General equilibrium

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CES Working Papers

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2008.05

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SCIENTIFIQUE

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://ces.univ-paris1.fr/cesdp/CES-docs.htm
ISSN : 1955-611X

GENERAL EQUILIBRIUM

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December 24, 2007

This is the draft of a contribution to appear in *Mathematical Models in Economics*, edited by Wei-Bin Zhang, in the *Encyclopedia of Life Support Systems*, EOLSS Publishers : Oxford, UK [http://www.eolss.net/E6-154-toc.aspx].

Key Words: Commodity space, Price space, Exchange economy, Production economy, Feasible allocation, Equilibrium, Quasi-equilibrium, Pareto optimum, Core, Edgeworth equilibrium allocations, Time and uncertainty, Continuum economies, Non-convexities, Public goods, Incomplete markets

Contents

Summary

General equilibrium is a central concept of economic theory. Unlike partial equilibrium analysis which study the equilibrium of a particular market under the clause "ceteris paribus" that revenues and prices on the other markets stay approximately unaffected, the ambition of a general equilibrium model is to analyze the simultaneous equilibrium in all markets of a competitive economy. Definition of the abstract model, some of its basic results and insights are presented. The important issues of uniqueness and local uniqueness of equilibrium are sketched; they are the condition for a predictive power of the theory and its ability to allow for statics comparisons. Finally, we review the main extensions of the general equilibrium model. Besides the natural extensions to infinitely many commodities and to a continuum of agents, some examples show how economic theory can accommodate the main ideas in order to study some contexts which were not thought of by the initial model.

1. Introduction

Since the second half of the twentieth century, the Walrasian model of general equilibrium is formulated in the concept of equilibrium of a so-called *private ownership economy*. In such an economy, finitely many price-taking consumers (who are endowed with initial holdings of the different commodities and who collect given shares of the profits from production) consume the commodities available on the market, optimizing their preferences among all possible consumption plans that satisfy their budget constraint. Finitely many producers, who also take prices as given, maximize their profit on their individual set of possible production plans; they produce the commodities which satisfy consumers' demand in competition with the initial resources not used by production. Market clearing determines equilibrium prices and the quantities actually consumed and produced at a state of equilibrium. Market clearing may result in the strict equality of supply and demand; we will then speak of (strict) *equilibrium*. But, conceivably, the conditions for equilibrium may require that excess demand be non-positive and that, for any commodity for which it is negative, the price be zero; we will then speak of *free-disposal equilibrium*.

Existence of equilibrium prices is the necessary test of consistency for a model which bases the coordination of plans of diverse economic agents on the fact that prices faced by all agents provide the common flow of information needed to coordinate the system. And, indeed, equilibrium exists under reasonable assumptions. Just as important is the relation between the existence of solutions and the problems of normative economics. The Paretian study of optimality considers the problem of efficient organization of an economy with an unspecified distribution of resources and shows that for any Pareto optimal feasible allocation there exists a price system to which consumers and producers are adapted. The Debreu-Scarf theorem shows that the same is true at the limit for an allocation that no coalition can block, in an economy with a specified distribution of resources when the number of consumers tends to infinity. These results have been first obtained in the 1950s and 1960s simultaneously by the three founders of the general equilibrium theory: K.J. Arrow, G. Debreu and, in a slightly different setting, L.W. McKenzie. Extended since then, these achievements strongly rely on convexity assumptions on the characteristics of the agents and on the use of convex analysis and fixed point theory. They confirm the new dependence of Mathematical Economics on an increasing list of mathematical tools such as functional analysis, measure theory, differential topology, non-smooth analysis, ordered sets theory … .

After 1970, the general equilibrium theory developed in several directions. First, in what we will call the classical model (finite number of commodities, finite number of agents), the conditions for equilibrium existence and optimality results have been strongly weakened. Besides, the model itself has been generalized in order to allow for an infinity of time periods and states of the world in the specification of commodities, and for an infinity of agents. The abstract model has also been modified in order to accommodate the analysis of a great variety of economic settings where properties of equilibrium may fail to hold. The different contexts share a same methodology with the classical model of general equilibrium: the assumption that individuals, who have perfect (imperfect in case of differential information) foresight for each future state of the world, make their transactions in one initial market for the whole future and all states of the world. This common methodology explains the central role of general equilibrium in economic theory and maybe some of their common drawbacks.

In what follows, Sections 2, 3, and 4 present the current state-of-the-art for the classical model. Section 5 sketch the issues of uniqueness of the equilibrium solution. Section 6 discusses some extensions.

2. The classical model

Commodities

In the classical model, only a finite number of commodities are exchanged, produced or consumed. The *commodity space* is thus R^ℓ, the real vector space of dimension ℓ. The vector $z = (z_k)_{k=1}^{\ell} \in R^\ell$ denotes a *commodity bundle* with sign conventions explained below.

Private ownership economies and their agents

On this commodity space, a *private ownership economy* is completely specified by:

$$E = \left(\left(X^i, P^i, \omega^i \right)_{i \in I}, \left(Y^j \right)_{j \in J}, \left(\theta_{ij} \right)_{i \in I, j \in J} \right)$$

where

- I is a finite set of consumers. Typically, a *consumer* is an individual but may be a household or a larger group with a common purpose. A consumer may even be a country in a model of international trade. For each consumer $i \in I$, a *consumption set* $X^i \subset R^\ell$ is the set of all consumption plans physically (or socially) possible for him. The consumption plan $x^i \in X^i$ is a list $\left(x_k^i \right)_{k=1}^{\ell}$ of the quantities of the various commodities that the consumer i consumes (positive numbers) or delivers (negative numbers). The *preference correspondence* $P^i : \prod_{h \in I} X^h \to X^i$ describes the tastes of consumer i. Under the condition that $x^i \notin P^i(x)$, the set $P^i(x)$ is interpreted as the set of consumption vectors x'^i strictly preferred to x^i when the consumption vectors of all consumers $h \neq i$ are x^h. Such a dependence may be justified by imitation or other psychological effects. It encompasses the case when consumer's preferences depend only on his own consumption vector and, a fortiori, the case when consumer i has a complete preference preorder $\geq^i$ on his consumption set and $P^i\left(x^i\right) = \left\{ x'^i \in X^i \middle| x'^i \succ^i x^i \right\}$. The vector ω^i is the ith consumer's *initial endowment* in each one of the commodities.
- J is a finite set of producers. For each *producer* $j \in J$, the *production set* Y^j is the set of all production plans technically possible for the jth producer. A production plan $y^j \in Y^j$ is a list $\left(y_k^j \right)_{k=1}^{\ell}$ of the quantities of the various commodities that the producer j consumes as *inputs* (negative numbers) or produces as *outputs* (positive numbers).
- For all $i \in I$ and $j \in J$, the *profit-share* θ_{ij} is the contractual claim of consumer i on the profit of producer j. By definition, the θ_{ij} are nonnegative and for every j, $\sum_{i \in I} \theta_{ij} = 1$.

A particular case of private ownership economy is the *exchange economy,* where there is no production, specified by:

$$E = \left(X^i, P^i, \omega^i \right)_{i \in I}$$

Feasibility and market clearing

An *allocation*

$$(x, y) \in \prod_{i \in I} X^i \times \prod_{j \in J} Y^j$$

is *feasible* (or *attainable*) if

$$\sum_{i\in I} x^i - \sum_{i\in I}\omega^i - \sum_{j\in J} y^j = 0$$

feasible (or *attainable*) *with free-disposal* if

$$\sum_{i\in I} x^i - \sum_{i\in I}\omega^i - \sum_{j\in J} y^j \leq 0$$

Equilibrium

Let $S = \{p \in R^l : \|p\| = 1\}$ be the sphere with center 0 y radius 1. If each $p \in S$ represents a possible list of the (normalized) prices of each commodity, an *equilibrium* of E is a point

$$\left(\overline{p}, \overline{x}, \overline{y}\right) \in S \times \prod_{i\in I} X^i \times \prod_{j\in J} Y^j$$

satisfying the following conditions:

1. For each $i \in I$, $\overline{p}.\overline{x}^i \leq \overline{p}.\omega^i + \sum_{j\in J}\theta_{ij}\,\overline{p}.\overline{y}^j$ and $x^i \in P^i\left(\overline{x}\right) \Rightarrow \overline{p}.x^i > \overline{p}.\overline{x}^i$
2. For each $j \in J$, for all $y^j \in Y^j$, $\overline{p}.y^j \leq \overline{p}.\overline{y}^j$
3. The allocation $\left(\overline{x}, \overline{y}\right)$ is feasible.

The point $\left(\overline{p}, \overline{x}, \overline{y}\right) \in S \times \prod_{i\in I} X^i \times \prod_{j\in J} Y^j$ is a *free-disposal equilibrium* if, in the previous definition, the third condition is replaced by:

3'. The allocation $\left(\overline{x}, \overline{y}\right)$ is attainable with free-disposal and $\overline{p}.\left(\sum_{j\in J}\overline{y}^j + \sum_{i\in I}\omega^i - \sum_{i\in I}\overline{x}^i\right) = 0$.

Condition 1 states that at equilibrium every consumer has chosen in his consumption set a consumption plan which best satisfies his preferences under his budget constraint. Summing over i the budget constraints, it follows then from Condition 3) or 3') that, at equilibrium, every consumer spends his equilibrium revenue: $\forall i \in I$, . $\overline{p}.\overline{x}^i = \overline{p}.\omega^i + \sum_{j\in J}\theta_{ij}\,\overline{p}.y^j$.

Condition 2 states that every producer chooses a production plan so as to maximize his profit in his production set.

Condition 3 states at equilibrium the equality between the total supply and the total demand. In case of free-disposal equilibrium, Condition 3' ensures that the cost of the disposal needed for achieving equilibrium is minimized and equal to zero.

A clever reader will have observed that the free-disposal equilibrium of an economy is thus the equilibrium of a fictitious economy identical to the original one but where an additional price-taker and profit-maximizing producer has the negative orthant $-R^l$ as his production set. It is easily deduced from this remark that, at a free-disposal equilibrium, the equilibrium vector price is non-negative. This need not be the case if the equilibrium is strict, that is, without free-disposal. Some commodities may be noxious or at least non-desired at equilibrium; the disposal needed for achieving equilibrium is not free anymore and may involve costly activities.

Quasi-equilibrium

A point $\left(\bar{p},\bar{x},\bar{y}\right)\in S\times\prod_{i\in I}X^i\times\prod_{j\in J}Y^j$ is a *quasi-equilibrium* (resp. *free-disposal quasi equilibrium*) if it satisfies the profit-maximization and feasibility conditions of the corresponding equilibrium definition and if Condition 1) is replaced by the weaker condition:

1'. For each $i\in I$, $\bar{p}.\bar{x}^i\leq\bar{p}.\omega^i+\sum_{j\in J}\theta_{ij}\bar{p}.\bar{y}^j$ and $x^i\in P^i\left(\bar{x}\right)\Rightarrow\bar{p}.x^i\geq\bar{p}.\bar{x}^i$.

The interpretation of Condition 1' is not very appealing since it means that, at a quasi-equilibrium, no consumer could be strictly better spending strictly less than his budget constraint. The interest of the quasi-equilibrium concept is purely mathematical. As we will see in the next section, the equilibrium existence proofs actually establish the existence of a quasi-equilibrium, allowing for a clear distinction between the quasi-equilibrium existence problem and the investigation of the conditions which guarantee that a quasi-equilibrium is an equilibrium.

3. Existence of equilibrium

Several equilibrium existence proofs are available in the literature. Among the three main approaches, the *excess demand approach* obtains the equilibrium price as a zero of the excess demand correspondence (a zero of the excess quasi-demand correspondence for the quasi-equilibrium price). This approach requires the preferences of each consumer to be formalized by a complete preorder on their consumption set.
The so-called *Negishi approach* bases the equilibrium existence on a fixed-point theorem applied in the *utility space*, that is in the vector space R^I whose dimension is equal to the finite number of consumers. This approaches requires the preferences of each consumer to be represented by a utility function but, since the first writings of Debreu, economists know that, at least when the commodity space is finite dimensional and under the usual assumptions of the equilibrium existence theorem, there is no loss of generality to assume that complete preference preorders on the consumption sets are represented by utility functions.
The approach presented below is the *simultaneous optimization approach*. The interest of this approach and of the theorems which will be presented is to be not too much demanding in terms of the rationality of consumers' choices (consumers' preferences need not be complete or even transitive) and to allow for some dependence of the individual preferences on the actions of the other agents.

