Social choice theory and the "Centre de Mathématique Sociale". Some historical notes

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Abstract. In this paper we describe some research directions in social choice and aggregation theory taken at the "Centre de Mathématique Sociale" since the fifties. We begin by presenting some institutional aspects concerning this center. Then we sketch a thematic history by considering the following questions about the "effet Condorcet" ("voting paradox"): What is it? How is it overcome? Why does it occur? These questions were adressed in Guilbaud's 1952 paper (*Les théories de l'intérêt général et le problème logique de l'agrégation*) which will mark the beginning of our inquiry. The conclusion outlines some more recent research developments linked to these questions.

Key words : distributive lattice, effet Condorcet, Guilbaud's theorem, median, metric, permutoedre, simple game, ultrafilter.
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1 Introduction

Arrow's book *Social Choice and Individual Values* appeared in 1951 and immediately caused a considerable interest as well as critical discussions. In May 1952 Arrow went to the Conference on Risk held at Marseille and then to Paris where François Perroux head of the "Institut des Sciences Économiques Appliquées (ISEA)" had asked him to make a presentation of his work. The lecture entitled *The rationality principle in collective decisions* was given on 9 June 1952 at the Institute. Translated into French under the title *Le principe de rationalité dans les décisions collectives* it was published in a special issue of *Économie appliquée*[1] devoted to welfare

[1] *Économie appliquée* 5(4): 469-484 (October-December 1952).

economics and called *L'avantage collectif*. This same issue contained Guilbaud's paper *Les théories de l'intérêt général et le problème logique de l'agrégation* (*Theories of the general interest and the logical problem of aggregation*)[2]. Guilbaud, a mathematician[3], was then one of the two assistant heads of the ISEA. Although Guilbaud's appointment to the 6th section of the "Ecole Pratique des Hautes Études" (EPHE now EHESS) took place only three years later, this paper must be taken as our departure point for our historical inquiry on the theory of social choice at the center created by Guilbaud. Indeed as we will see almost all the work in this field made at the center had its origin in ideas presented in this paper.

We have just spoken of "historical inquiry". But there are several ways to do the history of science. Here we will begin by the institutional history and then we will develop a thematic history. For the first one, we will first present the institutions and their members, then some scientific activities of these members (seminars, conferences, publications...) and in particular those related to the theories of decision and voting. The thematic history will be presented by considering the following questions about the "effet Condorcet": What is it? How is it overcome? Why does it occur? In our conclusion we will outline some developments of researches linked to these questions.

2 Institutional history

The "École des Hautes Études en Sciences Sociales (EHESS)[4]" is probably the most highly reputed French institution for social sciences researches (from history to linguistics through economy, demography, cognitive science, social psychology, sociology, anthropology, ethnology, geography....). Until 1975 when the École took its independance, it was just the sixth section of the "École Pratique des Hautes Études (EPHE)"[5]. In

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[2] *Économie appliquée* 5(4): 501-584 (October-December 1952).
(The other contributors of this issue were B. de Jouvenel, P. Streeten, I.M.D. Little, G. Nyblen, L. Buquet, G. Bernacer, P. Massé and J. Akerman). This paper was reprinted in *Eléments de la théorie mathématique des jeux*, Dunod Paris, 1967, and partially translated into English under the title *Theories of the general interest and the logical problem of aggregation*, in Lazarsfeld PF and Henry NW (eds) Readings in Mathematical Social Sciences, Science Research Associates, Inc., Chicago (1966), pp. 262-307. Since the original paper is difficult to find we will give the numbering of pages of this paper according to this translation.

[3] He was a student of the "Ecole Normale Supérieure" in the thirties.

[4] Below the "École des Hautes Études en Sciences Sociales" will be simply called the École.

[5] The EPHE was founded by Victor Duruy (then the french minister of public education) in 1868 as an institute of higher education initially charged to promote a more practical

1955 the president of the sixth section was the historian Lucien Febvre and secretary, the historian Fernand Braudel. Guilbaud, then 44 years old, was just leaving Perroux's Institute (ISEA). Claude Levi-Strauss and Charles Morazé, two members of the sixth section, asked him to join them as "directeur d'études" at the École (a position equivalent to a position of full professor at university) and he was elected this same year. Guilbaud presented to Febvre his project to do "de la mathématique sociale, à la Condorcet"[6]. He proposed to call his research group "Groupe de Mathématique Sociale", but Braudel asked him to add "et de Statistique" (so the acronym was GMSS)[7]. The group grew regularly by the addition of new members[8] of which we give only the names of those (more or less) concerned with social choice theory: Marc Barbut (1956), Pierre Rosenstiehl (1960), Bernard Monjardet (1963), Bruno Leclerc (1967) and Jacqueline Feldman (1967-1969). There were also (and there are still) more or less closely associated members like Jean-Pierre Barthélemy (now member) or Olivier Hudry. Note that all these persons, except J. Feldman who was a physicist, were mathematicians sometimes working also in computer sciences.

The first task of a "directeur d'études" to the École is to give seminars for advanced students and researchers. The title of Guilbaud's first seminar was *Modèles mathématiques dans les sciences sociales (Mathematical models in social sciences)* and he or others like Barbut gave such seminars up to now. In the sixties there was a great interest among many social scientists in the use of mathematical models[9] and several of them attended regularly these seminars. At the same time the mathematicians of the CAMS (see footnote 7) worked, or tried to work, since it was not necessarily easy, as well with historians (as Emmanuel Leroy-Ladurie),

teaching of the sciences than the one given by the universities. In 1975 it contained six sections of which the most important were the sixth (created in 1947) and in 2002 it has only three sections (life and earth sciences, history and philology sciences, religious sciences).

6 See *Mathématique sociale*, Entretien avec G.Th. Guilbaud, Savoir et Mémoire n°4, Éditions de l'EHESS, Paris, 1993, and Rosenstiehl P, La mathématique et l'École. In Revel J, Wachtel N (eds) *Une École pour les sciences sociales*. Cerf and Éditions de l'EHESS, Paris, 1996.

7 When the "groupe" grew it became a "centre", and after some terminological variations it is now called "Centre d'Analyse et de Mathématique Sociale (CAMS)". Henceforth we will use only (so sometimes anachronistically) this acronym CAMS in our text.

8 The center has presently 25 members, plus some retired former members but still working at the center.

9 See for instance Armatte M, Feldman J, Leclerc B, Monjardet B, Schiltz MA, Selz Laurière M, (1989) Mathématiques et Sciences Humaines : des années soixante aux années quatre vingts *La Vie des Sciences* 6(1): 59-76, 6(2): 139-165.

ethnologists (as Claude Levi-Strauss), linguists and psycholinguists (as François Bresson). The relations with economists either of the École as Edmond Malinvaud or associated member of the École as André Nataf were important during the sixties[10]. Traditional seminars with presentations by invited lecturers were also created like *Méthodes mathématiques dans les sciences sociales* (*Mathematical methods in social sciences)* by Guilbaud and Barbut (1960-1963), *Mathématiques discrètes et sciences sociales* (*Discrete mathematics and social sciences*) initiated in 1978 by Monjardet[11] and *Histoire du calcul des probabilités et de la statistique* (*History of probability and statistics*) initiated in 1982 by M. Barbut and Ernest Coumet[12].

