
(ii) and (iii) are straightforward. □

The collection of all $\top_i$'s is denoted by $\top_P$. A maximal chain of $L := \mathcal{O}(P)$ passing through all $\top_i$'s in $\top_P$ is called a *restricted maximal chain*. We denote the set of restricted maximal chains by $\mathcal{C}_r(L)$. From Proposition 1, $\mathcal{C}_r(L)$ is never empty.

Example 3. We consider the following poset P and its corresponding distributive lattice $\mathcal{O}(P)$.



$P = \{1, 2, 3, 4, 5\}$



$\mathcal{O}(P)$

Then

$$Q_1 = \{1, 4, 5\}, Q_2 = \{2\}, Q_3 = \{3\}.$$

$$S_1 = \{1, 4, 5, 14, 15, 45, 145\}, S_2 = \{12, 124, 125, 1245\}, S_3 = \{1234, 12345\}.$$

$$\top_1 = 145, \top_2 = 1245, \top_3 = 12345.$$

3 Games on distributive lattices

As said in the introduction, there are two main applications of games defined on distributive lattices, namely to model restriction on the set of feasible coalitions, and to allow for each player several possible (partially ordered) actions for participation to the game. Our development will follow the first stream, and so is close to the framework of Faigle and Kern [11]. We will comment briefly the second one, which is developed in [15], in Section 8, where we will indicate how our results can be straightforwardly applied to this case.

In the rest of the paper, $N = \{1, \ldots, n\}$ denotes the set of players, which we suppose to be endowed with a partial order $\leq$. The relation $i \leq j$, with $i, j \in N$, indicates that player i is below player j, or a subordinate of j (this is called *precedence constraint* by Faigle [10]). Hence, the relation $\leq$ describes a *hierarchy* among players. Practically, this means that, if j participates to the game, all subordinates of j must also participate to it. Therefore, a coalition $S \subseteq N$ is *feasible* if $j \in S$ and $i \leq j$ implies that $j \in S$. This has two important consequences, which can be drawn from Section 2:

(i) The set of feasible coalitions is precisely the set of all downsets of $(N, \leq)$, denoted by $\mathcal{O}(N)$.

(ii) The set of feasible coalitions is a distributive lattice.

Definition 1. Let $L := \mathcal{O}(N)$ be the collection of all feasible coalitions (all downsets of $(N, \leq)$). A *game on the distributive lattice* L is a real-valued function $v : L \to \mathbb{R}$ such that $v(\emptyset) = 0$.

Note that the classical definition of a TU-game is recovered when $(N, \leq)$ is an antichain, that is, when there is no hierarchy and all players are "on the same level". Then clearly no restriction on coalitions exist, and any $S \in 2^N$ is feasible.

We introduce the following property.

Definition 2. Let v be a game on $\mathcal{O}(N)$. The game v is *convex* if $v(S\cup T)+v(S\cap T)\geq v(S)+v(T)$, for all $S,T\in\mathcal{O}(N)$.

4 The core and the restricted core

We take the classical point of view for defining the core, that is, it is a set of pre-imputations satisfying some rationality condition, which prevent players to form sub-coalitions. A *pre-imputation* is a vector $\phi\in\mathbb{R}^n$ such that $\sum_{i=1}^n\phi_i=v(N)$, where ϕ_i is the amount of money given to player i. We use the usual shorthand $\phi(S):=\sum_{i\in S}\phi_i$ for any subset $S\subseteq N$.

4.1 The core

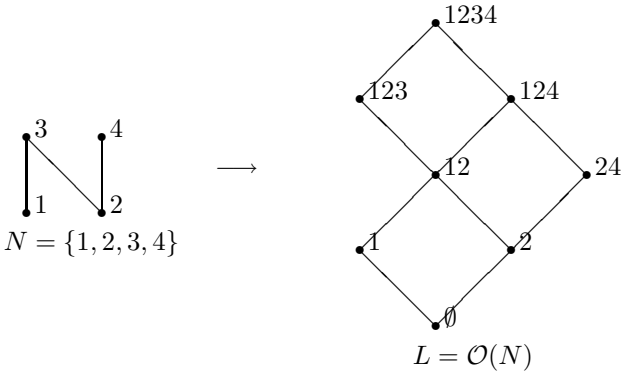
In the classical case, the rationality condition is $\phi(S)\geq v(S)$ for all coalitions S. Adapting it to our framework leads to the following definition.

Definition 3. The *core of a game v on $\mathcal{O}(N)$* is defined by the following set.

$$\mathcal{C}(v):=\{\phi\in\mathbb{R}^n\mid\phi(N)=v(N)\text{ and }\phi(S)\geq v(S),\ \forall S\in\mathcal{O}(N)\}.$$

Clearly, the core is a closed convex polyhedron. In the classical case (TU-games), the conditions $\phi(S)\geq v(S)$ for singletons suffice to ensure the boundedness of ϕ. However, in our framework, it may happen that some singletons are not feasible (because they are not subordinates). If i is such a singleton, there is no lower bound for ϕ_i. The consequence of this is that in general the core is unbounded (see next example)[4].

Example 4. We consider the poset $(N,\leq)$ of Example 2 (left), and its corresponding distributive lattice $\mathcal{O}(N)$ (right).