3.1. Equilibrium and quasi-equilibrium of abstract economies

An *abstract economy* (*generalized qualitative game, social system*) is completely specified by

$$\Gamma=\left(X^i,\alpha^i,P^i\right)_{i\in N}$$

where N is a finite set of *agents* (*players*) and for each $i\in N$,

- X^i is a *choice set* (or *strategy set*) that we will assume to be a subset of some finite dimensional vector space,
- The correspondence $\alpha^i:\prod_{h\in N}X^h\to X^i$ is called *constraint correspondence*. For $x\in X:=\prod_{h\in N}X^h$, the set $\alpha^i(x)$ is interpreted as the set of the possible strategies for i given the choices $\left(x^h\right)_{h\neq i}$ of the other agents,
- The correspondence $P^i:\prod_{h\in N}X^h\to X^i$ is a *preference correspondence*. For each

$x \in X$, under the condition that $x^i \notin P^i(x)$, the set $P^i(x)$ is interpreted as the set of elements of X^i strictly preferred by i when the choice of the other agents is $(x^h)_{h \neq i}$.

An *equilibrium* of Γ is a point $\bar{x} \in X$ such that for each $i \in N$,

1. $\bar{x}^i \in \alpha^i(\bar{x})$
2. $P^i(\bar{x}) \cap \alpha^i(\bar{x}) = \varnothing$.

The interpretation is that at each component of the equilibrium point, the corresponding agent best satisfies his preferences in his constraint correspondence.

As a special case, if for every $i \in N$ and for every $x \in X$, $\alpha^i(x) = X^i$, in other words if $\Gamma = (X^i, P^i)_{i \in N}$ is a *qualitative game*, an equilibrium of Γ is nothing other than a *Nash equilibrium* of the qualitative game Γ.

Now, let $\beta^i : X \to X^i$ be correspondences satisfying for every $i \in N$ and for every $x \in X$,

3. $\beta^i(x) \subset \alpha^i(x)$
4. $\beta^i(x) \neq \varnothing \Rightarrow \overline{\beta^i(x)} = \overline{\alpha^i(x)}$

where for a set A, $\bar{A}$ denotes the closure of A.

Given $\beta = (\beta^i)_{i \in N}$, a β*-quasi-equilibrium* is a point $\bar{x} \in X$ such that for each $i \in N$,

5. $\bar{x}^i \in \alpha^i(\bar{x})$
6. $P^i(\bar{x}) \cap \beta^i(\bar{x}) = \varnothing$.

β-quasi-equilibrium existence

The existence of a β*-quasi-equilibrium* can be deduced from the Gale–Mas-Colell lemma, lemma itself proved using Kakutani's theorem and a powerful selection theorem due to Michael.

Lemma (Gale–Mas-Colell) *Let N be a finite set of indices. Given $X = \prod_{i \in N} X^i$, where for each $i \in N$, X^i is a nonempty compact convex subset of some finite dimensional Euclidean vector space, let for each i, $\varphi^i : X \to X^i$ be a lower semi-continuous correspondence with convex (possibly empty) values. Then there exists $\bar{x} \in X$ such that for each $i \in N$, either $\varphi^i(\bar{x}) = \varnothing$ or $\bar{x}^i \in \varphi^i(\bar{x})$.*

The existence of a β- quasi-equilibrium for an abstract economy is proved under the following assumptions for each $i \in N$:

(a) X^i is a nonempty compact convex subset of some finite dimensional Euclidean vector space,

(b) α^i is an upper semi-continuous and non empty closed convex-valued correspondence,

(c) β^i is convex-valued,

(d) the correspondence $x \to \beta^i(x) \cap P^i(x)$ is lower semi-continuous,

(e) the correspondence P^i is convex-valued and for all $x \in X$, $x^i \notin P^i(x)$.

Proposition 1 *Under the conditions (a) – (e) and for* β *defined as above, the abstract economy* $\Gamma = \left(X^i, \alpha^i, P^i\right)_{i \in N}$ *has a* β *quasi-equilibrium. It is an equilibrium provided that for every* i, $\beta^i\left(\bar{x}\right) \neq \emptyset$.

3.2. Application to quasi-equilibrium existence for private ownership economies

Let

$$E = \left(\left(X^i, P^i, \omega^i\right)_{i \in I}, \left(Y^j\right)_{j \in J}, \left(\theta_{ij}\right)_{i \in I, j \in J}\right)$$

be a private ownership economy as defined in Section 2. In the following, $A(E)$ will denote, according to the case under consideration, the *set of feasible* (resp. *feasible with free-disposal*) *allocations* of the economy. For each $i \in I$, $j \in J$, $\hat{X}^i$, $\hat{Y}^j$ will denote the projections of this set on the corresponding consumption or production set, that is the *feasible sets* of each consumer and producer.

Quasi-equilibrium existence for the economy is proved under the following assumptions:

1. For each $i \in I$,
   (a) X^i is convex and closed, $\hat{X}^i$ is compact
   (b) P^i is lower semi-continuous with convex values and for all $x \in X$, $x^i \notin P^i(x)$,
   (c) $\omega^i \in X^i - \sum_{j \in J} \theta_{ij} Y^j$ ($\omega^i \in X^i - \sum_{j \in J} \theta_{ij} Y^j - R^l_+$ in case of free-disposal),
   (d) for each feasible allocation (x, y), $x^i \in \overline{P^i(x)}$;
2. For each $j \in J$, $0 \in Y^j$ which is convex and closed, $\hat{Y}^j$ is compact;
3. The *total production set* $Y := \sum_{j \in J} Y^j$ is closed ($Y + R^l_+$ in case of free-disposal).

Besides convexity, closedness and continuity assumptions, Assumption 1 (c) is a *weak survival* assumption: for each consumer, a consumption vector is possible using his own resources and, eventually, his share of the productive system. Assumption 1 (d) means that for each consumer, a feasible consumption vector is a point of *local no-satiation*: each consumer can be strictly better with a consumption vector as close as desired from his actual consumption in the feasible allocation. The compactness of all feasible sets, in 1 (a) and 2, can be obtained under, for example, the assumption that all consumption sets are bounded below ant that all production sets are bounded above. More sophisticated assumptions on the total consumption set $\sum_{i \in I} X^i$ and the total production set $\sum_{j \in J} Y^j$ can be found in the literature; eventually stated in terms of the recession cones of these sets, they guarantee the same result.

Sketch of the proof for a compact economy
In order to apply Proposition 1, the quasi-equilibrium existence for the private ownership economy is first proved assuming that, besides the previous assumptions, consumption and production sets are bounded. Agents of the abstract economy associated to the private owner ship economy are:

- the consumers, with their consumption set as strategy set, the same preferences as in the private ownership economy, the *budget correspondence*

$$\alpha^i(p)=\left\{x^i \in X^i \middle| p.x^i \le p.\omega^i + \sum_j \theta_{ij} p.y^j + \frac{1-\|p\|}{|I|}\right\}$$

as constraint correspondence, and

$$\beta^i(p)=\left\{x^i \in X^i \middle| p.x^i < p.\omega^i + \sum_j \theta_{ij} p.y^j + \frac{1-\|p\|}{|I|}\right\}$$

- the producers with their production set as strategy set, correspondences α^j and β^j whose values in reaction to the actions of the other agents are identically equal to their production set, preferences on their strategy set associated in a obvious way with the objective of profit maximization

- an additional agent, interpreted as the *Walras auctioneer*, who sets prices in reaction to demands and supplies of consumers and producers with the objective of maximizing the value of the excess demand. His strategy set is the compact convex set of prices $X^0 = \overline{B} = \{p \in R^l \,\|\, \|p\| \le 1\}$, the closed unit-ball centered at the origin and with radius equal to 1 (or the intersection of this ball with the positive orthant R^l_+ in case of free-disposal). In reaction to the actions of consumers and producers, correspondences α^0 and β^O have values identically equal to X^0.

In $\alpha^i(p)$ and $\beta^i(p)$, the addition to the regular revenue of consumers of a non-negative extra term depending on the norm of the price is a stratagem due to Bergstrom and has as objective to allow the different correspondences to satisfy the continuity properties assumed in Proposition 1. Applying this proposition, one gets a point $(\overline{p},\overline{x},\overline{y})$ satisfying the conditions for quasi-equilibrium of the economy, provided that $\|\overline{p}\| = 1$. The verification of this last condition completes the proof of the following proposition:

Proposition 2 *Under the conditions 1. (a) – (d), 2. and 3. a private ownership economy* $E = \left((X^i, P^i, \omega^i)_{i\in I}, (Y^j)_{j\in J}, (\theta_{ij})_{i\in I, j\in J}\right)$ *whose consumption and production sets are bounded admits a quasi-equilibrium.*

Quasi-equilibrium existence in the original economy

Taking advantage of the compactness of the feasible sets of consumers and producers, assumed in 1 (a) and 2, the idea of the proof is then to replace the original economy E by a compact one satisfying the same assumptions.

Let $K \subset R^l$ be a compact convex set containing all feasible sets in its interior. One associates with the original economy a compact economy with the sets $\breve{X}^i = X^i \cap K$ and $\breve{Y}^j = Y^j \cap K$ as consumption and production sets.

In order to facilitate passing from a quasi-equilibrium of the compact economy to a quasi-equilibrium of the original economy, the original consumer’s preference correspondences are also replaced by the *Gale–Mas-Colell augmented preference correspondences*:

$$\hat{P}^i(x) = \left\{z^i \in X^i \middle| z^i = \lambda x^i + (1-\lambda) z'^i, 0 \le \lambda < 1, z'^i \in P^i(x)\right\}$$

Note that the augmented preference correspondences are semi-continuous if the original preference correspondences are assumed to be semi-continuous, and that $x^i \in \overline{\hat{P}^i(x)}$ if

$P^i(x) \neq \emptyset$. The possible replacement of local no-satiation assumptions by no-satiation assumptions (at any component of a feasible allocation) is also a reason for introducing the Gale–Mas-Colell augmented preferences

Consider the compact economy

$$\breve{E} = \left(\left(\breve{X}^i, \hat{P}^i, \omega^i \right)_{i \in I}, \left(\breve{Y}^j \right)_{j \in J}, \left(\theta_{ij} \right)_{i \in I, j \in J} \right)$$

where the augmented preferences in the original economy and their obvious restrictions in the compact economy are noted identically. Applying the quasi-equilibrium existence result for compact economies, it is then shown that the economy

$$E' = \left(\left(X^i, \hat{P}^i, \omega^i \right)_{i \in I}, \left(Y^j \right)_{j \in J}, \left(\theta_{ij} \right)_{i \in I, j \in J} \right)$$

has a quasi-equilibrium $\left(\bar{p}, \bar{x}, \bar{y}\right) \in S \times \prod_{i \in I} X^i \times \prod_{j \in J} Y^j$. This quasi-equilibrium is a fortiori a quasi-equilibrium of the original economy E. Moreover, it follows from the remark above that the quasi-equilibrium existence has been obtained with Assumption 1 (d) replaced by the weaker assumption:

1. (d') For each $i \in I$ and for every feasible allocation (x, y), $P^i(x) \neq \emptyset$.

To summarize,

Proposition 3 *Under the conditions 1 (a) –(c), (d'), 2 and 3, a private ownership economy* $E = \left(\left(X^i, P^i, \omega^i \right)_{i \in I}, \left(Y^j \right)_{j \in J}, \left(\theta_{ij} \right)_{i \in I, j \in J} \right)$ *admits a quasi-equilibrium.*

Before closing this sub-section, let us remark that a glance to the proofs presented here shows that, under convenient assumptions, the quasi-equilibrium existence result could accommodate consumers' preferences depending on the current production vector and also **price-dependent preferences** which may make sense in economic models.

3.3. From quasi-equilibrium to quasi-equilibrium

More continuity of preferences

Now, let us set an additional continuity condition on consumers' preferences:

1. (e) For each $i \in I$ and for every $x \in X$, the set $P^i(x)$ is open in X^i.