To organize meetings is another classic scientific activity. Among the many conferences organised (completely or partially) by CAMS members the following were in the fields of choice and decision theory (the name of organizers and the place of the meeting are in parentheses): 1960, "La décision I" (Georges Darmois and G.Th. Guilbaud, Paris); 1967, "La décision II" (G.Th.Guilbaud and M. Barbut, Aix-en-Provence); 1971, "Ordres totaux finis" (M. Barbut, Aix-en-Provence); 1981, "TRAP I[13] : Analyse et Agrégation des Préférences" (Pierre Batteau, Eric Jacquet-Lagrèze and B. Monjardet, Aix-en-Provence); 1982 "TRAP II" (Marc Roubens, Mons); 1988, "TRAP III: Modélisation, Analyse et Agrégation des Préférences et des Choix" (Louis-André Gerard-Varet and B. Monjardet, Luminy)[14].

Many centers of the École have created journals in their fields of research. The CAMS journal *Mathématiques et Sciences humaines* was created by Barbut in 1962[15]. Among the special issues of this journal one

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[10] For the relations between the mathematicians and some social scientists of the École see the references given in footnote 6.

[11] The present organizers of this seminar are J.P. Barthélemy, O. Hudry, Marc Demange, B. Leclerc, and B. Monjardet.

[12] Presently Michel Armatte, Bernard Bru and Thierry Martin are also organizers of this seminar.

[13] TRAP is the acronym of "Table Ronde sur l'Agrégation des Préférences".

[14] Although J. Feldman was no longer a member of CAMS at this time, one can mention also the Conference organized in 1991 at the École: "Moyenne, Milieu, Centre" (J. Feldman, G. Lagneau and B. Matalon, Paris).

[15] An aim of this journal was also to promote exchanges between people teaching mathematics and statistics for social sciences and those teaching these sciences. In fact CAMS members and especially Guilbaud and Barbut played a significant role in the starting and the development of mathematical training courses for students in economics, psychology and sociology. In particular Barbut was one of the initiators of the creation of a new curriculum "Applied Mathematics and Social Sciences" presently existing in more than 30 french universities.

can quote: "Opinions et scrutins" (Opinions and votes, 1973), "Modélisation des préférences et quasiordres" (Preferences modelization and semiorders, 1978), "Métriques et relations" (Metrics and relations, 1979), "Condorcet" (1990). Note that another special issue "Cinquante ans de théorie du choix social" (Fifty years of social choice theory) will appear in 2003. On the other hand, for instance between the years 1962 and 1980, 49 papers were published in the fields "Décisions, préférences, procédures de vote".[16]

3 Thematic history

As said above our leading clue for this historic inquiry will be Guilbaud's 1952 paper *Les théories de l'intérêt général et le problème logique de l'agrégation*. But we have first to recall one of the major contributions of this paper. By dragging from the deep oblivion where it had fallen Condorcet's " *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix"*[17] (Paris, 1785) and by recalling Condorcet's significant approaches to the topic adressed by Arrow, Guilbaud was knotting again the broken thread of a history, which was in fact a long history[18]. Indeed the Essai had been read only by a few people including Daunou, Lhuillier[19] and Lacroix. But in France reputed mathematicians like Joseph Bertrand found the book unreadable and anyway without interest, an opinion shared by Todhunter who at least had read the Essai[20]. Guilbaud was the first to read again the Essai and more generally Condorcet's works on "la mathématique sociale". Moreover he suggested to his colleague epistemologist Gilles Granger to do researches on this topic, researches which led to Granger's book "La mathématique sociale du marquis de Condorcet" (1956).

[16] One must point out that during these years the quasi totality of these papers were published in French, a fact that didn't favor their knowledge by an international audience.

[17] Henceforth we will call Condorcet 's book simply the Essai.

[18] The famous "Annales’s school" of historians of the École (and in particular Braudel) supported the notion of the "long history".

[19] Daunou and Lhuillier worked on voting procedures at the end of 18th century and the beginning of 19th century and both quote the Essai (see McLean I (1995) The first golden age of social choice 1784-1803. In: Barnett W, Moulin H, Salles M, Schofield N (eds) *Social choice, Welfare and Ethics.* Cambridge University Press, pp 13-33).

[20] In his book *An history of the mathematical theory of probability from the time of Pascal to that of Laplace* (MacMillan, London, 1865), In fact Todhunter devotes a chapter to a detailed analysis of the Essai but he completely misses the significance of Condorcet's study on the systems of propositions and their possible contradictions ("these results however appear of too little value to detain us any longer", page 375).

Guilbaud's paper was published in "Économie Appliquée", an economics journal, but it was written both for economists and mathematicians, a not so easy task[21]. The paper, like Arrow's book, uses finite mathematical structures, in particular binary relations and families of sets on finite sets[22]. We recall now the classical notations used in social choice theory for such structures.

A = {x,y,z,...} is a finite set of m elements called *alternatives* (or issues, decisions, outcomes, candidates, objects, etc.).

N = {1,2,.....n} is a finite set of n elements called *voters* (or agents, persons, individuals, criteria, etc.).

The preference of a voter on the set A is given by a *linear order* [23]

$$L = x_1 ... x_k ... x_m,$$

where x_1 is the most preferred alternative, x_2 the second one, etc. If alternative x is preferred to alternative y in the linear order L, we write xLy or $(x,y) \in L$. More generally, if R is an arbitrary binary relation on A, we write xRy or $(x,y) \in R$ when x is in the relation R with y.

We denote by $\mathcal{L}$ the set of all linear orders on A:

$$\mathcal{L} = \{\textbf{linear orders L on A}\}.$$

A *profile* $\pi = (L_1,...,L_n)$ is a function π of N into $\mathcal{L}$, describing the state of preferences of the voters on the set of alternatives. $\mathcal{L}^N$ denotes the set of all such profiles.

A $\mathcal{L}$***-preference aggregation function*** is a map

$$f: \mathcal{L}^N \to \mathcal{L}$$

which assigns a linear order $f(\pi)$ -the collective preference- to each profile π of individual preferences.

For all $x,y \in A$, and $\pi \in \mathcal{L}^N$, we define

$$N_\pi(x,y) = \{i \in N : xL_iy\},$$

$$n_\pi(x,y) = |N_\pi(x,y)|.$$

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[21] Economists (with sufficient mathematical training) appreciated the paper (in the second edition of his book, Arrow describes it as a "remarkable exposition of the theory of collective choice and the general problem of aggregation"). But few mathematicians, -too often unaware or even contemptuous of social sciences- read it.

[22] It is interesting to mention that Arrow had been introduced to such mathematics by Tarski.

[23] Condorcet and most of his followers expressed the preference of a voter by such a linear order, i.e. by a transitive, antisymmetric and complete binary relation. Arrow expressed this preference by a *weak order* (called by him *ordering* and by others *quasi-ordering* or *complete preorder* or etc.), i.e. by a transitive and complete binary relation.

So $N_\pi(x,y)$ is the set of voters preferring x to y in the profile π and $n_\pi(x,y)$ is the number of these voters.
A subset S of N is called a *majority* if $|S| \geq n/2$ and a *strict majority* if $|S| > n/2$.
We define

$$\mathfrak{F}_{MAJ} = \{\text{majorities of } N\} = \{S \subseteq N : |S| \geq n/2\},$$
$$\mathfrak{F}_{SMAJ} = \{\text{strict majorities of } N\} = \{S \subseteq N : |S| > n/2\}.$$

We can now define Condorcet's majority rule(s) and the "effet Condorcet".

