Peano's Logical Language and Grassmann's Legacy

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Introduction and argument

One of the main difficulties every historian has to confront when he looks at works on the foundation of mathematics at the end of the XIXth Century is to get rid of the idea that these works exemplify a pre-Hilbertian stage of thought. We indeed fall into the habit of linking the stress put on language in mathematics with a kind of formalism, according to which the content of the primitive terms are fixed by the postulates. Thus, in Hilbert's *Grundlagen*, as it is well known, the terms points, lines and planes are just marks without any intuitive content. The idea that the axiomatic system itself gives the mathematical terms their meanings is so entrenched today that one feels some intellectual difficulty in imagining how it could be otherwise.

What I will suggest here is that, in a certain way, for Peano, the emphasis on language ran counter to the Hilbertian approach, and I will attempt to argue for that by stressing the importance of Peano's Grassmannian lineage[1]. Indeed, in Grassmann, the calculus of forms was not conceived as a system which implicitly defined the basic mathematical terms, but, on the contrary, the "theory of extension" was regarded as the sole means of grasping and displaying the content of certain given objects. I will sustain that the same hold for Peano. In other words, I will try to convince you that, in Peano, language and formalism were not so much viewed as a tool to *disconnect* symbols from their usual content, but as a means of *catching* new and until then hidden mathematical objects.

To flesh out this suggestion, I will briefly expound and discuss two examples. Firstly, I will speak about the use of Grassmann's calculus in the new definition of the area of a curved surface; secondly, I will study Peano's reading of Pasch's axiomatic geometry. In each of these cases, the Grassmannian influence is tied up with what we could call a kind of "mathematical realism", that is, with the idea that algebra gives us access to certain sorts of entities, the "forms", which cannot be reached by any other path.

In my last section, I will strengthen this first result by contrasting the way Peano presented his logical language with the one we find in the Boolean tradition. There, a difference between the empty calculi and interpreted algebras is frequently drawn. One does not find the equivalent in Peano's works. As is well known, Peano is very careless about the distinction between mention and use. I think this confusion should not be seen just as a defect, but also as a symptom of the strength of Peano's commitment to the Grassmanian tradition. When algebra is first and foremost viewed as a tool to grasp content which cannot be grasped otherwise, then a purely syntactic definition of the calculi seems very artificial.

1- On the definition on the area of a curved surface

[1] Peano seems to have read only the second version of the *Ausdehnungslehre*, that is, H. Grassmann, *Die Ausdehnungslehre. Vollständig und in strender Form bearbeitet*, Berlin, 1862. An English translation has recently been made by L. C. Kannenberg, in 2000 (*Extension theory*, Providence, American Mathematical Society, 2000).

I will be very brief on this matter, since Yvette Perrin has devoted her paper to this topic. I just want here to stress the fact that the way Peano uses geometrical algebra moves his approach away from the Hilbertian method[2].

The starting point of Peano's paper[3] was the "Schwarz paradox", that is the realization that Serret's definition was flawed. Serret considered the families of polyhedrons on a given curved surface, and he claimed that the area of the targeted surface was the limit of the area of the faces of the polyhedrons, when the sides of these faces tended toward zero. The definition did not work since, even for a surface as simple as a portion of a cylinder, a family of polyhedrons whose area tends toward infinity when the sides of the polyhedral faces tended toward zero can be constructed.

Despite its failure, the path taken by Serret seemed quite natural, because it resumed the Archimedian analogy between the rectification of a line and the quadrature of a surface. As the length is a limit of the length of a certain polygon, an area appears, in Serret's definition, as the limit of the area of a certain polyhedron. But how is the demand for rigor to be reconciled with the wish to stick to Archimedes' geometrical approach? Peano answered: by using geometrical algebra[4]. Thus[5]:

> Si può ottenere ad un tempo il rigore e l'analogia fra le definizioni relative all'arco e all'area, ove si faccia uso, oltrechè del concetto di retta limitata considerate in grandezza e direzione (segmento, vettore), anche del concetto dualitico di area piana considerate in grandezza e giacitura. Questi enti furono introdotti in geometria specialmente per opera di Chelini, Möbius, Bellavitis, Grassmann e Hamilton.