Then, if for all $p \in S$, we set

$$\gamma^i(p) = \left\{ x^i \in X^i \,\middle|\, p.x^i \leq p.\omega^i + \sum \theta_{ij} \max p.Y^j \right\}$$

the *budget set* of the ith consumer and

$$\delta^i(p) = \left\{ x^i \in X^i \,\middle|\, p.x^i < p.\omega^i + \sum \theta_{ij} \max p.Y^j \right\}$$

we can remark that $\delta^i(\bar{p}) \neq \emptyset$, $\bar{x}^i \in \gamma^i(\bar{p})$ and $P^i(\bar{x}) \cap \delta^i(\bar{p}) = \emptyset$ imply $P^i(\bar{x}) \cap \gamma^i(\bar{p}) = \emptyset$.

Thus, the investigation of conditions under which a quasi-equilibrium of E is also an equilibrium is reduced to the search for conditions under which $\delta^i(\bar{p}) \neq \emptyset$ for all $i \in I$, if $\bar{p}$ is a quasi-equilibrium price.

Strong survival versus non-triviality of the quasi-equilibrium and irreducibility of the economy

The *strong survival assumption*:

1. (f) For each $i \in I$, $\omega^i \in \operatorname{int}\left(X^i - \sum_{j \in J} \theta_{ij} Y^j\right)$ where $\operatorname{int} A$ denotes the interior of A

($\omega^i \in \operatorname{int}\left(X^i - \sum_{j \in J} \theta_{ij} Y^j + R^l_+\right)$ in case of free-disposal)

is such an assumption. It ensures that, at the quasi-equilibrium price, each consumer can consume without spending his entire revenue, that is $\delta^i(\bar{p}) \neq \emptyset$ for all $i \in I$.

To see this in the case of no-disposal, recall that $\bar{p} \neq 0$. For each $i \in I$, pick $u \in R^l$ such that $\bar{p}.u < 0$ and $\omega^i + u \in X^i - \sum_{j \in J} \theta_{ij} Y^j$. One can write $x^i = u + \omega^i + \sum_{j \in J} \theta_{ij} y^j$ for some $x^i \in X^i$, $y = (y^j) \in \prod_{j \in J} Y^j$, which implies $\bar{p}.x^i < \bar{p}.\omega^i + \sum_{j \in J} \theta_{ij}\, \bar{p}.y^j \leq \bar{p}.\omega^i + \sum \bar{p}.\bar{y}^j$. The proof is similar in case of free-disposal.

It is possible to weaken the strong survival assumption, replacing it by the following:

$$\omega := \sum_{i \in I} \omega^i \in \operatorname{int}\left(\sum_{i \in I} X^i - \sum_{j \in J} Y^j\right)$$

($\omega := \sum_{i \in I} \omega^i \in \operatorname{int}\left(\sum_{i \in I} X^i - \sum_{j \in J} Y^j + R^l_+\right)$ in case of free-disposal).

This is still a strong interiority assumption which insures that, at the quasi-equilibrium price, at least one consumer can consume without spending his entire revenue. The quasi-equilibrium is then said to be *non-trivial*.

It should be noticed that for the non-triviality of quasi-equilibrium, as shown by counter-examples, the interiority assumption cannot be dispensed with, without additional assumptions on the characteristics of the economy, for example assumptions analogous to the properness assumptions used in infinite dimensional economies.

The role of an *irreducibility assumption* is to guarantee that the non-emptiness of one $\delta^i(\bar{p})$ implies the non-emptiness of all. The general idea is to assume that, in some sense to be specified, it is always possible for any non-empty and proper group of consumers to benefit from the resources of the group of the other consumers. Various irreducibility assumptions have been proposed by Gale in the context of linear exchange economies, McKenzie, Debreu, Arrow-Hahn, Bergstrom and many others. To enter in the technical subtleties of the different formulations would exceed the objectives of this survey.

Collecting the findings of this section, the equilibrium existence theorem can now be stated.

Theorem 1 *Let* $E = \left(\left(X^i, P^i, \omega^i \right)_{i \in I}, \left(Y^j \right)_{j \in J}, \left(\theta_{ij} \right)_{i \in I, j \in J} \right)$ *be a private ownership economy satisfying the assumptions satisfying the assumptions 1. (a), (b), (c), (d') (e), 2 and 3, and let* $\left(\bar{p}, \bar{x}, \bar{y} \right) \in S \times \prod_{i \in I} X^i \times \prod_{j \in J} Y^j$ *be a quasi-equilibrium of this economy. If in addition,*

- *either* E *satisfies 1. (f), that is the strong survival assumption for every consumer,*
- *or the quasi-equilibrium is non-trivial and* E *is irreducible,*

then $\left(\bar{p}, \bar{x}, \bar{y} \right)$ *is an equilibrium.*

4. Optimality properties of equilibrium

The optimality notion which allows to establish that an equilibrium allocation is optimal goes back to Pareto. Roughly speaking, in the general equilibrium model, a feasible allocation is Pareto optimal if there is no other feasible allocation that each individual (actually each consumer) would prefer. In the same guise, the term of *Pareto optimality* is used nowadays in several problems of multi-criteria or multi-objective optimization in which a planner refuses to compare states that improve some criteria (e.g. the utility of some agents) decreasing some other ones (e.g. the utility of some other agents).
The cooperative game theoretic notion of *core* of a private ownership economy corresponds to the idea that no feasible allocation is sustainable as an equilibrium allocation if consumers are free to cooperate, in a way to be defined, and a coalition of consumers can obtain an allocation feasible for themselves and that they all prefer to the given allocation. As we will see, a core allocation is defined as an allocation blocked by no coalition.
Finally, the desirable persistency of this kind of stability against re-contracting in a private ownership economy when the relative weights of each consumer become infinitesimal leads to the definition of *limit-core* concepts.
After a precise definition of these notions which will be given below, basic results of welfare economics as the optimality of equilibrium allocations, belonging of these allocations to the core of the economy, and even to the core of all replica economies will become simple tautologies. More interesting are the converse issues. Statement of converse results and their proof constitute the main objective of this section. Before stating these theorems, we will verify with non-emptiness results the consistency of the newly introduced notions.

4.1 Optimality, core and limit-core concepts

Pareto dominance and Pareto optimality
Let

$$E = \left(\left(X^i, P^i \right)_{i \in I}, \left(Y^j \right)_{j \in J}, \omega \right)$$

be an economy for which we specify neither the repartition of the whole initial endowment ω among the consumers nor the profit shares corresponding to the ownership of the different firms by the different consumers. If (x, y) and (x', y') are two feasible allocations of E, we shall say that the feasible consumption allocation x' is *(strictly) preferred* to (or *strictly dominates*) the feasible consumption allocation x if for all $i \in I$, $x'^i \in P^i(x)$. Then, a feasible consumption allocation x is *Pareto optimal* if it is (strictly) dominated by no other feasible consumption allocation. In the case when each consumer i has a complete preorder of preferences $\geq^i$ on his consumption set X^i (recall that then $P^i(x) = \left\{ x'^i \in X^i \middle| x'^i \succ^i x^i \right\}$), one uses also a weaker dominance relation: the feasible consumption allocation x' *weakly*

dominates the feasible consumption allocation x if $x'^i \geq^i x^i$ for all consumers $i \in I$ and if the preference is strict for at least one of them. Then a feasible consumption allocation x is said to be a *strong Pareto optimum* if it is weakly dominated by no other feasible allocation.
In the following, the set

$$\hat{X} = \left\{ x \in \prod_{i \in I} X^i \,\middle|\, \exists y \in \prod_{j \in J} Y^j, \sum_{i \in I} x^i = \omega + \sum_{j \in J} y^j \right\}$$

will denote the set of all feasible consumption allocations of the economy.

Core of an economy

Consider now a private ownership economy

$$E = \left(\left(X^i, P^i, \omega^i \right)_{i \in I}, \left(Y^j \right)_{j \in J}, \left(\theta_{ij} \right)_{i \in I, j \in J} \right)$$

as defined in Section 2 and studied in Section 3. A *coalition* is any nonempty subset S of the set I of consumers. If $X^S = \prod_{i \in S} X^i$, a vector $x^S = \left(x^{iS} \right)_{i \in S} \in X^S$ is a *consumption assignment* for the coalition S. A *preference relation* $P^S : \prod_{i \in I} X^i \to X^S$ is defined by

$$P^S(x) = \left\{ x^S \in X^S \,\middle|\, x^{iS} \in P^i(x) \forall i \in S \right\}.$$

In other words, $P^S(x)$ is the set of consumption assignments $x^S = \left(x^{iS} \right)_{i \in S}$ which are unanimously preferred to x by the members of the coalition S. In order to define the productive power of the coalition S, it is assumed that the coalition has the technology set $Y^S = \sum_{j \in J} \sum_{i \in S} \theta_{ij} Y^j$ as *production set*. This kind of assumption, whose economic meaning is controversial, relies on the idea that the relative profit-shares θ_{ij} reflect consumers' stock holdings which represent proprietorships of production possibilities.

$$\hat{X}^S = \left\{ x^S \in X^S \,\middle|\, \exists y \in Y^S, \sum_{i \in I} x^{iS} = \sum_{i \in I} \omega^i + y \right\}$$

is the set of all *feasible consumption assignments for the coalition* S (resp . *feasible with free-disposal consumption assignments* if the equality is replaced by a large inequality).
Obviously, $\hat{X} = \hat{X}^I$; the set of feasible consumption allocations is the set of feasible consumption assignments for the *grand coalition* I, that is the coalition of all consumers.

A coalition S *blocks* the feasible consumption allocation x if there is some $x^S \in \hat{X}^S \cap P^S(x)$.
The *core* of the economy E is the set of all feasible consumption allocations blocked by no coalition. Notice that a feasible consumption allocation is Pareto optimal if and only if it is not blocked by the *grand coalition*. An element of the core is thus a Pareto optimal consumption allocation.

Replica economies and limit-core concepts

Let r be any positive integer. The r-*replica* of E is the economy composed of r sub-economies identical to the original one

$$E^r = \left(\left(X^{iq}, P^{iq}, \omega^{iq} \right)_{i \in I, q=1,\ldots,r}, \left(Y^{jq'} \right)_{j \in J, q'=1,\ldots,r}, \left(\theta_{iqjq'} \right)_{i \in I, j \in J, q, q'=1,\ldots,r} \right)$$

defined as follows:

- For each $j \in J$, r producers of type j have the same production set $Y^{j,q'} = Y^j$,

- For each $i \in I$, r consumers of type i have the same consumption set $X^{iq} = X^i$ and the same initial endowment $\omega^{iq} = \omega^i$,
- For externalities in preferences and profit shares, each consumer is restricted within his sub-economy: $P^{iq} : \prod_{i \in I} X^i \to X^i$ is defined by $P^{iq}(x) = P^i(x)$, $\theta_{iqjq} = \theta_{ij}$, $\theta_{iqjq'} = 0$ if $q \neq q'$.

In E^r, a consumption assignment $x^r \in \prod_{i \in I, q=1,\ldots,r} X^{iq}$ is said to have the *equal treatment property* if for every $i \in I$, $x^{iq} = x^i$ for all $q = 1,\ldots,r$. As easily verified, such a consumption assignment x^r is feasible in E^r if and only if $x = (x^i)_{i \in I}$ is a feasible consumption allocation in the original economy E.

In this paragraph, we are interested in the set $C^r(E)$ of the feasible consumption allocations x of the original economy whose r-replica x^r belongs to the core of E^r. For a feasible consumption allocation x in the original economy, and provided that each Y^j is convex, this is equivalent to the following condition: there is no coalition $S \subset I \times \{1,\ldots,r\}$ in E^r with some consumption assignment x^S such that

$$\sum_{(i,q) \in S} x^{iqS} - \sum_{i \in S} |S(i)| \omega^i \in \sum_{i \in I} |S(i)| \sum_{j \in J} \theta_{ij} Y^j$$
$$x^{iqS} \in P^i(x) \forall (i,q) \in S$$

where $S(i) := \{q \in \{1,\ldots,r\} | (i,q) \in S\}$ is the set of consumers of type i who belong to the coalition S and $|S(i)|$ denotes the number of elements of $S(i)$.