Let v be a game on $\mathcal{O}(N)$. By definition of the core, any element ϕ of the core must

[4]Derks and Reijnierse [9] provided necessary and sufficient conditions for the boundedness of the core of a game defined on a set of feasible coalitions, without special structure (set systems). See also the survey paper [14] for a general study of the core of games on set systems.

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satisfy:

$$\begin{aligned}\phi_1+\phi_2+\phi_3+\phi_4 &= v(\top)=v(1234)\\ \phi_1 &\geq v(1)\\ \phi_2 &\geq v(2)\\ \phi_1+\phi_2 &\geq v(12)\\ \phi_2+\phi_4 &\geq v(24)\\ \phi_1+\phi_2+\phi_3 &\geq v(123)\\ \phi_1+\phi_2+\phi_4 &\geq v(124).\end{aligned}$$

Whenever ϕ_1, ϕ_2 are large enough, we can always find out some ϕ_3, ϕ_4 to satisfy all conditions, i.e., ϕ_1, ϕ_2 can be arbitrarily large. Hence the core of this game has four rays (infinite directions): two positive rays for ϕ_1, ϕ_2 and two negative rays for ϕ_3, ϕ_4.

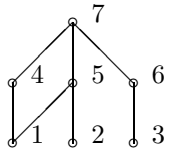
Since payoffs cannot attain infinite values, the core is of no practical use, and we have to add new constraints in its definition so that we obtain a bounded set. We denote the set of vertices of some convex set by $\mathrm{Ext}(\cdot)$, and the convex hull of some set by $\mathrm{co}(\cdot)$. We define the *convex part of the core* by $\mathcal{C}^F(v) := \mathrm{co}(\mathrm{Ext}(\mathcal{C}(v)))$. It is a polytope, and by the theory of polyhedra (see, e.g., Ziegler [25]), we know that $\mathcal{C}(v)$ is the Minkovski sum of its convex part and the conic hull of its rays (conic part).

A simple remedy to the above described drawback would be to impose that ϕ should be bounded from below by some quantity. In the sequel, we will provide a much less arbitrary and much better answer to this problem, both for mathematical properties (since we will see that we are able to keep many of the classical results on the core), and for the practical side, illustrated hereafter with an example of benefit sharing in a hierarchical structure, one of our main motivation.

4.2 How to share benefits in a hierarchical structure

The example we develop in this section will lead naturally to a new definition of the core.

We consider for illustration purpose a company with 7 employees $N = \{1, 2, 3, 4, 5, 6, 7\}$, and we represent the hierarchy among employees by the partial order $\leq$ on N. To be enough general, we may even consider that one employee may have more than one direct superior (it could be the case if the employee participates to several projects or belongs to several teams). Hence the partial order is not necessarily a tree. The poset below depicts the hierarchy in N.

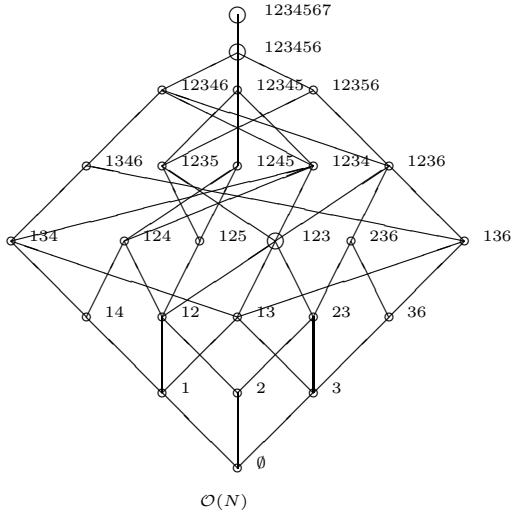


$$N = \{1,2,3,4,5,6,7 \mid 1<4<7, 2<5<7, 3<6<7 \text{ and } 1<5\}$$

We see that employee 1 has two direct superiors, namely 4 and 5.

As explained in Section 3, feasible coalitions are downsets of $(N, \leq)$. In this context, feasible coalitions correspond to *feasible teams* of the company, in the sense that the

presence of an employee in a feasible team implies the presence of all employees below. It must be remarked that in general a feasible team in the above sense may be formed of several teams in the usual sense, which we may call *elementary teams* (that is, a boss and all employees below): in terms of poset terminology, this amounts to say that a downset is the union of principal ideals (see Section 2). For example, the feasible team 12356 is formed of the two elementary teams 125 and 36, with bosses 5 and 6. Note also that 3 itself is a team reduced to a singleton. We give below the distributive lattice of all teams ordered by inclusion.



Computing the levels Q_k and top elements $\top_k$, we get

$$Q_1 = \{1, 2, 3\}, Q_2 = \{4, 5, 6\}, Q_3 = \{7\},$$

$$\top_1 = 123, \top_2 = 123456, \top_3 = 1234567 = N.$$

Level Q_k corresponds to employees having the same rank[5] k in the company, and $\top_k$ is the smallest feasible team containing all employees up to rank k. We call it the *principal team of rank k*. All feasible teams in S_k are called *feasible teams of rank k*. From Proposition 1 (iii), we know that $S_k = \downarrow \top_k \setminus \downarrow \top_{k-1}$.