3. 1 The Condorcet effect and its frequency

As is well known Condorcet proposed to adopt as a voting procedure the method retaining for each pair of alternatives the one (or the ones) supported by a majority of voters[24]. For a given profile π, we denote by $R_{MAJ}(\pi)$ (respectively $R_{SMAJ}(\pi)$) the binary relation obtained by using the majority (respectively the strict majority). Formally

$$\mathbf{x\ R_{MAJ}(\pi)\ y\ if\ N_\pi(x,y) \in \mathfrak{F}_{MAJ}} \text{ (i.e. if } n_\pi(x,y) \geq n/2\text{)},$$
$$\mathbf{x\ R_{SMAJ}(\pi)\ y\ if\ N_\pi(x,y) \in \mathfrak{F}_{SMAJ}} \text{ (i.e. if } n_\pi(x,y) > n/2\text{))}.$$

We call Condorcet's rule (respectively Condorcet's strict rule) the aggregation rule obtained by associating with each profile π the relation $R_{MAJ}(\pi)$ (respectively R_{SMAJ}). Note that when the number of voters is odd there is a single majority rule.

Now Condorcet's majority rules are not $\mathcal{L}$-preference aggregation functions since the majority preference relations can have *cycles* of length k (called *k-cycles*) greater than 2 (i.e. there can exist $k \geq 3$ distinct alternatives $x_1, x_2, \ldots. x_k$ such that for instance $x_1 R_{MAJ} x_2 R_{MAJ} x_3 \ldots\ldots. x_k R_{MAJ} x_1$).This fact discovered by Condorcet has been called "*l'effet Condorcet*" by Guilbaud. It is also known as the "paradox of voting". But we prefer the term "Condorcet effect" since as it will be shown in section 3.3 this effect is unavoidable and thus not really paradoxical.

[24] This method had been already proposed in the thirteenth century by Ramon Lull (see McLean I, London J (1990) The Borda and Condorcet principles: three medieval applications. *Social Choice and Welfare* 7: 99-108).

The simplest case of such an effect is obtained for three alternatives x,y,z and three voters with the preferences xyz, yzx and zxy. The collective preference is then the 3-cycle shown in Figure 1.

Figure 1. HERE
A 3-cycle

Before trying to deal with the Condorcet effect a preliminary question has to be asked: is it frequent?[25] Guilbaud studies the case of three alternatives. He gives the frequency of the Condorcet effect for a number of voters respectively equal to 3 (5,6%), 4 (7%), 9 (7,8%) and 25 (8,4%) and a formula for this value when n goes to infinity: $(1-3/\pi)\text{Arccos}(1/\sqrt{3}))$ = 0,08774. Since this formula was given in a footnote without explanation it intrigued somehow. But since Guilbaud's result there has been a considerable amount of work studying the probability of the effect or the weaker probability to have no Condorcet winner[26]. The general conclusion of these investigations is that the Condorcet effect is frequent especially when the numbers of alternatives and/or the number of voters is large. For instance if this frequency remains always small for three alternatives, it is 49% for 3 voters and 6 alternatives and about 96% for 25 voters and 9 alternatives (a situation quite possible in academic committees). Then if one is inclined to use majority rules as much as possible one must find ways to overcome this effect when it occurs. This was the real question raised by Condorcet and we will see below (in section 3.2.2) what was probably his answer. But more generally we will consider three approaches presented in Guilbaud's paper to overcome this effect.

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[25] In Black’s book (*The Theory of Committees and Elections* (1958) Cambridge University Press, Cambridge) the answer is given for three voters and three alternatives. In 1803 Daunou writes that the Condorcet effect "is by no means a rare occurence", but his assertion is based on wrong computations (See Mémoire sur les élections au scrutin, published in English translation In McLean I, Urken AB (eds) Classics of social choice. The University of Michigan Press, Ann Arbor, 1995, pp 237-276, page 243).

[26] See for instance a review of such works in Gehrlein WV (1983) Condorcet's paradox. *Theory and Decision* 15: 161-197. Note that the computation of the frequency is equivalent to computing the probability of the effet Condorcet under the so-called probabilistic model of "impartial culture" where each linear order has the same probability (1/m!) of being adopted by each voter. In this case Guilbaud's formula has been generalized (for instance in Gehrlein WV, Fishburn PC (1976) Probabilities of election outcomes for large electorates. *Journal of Economic Theory* 13: 14-25). More generally the probability of the effect has been studied when the preferences of the voters follow various probability models (see again Gehrlein's 1983 paper).

3.2 To overcome the Condorcet effect ?

3.2.1 To overcome the Condorcet effect ? the generalized majorities rules

Condorcet's majority rule is defined by taking in each pairwise comparison of alternatives the preference supported by a majority, where a majority has been defined as a set of at least half of the voters. One can ask if by changing the size of majorities one can get a rule which will avoid the Condorcet effect. It is certainly possible in some sense. For instance if one requires now unanimity in order to have x preferred to y, then the collective preference obtained has no cycles since it is the partial order intersection of the linear orders of the voters. But this collective preference has no reason to be complete, i.e. to be a linear order and so we don't get a $\mathcal{L}$-preference aggregation function. In fact such a unanimity rule can often lead to an indecisiveness that is not very satisfying. More generally Guilbaud proposes to see if by changing the form of majorities one can obtain $\mathcal{L}$-preference aggregation functions. He considers what Von Neumann and Morgenstern[27] have called a simple game[28]:

A ***simple game*** on N is a non-empty set $\mathfrak{F}$ of subsets of N satisfying :

$$[S \in \mathfrak{F} \text{ and } S \subseteq U] \Rightarrow [U \in \mathfrak{F}].$$

Guilbaud proposes to call such a simple game a family of *generalized majorities* and he considers the associated *generalized majority rules*:

The *preference aggregation function* $f_{\mathfrak{F}}$ associated with the simple game $\mathfrak{F}$ is given by:

$$\text{for every } \pi \in \mathcal{L}^N,\ f_{\mathfrak{F}}(\pi) = R_{\mathfrak{F}}(\pi), \text{ where}$$
$$\text{for all } x,y \in A,\ xR_{\mathfrak{F}}(\pi)y \Leftrightarrow N\pi(x,y) \in \mathfrak{F}.$$

Thus x is collectively preferred to y according to the generalized majority rule $f_{\mathfrak{F}}$ if and only if the set of voters preferring x to y in the profile π belongs to the family $\mathfrak{F}$ of generalized majorities. Obviously these rules generalize Condorcet's majority rules which are obtained with $\mathfrak{F} = \mathfrak{F}_{MAJ} = \{S \subseteq N : |S| \geq n/2\}$ and $\mathfrak{F} = \mathfrak{F}_{SMAJ} = \{S \subseteq N : |S| >$

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[27] in Von Neumann J, Morgenstern O (1944) *Theory of games and economic behaviour*. Princeton, University Press.