Let us define $t_i = \frac{|S(i)|}{r}$, $t = (t_i)_{i \in I}$, and set for each i such that $t_i > 0$, $x^{it} = \frac{1}{|S(i)|} \sum_{q \in S(i)} x^{iqS}$. The scalar t_i is a rational number in $[0,1]$ which can be thought of as the *rate of participation* of i *in the coalition* t and x^{it} is the mean consumption that consumer i achieves by participating in the coalition. Assuming that all consumption and production sets are convex and all preference correspondences are convex-valued, we thus can replace the above relations by

$$\sum_{\{i | t_i > 0\}} t_i x^{it} - \sum_{i \in I} t_i \omega^i \in \sum_{i \in I} t_i \sum_{j \in J} \theta_{ij} Y^j$$
$$x^{it} \in P^i(x) \forall i : t_i > 0.$$

The existence of some x^S satisfying the first two relations is obviously equivalent to the existence of some x^t satisfying the two last ones and $C^r(E)$ is the set of all feasible consumption allocations x in the original economy such that there exists no coalition t, $t \neq 0$, with some consumption assignment x^t satisfying these relations.

Let us now define $C^e(E) = \bigcap_{r \geq 1} C^r(E)$. A feasible consumption allocation $x \in \bigcap_{r \geq 1} C^r(E)$ will be said to be an *Edgeworth equilibrium* of E or to belong to its *Debreu–Scarf core*.

Clearly, $C^{r+1}(E) \subset C^r(E)$ and the feasible consumption allocation x is an Edgeworth equilibrium if and only if there is no vector $t = (t_i)_{i \in I}$ of rational rates of participation belonging to the interval $[0,1]$, $t \neq 0$, with some $x^{it} \in X^i$ for each $i \in I$, satisfying the above relations. Allowing as Aubin that the rates of participation take any real value in the interval $[0,1]$, we will say that the feasible consumption allocation x of the original economy belongs to the *fuzzy core* $C^f(E)$ of this economy if there is no vector $t \in [0,1]^I$, $t \neq 0$, with some $x^{it} \in X^i$, $i \in I$, satisfying the same relations.

4.2. Non-emptiness results

In the two first following results, the economy E under consideration is an economy for which are specified neither the distribution among consumers of the total initial endowment nor the relative profit-shares of consumers on the different firms.
The existence of a Pareto optimal allocation when preference correspondences have convex values and open lower sections is an easy consequence of an elementary fixed-point theorem.

Proposition 3 *An economy E has Pareto optimal allocations under the following conditions:*

1. *The set $\hat{X}$ of all feasible (resp. feasible with free-disposal) consumption allocations is nonempty, convex and compact*
2. *For each $i \in I$, the correspondence $P^i : \prod_{h \in I} X^h \to X^i$ is convex-valued*
3. *For each $i \in I$ and for every $z^i \in X^i$, the set $(P^i)^{-1}(z^i) := \{x \in X \mid z^i \in P^i(x)\}$ is open in X.*

Remarkably, when each P^i corresponds to the strict preference associated to a complete preorder $\geq^i$ of consumer i on his consumption set X^i, the convexity of the feasible set and of the values of the preference correspondences is not needed for the existence of strong Pareto optimal allocations

Proposition 4 *If $\hat{X}$ is nonempty and compact, under the conditions 3 of the previous proposition, an economy E whose consumers have complete preference preorders on their consumption sets has strong Pareto optimal allocation (and, a fortiori, Pareto optimal allocations).*

To understand this, it suffices to associate to each complete preorder of preferences $\geq^i$ an upper semi-continuous utility function $u^i : X^i \to R$ and to define $u(x) = \sum_{i \in I} \lambda_i u^i(x^i)$ for a system $\lambda = (\lambda_i)$ of strictly positive relative weights for each consumer. A maximum $\bar{x}$ of the function u on $\hat{X}$ corresponds to a strong Pareto optimal allocation.

Let us now consider a private ownership economy E as defined in Section 2 and let $\hat{X}$ be the set of all feasible (resp. feasible with free-disposal) consumption allocations. The non-emptiness of $C^r(E)$ for any positive integer r is proved under the following assumptions:

1. The set $\hat{X}$ is compact. Moreover, for each $i \in I$,
   (a) X^i is convex,
   (b) For each $x^i \in X^i$, $(P^i)^{-1}(x^i)$ is open in $X = \prod_{h \in I} X^h$,
   (c) For each $x \in X$, $P^i(x)$ is convex and $x^i \notin P^i(x)$,
   (d) $\omega^i \in X^i - \sum_{j \in J} \theta_{ij} Y^j$ (resp. $\omega^i \in X^i - \sum_{j \in J} \theta_{ij} Y^j + R^l_+$ in case of free-disposal);
2. For each $j \in J$, Y^j is convex.

Notice that the survival assumption 1(d) can be interpreted as an *autarky assumption*: for each consumer i, the coalition $\{i\}$ reduced to this consumer alone has a nonempty feasible set $\hat{X}^{\{i\}}$.

Rephrasing the previous assumptions, let us call *convex economy* an economy whose consumption sets and production sets are convex and preference correspondences have convex values. We have the following:

Proposition 5 *Let E be a convex economy. If the set $\hat{X}$ of feasible consumption assignments is compact and if each consumer satisfies the autarky assumption and has preference correspondences with open lower sections in X, then for any positive integer r, $C^r(E) \neq \emptyset$.*

The proof of Proposition 5 is based on a fixed point argument. When consumers' preferences correspond to complete preorders on their consumption set, a proof can be based on the Scarf theorem of non-emptiness of the core of a NTU game.

Under the assumptions of Proposition 5, it is easily verified that $C^r(E)$ is a closed subset of $\hat{X}$. It was previously noticed that for all $r \geq 1$, $C^{r+1}(E) \subset C^r(E)$. Coming back to the definition of $C^e(E) = \bigcap_{r \geq 1} C^r(E)$, it follows from an obvious compactness argument that the Debreu–Scarf core, that is the set of Edgeworth equilibria, is nonempty.

Proposition 6 *Let E be a convex economy. Under the same assumptions as in Proposition 5, the set $C^e(E)$ is nonempty.*

To get the non-emptiness of the fuzzy core of the economy E, one adds the following continuity assumption:

3. For each $i \in I$, for every $x \in \hat{X}$, the set $P^i(x)$ is open in X^i.

Then, exploiting the density in $[0,1]^I$ of the subset of all elements of $[0,1]^I$ with rational coordinates, one gets the following:

Proposition 7 *For an economy E satisfying the conditions 1–3, the fuzzy core is nonempty.*

When consumers' preferences are represented by utility functions on their consumption set, the previous proposition can be translated into:

Proposition 8 *For a convex economy whose set of feasible consumption assignments is compact and consumers satisfy the autarky assumption and have continuous (quasi-concave) utility functions, the fuzzy core is nonempty.*

4.3 Price-decentralization results

For a private ownership economy, it follows from the definitions that an equilibrium consumption assignment is an element of the fuzzy core, thus an Edgeworth equilibrium, an element of the core and a Pareto optimal feasible consumption allocation.

Proposition 9 *Let* $(\bar{x},\bar{y},\bar{p})\in \prod_{i\in I} X^i \times \prod_{j\in J} Y^j \times S$ *be an equilibrium of a private ownership economy* E. *Then* $\bar{x}$ *is an element of* $C^f(E)$.

Indeed, assume on the contrary that there exist $t\in [0,1]^I$, $t\neq 0$, and $x^t \in \prod_{\{i|t^i>0\}} X^i$ such that

$$\sum_{\{i|t_i>0\}} t_i x^{it} - \sum_{i\in I} t_i\omega^i \in \sum_{i\in I} t_i \sum_{j\in J}\theta_{ij}Y^j$$

$$x^{it} \in P^i(x) \forall i : t_i > 0.$$

From the first relation and the equilibrium definition, we deduce

$$\bar{p}.\left(\sum_{\{i\in I|t_i>0\}} t_i x^{it} - \sum t_i\omega^i\right) \leq \bar{p}.\sum_{i\in I} t_i \sum_{j\in J}\theta_{ij}\bar{y}^j.$$

From the second relation and the equilibrium definition, we deduce

$$\bar{p}.x^{it} > \bar{p}.\omega^i + \sum \theta_{ij}\bar{p}.\bar{y}^j \text{ for each } i \text{ such that } t_i > 0$$

and

$$\bar{p}.\left(\sum_{\{i\in I|t_i>0\}} t_i x^{it} - \sum t_i\omega^i\right) > \bar{p}.\sum_{i\in I} t_i \sum_{j\in J}\theta_{ij}\bar{y}^j,$$

a contradiction.

The previous proposition is not surprising. The optimality, core and limit-core definitions were tailor–made for this result.
On the contrary, the price-decentralization results that we will state as converse results are celebrated theorems of general equilibrium theory.

The second welfare theorem

By price-decentralization of a Pareto optimal allocation $(\bar{x},\bar{y})\in \prod_{i\in I} X^i \times \prod_{j\in J} Y^j$ we mean the definition of a price vector $\bar{p}\in S$ such that

- For each $i\in I$, $x^i \in P^i(\bar{x}) \Rightarrow \bar{p}.x^i \geq \bar{p}.\bar{x}^i$,
- For each $j\in J$, $y^j \in Y^j \Rightarrow \bar{p}.y^j \leq \bar{p}.\bar{y}^j$.

If $\bar{p}\in S$ satisfies these two conditions, we say that $(\bar{x},\bar{y})$ *is a quasi-equilibrium allocation relative to the price system* $\bar{p}$, or that $\bar{p}$ is a *price equilibrium with transfers*.

Theorem 2 *Let* $(\bar{x},\bar{y})$ *be a Pareto optimal allocation of an economy* E *whose are specified neither the distribution of the total resources* ω *nor the relative profit-shares of consumers in the different producers. Under the conditions:*

1. *For each* $i\in I$, *the preferred set* $P^i(\bar{x})$ *is convex and* $\bar{x}^i \in \overline{P^i(\bar{x})}$ *(the closure of* $P^i(\bar{x})$*),*
2. *The total production set* $Y=\sum_{j\in J} Y^j$ *is convex,*

there exists $\bar{p} \in S$ *such that* $(\bar{x}, \bar{y})$ *is a quasi-equilibrium allocation relative to the price system* $\bar{p}$.

The proof of Theorem 2 is done using a separation argument in R^l for the two disjoint nonempty convex sets $\left(\sum_{i\in I} P^i(\bar{x}) - \sum_{j\in J} Y^j - \omega\right)$ and $\{0\}$ (resp. $-R^l_+$ in case of free-disposal).

Let $\bar{p} \neq 0$ be such that

- $\bar{p}.\left(\sum_{i\in I} P^i(\bar{x}) - \sum_{j\in J} Y^j - \omega\right) \geq 0$ (no-disposal case)
- $\bar{p}.\left(\sum_{i\in I} P^i(\bar{x}) - \sum_{j\in J} Y^j - \omega\right) \geq \bar{p}.(-R^l_+)$ (free-disposal case).

In both cases, we get $\bar{p}.\left(\sum_{i\in I}\left(P^i(\bar{x}) - \bar{x}^i\right) - \sum_{j\in J}\left(Y^j - \bar{y}^j\right)\right) \geq 0$, which, with the local non-satiation assumption made at each component of $\bar{x}$ completes the proof. There is no loss of generality to assume that $\bar{p} \in S$. Notice in addition that, in case of free-disposal, $\bar{p}.(-R^l_+) \leq 0$ implies $\bar{p} \geq 0$ and $\bar{p}.\left(\sum_{i\in I} \bar{x}^i - \sum_{j\in J} \bar{y}^j - \sum_{i\in I} \omega^i\right) = 0$

The economic meaning of the second welfare theorem is the following: If $(\bar{x}, \bar{y})$ is a Pareto optimal allocation of E where each consumer is locally non-satiated, there exists a price vector $\bar{p}$ such that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of the private ownership economy E' obtained from E by giving to each consumer i the initial endowment $\omega^i = \bar{x}^i - \frac{1}{|I|}\sum_{j\in J} \bar{y}^j$ and the profit shares $\theta_{ij} = \frac{1}{|I|}$. In other words, the responsibility of a benevolent planner can be limited to the choice between different social (Pareto) optima and to implementing the distributions of resources and profit shares associated with the chosen optimum.