At the end of each year, the total benefit (or a fixed proportion of it) has to be distributed among all employees as a bonus. We denote it by $v(N)$. For a given feasible team S, we denote by $v(S)$ the benefit achieved by S (and only by S) which is brought to the company, and we denote by $\phi(S)$ the bonus or reward given to S. To achieve the sharing, we propose to perform a local sharing at each hierarchical level Q_k. More precisely:

- For hierarchical level Q_k, the amount to be shared among the employees of this level is $v(\top_k) - v(\top_{k-1})$, that is, roughly speaking, the difference between the benefit achieved by all employees up to level k, and the benefit achieved by all employees of level strictly lower than k. In a sense, this is the genuine contribution of level k.

[5]Mathematically speaking the same height, see Section 2.

- Inside a given level Q_k, the sharing is done freely, up to the condition that for each feasible team $S \in S_k$, $\phi(S) \geq v(S)$. Otherwise, if for some S, $\phi(S) < v(S)$, then the team S may split from N and build a new independent company, because the benefit achieved by S alone is more than that S will receive.

Asuming there are l hierarchical levels, this gives the linear system in ϕ

$$\begin{aligned}\phi(Q_l) &= v(N) - v(\top_{l-1})\\ \phi(Q_{l-1}) &= v(\top_{l-1}) - v(\top_{l-2})\\ \vdots &= \vdots\\ \phi(Q_1) &= v(\top_1)\end{aligned}$$

and since $\top_k = \cup_{i=1}^k Q_i$, and the Q_k's are pairwise disjoint, we deduce that $\phi(\top_k) = \sum_{i=1}^k \phi(Q_i) = v(\top_k)$ for $k = 1, \ldots, l$. Conversely, imposing $\phi(\top_k) = v(\top_k)$ for $k = 1, \ldots, l$ leads to the above system.

Applying this procedure to our example, we get

$$\begin{aligned}v(N) - v(123456) &\text{ is given to the group } \mathcal{Q}_3 = \{7\},\\ v(123456) - v(123) &\text{ is given to the group } \mathcal{Q}_2 = \{4,5,6\},\\ v(123) &\text{ is given to the group } \mathcal{Q}_1 = \{1,2,3\}.\end{aligned}$$

4.3 The restricted core

From the previous development, we are led to the following definition.

Definition 4. *The restricted core of a game v on $\mathcal{O}(N)$* is defined by

$$\mathcal{RC}(v) := \{\phi \in \mathcal{C}(v) \mid \phi(\top_i) = v(\top_i), \forall \top_i \in \top_N\}.$$

Hence, the normalization condition is imposed at each level of the hierarchy.

Evidently, the restricted core is a closed convex polyhedron.

Theorem 1. The restricted core of a game v on $\mathcal{O}(N)$ is bounded, hence it is a polytope.

Proof. Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ be the collection of levels of the poset $(N, \leq)$. We show by induction on the level number that $\phi_i, \forall i \in N$ is lower bounded.

In the first level, any element $i_1 \in Q_1$ corresponds to the singleton $\{i_1\}$ of $\mathcal{O}(N)$. Hence $\phi_{i_1} \geq v(i_1)$ for all $\phi \in \mathcal{RC}(v)$.

Suppose that the property holds till the k-th level.

In the $(k+1)$-th level, by definition of levels, any element $i_{k+1} \in Q_{k+1}$ corresponds to some subsets $L^1 \subseteq Q_1, \ldots, L^k \subseteq Q_k$ such that $\{i_{k+1}\} \cup (\cup_{i=1}^k L^i) = \downarrow i_{k+1} \in \mathcal{O}(N)$. Hence $\phi(\downarrow i_{k+1}) \geq v(\downarrow i_{k+1})$ for all $\phi \in \mathcal{RC}(v)$. We have

$$\begin{aligned}\phi_{i_{k+1}} &= \phi(\downarrow i_{k+1}) - \phi(\cup_{i=1}^k L^i)\\ &\geq v(\downarrow i_{k+1}) - \phi(\cup_{i=1}^k L^i)\\ &= v(\downarrow i_{k+1}) - \phi(\top_k) + \phi(\top_k \setminus (\cup_{i=1}^k L^i))\\ &= v(\downarrow i_{k+1}) - v(\top_k) + \phi(\top_k \setminus (\cup_{i=1}^k L^i))\end{aligned}$$

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By induction, $\phi(\top_k \setminus (\cup_{i=1}^{k} L^i))$ is lower bounded, so $\phi_{i_{k+1}}$ is also lower bounded.

Hence, every coordinate has a lower bound. Finally since $\phi(\top) = v(\top)$, the restricted core is bounded. □

An important remark is that our definition collapses to the classical one if the set of feasible coalitions is 2^N. Indeed, in this case, $(N, \leq)$ is an antichain, so that there is only one level $Q_1 = N$, and $\top_1 = N$.

To end this section, we come back to Example 4, and compute its restricted core. We have

$$Q_1 = \{1, 2\}, Q_2 = \{3, 4\}, \quad \top_1 = 12, \top_2 = 1234.$$

Hence to the previous system, we add the following equation:

$$\phi(1) + \phi(2) = v(12).$$

Clearly, $\phi(1), \phi(2)$ can no more take infinite values.

5 Balancedness

It is well known that the necessary and sufficient condition for the nonemptiness of the core of a game on 2^N is balancedness of the game. Now, this result can be directly adapted to our case, and in fact, to any structure of feasible coalitions.

Definition 5. (i) A collection $\mathcal{B}$ of elements of $\mathcal{O}(N)\setminus\{\emptyset\}$ is *balanced* if it exist positive coefficients $\mu(S)$, $S \in \mathcal{B}$, such that $\sum_{S:S\ni i} \mu(S) = 1$, for all $i \in N$.