[28] The term simple game to name this mathematical structure was justified in the context of Von Neumann and Morgenstern's book. But in fact such a structure appears in many fields of mathematics where to use the term simple game would be absurd. Unfortunately such an terminological absurdity has became unavoidable in social choice theory (note that from a mathematical point of view a simple game is nothing more than an order filter in the Boolean lattice of all subsets of S).

n/2}. The unanimity rule is also a generalized majority rule since it is obtained by taking $\mathfrak{F} = \{N\}$. More generally one can call "*oligarchic*" the generalized majority rule obtained for the simple game $\mathfrak{F}_V = \{W \subseteq N : V \subseteq W\}$ where $\varnothing \subset V \subseteq N$. Indeed such a rule amounts to take as collective preference the unanimous preferences of the voters belonging to V and ignoring the preferences of the other voters. A particular case is obtained for V = {i}. Then $\mathfrak{F}_{\{i\}}$ is the family of subsets of N containing i and the associated rule can be called a *dictatorial rule* since the collective preference is always the preference of the voter i. Now the problem is to determine among all these generalized majority rules those that always induce a linear order, i. e. to determine the preference aggregation functions $f_{\mathfrak{F}}$ that are $\mathcal{L}$-preference aggregation functions. The answer is given in Guilbaud's paper and we will call it Guilbaud's theorem [29]:

Let $\mathfrak{F}$ be a simple game on N and $\mathcal{L}$ be the set of all linear orders on A (n, m ≥ 3). The map $f_{\mathfrak{F}}$ is an $\mathcal{L}$-preference aggregation function if and only if $\mathfrak{F}$ is dictatorial (i.e., there exists $i \in N$ such that $\mathfrak{F} = \mathfrak{F}_{\{i\}} = \{W \subseteq N : i \in W\}$).

Observe that $\mathfrak{F} = \mathfrak{F}_{\{i\}}$ is equivalent to saying that $\mathfrak{F}$ is an ultrafilter[30] since when N is finite the ultrafilters on N are exactly the n families $\mathfrak{F}_{\{i\}}$ ($i \in N$).

It is instructive to sketch the very simple proof. First one checks that the collective preference $R_{\mathfrak{F}}(\pi)$ is antisymmetric and complete (i.e. a so-

[29] In fact this theorem is a consequence of a more general result proved by Guilbaud and concerning a logical problem already raised by Condorcet. Let a set of binary (“yes or no“) questions be logically linked in the sense that the answers to some imply the answers to others. A *coherent opinion* of an individual is defined as a set of answers to these questions respecting their links. What are the rules allowing one to aggregate several coherent individual opinions into a coherent collective opinion ? Guilbaud proves that if the aggregation rule must preserve all the possible logical links between these questions, then it must be dictatorial (page 306). A contrario, the use of, for instance, the majority rule can lead to an incoherent collective opinion, a fact called also "Condorcet effect" by Guilbaud. It is interesting to observe that this much more general Condorcet effect has been rediscovered in the eighties under the name of the "doctrinal paradox" and has led to results similar to Guilbaud's results (see List C, Pettit P (2002) Aggregating Sets of Judgments: An Impossibility Result. *Economics and Philosophy* 18: 89-110 and a bibliography at http://www.nuff.ox.ac.uk/users/list/doctrinalparadox.htm).

[30] A filter $\mathfrak{F}$ on a set N is an *order ideal* ($U \in \mathfrak{F}$ and $U \subseteq V$ imply $V \in \mathfrak{F}$), is stable by intersection ($U, V \in \mathfrak{F}$ imply $U \cap V \in \mathfrak{F}$) and does not contain the empty set. An *ultrafilter* is a maximal filter, i.e. a filter which is not strictly contained in another filter.

called *tournament*) for all profiles π if and only if $\mathfrak{F}$ is a so-called *proper* and *strong* simple game, i.e. satisfies the following condition:

$$U \notin \mathfrak{F} \Leftrightarrow N \setminus U \in \mathfrak{F} \quad (1)$$

Now easy and well known results on tournaments say that a tournament is transitive (i.e. a linear order) if and only if it has no cycles and if and only if it has no 3-cycles (as in Figure 1). But to avoid a 3-cycle in $R_{\mathfrak{F}}(\pi)$ it is necessary and sufficient that $\mathfrak{F}$ satisfies the following condition:

$$\text{for all } U, V, W \in \mathfrak{F},\ U \cap V \cap W \neq \emptyset \quad (2)$$

Indeed in this case one cannot have three (generalized) majorities U,V,W and three alternatives x,y,z, with x preferred to y (respectively y preferred to z and z preferred to x) for each voter of U (respectively of V and W) since then a voter belonging to $U \cap V \cap W$ would have a 3-cycle in his (her) linear order of preference. And conversely if there exists U, V, W $\in \mathfrak{F}$ such that $U \cap V \cap W = \emptyset$, it is easy to construct a profile for which there will be a 3-cycle on a set {x,y,z} of three alternatives. It remains now to prove that a simple game satisfying condition (1) and (2) is an ultrafilter, which is easy.

Remarks

1) Recall that Arrow's theorem says that an independent and Paretian [31] social welfare function $\mathcal{W}^N \rightarrow \mathcal{W}$ (set of all weak orders on A) is dictatorial (in the sense that the strict preference of the dictator is the collective strict preference). One can get Arrow's theorem from Guilbaud's theorem. First one shows that an independent and Paretian $\mathcal{L}$-preference aggregation function f is a preference aggregation function $f_{\mathfrak{F}}$ associated with a simple game $\mathfrak{F}$ ($\mathfrak{F}$ is the family of the *decisive sets* of f). By applying Guilbaud's theorem one gets that f is dictatorial. This result is extended to weak orders by using a domain restriction standard argument, probably found in Blau[32] for the first time. But obviously Arrow's proof in the second edition of his book or in his 1952 paper in French is direct and quicker.

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[31] A social welfare function is independent (of irrelevant alternatives) if the social preference on two alternatives depends only on the individual preferences on these alternatives and it is Paretian if x is socially (strictly) preferred to y if all the voters prefer (strictly) x to y.

[32] See Blau JH (1979) Semiorders and collective choice. *Journal of Economic Theory* 29: 195-206. Wilson (Wilson RB (1972) Social choice theory without the Pareto principle. *Journal of Economic Theory* 5: 478-486) proved also that Arrow's theorem could be obtained from the linear order version of this theorem, but his proof of this version which results from his general theory of aggregation is remarkably complicated.

2) The term ultrafilter does not appear explicitly in Guilbaud's paper. But it suffices to read his proof to see that "followers of Bourbaki will notice an ultrafilter in the background" as Blau[33] would have said and as I observed it later: "then it is immediate that in the Boolean algebra of subsets of N, $\mathfrak{F}$ must be a maximal filter"[34].

3) The characterization of ultrafilters given above can be improved. Indeed one has the following result: a family $\mathfrak{F}$ on an arbitrary set N is an ultrafilter if and only if it is strong and satisfies condition (2)[35]. Note also that this last condition can be written $\upsilon(\mathfrak{F}) > 3$, where $\upsilon(\mathfrak{F})$ is Nakamura's number of $\mathfrak{F}$ so that Nakamura's theorem on simple games can be seen as a generalization of Guilbaud's theorem (a point of view developped in my 2003 paper quoted in the Annex).

4) Guilbaud's theorem is in fact a characterization of the functions $\mathcal{L}^N \rightarrow \mathcal{L}$ that are projections. This is an interesting mathematical result which can have links with other such projection characterization results for mathematical structures like vector spaces and posets[36].