The equivalence theorem

We now start from a feasible allocation $(\bar{x}, \bar{y}) \in \prod_{i\in I} X^i \times \prod_{j\in J} Y^j$ of a private ownership economy E such that $\bar{x} \in C^f(E)$, the fuzzy core of E.

Theorem 3 *Let* $(\bar{x}, \bar{y}) \in \prod_{i\in I} X^i \times \prod_{j\in J} Y^j$ *be a feasible allocation of a private ownership economy* E *such that* $\bar{x} \in C^f(E)$. *Under the conditions:*

1. *For each* $i \in I$, *the preferred set* $P^i(\bar{x})$ *is convex and* $\bar{x}^i \in \overline{P^i(\bar{x})}$,
2. *For each* $j \in J$, *the production set* Y^j *is convex,*

there exists $\bar{p} \in S$ *such that* $(\bar{x}, \bar{y}, \bar{p})$ *is a quasi-equilibrium of* E.

As for the second welfare theorem, the proof is done using a separation argument between the two nonempty convex sets $co\left(\bigcup_{i\in I}\left(P^i\left(\bar{x}\right)-\sum_{j\in J}\theta_{ij}Y^j-\omega^i\right)\right)$ and $\{0\}$ (resp. $-\mathbb{R}^\ell_+$). It follows from the fact that $\bar{x}\in C^f(E)$ that the sets are disjoints. Let $\bar{p}\neq 0$ be such that

- For each $i\in I$, $\bar{p}.\left(P^i\left(\bar{x}^i\right)-\sum_{j\in J}\theta_{ij}Y^j-\omega^i\right)\geq 0$ (no-disposal case)
- For each $i\in I$, $\bar{p}.\left(P^i\left(\bar{x}\right)-\sum_{j\in J}\theta_{ij}Y^j-\omega^i\right)\geq \bar{p}.\left(-\mathbb{R}^\ell_+\right)$ (free-disposal case).

In both cases, for each $i\in I$, $\bar{p}.P^i\left(\bar{x}\right)\geq \bar{p}.\sum_{j\in J}Y^j+\bar{p}.\omega^i$. Using the local non-satiation at each component of the consumption feasible allocation and summing over i, one gets successively $\bar{p}.\left(\sum_{i\in I}\bar{x}^i-\sum_{j\in J}\bar{y}^j-\sum_{i\in I}\omega^i\right)=0$, $\bar{p}.\bar{x}^i=\bar{p}.\omega^i+\sum_{j\in J}\theta_{ij}\bar{p}.\bar{y}^j\,\forall i\in I$, $\bar{p}.Y^j\leq \bar{p}.\bar{y}^j\,\forall j\in J$, $\bar{p}.P^i\left(\bar{x}\right)\geq \bar{p}.\bar{x}^i,\forall i\in I$, and the proof is complete. Notice, as in the second welfare theorem, that, in the free-disposal case, $\bar{p}\geq 0$.

The economic interpretation of this theorem is that the core of an economy shrinks to the set of equilibrium allocations when the number of consumers tends to infinity and the market power of each of them tends to zero. This idea that traces back to Edgeworth is probably better formalized with the equivalence theorem for continuum economies.

One of the main interests of the previous theorem is elsewhere. Adding the equivalence theorem to the non-emptiness result for the fuzzy core yields an alternative but close statement and a fourth proof of the existence of equilibrium in private ownership economies. We leave the reader to formulate this equilibrium existence result.

5. Uniqueness properties of equilibrium

The equilibrium defined until now has no reason to be unique, so that general equilibrium theory until now is not deterministic. Some examples show that there may even exist a continuum of equilibrium prices. In this section, we look at conditions which guarantee *uniqueness* or at least *local uniqueness* of equilibrium.

In order to apply general equilibrium to comparative statics analysis or to examine the dynamic behavior of general equilibrium systems, uniqueness of equilibrium would be the most desirable property. It is generally admitted that uniqueness of equilibrium requires strong additional properties of the aggregate excess demand as the *gross substitute property* or the satisfaction of the (revealed preference) *weak axiom*. The plausibility of these properties is controversial and still discussed.

We will first establish in this section local uniqueness of equilibrium as a “generic” (in a sense to be made precise below) property of the classical model depending on the parameters of the model. In order to state easily understandable statements, we will stay voluntarily below the current state-of-the-art in this question. In particular, we will restrict the analysis to exchange economies where consumers have utility functions defined on consumption sets identical to the positive orthant R^{ℓ}_{+} of the commodity space and strictly positive initial endowments. Considering consumer's tastes as fixed and individual resources as variable, we parametrize by the vector $\omega = \left(\omega^i\right)_{i \in I}$ of the strictly positive initial consumer's endowments the economies under considerations and make on the exchange economy $E(\omega) = \left(X^i, u^i, \omega^i\right)_{i \in I}$ the following assumptions:

For each $i \in I$,

(a) $X^i = R^{\ell}_{+}$ and $\omega^i >> 0$

(b) $u^i : R^{\ell}_{+} \to R$ is continuous on R^{ℓ}_{+}, twice continuously differentiable on R^{ℓ}_{++}, the interior of R^{ℓ}_{+}, with in addition:

(c) u^i is *differentiably strictly increasing*, that is, for all $x^i \in R^{\ell}_{++}$,

$$\nabla u^i\left(x^i\right) >> 0$$

(d) u^i is *differentiably strictly quasi-concave*, that is,

$$\nabla u^i\left(x^i\right)v = 0 \Rightarrow vD^2u^i\left(x^i\right)v < 0, \forall x^i \in R^{\ell}_{++}, \forall v \in R^{\ell} : v \neq 0$$

(e) For each $i \in I$ and for any $u \in R^{*}_{+}$, the closure of the set $\left\{x^i \in R^{\ell}_{++} \middle| u^i\left(x^i\right) \geq u\right\}$ is contained in R^{ℓ}_{++}.

It easily follows from Theorem 1 and the different above assumptions that $E(\omega)$ has an equilibrium $(\bar{x}, \bar{p})$ with, necessarily, $\bar{p} >> 0$ and $\bar{x} \in \left(R^{\ell}_{++}\right)^I$. Without loss of generality $\bar{p} \in ri(\Delta)$, the relative interior of the unit-simplex of R^{ℓ}. On the other hand, if $p \in R^{\ell}_{++}$, letting for each $i \in I$, $w^i = p.\omega^i$, the budget set $\alpha^i\left(p, w^i\right) = \left\{x^i \in R^{\ell}_{+} \middle| p.x^i \leq w^i\right\}$ is compact and the demand set

$$x^i\left(p, w^i\right) = \left\{x^i \in \alpha^i\left(p, w^i\right) \middle| u^i\left(x^i\right) \geq u^i(z) \forall z \in \alpha^i\left(p, w^i\right)\right\}$$

is well-defined, actually is a singleton. Each consumer has a continuous demand function on R^{ℓ}_{++}. Writing the first order conditions for the utility function maximization, one deduces from the above assumptions and the *implicit function theorem* that each demand function is also differentiable, satisfies $p.x^i\left(p, w^i\right) = w^i$ (*Walras law*) and .that if the sequence $\left(p^{\nu}, w^{i\nu}\right)$ in $ri(\Delta) \times R^{*}_{+}$ converges to (p, w) with $p \notin ri(\Delta)$, $w \in R^{*}_{+}$, then $\left\|x^i(p, w)\right\|$ converges to $+\infty$. By definition, given $\omega \in \left(R^{\ell}_{++}\right)^I$, $p \in ri(\Delta)$ is an equilibrium price vector of $E(\omega)$ if and only if $\sum_{i \in I} x^i(p, \omega) = \sum_{i \in I} \omega^i$. The set of such price vectors is denoted by $W(\omega)$

The main result of this section is the following:

Theorem 4 *Under the above assumptions, the set $W(\omega)$ is finite, except for a (relatively) closed subset of $\left(R^{\ell}_{++}\right)^I$ with Lebesgue measure zero in $\left(R^{\ell}\right)^I$. Moreover, on some neighborhood of ω outside of this exceptional set, equilibrium prices are continuously differentiable functions of the vector of individual initial resources.*

The proof of Theorem 4 relies on the application of *Sard's theorem* and of the *inverse function theorem* to the function $F : ri(\Delta) \times R^{*}_{+} \times \left(R^{\ell}_{++}\right)^{|I|-1} \to \left(R^{l}\right)^{|I|}$ defined by:

$$F\left(p,w^1,\omega^2,\ldots,\omega^{|I|}\right)=\left(\left(x^1\left(p,w^1\right)+\sum_{i\neq 1}x^i\left(p,p.\omega^i\right)-\sum_{i\neq 1}\omega^i\right),\omega^2,\ldots,\omega^{|I|}\right)$$

and on the remark that $p\in W(\omega)\Leftrightarrow F\left(p,p.\omega^1,\omega^2,\ldots,\omega^{|I|}\right)=\omega$.

We will not enter in the details of Debreu’s proof whose main steps are:

- The set C of the $\omega=F\left(p,p.\omega^1,\omega^2,\ldots\,\omega^{|I|}\right)$ such that the Jacobian matrix of F at $\left(p,p.\omega^1,\omega^2,\ldots,\omega^{|I|}\right)$ has rank smaller than $\ell\,|I|$ has Lebesgue measure zero;
- $C\cap\left(R^{\ell}_{++}\right)^{|I|}$ is closed relative to $\left(R^{\ell}_{++}\right)^{I}$
- Outside the set C, $W(\omega)$ is finite and each equilibrium is locally unique.

One says that equilibrium is *locally determinate* outside the *exceptional* set C.

A result which is verified on a open and of full Lebesgue measure subset of the set of initial data is said to be *generic*. Genericity analysis and the use of differential analysis have been extended to production economies, to more general formalizations of the preference of the agents, and also from the point of view of the dependence of equilibrium prices on the other primitive data of the model. Besides obvious differential assumptions on the characteristics of the agents, such an analysis requires in general strong assumptions on the model that contrast with the generality researched in the part of general equilibrium analysis which deals with existence and optimality of equilibrium.

However, because they open a room for comparative statics and dynamic equilibrium studies, genericity analysis belong to general equilibrium theory not only for the classical model but also for its extensions.

6. Extensions of the classical model

The list of commodities, the list of agents, the ubiquitous convexity assumptions delineate the economic context the general equilibrium model is supposed to deal with. The same is true for other properties of the characteristics of the agents and the definition of their behavior. In this section, we indicate at what extent discarding one or another assumption of the classical model changes its conclusions about equilibrium existence and the ability of market equilibrium to yield Pareto optimal allocations.

6.1. Extension to infinitely many commodities, to a continuum of agents

Infinite dimensional economies

In general equilibrium, a commodity is described not only by its physical properties but also by the date, the location and the state of nature that precise the conditions of its availability. The classical general equilibrium model hypothesis of a finite number of commodities implies that the economic activity extends over finitely many dates, locations and events. Such an hypothesis limits dramatically the application of the results to understanding the real economic life.

Assume on the contrary that, from the point of view of their physical properties, there are ℓ different goods (or services) but that a commodity bundle should specify the quantity of each good depending on (possibly) infinitely many dates, locations or states of nature. Such a statement suggests that a commodity bundle should be a vector-valued function and that an admissible *commodity space* should be a function space. Actually, on a measure space (Ω, Σ, μ) formalizing time or uncertainty or on the interval $[0,1]$ with its Lebesgue measure, spaces $L^{\ell}_p(\Omega, \Sigma, \mu)$, $L^{\ell}_p[0,1]$, $1 \leq p \leq \infty$, are among the most currently used commodity spaces in economics. As usual, two measurable functions from Ω to R^{ℓ} which coincide μ-almost everywhere on Ω (resp. two measurable functions from $[0,1]$ to R^{ℓ}) are considered as the same function. Spaces $L^{\ell}_{\infty}(\Omega, \Sigma, \mu)$ (resp. $L^{\ell}_{\infty}[0,1]$), that are spaces of (classes of) essentially bounded measurable functions, are relevant to the allocation of resources over time or states of nature. Spaces $L^{\ell}_p(\Omega, \Sigma, \mu)$ (resp. $L^{\ell}_p[0,1]$), $p \geq 1$, that are spaces of (classes of) p-integrable measurable functions, arise in Finance or in any uncertainty setting where mean, variance or eventually the moments of superior order of commodity bundles matter.