(ii) A game v on $\mathcal{O}(N)$ is *balanced* if for every balanced collection $\mathcal{B}$ of elements of $L \setminus \{\emptyset\}$ with coefficients $\mu(S)$, $S \in \mathcal{B}$, it holds

$$\sum_{S\in\mathcal{B}} \mu(S)v(S) \leq v(N).$$

Proposition 2. A game on $\mathcal{O}(N)$ has a nonempty core if and only if it is balanced.

We omit the proof of this result, since it is identical to the classical case. We mention that Faigle [10] has found other conditions for nonemptiness of the core in the general case of set systems. We turn now to the case of the restricted core.

Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ be the collection of levels of N and $\top_N = \{\top_1, \ldots, \top_q\}$ be the collection of top elements of every level of $\mathcal{O}(N)$. Similarly, we introduce the notion of level-balancedness as follows.

Definition 6. (i) A collection $\mathcal{B}$ of elements of $\mathcal{O}(N) \setminus \{\emptyset\}$ is *level-balanced* if it exist positive coefficients $\mu(S)$, $S \in \mathcal{B}$, such that $\sum_{S:S\ni i} \mu(S) = q - k + 1$, for all $i \in Q_k, k = 1, \ldots, q$.

(ii) A game v on $\mathcal{O}(N)$ is *level-balanced* if for every balanced collection $\mathcal{B}$ of elements of $\mathcal{O}(N) \setminus \{\emptyset\}$ with coefficients $\mu(S)$, $S \in \mathcal{B}$, it holds

$$\sum_{S\in\mathcal{B}} \mu(S)v(S) \leq \sum_{k=1}^{q} v(\top_k).$$

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Let us come back to Example 4. The conditions for level-balancedness read

$$\sum_{S\ni 1}\mu(S)=\sum_{S\ni 2}\mu(S)=2,\quad \sum_{S\ni 3}\mu(S)=\sum_{S\ni 4}\mu(S)=1.$$

The sum for elements of lower height have a higher value since the more an element is in the bottom of the hierarchy, the more it is frequent in coalitions. Examples of level-balanced collections are

$$\mathcal{B}=\{123,124\} \text{ with weights } 1,\ 1$$
$$\mathcal{B}=\{1234,12,1,2\} \text{ with weights } 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}.$$

Proposition 3. A game on $\mathcal{O}(N)$ has a nonempty restricted core if and only if it is level-balanced.

Proof. Nonemptiness of the restricted core of a game is equivalent to find out a vector $\phi\in\mathbb{R}^n$ satisfying the following conditions:

$$\phi(\top_i)=v(\top_i),\forall\top_i\in\top_N \text{ and } \phi(S)\geq v(S),\forall S\in\mathcal{O}(N)\setminus\{\emptyset\}.$$

Consider the following linear program with the variables $\phi_i\in\mathbb{R}$, $i\in N$:

$$\min z=\sum_{\top_i\in\top_N}\phi(\top_i)=\sum_{i=1}^{q}\sum_{k=1}^{i}\phi(Q_k)$$

under

$$\sum_{i:i\in S}\phi_i\geq v(S),\forall S\in\mathcal{O}(N)\setminus\{\emptyset\}.$$

Its optimal value is $z=\sum_{\top_i\in\top_N}v(\top_i)$ if and only if the restricted core is nonempty.

Its dual problem is

$$\max w=\sum_{S\in L\setminus\{\emptyset\}}\mu(S)v(S)$$

under

$$\sum_{S:S\ni i}\mu(S)=q-k+1,\ \forall i\in Q_k,k=1,\ldots,q$$
$$\mu(S)\geq 0,\ \forall S\in\mathcal{O}(N)\setminus\{\emptyset\}.$$

By the duality theorem, it has the same optimal value $w=\sum_{\top_i\in\top_N}v(\top_i)$ if we can find out some μ satisfying all conditions. This is the desired result. □

6 Marginal worth vectors

Since $\mathcal{O}(N)$ is a distributive lattice with n join-irreducible elements, we know from Section 2 that any maximal chain has length n. Therefore, let $C=\{S_0:=\emptyset\prec S_1\prec\cdots\prec S_n:=N\}$ be a maximal chain in $L:=\mathcal{O}(N)$. To each maximal chain we associate a permutation on N, $\pi:N\to N$, such that the additional element between any two consecutive coalitions S_{i-1},S_i of C is $\pi(i)$. So we have $S_i=\{\pi(1),\pi(2),\ldots,\pi(i)\}$.

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It is easy to see that π defines a linear extension of $\leq$ on N, and moreover, any linear extension of $\leq$ corresponds to such a permutation π. Indeed, $i < j$ implies that $\pi(i) > \pi(j)$ will never happen, for any permutation. Conversely, if $i_1, \ldots, i_n$ is a linear extension, then $k < l$ implies that $i_k > i_l$ cannot happen. Hence $\{\{i_1\}, \{i_1, i_2\}, \ldots, \{i_1, \ldots, i_n\}\}$ is a chain of downsets, defining a permutation π on N.

Definition 7. The *marginal worth vector* $\psi^C \in \mathbb{R}^n$ associated to C and v is defined by

$$\psi^C_j := v(S_i) - v(S_{i-1}), \quad \forall i \in N,$$

with $j = S_i \setminus S_{i-1}$.