3.2.2 To overcome the Condorcet effect ? the Condorcet (median) procedure

A big problem of Condorcet was to remedy the defect of his majority rule and he addressed it in several places in the Essai. Since his proposals are not necessarily very clear and can vary, several interpretations of what he has in mind have been given[37]. Let us quote Guilbaud's interpretation: "Condorcet could not resign himself to conclude that it is impossible to attribute any coherent opinion[38] to the electoral body (......). He looks for lesser evil, that is to say among all the *coherent* opinions the one which is supported by the largest possible number of votes"[39] .

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[33] See Blau's paper in footnote 32.

[34] Translation of my sentence "Il est alors immédiat que dans l'algèbre de Boole des parties de N, $\mathfrak{F}$ doit être un filtre maximal" in Monjardet B (1969) Remarques sur une classe de procédures de vote et les théorèmes de possibilité In *La décision*, Éditions du CNRS, Paris, pp 177-184.

[35] See Monjardet B (1978) Une autre preuve du théorème d'Arrow. *R.A.I.R.O.* 12: 291-296 and Monjardet B (1981) On the use of ultrafilters in social choice theory In Pattanaik PK, Salles M (eds) *Social Choice and Welfare*. Amsterdam, North-Holland, pp 73-78.

[36] See for instance Pouzet M (1998) A projection property and Arrow's impossibility theorem. *Discrete Mathematics* 192(1-3): 293-308.

[37] See for instance Black's book quoted in footnote 25.

[38] In the Essai "coherent opinion" means linear order.

[39] Page 265 in the English translation of Guilbaud's paper quoted in footnote 2.

One can give a formal description of this procedure. For $\pi = (L_1,...,L_n)$ an arbitrary profile in $\mathcal{L}^N$ and L an arbitrary linear order in $\mathcal{L}$, one sets:

$$\alpha(\pi,L) = \Sigma\{n_\pi(x,y), (x,y) \in L\}.$$

$\alpha(\pi,L)$ is a measure of ***agreement*** between the profile π and the possible collective preference L, since it counts the number of pairwise agreements between the preferences of the voters and the linear order L. Now the procedure described in Guilbaud's sentence consists to:

Take as the collective preference a linear order L that solves the discrete optimisation problem:

$$\mathbf{MAX\{\alpha(\pi,L), L \in \mathcal{L}\}}.$$

For reasons that we will explain below we call this procedure the ***median procedure*** and we call ***median orders*** of a profile the corresponding linear orders obtained by applying it to this profile.

One can make several remarks on this median procedure. First this optimisation problem has at least one solution (since the set $\mathcal{L}$ is finite) but it can have several (an even many) solutions. It is easy to see that if there is no Condorcet effect, these solutions are given by Condorcet's majority rules. More precisely in this case and when the number of voters is odd there is a unique median order given by the two Condorcet's rules. Always in this case and when the number of voters is even one notes first that the strict majority relation $R_{SMAJ}(\pi)$ is a partial order contained in the majority relation $R_{MAJ}(\pi)$. Then there exists (at least) one linear order between (in terms of set-inclusion) these two majority relations and it is easy to understand that the median orders are all such linear orders. This is the "easy" case, whereas in the general case the computation of the solutions is a "difficult" problem[40].

Guilbaud's quotation (above) identifies Condorcet's procedure to overcome the Condorcet effect as the median procedure. In the case of three alternatives and of a Condorcet effect, Condorcet proposes to take as collective preference the linear order obtained by inverting the preference supported by the weakest majority. This is clearly equivalent to adopting the median procedure. When there are more than three alternatives, Condorcet's ambiguous proposals lead to several possible algorithms. And

[40] The fact that the median problem is NP-complete can be found for example in Orlin (1981) unpublished note or Hudry O (1989) *Recherche d'ordres médians : complexité, algorithmique et problèmes combinatoires*. Thèse ENST, Paris.

one can find profiles with a Condorcet effect for which none of these algorithms lead to a median order. But this is not very surprising considering the difficulty of the problem to find the median orders. On the other hand for Condorcet there is an objective linear order between the alternatives (for instance a true order of merit between candidates). The aim of the voting procedure is to find this objective order from those given by the voters and containing errors. Then, in the search for this true order, he introduces a probabilistic model to find what we would call the "maximum likelihood" order. Young's analysis of this Condorcet model led him to conclude that the orders obtained by the corresponding procedure are the median orders[41]. So following Guilbaud's and Young's interpretations we will consider that Condorcet has been the creator of the median procedure for profiles of linear orders and we will call this procedure the *Condorcet generalized rule*.

Now we should explain why we have called it the median procedure. To do that we have to define what is a median in a metric space.

Let (E,d) be an arbitrary (finite) metric space, and $(x_1,...,x_n)$ an n-tuple of points in this space.

A (*metric*) *MEDIAN* of $(x_1,...,x_n)$ in the metric space (E,d) is any point m of E minimizing the sum

$$\Sigma\{d(x_i,x),\ i = 1,...n\}$$

Here also since E is finite it is clear that the n-tuple $(x_1,...,x_n)$ has at least one median. In fact it has often several medians.

It remains then to show that the median orders described above are medians in a metric space, which is easy. We take as the metric space E the set of all linear orders $\mathcal{L}$ endowed with the following distance:

for $L,L' \in \mathcal{L}$, $d_K(L,L')$ = |{pairs {x,y} such that xLy and yL'x, or xL'y and yLx}|.

Thus this quantity measures the disagreement between the two linear orders L and L' and it is a distance since it is nothing more than half of the classical symmetric distance |L\L'| + |L'\L| between these two orders[42].

[41] See Young HP (1988) Condorcet's Theory of Voting. *American Political Science Review* 82: 1231-1244.

[42] We denote this distance by d_K since it has been implicitly used by Kendall as early as 1938 (the well-known Kendall "correlation coefficient" tau is nothing more than the normalization of this distance between -1 and +1) as well that by Kemeny (see below).

Now an old result of Barbut [43] proves our claim:
For $\pi = (L_1,...,L_n) \in \mathcal{L}^N$, the medians of π in the metric space $(\mathcal{L}, d_K)$ are the solutions of the discrete optimisation problem: $MAX\{\alpha(\pi,L), L \in \mathcal{L}\}$.

One can note that as early as 1959 Kemeny proposed to take as collective preference(s) for a profile of linear orders its medians in the metric space $(\mathcal{L}, d_K)$ defined above[44]. This proposal was later published in Kemeny and Snell's famous book *Mathematical Models in the Social Sciences* and so this procedure is widely known as Kemeny's procedure [45]. One can now add that Kemeny's procedure, the median procedure and Condorcet's generalized rule are all the same.