On the other hand, the physical properties of a commodity can vary continuously depending on some characteristics. In such a setting, a commodity can be defined as a point in a space K of characteristics, for example its location or its content of more fundamental constitutive elements. The space of characteristics is equipped with a metric defining closeness of commodities. The definition of commodity bundles as measures on this space corresponds to the idea that commodities with close characteristics should be considered as close substitutes. The space $M(K)$ of all finite countably additive signed measures on a compact metric space of characteristics is used in models of location or for the analysis of commodity differentiation. In this setting, a point of K is to be understood as the complete description of a unit of a commodity. A commodity bundle is not a function from K to some vector space but a finite measure on K, that is a countably additive function $m : B(K) \to R$ from the Borel σ-field of subsets of K to R; for $B \in B(K)$, $m(B)$ denotes the number of units of commodities having characteristics in B.

A *price vector* is a linear functional on the commodity space and the *price space* is a vector space in duality with the commodity space, thus a subspace of its algebraic dual. In the previous examples, good candidates as price spaces are the norm duals of the corresponding commodity spaces, and also the pre-duals $L^{\ell}_1(\Omega, \Sigma, \mu)$, $L^{\ell}_1[0,1]$ of $L^{\ell}_{\infty}(\Omega, \Sigma, \mu)$, $L^{\ell}_{\infty}[0,1]$ and $C(K)$, the space of continuous real functions defined on K, the pre-dual of $M(K)$.

One time defined a *commodity-price duality* $\langle L, L' \rangle$, as in the classical model, a private ownership economy is given by

$$E = \left(\langle L, L' \rangle, \left(X^i, P^i, \omega^i \right)_{i \in I}, \left(Y^j \right)_{j \in J}, \left(\theta_{ij} \right)_{i \in I, j \in J} \right)$$

where X^i, Y^j are now subsets of L and $\omega^i \in L$. Feasible, Pareto optimal, core and fuzzy core allocations are defined as in the classical model. The same is true for quasi-equilibrium and equilibrium, with the exception that the price vector is now an element of L'.

When the commodity space is an L_{∞}-type space (characterized by non-emptiness of the interior of the positive cone and by the existence of a pre-dual), one has the following equilibrium existence result where open lower sections of preference correspondences and compactness of attainable sets are stated using the weak-star topology relative to the pre-dual of the commodity space while the other topological assumptions use the norm topology of the commodity space. In the following proposition, if τ is a topology on all X^i, τ^I denotes the product topology on their Cartesian product.

Proposition 10 *Assume either free-disposal or that at least one consumer's preference correspondence has open values. The economy has a quasi-equilibrium with a price in the dual of the commodity space under the following assumptions:*

1. *For each* $i \in I$*,*
   (a) X^i *is convex,*
   (b) P^i *has* $\sigma^I(L_\infty, L_1)$*-open lower sections and norm-open convex values with* $x^i \notin P^i(x)$*,*
   (c) The autarky condition is satisfied (eventually with free disposal),
   (e) For every feasible allocation (x,y)*,* $x^i \in \overline{P^i(x)}$*, the closure of the preferred set;*
2. *For each* $j \in J$*,* Y^j *is convex;*
3. $\hat{X}$ is $\sigma^I(L_\infty, L_1)$*-compact.*

The quasi-equilibrium is an equilibrium if

- *either the interiority assumption which defines strong survival is satisfied for every consumer*
- *or the interiority assumption is satisfied for the whole economy and the economy is irreducible.*

Notice that a price in the dual of an L_∞-type space is of difficult economic interpretation. Obtaining a quasi-equilibrium or an equilibrium with a price in the pre-dual of the commodity space requires additional assumptions.

Proposition 10 or any analogous equilibrium existence result for an economy whose commodity space is of the same type can be proved using any of the three main approaches to the equilibrium existence problem. It can even be proved using, as in the proof published by Bewley in 1972, a limiting-process on the equilibria of a net of finite dimensional economies satisfying the assumptions of Theorem 1. Maybe, the limitation to L_∞-type commodity spaces will be better understood looking at the equilibrium existence proof, used in this survey, based on price decentralization of elements of the fuzzy core.

Indeed, non-emptiness of the fuzzy core of a private ownership economy can be obtained, using the same arguments as in Propositions 3–4–5–6–7, provided that, in their assumptions, the weak-star topology of the commodity space relative to its pre-dual replaces the original topology for formulating compactness assumptions and the openness of lower sections of preference correspondences.

In counterpart, the price decentralization of the elements of the fuzzy core uses a separation argument which requires, when the commodity space is infinite dimensional, that either the positive orthant of the commodity space (in case of free-disposal) or the set $\bigcup_{i \in I}\left(P^i(x) - \sum_{j \in J}\theta_{ij}Y^j - \omega^i\right)$ have a non-empty interior. Non-emptiness of the interior of the positive orthant is obviously a characteristic of the L_∞-type spaces but not of the other commodity space referred to as possible commodity spaces in infinite dimensional economies. This interiority assumption is also required if the openness of values for some P^i is guaranteed by some monotonicity assumption of this correspondence or if the consumption sets are assumed to be bounded below.

In the absence of interiority assumptions, usual equilibrium existence proofs rely on *properness assumptions* on the characteristics of the economy whose role is to restore the missing interiority, and on lattice theoretic assumptions on the commodity space and/or the price space. In addition, consumption sets are assumed to coincide with the positive orthant.

Notice immediately that, when endowed with their canonical order, the examples of commodity spaces given in this section and their duals or pre-duals are ordered vector spaces and more precisely vector lattices or Riesz spaces. It is not anymore the case when the commodity space is re-ordered by an ordering more significant in some economic contexts.

A thorough study of the different properness assumptions to be found in the literature would go far beyond the objective of this contribution. The same is true for the attempts to use properness assumptions in economies defined on commodity spaces without lattice structure. Such a study, useful for example in models of portfolio trading, is the object of very active researches.

Continuum economies

Let us come back to economies with finitely many commodities. Considering atomless measure spaces of agents rather than finite numbers of them formalizes the basic requirement that, in a competitive model, individuals have no power to influence market prices. To understand how are translated in this new framework the definition and properties of equilibrium, let us review the pioneering papers of Aumann.

Consider on R^l as commodity space an exchange economy with an atomless, positive bounded measure space $(T, \mathcal{A}, \nu)$ of agents

$$E = \left((T, \mathcal{A}, \nu), \left(\succ^t, \omega(t)\right)_{t \in T}\right).$$

For simplicity, we assume that T is the real interval $[0,1]$, $\mathcal{A}$ is the Borel σ-algebra of subsets of T and ν is the Lebesgue measure. Each agent $t \in T$, whose consumption set is the positive orthant $X^t = R^l_+$, is characterized by the *initial endowment* $\omega(t) \in R^l_+$ and the (strict) *preference relation* $\succ^t$ on R^l_+. The function $\omega : (T, \mathcal{A}, \nu) \to R^l_+$, $t \to \omega(t)$, is assumed to be integrable.

Definitions of feasible, equilibrium, core allocations are the same as in the classical model of an exchange economy, except that sums over a finite number of individual consumers are replaced by Lebesgue vector integrals. An *allocation* is an integrable function $x : (T, \mathcal{A}, \nu) \to R^l_+$. The allocation is *feasible* if $\int_T x = \int_T \omega$, where $\int_T x$ means $\int_T x(t) dt$.

A *competitive equilibrium* is a pair $(\bar{p}, \bar{x})$ consisting of a non-zero price vector $\bar{p}$ and a feasible allocation $\bar{x}$ such that for almost every $t \in T$, $\bar{x}(t)$ is optimal with respect to $\succ^t$ in t' s *budget set* $\{x \in R^l_+ | p.x \leq p.\omega(t)\}$. An *equilibrium allocation* is an allocation $\bar{x}$ for which there exists a price vector $\bar{p}$ such that $(\bar{p}, \bar{x})$ is a competitive equilibrium.

A *coalition* is a Lebesgue measurable subset S of T with non-zero Lebesgue measure. An allocation y *dominates* an allocation x *via a coalition* S if $y(t) \succ^t x(t)$ for each $t \in S$, and S is *effective* for y, that is, $\int_S y = \int_S \omega$. The *core* of E is the set of all feasible allocations that are not dominated via any coalition.

Not surprisingly, the firs result is the version of the Edgeworth conjecture given by the Aumann *equivalence theorem*.

Proposition 11 *Assume on* E *the following conditions:*

(a) $\int_T \omega >> 0$,

(b) For each $y \in R^{\ell}_{+}$*, the sets* $\{x \in R^{\ell}_{+} | x \succ_t y\}$ *and* $\{x \in R^{\ell}_{+} | y \succ_t x\}$ *are open in* R^{ℓ}_{+},

(c) In R^{ℓ}_{+}*,* $x > y \Rightarrow x \succ_t y$,

(d) If x *and* y *are allocations, then the set* $\{t \in T | x(t) \succ_t y(t)\}$ *is Lebesgue measurable in* T.

Then the competitive equilibrium allocations of E *are precisely the core allocations of* E.

In other words, if preference relations are strictly monotone, continuous and measurable and if every commodity is present in the economy, the core coincide with the set of equilibrium allocations.

As in the classical model with a finite number of agents, that every equilibrium allocation is in the core simply follows from the definition. The proof of the converse statement parallels the one given in the classical model for price-decentralization of the elements of the fuzzy core (Theorem 3). Using the same notations, if x is a core allocation and if $U \subset T$ is a set of agents, let $G_U = co\left(\bigcup_{t \in U}\left(P^t(\bar{x}(t)) - \omega(t)\right)\right)$. It is first proved that there exists a set U, whose complement in T has a zero Lebesgue measure, such that $0 \notin \operatorname{int} G_U$, the interior of G_U. Then a separation argument shows the existence of $\bar{p} >> 0$ such that $(\bar{p}, \bar{x})$ is a competitive equilibrium.

The equilibrium existence result given by Aumann adds to the assumptions (b), (c), (d) of Proposition 11 the additional assumption that consumer's preference relations are complete preorders whose $\succ_t$ is the strict relation. The equilibrium existence result is thus:

Proposition 12 *If consumers's preference relations are complete preorders on their consumption set, then, under the assumptions of Proposition 11, the economy* E *has a competitive equilibrium.*

Because, in particular, no boundedness of the set of feasible allocations allows for compactifying the economy, the proof of Proposition 12 is considerably more complex than any equilibrium existence proof in the classical model with a finite number of agents, and requires a lot of integration theory.

In counterpart, the fact that the set of consumers is a measure space without atoms allows to discard convexity of individual preferences, an assumption made neither in Proposition 11, nor in Proposition 12. For the proof of Proposition 12, this is due to the definition of the integral of a correspondence and to the fact, based on Liapunov's theorem, that the integral of a correspondence from an atomless measure space into R^{ℓ} is a convex subset of R^{l}.

If we add that it is proved by Hildenbrand at the end of his book that for the same economy with an unspecified distribution of resources, *Pareto optimal allocations* exist and can be decentralized as *quasi-equilibrium allocations relative to a price system*, we see that the research program and the results for economies with a continuum of agents exactly parallel the one for the classical model with a finite number of agents. And, in effect, the results reported here have been extended to economies with production, with externalities in preferences, and to several other contexts, including economies with atoms, and "large" economies, that is, economies with measure spaces of agents and infinitely many commodities.

To conclude this sub-section, it is necessary to emphasize that the full validity of all properties of equilibrium is depending on the continuum hypothesis (the space of agents is an atomless, positive bounded measure space). In economies with countably many agents, whose overlapping generations economies are the most significant example, equilibrium allocations which are shown to exist, are not necessarily Pareto optimal. Sufficient conditions, for example on the relation of the equilibrium price with the distribution of individual resources, exist for Pareto optimality of equilibrium.

6.2. Some market failures

We turn now to extensions of the classical model to contexts in which some of the assumptions of the classical model do not hold and market equilibrium, when it exists, cannot be relied on to yield Pareto optimal outcomes. It is that we mean by market failures.