The set of all marginal worth vectors ψ^C for all maximal chains is denoted by $\mathcal{M}(v)$. We can easily get

$$\psi^C(S_i) := \sum_{k=1}^{i} \psi^C_{\pi(k)} = \sum_{k=1}^{i} (v(S_k) - v(S_{k-1})) = v(S_i), \forall S_i \in C.$$

Definition 8. The *Weber set* $\mathcal{W}(v)$ of v is defined as the convex hull of all vectors in $\mathcal{M}(v)$:

$$\mathcal{W}(v) := \text{co}(\mathcal{M}(v)).$$

Theorem 2. For any game v on $\mathcal{O}(N)$, the convex part of the core is included in the Weber set, i.e, $\mathcal{C}^F(v) \subseteq \mathcal{W}(v)$.

Proof. We show that all vertices of the core are included in the convex hull of the set of marginal worth vectors by induction on the number of players $|N|$.

(i) If $N = \{1\}$, then $\mathcal{C}(v) = \mathcal{W}(v) = \mathcal{M}(v)$. The statement is true.

(ii) Suppose that the statement is true whenever $|N| < n$.

(iii) Let $N = \{1, \ldots, n\}$ and $\phi \in \text{Ext}(\mathcal{C}(v))$. Then $\exists S \in \mathcal{O}(N) \setminus \{N, \emptyset\}$ such that $\phi(S) = v(S)$.

Let $u(T) := v(T), \quad \forall T \subseteq S, T \in \mathcal{O}(N)$. We have clearly $\phi_{|S} \in \mathcal{C}(u)$, and by induction,

$$(\phi_{|S})_i = \sum_{\psi^k \in \mathcal{M}(u)} \alpha_k \psi^k_i \text{ whith } \sum_{k:\psi^k \in \mathcal{M}(u)} \alpha_k = 1, \alpha_k \in [0,1], \quad \forall i \in S.$$

Let $w(T) := v(S \cup T) - v(S), \quad \forall T \subseteq N \setminus S, S \cup T \in \mathcal{O}(N)$. Evidently, w is a game on the distributive lattice $\mathcal{O}(N \setminus S)$, the latter being isomorphic to the distributive sublattice $\uparrow S = \{T \supseteq S \mid T \in \mathcal{O}(N)\}$ by the mapping $\theta : T \to T \cup S, \forall T \subseteq N \setminus S, S \cup T \in \mathcal{O}(N)$. We have, for all $T \subseteq N \setminus S, S \cup T \in \mathcal{O}(N)$,

$$\phi_{|N\setminus S}(T) = \phi(T) = \phi(S \cup T) - \phi(S) \geq v(S \cup T) - v(S) = w(T)$$

and

$$\phi_{|N\setminus S}(N \setminus S) = \phi(N) - \phi(S) = v(N) - v(S) = w(N \setminus S).$$

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Hence $\phi_{|N\setminus S} \in \mathcal{C}(w)$, i.e., $(\phi_{|N\setminus S})_i = \sum_{\psi^k \in \mathcal{M}(w)} \beta_k \psi_i^k$ where $\sum_{k:\psi^k \in \mathcal{M}(w)} \beta_k = 1, \beta_k \in [0,1] \quad \forall i \in N \setminus S$.

Any $\psi^i \in \mathcal{M}(u)$ corresponds to a maximal chain C^i from $\emptyset$ to S. Any $\psi^j \in \mathcal{M}(w)$ corresponds to a maximal chain C^j from $\emptyset$ to $N \setminus S$ in $\mathcal{O}(N \setminus S)$. By the mapping θ, each element $T \in C^j$ corresponds to an element $\theta(T) = T \cup S \in \mathcal{O}(N)$, i.e., the maximal chain C^j corresponds to a maximal chain $C'^j := \theta(C^j)$ of $\mathcal{O}(N)$ from S to N. Let

$$\psi_k^{(i,j)} := \begin{cases} \psi_k^i, & \text{if } k \in S \\ \psi_k^j, & \text{if } k \in N \setminus S. \end{cases}$$

Then $\psi^{(i,j)}$ corresponds to the maximal chain $C = (C^i, C'^j)$ from $\emptyset$ to N, i.e., $\psi^{(i,j)} \in \mathcal{M}(v)$. We can show that, for all i such that $\psi^i \in \mathcal{M}(u)$ and all j such that $\psi^j \in \mathcal{M}(w)$,

$$\begin{aligned} \phi_k &= \sum_i \alpha_i \psi_k^i = \sum_i \alpha_i \psi_k^{(i,j)} \\ &= \sum_j \beta_j (\sum_i \alpha_i \psi_k^{(i,j)}) = \sum_j \sum_i \alpha_i \beta_j \psi_k^{(i,j)}, \quad \forall k \in S, \end{aligned}$$

and

$$\begin{aligned} \phi_k &= \sum_j \beta_j \psi_k^j = \sum_j \beta_j \psi_k^{(i,j)} \\ &= \sum_i \alpha_i (\sum_j \beta_j \psi_k^{(i,j)}) = \sum_i \sum_j \alpha_i \beta_j \psi_k^{(i,j)}, \quad \forall k \in N \setminus S, \end{aligned}$$

i.e., $\phi = \sum_i \sum_j \alpha_i \beta_j \psi^{(i,j)}$ where $\sum_i \sum_j \alpha_i \beta_j = \sum_i \alpha_i (\sum_j \beta_j) = 1$. Hence $\phi \in \mathcal{W}(v)$.