And Peano set down[6]:

> Lunghezza d'un arco di curva è il limite superiore della somma delle grandezze dei vettori delle sue parti.
> Area d'una porzione di superficie è il limite superiore della somma delle grandezze dei bivettori delle sue parti.

The analogy was then respected, but the Grassmannian notions of vectors and bivectors occurred in the definition.

I have not the time to explain in detail what is a vector or a bivector. Let me just say that vectors and bivectors, like all geometrical forms, have a magnitude and an orientation, and that, at the same time, these entities occur as factors in certain product operations. Vectors and bivectors were then conceived at the same time as both spatial and formal entities. Now, why is Peano's definition better than Serret's one? According to Peano, Serret made a mistake because he did not take into account the orientation of the sides of the approaching polyhedrons. Now, since the Grassmannian notion of bivector encapsulated an orientation, this source of error is discarded from the very beginning.

I will say nothing more about Peano's construction. What I want to emphasize is that the way Peano used geometrical algebra is completely at odds with Hilbert's axiomatic approach. In the *Grundlagen*, the interpretation of the primitive terms could be changed at will, as long as the relational network set out in the axioms was preserved. Nothing is more remote from Peano's view. Indeed, for avoiding Serret's mistake, while preserving the link

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[2] See, as well, Gandon S. et Perrin Y., Le problème de la définition de l'aire d'une surface gauche: Peano et Lebesgue, submitted to *Archive for History of Exact Sciences*.

[3] Sulla definizione dell'area d'una superficie, *Atti della Reale Accademia dei Lincei: Rendiconti*, 4, 1890, 54-57.

[4] Peano has devoted a book to the geometrical algebra : *Calcolo geometrico secondo l'Ausdehnungslehre di Hermann Grassmann, preceduto dalle operazioni della logica deduttiva*, Bocca, 1888. A English translation, made by L. C. Kannenberg, has recently been published : *Geometric Calculus*, Birkhäuser, 2000.

[5] *Ibid.*, 55.

[6] *Ibid.*, 56.

between rectification and quadrature, one has to renounce the familiar notions of polygon or polyhedron for a new "ontology" of vectors and bivectors. Grassmannian algebra (as used by Peano) appears then as a tool that allows a reshaping of the basic geometrical concepts – as a means to replace the intuitive Euclidean objects with abstract entities, which are more apt to express the features of geometrical magnitudes. In *Sulla definizione dell'area di una superficie*, the role of the calculus is then to *bring to light* a certain layer of geometrical reality.

2- Peano's reading of Pasch's axiomatic

In 1890, Peano wrote *I Principii della Geometria*[7], an axiomatic presentation of absolute geometry, inspired by the first two sections of Pasch's *Vorlesungen über neure Geometrie* (1882). This last book is celebrated in the literature as the first complete axiomatization of a geometry, and thus as an important step toward Hilbert's *Grundlagen*. Peano translated Pasch's axioms into his new logical notation, and proved some independence results by using the same kind of model-theoretical method we still use today. Thus, here at least, we could believe we have found a purely axiomatic approach to geometry, foreshadowing Hilbert's work. But a closer look at the article, and notably at its sections 2 and 3, will reveal how much Peano was still under the influence of Grassmann[8].

In Pasch's work, the only indefinables are the points and the relation "between" (the notion of "plane" is as well taken as an indefinable, but I leave this point aside). In his section 2, Peano devised a notation to speak about the various parts of a line divided by two points. If *a* and *b* are two points, *ab* will refer to the segment whose extremities are *a* and *b*. He then introduces the sign " ' ", and explains that *a'b* is "the shadow of *b* when lit up from *a*", that is, *a'b* is the half line whose extremity is in *b* and which does not contain *a*; while *ab'* is the half line whose extremity is in *a* and which does not contain *b*.