In what follows, we will treat only three examples. The two first ones correspond to long-standing problems of economic policy related with the second welfare theorem: how to regulate firms which produce with increasing returns so as to achieve Pareto optimal states in a decentralized way? How to optimize the production and the provision of public goods? In each case, the solution is found in the definition of a new equilibrium concept which coincide with the classical one under the conditions of the classical model. The third example, raised in the context of an increasing role of market institutions, is maybe the translation in the theory of a massive phenomena of the real modern economies: the prevalence of the role of financial markets.

Non-convex economies
As just stated, the interest for non-convex economies arises on the production side from the consideration of increasing returns to scale and/or certain externalities. On the consumption side, besides all forms of indivisibilities which have given rise to a wide literature and that we postulate away in this paragraph, non-convex preferences correspond to anti-complementarities between commodities, for example to non-aversion to risk for agents facing uncertainty.
When preferred and production sets are non-convex, it is well known that Pareto optimal allocations of the economy are no longer equilibria for a price-system where consumers would optimize their preferences under their budget constraint and producers would maximize their profit. Statements of the second welfare theorem for a non-convex economy $E = \left(\left(X^i, P^i\right)_{i\in I}, \left(Y^y\right)_{j\in J}, \omega\right)$, where individual resources are non specified, have been provided, starting with a seminal paper by Guesnerie. They are proved using a separation argument which parallels the proof of Theorem 2, replacing the normal cone of convex analysis by a definition of normal cone borrowed from non-smooth and non-convex analysis. Roughly speaking, they state that if $(\bar{x},\bar{y})$ is a feasible allocation where consumers are locally non-satiated, then there exist a price vector $\bar{p} \neq 0$ such that

$$-\bar{p} \in N_{P^i(\bar{x})}\left(\bar{x}^i\right) \forall i \in I \text{ and } \bar{p} \in N_{Y^j}\left(\bar{y}^j\right) \forall j \in J,$$

where $N_X(x)$ denotes the *normal cone to a set X at a point $x \in \overline{X}$*, for some definition of this notion. Writing $-\bar{p} \in N_{P^i(\bar{x})}\left(\bar{x}^i\right)$ in place of $\bar{p}.P^i(\bar{x}) \geq p.\bar{x}^i$ as stated in Theorem 2, and

$\bar{p} \in N_{Y^j}(\bar{y}^j)$ in place of $\bar{p}.Y^j \leq \bar{p}.\bar{y}^j$, we see that the notion of *normal cone* generalizes the definition of normal cone in convex analysis. It generalizes also the first order conditions for Pareto optimality that we would write for an economy with differentiable utility and production functions.

We will not give a precise mathematical definition of the normal cone. Actually, several alternative definitions allow for the above characterization of feasible Pareto optimal allocations. Depending on the assumptions made on the production sets, each of them has a different implication for the economic significance of the price-decentralization result. For non convex preferences, economists prefer to use the Shapley–Folkman theorem to exhibit an approximate expenditure minimizing behavior. For production with increasing returns, Clarke's normal cone, considered as a generalization of the marginal cost pricing rule, is most often used. What should be understood is that, whatever be the chosen notion, in order to decentralize Pareto optimal allocations of a non-convex economy, *non-convex firms must be instructed to behave in conformity with the (necessary) conditions of Pareto-optimality*.

This idea will help us to understand the model and the equilibrium concept used for non-convex economies. From now on, we restrict ourselves to economies where only production sets may be non convex. Let

$$E = \left(\left(X^i, \geq^i, \omega^i, r^i \right)_{i \in I}, \left(Y^j, \phi^j \right)_{j \in J} \right)$$

be an economy almost standardly defined.

For each consumer $i \in I$, the function $r^i : \prod_{j \in J} Y^j \times R^l \to R$, continuous, homogeneous of degree one with respect to the price vector, and satisfying $\sum_{i \in I} r^i(y,p) = p\left(\sum_{j \in J} y^j + \sum_{i \in I} \omega^i \right)$, defines the *wealth* of consumer i. This definition obviously encompasses the particular case $r^i(y,p) = p.\omega^i + \sum_{j \in J} \theta_{ij} p.y^j$ for consumer's wealth in a classical private ownership economy where producers maximize profit. A more general structure of revenues, on which will be done survival assumptions, is necessary in an economy with non-profit maximizing producers where survival of consumers at equilibrium is not anymore a consequence of the autarky assumptions made in private ownership economies.

For each producer $j \in J$, $\phi^j : \partial Y^y \to R_+^l$ is the *pricing rule* followed by the producer. In view of a free-disposal assumption made for each producer in this model, the pricing rule associates a set of non-negative prices to efficient production plans of the producer. The interest of the notion of pricing rule is to be compatible with various behaviors that are currently considered in the economics literature; profit maximization, but also average or marginal cost pricing rules.

A pair $(p,y) \in R_+^l \times \prod_{j \in J} \partial Y^j$ such that for every $j \in J$, $p \in \phi^j(y^j)$ is called *production equilibrium*.

An *equilibrium* of E is a t-uple $(\bar{x}, \bar{y}, \bar{p})$ in $\prod_{i \in I} X^i \times \prod_{j \in J} Y^j \times R^l$ such that $\bar{p} \neq 0$ and

1. Each consumer optimizes his preferences in his budget set
2. For every $j \in J$, $\bar{p} \in \phi^j(\bar{y}^j)$

3. All markets clear, that is, $\sum_{i\in I}\bar{x}^i = \sum_{i\in I}\omega^i + \sum_{j\in J}\bar{y}^j$.

Usual assumptions guarantee compactness of all feasible sets. Besides local non-satiation of consumers, the already mentioned assumption of free-disposal in all production sets (Y^j is closed, contains 0 and $Y^j - R^l_+ \subset Y^j$), and a strong survival assumption made as well for the whole economy as for individual consumers: if (p,y) is a production equilibrium, then

$$p.\left(\sum_{j\in J} y^j + \sum_{i\in I}\omega^i\right) > \inf p.\sum_{i\in I} X^i \text{ and for each } i\in I,\ r^i(y,p) > \inf p.X^i$$

the key assumption concerns the correspondences ϕ^j assumed to have a closed graph and values equal to closed convex cones not reduced to $\{0\}$, an assumption verified in particular by the Clarke normal cone.
Under the additional assumption that the pricing rule of each producer has bounded losses, the non-convex economy E has an equilibrium.

Public goods
A good is said to be a (pure) *public good* (by opposition to a *private good*) when consumption of a unit of the good by one agent does not prevent its availability to other consumers.

In order to introduce pure public goods in the general equilibrium model, let us consider a private ownership economy where finitely many firms, owned by consumers as in the classical model, jointly produce private and public goods. Finitely many consumers have an initial endowment of private goods only and jointly consume private goods and *a same amount of public goods*. If l is the finite number of private goods and k is the finite number of public goods, the economy is thus described by

$$E = \left(\langle R^l \times R^k, R^l \times R^k\rangle, (X^i, u^i, \omega^i)_{i\in I}, (Y^j)_{j\in J}, (\theta_{ij})_{i\in I, j\in J}\right)$$

where $\langle R^l \times R^k, R^l \times R^k\rangle$ is the commodity–price duality of the model, and for each $i\in I$, $\omega^i = (e^i, 0) \in R^l \times \{0\}$ is his initial endowment and $u^i : R^l \times R^k \to R$ his utility function depending on his joint consumption of private and public goods. Two equilibrium concepts make sense for such a model, with different consequences on optimality of equilibrium.

Private provision equilibrium
A first equilibrium concept emphasizes the personal decision of each consumer to *provide* some amount of public goods that all consumers buy at a same market price. The utility a consumer gets from his composite vector of private goods consumption–public goods provision depends simultaneously on his private goods consumption and on the *total provision* of public goods by all consumers. Charities, subscriptions are examples of such a private provision.
If we set (x^i, x^i_g) for a consumption–private provision vector of i, (y^j, y^j_g) for a production vector of j, (p, p_g) for a market price vector, a *private provision equilibrium* of E is a t-uple

$$\left((\bar{x}^i, \bar{x}^i_g)_{i\in I}, (\bar{y}^j, \bar{y}^j_g)_{j\in J}, (\bar{p}, \bar{p}_g)\right)$$

in $\left(R_+^l \times R_+^k\right)^I \times \prod_{j\in J} Y^j \times \left(R^l \times R^k\right)$ such that $\left(\overline{p},\overline{p}_g\right) \neq (0,0)$ and:

1. For each $i \in I$, $\left(\overline{x}^i,\overline{x}_g^i\right)$ maximizes $u^i\left(x^i, x_g^i + \sum_{h\neq i} \overline{x}_g^h\right)$ in the budget set
$$B^i\left(\overline{p},\overline{p}_g\right) = \left\{\left(x^i,x_g^i\right) \in R_+^l \times R_+^k \middle| \overline{p}.x^i + \overline{p}_g.x_g^i \leq \overline{p}.e^i + \sum_{j\in J}\theta_{ij}\left(\overline{p}.\overline{y}^j + \overline{p}_g.\overline{y}_g^j\right)\right\}$$
2. For each $j \in J$ and for all $\left(y^j,y_g^j\right) \in Y^j$, $\overline{p}.y^j + \overline{p}_g.y_g^j \leq \overline{p}.\overline{y}_j + \overline{p}_g.\overline{y}_g^j$
3. $\sum_{i\in I}\left(\overline{x}^i,\overline{x}_g^i\right) = \sum_{i\in I}\left(e^i,0\right) + \sum_{j\in J}\left(\overline{y}^j,\overline{y}_g^j\right)$.

For a *private provision quasi-equilibrium*, the first condition is replaced by:

1'. For each $i \in I$, $u^i\left(x^i, x_g^i + \sum_{h\neq i}\overline{x}_g^h\right) > u\left(\overline{x}^i, \overline{x}_g^i + \sum_{h\neq i}\overline{x}_g^h\right)$ implies $\overline{p}.x^i + \overline{p}_g.x_g^i \geq \overline{p}.\overline{x}^i + \overline{p}_g.\overline{x}_g^i$.

Because, in their appreciation of a composite vector of consumption of private goods–private provision of public goods, consumers take as given the private provisions of public goods of the other consumers, this model is a particular case of the classical model where this type of externalities in preferences was assumed. Quasi-equilibrium existence thus follows from quasi-equilibrium existence in the classical model, under analogous assumptions which require some local non-satiation assumption in terms of private or public goods.

Assume moreover that there exists some $\left(y,y_g\right) \in \sum_{j\in J} Y^j$ with $y_g >> 0$, that is, there is possible to produce simultaneously a strictly positive quantity of each public good. Then the strong survival assumption for the whole economy is satisfied and, if consumers' utility functions are continuous and under some irreducibility condition on the economy, the quasi-equilibrium is an equilibrium.

Notice that, as in the classical model, the equilibrium allocation of the private provision model is optimal for a *constrained optimality concept* where the utility for each consumer of a feasible provision of public goods is appreciated, taking as given the provisions of the other consumers. This constrained optimality leads by no way to an optimal level of the feasible total supply of public goods. Sub-optimality of the equilibrium provision of public goods is the main drawback of the model of private provision of public goods.

Versus Lindahl–Foley equilibrium

On the contrary, in the Lindahl-Foley equilibrium approach, personalized prices paid by each consumers for the total supply of public goods restore optimality of equilibrium allocation.