By induction, all vertices of the core belong to the Weber set. Therefore the convex part of the core is included in the Weber set. □

We consider marginal worth vectors ψ^{C_r} associated to restricted chains C_r. The set of all such marginal worth vectors is denoted by $\mathcal{RM}(v)$.

Definition 9. The *restricted Weber set* is defined as the convex hull of all marginal worth vectors associated to restricted maximal chains:

$$\mathcal{RW}(v) := \text{co}(\mathcal{RM}(v)).$$

Theorem 3. For any game v on $\mathcal{O}(N)$, the restricted core is included in the restricted Weber set, i.e., $\mathcal{RC}(v) \subseteq \mathcal{RW}(v)$.

Proof. We prove it by induction on the level number. If a poset N has only one level, then the proof is the same as the one of Theorem 2.

Suppose that the statement is true for all posets having at most k levels. We assume now that the poset has $k+1$ levels. Let $v'(T) := v_{|\top_k}(T) = v(T), \forall T \subseteq \top_k, T \in \mathcal{O}(N)$, and $\phi \in \text{Ext}(\mathcal{RC}(v)) \subseteq \mathcal{RC}(v)$. Clearly, $\phi_{|\top_k} \in \mathcal{RC}(v_{|\top_k}) = \mathcal{RC}(v')$. Then $\phi_{|\top_k} \in$

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$\text{Ext}(\mathcal{RC}(v'))$. Indeed, if $\phi_{|\top_k} \notin \text{Ext}(\mathcal{RC}(v'))$, then $\exists \phi^1 \neq \phi^2 \in \mathcal{RC}(v'), \exists \lambda \in (0,1)$ such that $\phi_{|\top_k} = \lambda\phi^1 + (1-\lambda)\phi^2$. Let

$$\phi_i'^j := \begin{cases} \phi_i^j & \forall i \in \top_k, \\ \phi_i & \forall i \in \top \setminus \top_k \end{cases}$$

for $j = 1, 2$. Then $\phi = \lambda\phi'^1 + (1-\lambda)\phi'^2$, which contradicts $\phi \in \text{Ext}(\mathcal{RC}(v))$.

Similarly, let $v''(T) = v(\top_k \cup T) - v(\top_k), \forall T \subseteq \top \setminus \top_k$, then

$$\phi_{|\top\setminus\top_k}(T) = \phi(T \cup \top_k) - \phi(\top_k) \geq v(T \cup \top_k) - v(\top_k) = v''(T), \forall T \subseteq \top_k.$$

So $\phi_{|\top\setminus\top_k} \in \mathcal{RC}(v'')$. Then

$$\phi_k = \begin{cases} \sum_{\psi^i \in \mathcal{PM}(v')} \alpha^i \psi_k^i & \forall k \in \top_k \\ \sum_{\psi^j \in \mathcal{PM}(v'')} \beta^j \psi_k^j & \forall k \in \top \setminus \top_k. \end{cases}$$

By the proof of Theorem 2, let $\psi^{(i,j)} = (\psi^i, \psi^j)$ where $\psi^i \in \mathcal{RM}(v'), \psi^j \in \mathcal{RM}(v'')$, then $\phi = \sum_i \sum_j \alpha_i \beta_j \psi^{(i,j)}$, i.e., $\phi \in \mathcal{RW}(v)$. This is the desired result. □

7 The restricted core of convex games

Theorem 4. Let v be a game on $\mathcal{O}(N)$. Then v is convex if and only if $\text{Ext}(\mathcal{C}(v)) = \mathcal{M}(v)$, i.e., $\mathcal{C}^F(v) = \mathcal{W}(v)$.

To prove this theorem, we must show some lemmas.

Lemma 1. If a game v on $\mathcal{O}(N)$ is convex, then the Weber set is a subset of the core:

$$\mathcal{W}(v) \subseteq \mathcal{C}(v).$$

Proof. Because $\mathcal{W}(v) = \text{co}(\mathcal{M}(v))$, if all vectors of $\mathcal{M}(v)$ belongs to the core, by the convexity of the core, all elements of the Weber set must be contained in the core. Now we show $\mathcal{M}(v) \subseteq \mathcal{C}(v)$.

Let $C = \{S_0 := \emptyset \prec S_1 \prec \cdots \prec S_n := N\}$ be a maximal chain in $\mathcal{O}(N)$. Because $\psi^C(S_i) = v(S_i), \forall S_i \in C$, it remains to show that $\psi^C(S) \geq v(S), \forall S \in \mathcal{O}(N) \setminus C$. We prove it by induction on $|S|$.

If $S = \{i\}$, then $\exists j$ such that $S_{j+1} = S_j \cup \{i\}$. By the convexity of v, we have $\psi_i^C = v(S_{j+1}) - v(S_j) \geq v(i) - v(\emptyset) = v(i)$.