Peano then generalizes the notation in this way: *h* and *k* being some figures (classes of points), *ak* designates the set of segments relating *a* to a point of *k*:

$$a\in\mathbf{1}\ .\ k\in \mathrm{K}\mathbf{1}\ .\supset:.\ ak =:\ \mathbf{1}\ .\ [x\in](y\in k\ .\ x\in ay :-=_y \Lambda)$$

The sign *hk* designates the set of segments relating the points of *h* to the points of *k*:

$$h,\ k\in \mathrm{K}\mathbf{1}\ .\supset:.\ hk =:\ \mathbf{1}\ .\ [x\in](y\in h\ .\ x\in yk :-=_y \Lambda).$$

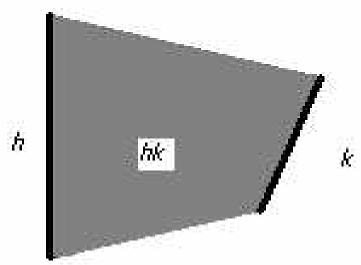

Figure 1 : *hk* is the shaded quadrilateral.

Peano goes on by defining *a'k*, "the shadow of *k* when lit up from *a*" and *ak'*:

[7] *I principii di geometria logicamente esposti*, Bocca, 1889.

[8] I summarize here a development first made in La réception des *Vorlesungen über neuere Geometrie* de Pasch par Peano, *Revue d'Histoire des Mathématiques*, 12, 2006, 249-290.

$$a\in\mathbf{1}\,.\,k\in\mathrm{K}\mathbf{1}\,.\supset\!:.\, a'k =:\, \mathbf{1}\,.\,[x\in](y\in k\,.\,x\in a'y :\!-\!=_y \Lambda)$$
$$a\in\mathbf{1}\,.\,k\in\mathrm{K}\mathbf{1}\,.\supset\!:.\, ak' =:\, \mathbf{1}\,.\,[x\in](y\in k\,.\,x\in ay' :\!-\!=_y \Lambda).$$

He introduces as well *h'k*, "the shadow of *k* when lit up from *h*". The following figure represents the two area *hk'* and *h'k* :

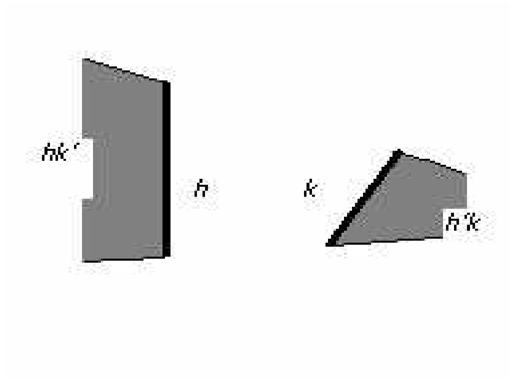

Figure 2 : *hk'* is the shaded area to the left of *h* ; *h'k* is the shaded area to the right of *k*.

Finally, Peano defines *h''*, "the shadow of *h* lit up from *h* itself":

$$h\in\mathrm{K}\mathbf{1}\,.\supset.\, h'' = hh.$$

All these rules[9] give a meaning to the sign " ' ": when a letter is followed by an apostrophe, this means that the point designated by the letter should be regarded as an emitting source, and that the whole expression refers to the shadow engendered by the light, when projected on the objects named by the other letters of the formula. One important feature of this notational device is that the same figure can be seen as both an emitting source and as an obstacle to the light.

Peano uses the apostrophe to give a neat definition of the plane: a plane containing three non collinear points is the triangle *abc* lit up by itself, that is *(abc)''*. I give, in figure 3, taken from Whitehead[10], the Peanian names of the different planar regions defined by a triangle:

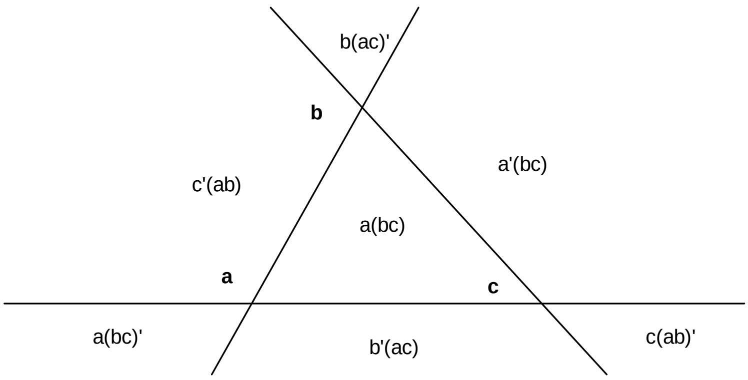

Figure 3.