A *Lindahl–Foley equilibrium* of E is a t-uple
$$\left(\left(\overline{x}^i\right)_{i\in I}, \left(\overline{t}^i\right)_{i\in I}, \overline{G}, \left(\overline{y}^j,\overline{y}_g^j\right)_{j\in J}, \left(\overline{p},\overline{p}_g\right)\right)$$
in $\left(R_+^l\right)^I \times [0,1]^I \times R_+^k \times \prod_{j\in J} Y^j \times \left(R^l \times R^k\right)$ such that $\left(\overline{p},\overline{p}_g\right) \neq (0,0)$, $\sum_{i\in I}\overline{t}^i = 1$, and :

1. For each $i \in I$, $\left(\bar{x}^i, \bar{G}\right)$ maximizes $u^i\left(x^i, G\right)$ in the budget set

$$B^i\left(\bar{p}, \bar{t}^i \bar{p}_g\right) = \left\{\left(x^i, G\right) \in R_+^l \times R_+^k \middle| \bar{p}.x^i + \bar{t}^i \bar{p}_g.G \leq \bar{p}.e^i + \sum_{j \in J} \theta_{ij}\left(\bar{p}.\bar{y}^j + \bar{p}_g.\bar{y}_g^j\right)\right\}$$

2. For each $j \in I$, and for all $\left(y^j, y_g^j\right) \in Y^j$, $\bar{p}.y^j + \bar{p}_g.y_g^j \leq \bar{p}.\bar{y}^j + \bar{p}_g.\bar{y}_g^j$
3. $\left(\sum_{i \in I} \bar{x}^i, \bar{G}\right) = \sum_{i \in I}\left(e^i, 0\right) + \sum_{j \in J}\left(\bar{y}^j, \bar{y}_g^j\right)$.

As in the private provision equilibrium, Condition 2 means that each producer maximizes his profit when facing the equilibrium price vector $\left(\bar{p}, \bar{p}_g\right)$. As in the private provision equilibrium, Condition 3 states that the allocation $\left(\left(\bar{x}^i\right)_{i \in I}, \bar{G}\right)$, where $\bar{G}$ is the total supply of public goods, is feasible. Condition 1 states that each consumer i maximizes his utility when facing the equilibrium private goods price $\bar{p}$ and the equilibrium personalized public goods price $\bar{t}^i \bar{p}_g$.

It simply follows from the equilibrium definition that a Lindahl-Foley equilibrium allocation is Pareto optimal, that is, there is no consumption allocation $\left(\left(x^i\right)_{i \in I}, G\right)$ in $\left(R_+^l\right)^I \times R_+^k$ such that $\left(\sum_{i \in I} x^i, G\right) \in \sum_{i \in I}\left(e^i, 0\right) + \sum_{j \in J} Y^j$ and $u^i\left(x^i, G\right) > u^i\left(\bar{x}^i, \bar{G}\right) \forall i \in I$. First, second welfare theorems, equivalence theorem with the limit-core allocation are satisfied in the Lindahl–Foley approach.

Existence of Lindahl–Foley equilibrium is generally proved by associating to the original public goods economy an auxiliary economy with only private goods, defined on a commodity space of enlarged dimension. Non-emptiness of the core, and of the Debreu-Scarf core, equilibrium existence require more stringent assumptions than the ones used for existence of private provision equilibrium. As in the private provision model, it is assumed that there is no initial endowment in public goods. In addition, public goods are never inputs, are free-disposable in production, and consumer's utilities are assumed to be monotone with respect to public goods.

Incomplete markets

The role of a general equilibrium model with incomplete markets is to understand the role of assets for the allocation of resources in a world in which time and uncertainty enter in an essential way. In the simplest model, there are two time periods ($t = 0$ and $t = 1$) and an a priori uncertainty at the first period about which of a finite number of states of the world will prevail at the second period. Finitely many consumers exchange at each period and in each state of the world a same finite number l of goods. There is, in addition, at the first period a financial market for a finite number of assets bought (or sold) at period 0, which deliver at $t = 1$ a random return across the states of the world.

Let J be the finite set of assets and S be the finite set of states of the world. The return of one unit of the asset $j \in J$ in each state can be denominated

- in units of account (*nominal assets*)
- in units of one good or of a commodity bundle $e \in R^l$ called "numeraire" (*numeraire assets*)

- in units of a vector $a^j \in R^\ell$ (*real assets*).

Given a price vector $p \in R^{\ell(1+S)}$ for commodities at time 0 and in each state s of period 1, the financial return of asset j is $v^j(p,s)$, differently calculated according to the nominal, numeraire or real nature of the asset. The vector map $p \to \left(v^j(p,s)\right)_{s\in S}$ describes the financial returns of asset j. The $S \times J$ matrix map

$$p \to V(p) = \left(v^j(p,s)\right)_{s\in S,\, j\in J}$$

summarizes the *financial structure* of the model.

The financial economy is described by the list

$$E = \left(\left(X^i, P^i, \omega^i\right)_{i\in I}, V\right)$$

As in the classical model of general equilibrium, each agent i of a finite set I is characterized by a *consumption set* $X^i \subset R^{\ell(1+S)}$, a preference correspondence $P^i : \prod_{h\in I} X^h \to X^i$ and an *initial endowment* in commodities $\omega^i \in R^{\ell(1+S)}$. We assume in addition that each consumer has no initial endowment of assets, and that unlimited short selling of assets is possible. Given commodity and asset prices $(p,q) \in R^{\ell(1+S)} \times R^J$ measured in units of account, the budget set of i is defined by:

$$B^i(p,q) = \left\{ x^i \in X^i \,\middle|\, \begin{array}{c} \exists z^i \in R^J \\ p(0).x^i(0) + q.z^i \le p(0).\omega^i(0) \\ p(s).x^i(s) \le p(s).\omega^i(s) + v(p,s).z^i, \forall s \in S \end{array} \right\}$$

It is the multiplicity of the $(S+1)$ budget constraints of the agents that characterizes models of financial markets. If these multiple budget constraints cannot be proved to be equivalent to a unique budget constraint defined by the sum over s of the different inequalities, each one weighted by a positive coefficient called *state price*, markets are said to be *incomplete*. It is always the case if there are strictly less assets than states of nature, that is, if $J < S$. The multiplicity of budget constraints expresses that consumers have a (limited) possibility of transferring wealth across dates and states, a possibility that is unlimited in case of only one budget constraint and null if there is no asset and consumers are obliged to satisfy in each state the budget constraint corresponding to this state.

Equilibrium is defined as a pair of admissible actions and prices $\left(\left(\bar{x}^i, \bar{z}^i\right)_{i\in I}, (\bar{p}, \bar{q})\right)$ such that

1. For each $i \in I$, $\bar{x}^i \in B^i(\bar{p},\bar{q})$, $\bar{z}^i$ is the corresponding *portfolio*, and $P^i(\bar{x}) \cap B^i(\bar{p},\bar{q}) = \varnothing$
2. $\sum_{i\in I}\left(x^i - \omega^i\right) = 0$ and $\sum_{i\in I} \bar{z}^i = 0$.

Classically, the first condition expresses that with $\left(\bar{x}^i, \bar{z}^i\right)$, consumer i optimizes his preferences in his budget set. The second condition expresses market clearing under the implicit hypothesis that no production or intertemporal storage is possible and assets are in zero net supply.

The equilibrium existence problem has been solved in the eighties. When assets are nominal, equilibrium existence can be proved with similar methods to the ones used in the classical model. Assumptions are strengthened. In particular, agents satisfy the individual strong survival (papers exist weakening this assumption); agent's preferences satisfy the additional convexity assumption

$$y^i \in P^i(x) \text{ and } 0 < \lambda \leq 1 \text{ imply } x^i + \lambda\left(y^i - x^i\right) \in P^i(x)$$

and a kind of local non-satiation at each date-event pair and at every component of a feasible consumption. These assumptions on preferences are satisfied when each agent i has on his consumption set $X^i = R_+^{l(1+S)}$ a von Neumann–Morgenstern utility function

$$U^i(x) = \sum_{s \in S} \rho_s^i u^i\left(x^i(0), x^i(s)\right)$$

where $u^i : R_+^l \times R_+^l \to R$ is a strictly quasi-concave and strictly increasing function, each $\rho_s^i > 0$ denotes the (subjective) probability of state s for agent i, and $\sum_{s \in S} \rho_s^i = 1$.

When assets are numeraire, the same result holds under an assumption of strict desirability of the numeraire at each date-event pair and at every component of a feasible consumption. Equilibrium exists and under appropriate assumptions can be proved to be generically locally unique.

With real assets, the continuity properties of budget correspondences may fail, due to the fact that the rank of the return matrix $V(p)$ is not anymore constant when the commodity price p varies. For this reason, equilibrium existence has been proved to be only generic.

In all cases, equilibrium allocations have no reason to be Pareto optimal. In particular, in economies without assets but with multiple budget constraints, it is easy to construct examples where some equilibrium allocation Pareto dominates another equilibrium allocation.

7. Concluding remarks

Obviously, this presentation of the extensions of the classical model is far from being exhaustive. A number of other frameworks analyze, in a general equilibrium approach, the problems of public policy that have given rise to the study of non-convex economies. The same is true for economies with public goods where researches on implementation of desirable allocations have considerably developed, in continuation to the equilibrium definition problem. And there is an increasing literature on financial markets.

More generally, an exhaustive analysis of all the contexts that general equilibrium can deal with would be impossible. With its sophisticated methods, general equilibrium is, for its specialists, a great architecture with still incomplete and imperfect pieces of building, but always on the way of giving foundations to new themes of economic theory.

Actually, the abstract model of general equilibrium is what is called a "paradigm" whose one may and one should ask at what extent it is or not well suited for our understanding of the real world.

Glossary

Commodity space: Real vector space whose dimension is equal to the number of commodities present in the economy under consideration.

Continuity properties of a correspondence: A correspondence from a topological space X to a topological space Y has *open lower sections* if every element of Y has an open inverse image. The correspondence is *upper semi-continuous* if for every open subset of Y, the set of all elements of X whose image is contained in this set is open. The correspondence is *lower semi-continuous* if for every open subset of Y, the set of all elements of X whose image intersects this set is open.

Correspondence: Sometimes called point-set function, a correspondence associates to each element of the definition space a subset (which may be empty) of the arrival space.

Excess demand correspondence: This correspondence associate to each vector price the subset of the commodity space which is equal to the vector difference between the total demand and the total supply.

Free-disposal: Disposal of the excess supply without cost in terms of the use of additional inputs. In the real economies, disposal of by-products of a production activity, whatever be the form of this disposal (destruction or stocks), is a problematic and costly activity.

Implicit function theorem: Let U be an open subset of $R^n \times R^m$ and let f be a continuously differentiable function from $R^m \times R^q$ to R^n. Let $(a,b) \in U$ be such that $f(a,b) = 0$. If the partial derivative $f'_y(a,b)$ defines a linear isomorphism from R^q onto R^n, then there exists in $R^m \times R^q$ an open neighborhood V of (a,b) contained in U, there exists in R^m an open neighborhood W of a, there exists a function a continuously differentiable function

$$g : W \to R^q$$

such that the assertions $(x,y) \in V$ and $f(x,y) = 0$ are equivalent to $x \in W$ and $y = g(x)$.

Inverse function theorem: Let U be an open subset of R^n and $f : U \to R^n$ be a continuously differentiable function whose Jacobian matrix at the point $a \in U$ has a non-zero determinant. Then there exists an open neighborhood $V \subset U$ of a and an open neighborhood W of $b = f(a)$ such that f is a C^1-diffeomorphisme of V onto W, that is, there exists a continuously differentiable function $g : W \to V$ which is the inverse function of f.

Kakutani's theorem: Let X be a convex compact subset of R^n. An upper semi-continuous correspondence $\varphi : X \to X$ with nonempty closed convex values has a fixed point, that is there is some $\bar{x} \in \varphi(\bar{x})$.

Lebesgue measure: On R, the Lebesgue measure is the unique measure on the Borel sets of R (that is on the σ-algebra generated by the family of open sets) whose value on every interval is its length. Using product measures, the Lebesgue measure can be generalized to Euclidean spaces of any dimension.

Pareto optimality: A feasible allocation of resources is Pareto efficient (or Pareto optimal) if there is no other feasible allocation which will make every consumer better off (or, in case of strong Pareto optimality, everyone at least as well off and at least one consumer better off).

Portfolio: If there is a finite number of assets and if R^J is the asset space, an asset bundle is called portfolio.

Qualitative game: A game in normal form where the payment function is replaced by a preference relation on the outcomes of the game.

Sard's theorem: Let U be an open subset of R^m and let F be a r-times continuously differentiable function from U to R^n with $r > \max\{0, m - n\}$. Then the set of points $y = F(x)$ such that the Jacobian matrix of F at x has a rank smaller than n has Lebesgue measure zero in R^n.

Utility space: Real vector space whose dimension is equal to the number of consumers.

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