We have not yet finished with medians since we are turning now to the notion of algebraic medians, a notion closely related to the notion of majority. Note first that a profile $\pi = (L_1,...,L_n)$ of linear orders is in particular a profile of n binary relations on A. Take now as a metric space (E,d) the set 2^{A^2} of all binary relations on A endowed with the symmetric difference distance $d_\Delta(R,R') = |(R\backslash R') \cup (R'\backslash R)|$. What are the medians of π in this metric space? The answer is easily found. Since one always has $R_{SMAJ}(\pi) \subseteq R_{MAJ}(\pi)$ one can define in 2^{A^2} the interval $[R_{SMAJ}(\pi), R_{MAJ}(\pi)]$ of all binary relations contained between the two majority relations. Then one has:

The medians of a profile π of linear orders in the metric space $(2^{A^2}, d_\Delta)$ are all the binary relations of the interval

$$[R_{SMAJ}(\pi), R_{MAJ}(\pi)] = \{R \in 2^{A^2} : R_{SMAJ}(\pi) \subseteq R \subseteq R_{MAJ}(\pi)\}.$$

Now this result is a special case of a general result on distributive lattices. Recall that a lattice $(L, \leq)$ is a partially ordered set such that each pair {x,y} of elements of L has a *greatest lower bound* (or *meet*) denoted by $x \wedge y$ and a *least upper bound* (or *join*) denoted by $x \vee y$. Then if L is

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[43] Barbut M (1967) Médianes, Condorcet et Kendall. *Note SEMA*, Paris and (1980) *Mathématiques et Sciences Humaines* 69: 5-13.

[44] Kemeny JG (1959) Mathematics without numbers. *Daedalu*s 88: 577-591. Kemeny JG, Snell JC (1961) *Mathematical Models in the Social Sciences*. Ginand Co, New York.

[45] In fact this procedure has many other equivalent forms and so it has been very often (re)discovered, in particular by Brunk and independently by Hays as early as in 1960. See Monjardet B (1991) Sur diverses formes de la "Règle de Condorcet" d'agrégation des préférences. *Mathématiques et Sciences humaines* 111: 61-71 or Monjardet B (1997) Concordance between two linear orders: The Spearman and Kendall coefficients revisited. *Journal of Classification* 14 (2): 269-295.

finite any subset X of L has a meet denoted by $\wedge X$ and a join denoted by $\vee X$. A lattice is *distributive* if for all $x,y,z \in L$,

$$x\wedge(y\vee z) = (x\wedge y) \vee(x\wedge z).$$

The lattice $(2^E, \subseteq)$ of all the subsets of a set ordered by set inclusion is a distributive lattice with intersection and union as meet and join operations (in fact it is a special kind of distributive lattice, namely a Boolean lattice). In particular $(2^{A^2}, \subseteq)$ is the distributive lattice of all the binary relations defined on the set A. Now in a distributive lattice L there exists a "natural" distance d_L generalizing the symmetric difference distance in the lattice of the subsets of a set[46]. So one can consider the metric space (L,d_L) and the medians of an n-tuple $\pi = (x_1,...,x_n)$ of elements of L. The following proposition generalizing Birkhoff and Kiss's result is due to Barbut and Monjardet[47]:

The medians of an n-tuple $\pi = (x_1,...,x_n)$ of elements of the metric space (L, d_L), L being a distributive lattice, are all the elements of the interval

$$[m_1(\pi), m_2(\pi)] = \{x \in L : m_1(\pi) \leq x \leq m_2(\pi)\} \text{ where}$$

$$m_1(\pi) = \vee\{\{\wedge\{x_i, i \in W\}, W \subseteq \{1,...n\}, |W| > n/2\}\} \text{ and}$$

$$m_2(\pi) = \vee\{\{\wedge\{x_i, i \in W\}, W \subseteq \{1,...n\}, |W| \geq n/2\}\}.$$

These two elements $m_1(\pi)$ and $m_2(\pi)$ (which are equal if n is odd) can be called the *algebraic medians* of π. The result on the medians of a profile π of linear orders in the metric space $(2^{A^2}, d_\Delta)$ is a particular case of the above result since in the Boolean lattice $(2^{A^2}, \subseteq)$ one has clearly $R_{SMAJ}(\pi) = \cup\{\{\cap\{L_i, i \in W\}, W \subseteq N, |W| > n/2\}\}$ and $R_{MAJ}(\pi) = \cup\{\{\cap\{L_i, i \in W\}, W \subseteq N, |W| \geq n/2\}\}$. Another particular case of this result is the case where L is totally ordered. Then the meet and join operations are the minimum and maximum operations and one gets the usual statistical median of numbers. For instance if $x_1<x_2<x_3$ the median of these three

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[46] This distance is nothing more than the "path length" distance in the unoriented graph associated with the covering relation of the lattice (see for instance Birkhoff G (1967) *Lattice theory*. Amer. Math. Soc., Providence or Barbut M, Monjardet B (1970) *Ordre et Classification, Algèbre et Combinatoire, tomes I et II*. Hachette, Paris).

[47] Barbut M (1961) Médianes, distributivité, éloignements. Note CAMS, EHESS, Paris and (1980) *Mathématiques et Sciences Humaines* 70: 5-31, Monjardet B (1980) Théorie et application de la médiane dans les treillis distributifs finis. *Annals of Discrete Mathematics* 9: 87-91.

numbers is $x_2 = \mathrm{Max}\{\mathrm{Min}\{x_1,x_2\}, \mathrm{Min}\{x_2,x_3\}, \mathrm{Min}\{x_3,x_1\}\} = \mathrm{Min}\{\mathrm{Max}\{x_1,x_2\}, \mathrm{Max}\{x_2,x_3\}, \mathrm{Max}\{x_3,x_1\}\}$.

The fact that the majority rule is a median rule appears informally several times in Guilbaud's paper (see for instance pages 279 and 291). But it induces consequences linked to the third approach to overcome the Condorcet effect, the so-called *Condorcet* (or *restricted* or *coherent*) *domains*.

3.2.3 To overcome the Condorcet effect ? the restricted domains

We will use the following definition of a restricted domain of linear orders.

A set $\mathcal{D} \subset \mathcal{L}$ of linear orders is a *Condorcet domain* if the use of Condorcet's strict majority rule for profiles of $\mathcal{D}$ never induces the Condorcet effect, i.e.:

$$\forall \text{ N finite set}, \forall \pi \in \mathcal{D}^N, R_{SMAJ}(\pi) \text{ is without cycles.}$$

Thus a set of linear orders is a Condorcet domain if the strict majority rule applied to any profile of linear orders of the domain leads always to an asymmetric relation without cycles. Note that if |N| is odd such a relation is a linear order and that in the general case it can be always completed into a linear order. Moreover an easy consequence of this definition is that $R_{SMAJ}(\pi)$ is always a partial order.

One can show that a set of linear orders is a Condorcet domain if and only if the strict majority rule applied to any profile of n linear orders of the domain with n an odd integer leads always to a linear order[48]. In fact we are going to consider Condorcet domains satisfying the stronger condition of *stability*, namely that this linear order must belong to the domain.

The best known example of stable domain is the *single-peaked domain* $\mathcal{B}$ defined by Black[49]. Since its definition needs to take a reference linear order L considered as the "objective" order between the alternatives

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[48] This definition is also equivalent to saying that $\mathcal{D}$ has no *cyclic triples*, i.e. that there do not exist a subset {x,y,z} of three alternatives and three linear orders in $\mathcal{D}$ such that the restrictions of these orders to {x,y,z} is a cyclic permutation like xyz, yzx and zxy (such a set $\mathcal{D}$ has been also called a *consistent* or an *acyclic* or a *majority-consistent* set, see the references in footnotes 53 and 54).

[49] See Black D (1948) On the rationale of group decision-making. *Journal of Political Economy* 56: 23-34 and Black's book quoted in footnote 25.

it is also called the domain of *L-unimodal* linear orders. Now one finds in Guilbaud's paper an analysis of the single-peaked domain showing that the set of single-peaked linear orders has a distributive lattice structure and that the majority relation of a profile in this domain is the median of the elements of the profile in this lattice (see page 289 and the figure showing the distributive lattice of the 16 single-peaked linear orders on a set of cardinality 5).