[9] Peano provides us with some brackets rules: *abc* means *a*(*bc*), that is "the set of points between *a* and the segment *bc*", and *abcd* designates *a*(*bcd*) (and so on).
[10] *The Axioms of Descriptive Geometry*, Cambridge University Press, 1907.

In section 3 of his treatise, Peano studies the formal properties of this notation. He notably examines how the apostrophe symbolism interacts with the set-theoretical operation of union and inclusion. Peano first shows that, if h is a part of k, then, a being a point:

$$ah \subset ak,\ a'h \subset a'k,\ ah' \subset ak'.$$

He then proves that:

$$a(h \cup k) = ah \cup ak$$
$$a'(h \cup k) = a'h \cup a'k$$
$$a(h \cup k)' = ah' \cup ak'.$$

At last, if k is an empty class, then:

$$ak = a'k = ak' = \Lambda.$$

He extends these results to the relations of any three figures, h, l et k :

- if $h \subset k$, then $lh \subset lk$, $l'h \subset l'k$, $lh' \subset lk'$
- $l(h \cup k) = lh \cup lk$, $l'(h \cup k) = l'h \cup l'k$, $l(h \cup k)' = lh' \cup lk'$
- if k is the empty class, $hk = h'k = hk' = \Lambda$.

In this section 3 then, Peano analyzes the formal properties of what he seems to regard as three operations: hk, $h'k$ and hk'. More precisely, he shows that these operations can all be viewed as products, that is, according to Grassmann's definition, as operations distributive over addition (set-theoretic union). Moreover, these products preserve the (partial) order engendered by set-theoretic inclusion, and they also have a zero element, the empty set. In other words, Peano considers Pasch's theory *as a new kind of geometrical calculus, founded on the formal properties of three geometrical products*. That is to say, Peano reads the *Vorlesungen* through Grassmannian spectacles.

This reading is extremely surprising. Indeed, according to Pasch, a segment is not the result of the product of points – a segment is an observable entity, and its extension is the result of a certain geometrical construction. The algebraic standpoint is completely absent in the *Vorlesungen*. Thus, all happens as if his former Grassmannian works led Peano to spontaneously interpret Pasch's axiomatization as the definition of a new geometrical algebra, similar to Grassmann's one. Seen in this light, Pasch's treatise is indeed very interesting. Grassmann's own regressive and progressive products formalized only the symmetrical operations of geometrical projection and intersection. The new Peanian products are thus more powerful than Grassmann's ones – the apostrophe sign introduces an asymmetry, which reflects, in the symbolism, the asymmetrical orientation of the affine line[11].

I principii di geometria was thus not a mere translation into a logical language of the axiomatic system set out in the first two sections of the *Vorlesungen*. Unlike Pasch, Peano did not really believe that his axioms described the content of an empirical experience, and one has to acknowledge that this feature brought Peano close to Hilbert. But at the same time, the shape Peano gave to his system was very peculiar: Peano wanted to define a new geometrical product, he did not contend himself with having formulated a mere axiomatic system. For Hilbert, the free mathematician could postulate what he wanted; his power was only limited

[11] The new calculus is then expressively more powerful than the old one. Thus, the concept of a convex envelope of n points $A_1, \ldots, A_n$ was very complicated to express in the Grassmanian calculus (see *Calcolo*…, 132-133); it is very easily formulated as the product $A_1 \ldots A_n$ in the new "calculus" (see *I Principii*…, 11).

by contradiction. But the axiomatic system of *I principii* was not *any* system of rules; it was conceived of as a product algebra. The Grassmannian framework puts then strong restrictions on the possible shape of an axiomatic system and on the freedom of the axiomatic mathematician.