Assume that $\psi^C(S) \geq v(S)$ for any $S \in \mathcal{O}(N) \setminus C$ and $|S| < s$. Let $S \in \mathcal{O}(N) \setminus C$ and $|S| = s$. Denote by π the permutation associated to C, such that $S_i = \{\pi(1), \ldots, \pi(i)\}$ and $j := \pi(i)$. Then for S, we can get a sequence $i_1 < \cdots < i_s$ such that $i_k = \pi^{-1}(j_k)$ for all $j_k \in S$. Hence by the convexity of v, we have

$$v(S_{i_s}) - v(S_{i_s - 1}) \geq v(S) - v(S \setminus \{\pi(i_s)\}),$$

then

$$\psi_{\pi(i_s)}^C = v(S_{i_s}) - v(S_{i_s - 1}) \geq v(S) - v(S \setminus \{\pi(i_s)\}).$$

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By induction, for $S \setminus \{\pi(i_s)\} = \{\pi(i_1), \ldots, \pi(i_{s-1})\}$, we have

$$\psi^C(S \setminus \{\pi(i_s)\}) \geq v(S \setminus \{\pi(i_s)\}).$$

Finally

$$\psi^C(S) = \psi^C_{\pi(i_s)} + \psi^C(S \setminus \{\pi(i_s)\}) \geq v(S).$$

Hence ψ^C belongs to the core. □

Lemma 2. If a game v on $\mathcal{O}(N)$ is convex, then any marginal worth vector in $\mathcal{M}(v)$ is a vertex of the core:

$$\mathcal{M}(v) \subseteq \text{Ext}(\mathcal{C}(v)).$$

Proof. By Lemma 1, we have $\mathcal{M}(v) \subseteq \mathcal{C}(v)$, it remains to show that every ψ^C is a vertex of the core. Suppose there exist vectors $\phi^1, \phi^2 \neq \psi^C \in \mathcal{C}(v)$, and $\lambda \in (0,1)$ such that $\psi^C = \lambda\phi^1 + (1-\lambda)\phi^2$. Because we have $\psi^C(S_i) = v(S_i)$ for any $S_i \in C$, we have $v(S_i) = \lambda\phi^1(S_i) + (1-\lambda)\phi^2(S_i)$. But $\phi^k(S_i) \geq v(S_i)$ for all $S_i \in C$, $k = 1, 2$, hence necessarily $\phi^1(S_i) = \phi^2(S_i) = v(S_i)$, i.e., $\phi^1 = \phi^2 = \psi^C$, a contradiction. Hence, ψ^C is a vertex of the core. □

Lemma 3. If a game v on $\mathcal{O}(N)$ is convex, then $\text{Ext}(\mathcal{C}(v)) = \mathcal{M}(v)$, or equivalently $\mathcal{C}^F(v) = \mathcal{W}(v)$.

Proof. By Lemma 2, we know that for a convex game v, any vertex of the restricted Weber set is a vertex of the core, also of the convex part of the core. Since the convex part of the core is included in the restricted Weber set by Theorem 2, it follows that the vertices of the two sets coincide. □

Now let us prove Theorem 4.

Proof. We have already shown in Lemma 3 that, if v is convex, then $\text{Ext}(\mathcal{C}(v)) = \mathcal{M}(v)$. Conversely, suppose $\text{Ext}(\mathcal{C}(v)) = \mathcal{M}(v)$. For any $S = \{s_1, \ldots, s_k, p_1, \ldots, p_s\}$, $T = \{s_1, \ldots, s_k, q_1, \ldots, q_t\} \in \mathcal{O}(N)$ and $S \cap T, S \cup T \in \mathcal{O}(N)$, we can always find out a maximal chain C passing through the points $S \cap T, S, S \cup T$. Hence $v(S \cup T) - v(S) = \psi^C(S \cup T) - \psi^C(S) = \psi^C(q_1, \ldots, q_t) = \psi^C(T) - \psi^C(S \cap T) \geq v(T) - v(S \cap T)$. It implies the convexity of v. □

For the restricted core, we have a similar result.

Theorem 5. If a game v on $\mathcal{O}(N)$ is convex, then any marginal worth vector in $\mathcal{RM}(v)$ is a vertex of the restricted core:

$$\mathcal{RM}(v) \subseteq \text{Ext}(\mathcal{RC}(v)).$$

Proof. Consider a restricted maximal chain C_r and its associated marginal worth vector ψ^{C_r}. We know by Theorem 4 that it is a vertex of the core, and since ψ^{C_r} coincide with v on C_r, it has the property $\psi^{C_r}(x) = v(x)$, $\forall x \in \top_N$, hence it belongs to the restricted core and is a vertex of it. □

Corollary 1. If a game v on $\mathcal{O}(N)$ is convex, then $\text{Ext}(\mathcal{RC}(v)) = \mathcal{RM}(v)$, or equivalently $\mathcal{RC}(v) = \mathcal{RW}(v)$.

Proof. using Theorem 3 and 5, we can similarly prove it like Lemma 3. □

Remark that $\mathcal{RC}(v) = \mathcal{RW}(v)$ does not imply that v is convex. Put differently, $\mathcal{RC}(v) = \mathcal{RW}(v)$ is not equivalent to $\mathcal{C}^F(v) = \mathcal{W}(v)$. This is shown by the following counterexample.

Example 5. Let v be a game on $\mathcal{O}(N)$ with $N = \{1, 2, 3, 4, 5\} : 1 < 2 < 3, 4 < 5$. Consider v satisfying $v(S) = \sum_{s \in S} s$ for any $S \neq \{12\}$ and $v(12) = 1$. We have $\mathcal{RC}(v) = \mathcal{RW}(v) = \{(1, 2, 3, 4, 5)\}$ but $v(12345) + v(12) = 16 < v(1245) + v(123) = 18$. Therefore v is not convex.