Using another paper by Guilbaud and Rosenstiehl[50], one can generalize this result. Indeed Guilbaud and Rosenstiehl show that the set $\mathcal{L}$ of linear orders can be endowed with a lattice structure called the "*permutoèdre*" *lattice*. This lattice has an arbitrary linear order L as the greatest element and the dual of L as the least element. The unoriented covering relation of this lattice is the adjacency relation where the linear order L is adjacent to the linear order L' if they differ on a unique pair of elements. The permutoèdre lattice is not distributive but it contains distributive sublattices. We say that a sublattice of $\mathcal{L}$ is *covering* if the covering relation in this sublattice is the same as the covering relation in $\mathcal{L}$. The following result is due to Chameni-Nembua[51]:

Any distributive covering sublattice of the permutoèdre lattice $\mathcal{L}$ is a stable domain and thus a Condorcet domain.

Indeed in this distributive lattice to take the majority relation of a profile of n (odd integer) linear orders reduces to taking the (metric or algebraic) median of these n elements. More generally the lattice interval $[m_1(\pi), m_2(\pi)]$ associated with an arbitrary profile π of linear orders in this lattice is exactly the set of linear orders containing the strict majority relation $R_{SMAJ}(\pi)$.

Figure 2 represents the permutoèdre lattice $\mathcal{L}$ on a set A = {a,b,c,d}. The covering sublattice of the single-peaked orders (w.r.t. the linear order abcd) is represented on this Figure by squares. It has 8 elements. One can find another covering distributive sublattice with 9 elements (search for it!).

Figure 2 HERE

The permutoèdre lattice on four elements

[50] Guilbaud GT, Rosenstiehl P (1970) Analyse algébrique d'un scrutin. *Mathématiques et Sciences Humaines* 4: 9-33.

[51] Chameni-Nembua C (1989) Règle majoritaire et distributivité dans le permutoèdre. *Mathématiques et Sciences Humaines* 108: 5-22.

Figure 3 shows a covering distributive sublattice of the permutoèdre lattice on 6 elements. It has 45 elements. When I discovered it in 1988[52] it was interesting since the best lower bound on the size of a Condorcet domain was then given by an Abello and Johnson's construction[53] which for n = 6 was 44. On the other hand Figure 3 illustrates in the case of n = 6 a general construction of Condorcet domains found later independently by Craven and Fishburn[54].

Figure 3 HERE

A Condorcet domain sublattice of the permutoèdre lattice on 6 elements.

3.3 Why the Condorcet effect ?

Another significant contribution of Guilbaud's paper was to bring together the Condorcet effect and some other "paradoxes" like the paradox raised by Quetelet's "homme moyen". The statistician and social scientist Quetelet proposed to extend the notion of mean to a population of individuals[55].To do that he considers the measures of several characteristics of these individuals (size, weight, strength...) and he calculates the means of these measures for each characteristic. The problem is that a mean man so defined could be an impossible man. This fundamental objection is well presented by Cournot who writes: "When one applies mean operations to various parts of a complicated system, one must be aware that these mean values can be incompatible: the state of the system where all elements take the mean values separately determined for each could be an impossible state"[56]. And Cournot gives the example of the triangle obtained by taking the means of the lengths of the three sides of rectangular triangles, a triangle which in general is not rectangular.

More generally Guilbaud considers a method of aggregation of complex objects that can be called a *component-wise (algebraic) mean method.* It consists first of decomposing complex objects into their simple elements, then of applying to each series of such elements an algebraic

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[52] It was published in Chameni-Nembua's paper quoted in the above footnote.

[53] Abello JM, Johnson CR (1984) How large are transitive simple majority domains? *SIAM J. Algebraic and Discrete Methods* 3(4): 603-618 (the bound was $3.2^{(n-2)} - 4$).

[54] Craven J (1996) Majority consistent preference orderings. *Social Choice and Welfare* 13: 259-267. Fishburn PC (1997) Acyclic sets of linear orders. *Social Choice and Welfare* 14: 113-124.

[55] Quetelet A (1835) *Sur l'homme et le développement de ses facultés ou Essai de physique sociale*. Paris.

[56] Free translation of the Cournot sentence quoted by Guilbaud and taken from Cournot's book *Exposition de la théorie des chances,* Paris, 1843.

mean operation. By definition an essential property of this method is its property of independence: each series of simple elements is aggregated (by the same or by different mean operations) independently of the other series. Now as soon as the complex objects considered are defined by some relations between their simple elements, the aggregated complex object does not necessarily satisfy these same relations. In Condorcet's case the simple elements are the ordered pairs forming the linear orders and the mean operation is the majority relation taken on each pair. One can represent any binary relation R on $A = \{a_1, ... a_m\}$ by a boolean vector b_{ij}, where $b_{ij} = 1$ (respectively 0) if $(a_i,a_j) \in R$ (respectively $(a_i,a_j) \notin R$). Then linear orders are defined by the properties $b_{ii} = 1$ (reflexivity), $b_{ij} + b_{ij} = 1$ (antisymmetry), $b_{ij} + b_{jk} + b_{k1} \leq 2$ (transitivity). To take the majority on each pair keeps always the first (trivial) reflexivity property. It keeps always the second one if the number of voters is odd, but it need not maintain the transitivity one, which is exactly the Condorcet effect. Note that it is no more paradoxical than the fact that the majority rule applied to a even number of complete relations can induce an incomplete relation. Formally one can consider that the objects to be aggregated are defined as p-tuples of a set X of elementary objects (or more generally as elements of a direct product ΠX_i).The set of all the possible objects is then X^p. If all these objects are admissible the component-wise mean method for aggregating them causes no problem. But if the set S of admissible objects is a subset of X^p defined by a set Φ of formulas linking the components of the objects, then to use this method requires closure properties of the aggregation operator m with respect to Φ : $x_1, ... x_n \in S$ (i.e. satisfy Φ) implies $m(x_1, ... x_n) \in S$ (i.e. satisfy Φ). If not the contradictions (the so-called "paradoxes") are unavoidable.

Another merit of Guilbaud's analysis was to show that the same logical problem of aggregation was occurring in several domains. Nevertheless it was only more than thirteen years later that it was shown that the use of an independence axiom in consensus problems arising in data analysis leads to state impossibility theorems[57] resulting from similar unavoidable contradictions.

Finally, Guilbaud mentioned as a possible solution to the general aggregation problem the metric method illustrated above in the case of

[57] The following first such result has been followed by many others: Mirkin BG (1975) On the problem of reconciling partitions In *Quantitative Sociology, International Perspectives on mathematical and Statistical Modelling*. Academic Press, New-York, pp 441-449 (in the case of partitions as in the case of partial orders the independence and unanimity axioms lead to "oligarchic" consensus functions).

aggregating linear orders: to calculate (metric) medians. In fact this method belongs to the classes of methods proposed by Maurice Fréchet in the forties. In his paper *Réhabilitation de la notion statistique de l'homme moyen*[58], quoted by Guilbaud, Fréchet presented informally how to apply some of his mathematical results on the "typical elements" (a generalization of the central values of random variables) of arbitrary random elements in an arbitrary abstract metric space[59]. It is interesting to add that as early as 1914 the median method has been also proposed in the case of subsets of (what are now called) Euclidean spaces by the well-known Italian statistician Corrado Gini and still about "l'homme moyen" [60].