3- Peano's alleged confusion between mention and use

In section 1, I have shown how Peano used geometrical calculus in a "realistic way": the algebraic notions of vector and bivector were supposed to reveal the features of a certain layer of the geometrical reality. In section 2, I have shown that, when Peano explicitly adopted an axiomatic standpoint, he sometimes regarded the axiomatic system itself as a means to define a new and extended "calculus of forms", in the sense of Grassmann. Thus, not only his use of algebraic calculus moved Peano away from a Hilbertian framework, but even his "official" adoption of an axiomatic standpoint seemed, in certain cases at least, to conceal a more fundamental adhesion to the Grassmannian program.

In this last section, I will not focus on some features of Peano's reasoning. Instead, I will speak about what I regard as a significant lack in Peano's work. Many people have noted the multifarious influences of the "Boolean" school on Peano. It seems to me, however, that, on one point, Peano is completely foreign to this tradition. The logicians coming from the "logical algebra" usually draw a clear-cut distinction between a pure calculus (a mere formal combinatorics), and a full interpreted algebra. Of course, the nature of this distinction was not always understood in the same way by the logicians, but there is no doubt that the distinction itself pertained to the folklore of this tradition. To flesh that out, let me quote two texts. First, one from A. de Morgan, one of the trailblazers of this school[12]:

> The object of this book is the construction of Algebra upon a basis which will enable us to give a meaning to every symbol and combination of symbols before it is used, and consequently to dispense, first, with all unintelligible combination, secondly, with all search after interpretation of combinations subsequently to their first appearance. […] **Nothing can be clearer than the possibility of dictating the symbols with which to proceed, and the mode of using them, without any information whatever on the meaning of the former, or the purpose of the latter**. […] A person who should learn how to put together a map of Europe dissected before the paper is pasted on, would have symbols, various shaped pieces of wood, and rules of operation, directions to put them together so as to make the edges fit and the whole form an oblong figure. Let him go on until he can do this with any degree of expertness, and he has no consciousness of having learnt anything: but paste on the engraved paper, and he is soon made sensible that he has become master of the forms and relative situations of the European countries and seas.

Second, one from L. Couturat, who is at the other extremity of the stage[13]:

> Les lois fondamentales de [l'algèbre de la logique] ont été inventées pour exprimer les principes du raisonnement, les « lois de la pensée » ; **mais on peut considérer ce calcul au point de vue purement formel […] comme une Algèbre reposant sur certains principes arbitrairement posés.** C'est une question philosophique de savoir si, et dans quelle mesure, ce calcul répond aux opérations réelles de l'esprit, et est propre à traduire ou même à remplacer le raisonnement ; nous n'avons pas à la traiter ici. La valeur formelle de ce calcul et son intérêt pour le mathématicien sont absolument indépendants de l'interprétation qu'on en donne et de l'application qu'on peut en faire aux problèmes logiques.

Of course, one can find in Peano's works some places where a distinction between an empty symbolism and a full interpreted formalism is drawn (for instance, a very early occurrence of a model-theoretical proof of independence can be found in *I principii*). But we do not find

[12] *Trigonometry and double algebra*, Taylor, Walton and Maberly, 1849, II, 1.
[13] *L'algèbre de la Logique*, Gauthiers-Villars, 1905, 1.

there the equivalent of the general methodological developments which swamp the prefaces of the books belonging to the Boolean tradition. On the contrary, Peano is often very careless about the distinction between metalanguage and language object.

Take, for instance, his distinction between the so-called intensional and extensional interpretation of Boolean calculus. Instead of saying that the variable letter can be interpreted sometimes as a class and sometimes as a propositional function, Peano prefers to say that both the class and the propositional function represent the same idea, expressed in a univocal way by the logical language[14]:

> Les classes et les propositions conditionnelles ne sont donc que deux formes pour représenter la même idée. Nous préférons opérer sur les classes. Une proposition conditionnelle, contenant une variable x, sera considérée sous la forme $x \in a$, où a est une classe.

For Peano then, the usual semantic distinction should step aside in favour of what is suggested by the logical syntax. Evgueni Zaitsev, in his paper from 1994[15], observed that the same phenomenon occurs in the context of the distinction between implication and deduction. In a letter to Frege (14/10/1896), Peano wrote:

> I have given two names to the sign $\supset$ "we deduce" and "is contained", and it also may be read many other ways. This does not mean that the sign has several meanings. My idea is better expressed by saying that the sign has a single meaning, but that in ordinary language this meaning is represented by several different words, according to the circumstances.