To end this section, we come back to Example 4 and compute its restricted core. The four restricted maximal chains are

$$C_1 := \{\emptyset, 1, 12, 123, 1234\}, \quad C_2 := \{\emptyset, 1, 12, 124, 1234\}$$
$$C_3 := \{\emptyset, 2, 12, 123, 1234\}, \quad C_4 := \{\emptyset, 2, 12, 124, 1234\}.$$

Under convexity of v, the restricted core of v is the convex hull of the four following vectors:

$$\phi^1 := (v(1), v(12) - v(1), v(123) - v(12), v(N) - v(123))$$
$$\phi^2 := (v(1), v(12) - v(1), v(N) - v(124), v(124) - v(12))$$
$$\phi^3 := (v(12) - v(2), v(2), v(123) - v(12), v(N) - v(123))$$
$$\phi^4 := (v(12) - v(2), v(2), v(N) - v(124), v(124) - v(12)).$$

In general, it is a 3-dimensional polytope with 4 vertices, hence a 3-dimensional simplex.

8 Games with a partially ordered set of actions

We give a brief indication about games where each player has at disposal a partially ordered set of (elementary) actions. This notion of game is described in [15]. Consider a set of players N, and for each $i \in N$, define P_i the partially ordered set of possible actions of player i. A simple but useful example is to take the case of multichoice games [17]. Then the P_i's are totally ordered sets $P_i := \{0 =: a_0, a_1, \ldots, a_m\}$, where $a_0 < a_1 < \cdots < a_m$ indicate levels of participation.

We consider the distributive lattices $L_i := \mathcal{O}(P_i)$, $i \in N$. They represent all possible combinations of elementary actions, where if action x is performed and $y \leq x$ in the poset of actions, then y must be performed too. Considering all players together, a given *profile* of actions is an element of the product lattice $L := L_1 \times \cdots \times L_n$.

Since L is again distributive, all previous definitions and results can be applied to L. In particular, the restricted core of v is defined as the set of pre-imputations ϕ on L such that ϕ dominates v on L, and coincides with v on each element of L of the form $(\top_1^k, \top_2^k, \ldots, \top_n^k)$, where $\top_i^k$ is the top element associated to the k-th level of P_i. When L corresponds to a multichoice game, we recover the results shown previously by the authors in [16].

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9 Related works

There is a substantial amount of research devoted to the core of games defined on a set of feasible coalitions (see a survey on this topic by the first author [14]), and it will be out of the scope of this paper to detail this. Our proposition solves the problem of unboundedness of the core, by imposing some additional normalization conditions. Up to our knowledge, there is no work taking the same philosophy.

On the other hand, the notion of hierarchy has received some attention by several authors, in particular by Gilles et al. [12] who propose permission structures, Demange [8], and van den Brink et al. [23].

A (conjunctive) permission structure is a mapping $\sigma : N \to 2^N$ such that $i \notin \sigma(i)$. The players in $\sigma(i)$ are the direct subordinates of i. "Conjunctive" means that a player i has to get the permission to act of all his superiors. Consequently, an *autonomous* coalition contains all superiors of every member of the coalition, i.e., the set of autonomous coalitions generated by the permission structure σ is

$$\mathcal{F}_\sigma = \{S \in 2^N \mid S \cap \sigma(N \setminus S) = \emptyset\}.$$

This collection is closed under union and intersection (and conversely, any collection of feasible coalitions closed under union and intersection corresponds to a permission structure). Clearly, our collection $\mathcal{O}(N)$ is closed under union and intersection, and so should correspond to a permission structure. However, the notions of team in our sense and of autonomous coalition are quite opposite, since a team must contain all subordinates of its members, and an autonomous coalition must contain all superiors. These are two different viewpoints of a hierarchy. A team S is an entity able to perform some work giving rise to some profit $v(S)$. It is considered that the work cannot be achieved if one subordinates is missing. This view is suitable for projects, companies, etc. An autonomous coalition T is able to achieve some work because they have the permission of all their superiors, this permission being represented simply by the presence of the superiors in the coalition. Therefore, $v(T)$ has not the meaning of some profit achieved by the coalition.

In the work od Demange, a hierarchy is the same as our partial order defined on N, up to the difference that a greatest element exists (called the principal), so that each player is a subordinate of the principal. Also, the notion of team differs: any singleton is a team, and if a team has at least two members, any two members have a superior in the team, and if i is a superior of j, all intermediates between i and j must be present. Therefore, any "interval", i.e, a chain in the hierarchy, is a team. Again, this definition does not fit our idea of defining team as entities being able to produce something. Clearly, a single player, unless he has no subordinate, cannot produce something by himself. The same remark is valid for intervals.

The work of van den Brink et al. concerns oriented communication graphs. Most of the research on communication graphs do not consider orientation, since it is generally assumed that communication is in both directions. Defining an orientation implicitely defines some order among players, hence some hierarchy. The philosophy adopted in this work is that if player i is higher in the hierarchy than j (i.e., there is an oriented path from i to j), the payoff (or cost) given to player i should be higher than the one of player j. This is well suited to the well known water distribution problem of Ambec

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and Sprumont [1], also considered by van den Brink et al. [22]. However, it is not suited for our view, since assuming $S \supset S'$, it may be the case that $v(S) = v(S')$, which means that the superior(s) in $S \setminus S'$ do not really add some value to the team. Therefore, their payoff should be zero or very low.

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