The horseshoe has a single meaning and expresses a unique form, which, however, is instanced in different ways in the usual notation.

This feature is not only at odds with what we find in the Boolean tradition. It also reminds us very much of Grassmann's actual practice. In the second edition of the *Ausdehnungslehre*, Grassmann, after having presented his calculus of form, applies his algebra to geometry[16]. This application is not introduced as an interpretation of an empty system, but as a special way of displaying the theory of forms. For instance, Grassmann explains that, when spatial magnitudes are considered, first-order forms can take two different appearances – they can appear as points and they can also appear as "displacements", that is, as points at infinity. Grassmann does not say that some first-order forms can be interpreted as points, and that some others as points at infinity. To recast the reasoning in these terms would be very misleading, for, according to Grassmann, one of the strengths of the new algebra consists in showing that what ordinary geometry took to be two separate things are in reality two facets of the very same formal object.

I am not sure that Peano's considerations about implication or about the variable letter could be rendered as convincing as Grassmann's analysis. But this is off the point. My sole claim is to say that, if we assume that logical algebra provides us with the sole access to a certain kind of very fundamental object, then the distinction between an empty formalism on the one hand, and an interpreted calculus on the other, becomes vain and artificial: for a Grassmannian, far from being empty, algebra is always object-oriented, since it is the only way to reach a certain sort of content. Seen from this angle, the general methodological discourse of a Couturat or a de Morgan becomes just unintelligible. The syntactical articulation of the calculus of forms is directly seen as a window, which gives us access to a type of reality that we cannot see otherwise.

[14] Formules de logique mathématiques, *Rivista di matematica*, VII, 1900-1901, *Opere scelte*, II, 1958, 315.
[15] An interpretation of Peano's logic, *Archive for History of Exact Science*, 46, 4, 1994, 367-383.
[16] See *Die Ausdehnungslehre…*, I, 5.

Conclusion

I have argued that some of the thesis defended by Peano can be better understood if put back in a Grasmannian context. More precisely, I have argued that today's lost idea that a calculus can be the sole access to a certain kind of formal content plays an important role in Peano's thought. In conclusion, I would like to make two remarks about the scope of my claim.

1) Firstly, I would like to restrict the scope. I am not saying here that a Grassmanian thread can unify the very various and intricate developments one finds in Peano. I. Grattan-Guinnes portrays Peano as a pragmatist (opportunist, says he) mathematician[17]. I agree. One does not find in Peano the equivalent of Frege's or Russell's programme, or even the equivalent of Poincaré's conventionalist approach, which could unify the various reasonings. Peano's sources of inspirations are irreducibly multifarious, and Peano's works are, I think, irremediably heterogeneous. My project here was not to reduce this complexity; on the contrary, by emphasizing the importance of Grassmann's influence, my aim was to add a new layer to the already existent accumulation of conceptual strata.

2) Secondly, I would like to extend the scope of my claim. The idea that certain formalisms represent a unique access to a certain kind of content played an important role in the works of other contemporary writers as well. I especially think of Whitehead's *Universal Algebra*, which is nothing else than an effort to generalize Grassmann's approach, so as to account for the new non-Euclidean geometries. Now, if you mention Peano's and Whitehead's works, you cannot avoid reference to Russell's theory of relations. Indeed, Peano and Whitehead were Russell's main sources of inspiration. As is well documented, Russell did not view his own algebra as an empty calculus which can be variously interpreted[18]. Of course, one can find many different roots to this Russellian object-oriented conception of logic, and, among them, the indirect Grassmannian influence is surely not the most important one. But I think that, at least, it could have contributed to leading Russell away from the more standard Boolean tradition.

[17] I. Grattan-Guinness, *The search for mathematical roots 1870-1940*, Princeton University Press, 2000.
[18] See especially, G. Landini, *Russell's hidden substitutional theory*, Oxford UP, 1998.