4 Conclusion

As we said in our introduction most of the researches conducted at CAMS in the domain of social choice theory and more generally of consensus theories were developments of ideas found in Guilbaud's paper. But as it is usual in science these researches of CAMS members met researches led independently elsewhere or/and inspired researches led by other people (in cooperation or not with CAMS members).

Let us mention quickly some of the research directions and results. A first direction concerns preferences and social choice theory with for instance the study of the permutoèdre and of restricted domains, or the use of ultrafilters for Arrowian or Gibbard and Satterthwaite theorems. A second direction concerns axiomatic or metric consensus theories in other fields and particularly in data analysis (consensus of partitions, classification trees etc.). Transversely to these directions many researches have concerned medians. In particular the link has been made with works made completely independently since the forties in "pure" lattice or graph theories. It is now clear what are the "good" discrete metric spaces for medians, i.e. the spaces where medians can be easily computed from algebraic formulas: they are the so-called *median semilattices* which contain in particular the distributive lattices and the tree semilattices[61]. The computation of medians in the "bad" cases (like those of linear orders) lead

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[58] Les Conférences du Palais de la Découverte, Paris, 1949.

[59] Recall that the notion of distance i.e. of metric space goes back to Frechet in 1904.

[60] Gini C (1914) L'uomo medio. *Giornali degli economiste e revista de statistica* 48: 1-24. The notion of the metric median in Euclidean spaces goes back to a problem raised by Fermat in 1629 and it has been used for location problems since Weber's 1909 book *Uber den Standort der Industrien* (see Monjardet B., Éléments pour une histoire de la médiane métrique, In *Moyenne, Milieu, Centre. Histoires et usages* (1991) Coll. Histoire des Sciences et Techniques, n°5. Éditions de l'EHESS, Paris).

[61] See references in Barthélemy, Leclerc and Monjardet's 1986 paper in the Annex.

to many researches to find exact algorithms and (since the problem is NP-complete) "good" heuristics. From an axiomatic point of view the median rule has been axiomatized in many spaces[62]. On the other hand it has for instance been shown only recently that this rule satisfies the Paretian property in the metric space defined by the symmetric difference distance between partial orders on a set, although it does not satisfy it for other distances defined between partial orders[63]. More abstractly, axiomatic or metric consensus theories have been developed in lattices, semilattices or posets, since for instance these theories allow one to derive from an Arrowian theorem in such "abstract" structures several known or new Arrowian theorems in various "concrete" domains[64].

The annex below contains the papers on the above topics written by at least a CAMS member or associate member since Guilbaud's paper in 1952. They are ranked by year of publication.

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ANNEX

1952
G.Th. Guilbaud, Les théories de l'intérêt général et le problème logique de l'agrégation, Economie appliquée, 5, 501-584. (Partial) English translation: Theories of the general interest and the logical problem of aggregation. In , P.F. Lazarsfeld and N.W. Henry (eds) Readings in Mathematical Social Sciences, Science Research Association, Inc., Chicago, 1966, pp. 262-307.
1959
M. Barbut Quelques aspects mathématiques de la décision rationnelle, *Les temps modernes*, 15, 725-745. Translated as: Does the majority ever rule? *Portfolio and Art News Annual*, 4, 1961, 79-83 and 161-168.
1961
M. Barbut, Médianes, distributivité, éloignements, Note CAMS, Paris, reprinted (1980) in *Mathématiques et Sciences Humaines* 70, 5-31.
1963
G.Th. Guilbaud and P. Rosenstiehl, Analyse algébrique d'un scrutin, *Mathématiques et Sciences Humaines* 4, 9-33.
1966

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[62] See for instance McMorris FR, Mulder HM, Powers RC (2000) The median function on median graphs and semilattices. *Discrete Applied Mathematics* 101: 221-230.

[63] See Leclerc's 2002 paper in the Annex.

[64] For instance the oligarchic results mentioned in footnote 57 are special cases of a meet-projection result on a meet semilattice (see Leclerc and Monjardet's 1995 paper in the Annex).

M. Barbut, Note sur les ordres totaux à distance minimum d'une relation binaire donnée, *Mathématiques et Sciences Humaines* 17, 47-48.
1967
M. Barbut, Médianes, Condorcet et Kendall, *note SEMA*, Paris, reprinted (1980) in *Mathématiques et Sciences Humaines* 69, 5-13.
1969
M. Barbut, Une classe de demi-treillis ordonnés pouvant servir à des agrégations de critères. In *La décision, Agrégation et dynamique des ordres de préférence,* Éditions du CNRS, Paris, pp 27-35.
J. Feldman, Ordres partiels et permutoèdre, *Mathématiques et Sciences Humaines* 28, 27-38.
B. Monjardet, Remarque sur une classe de procédures de vote et les “théorèmes de possibilité”. In *La décision, Agrégation et dynamique des ordres de préférence,* Éditions du CNRS, Paris, pp 177-184.
1970
M. Barbut and B. Monjardet, *Ordre et Classification, Algèbre et Combinatoire, tomes I et II*, Hachette, Paris.
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Social choice theory and the "Centre de Mathématique Sociale": Some historical notes.

Bernard Monjardet

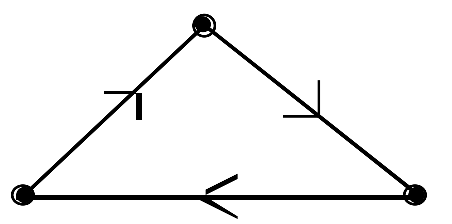

Figure 1

A 3-cycle

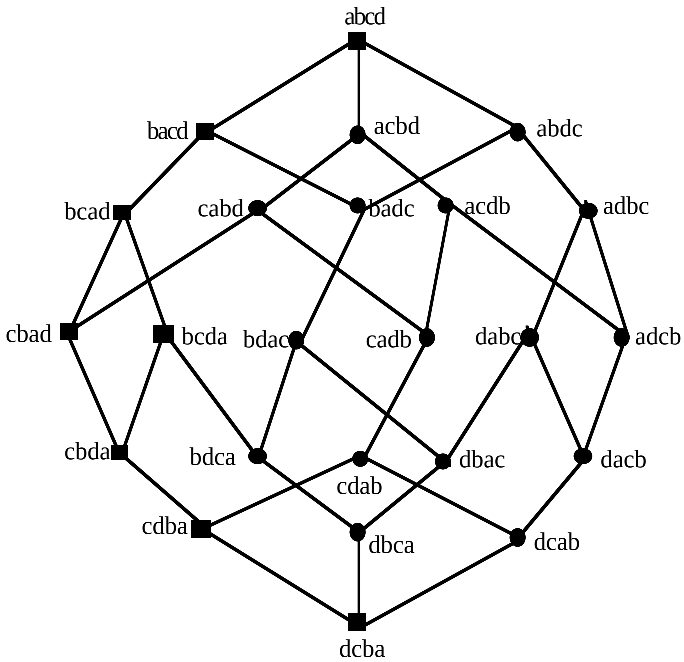

Figure 2

The permutoèdre lattice on four elements

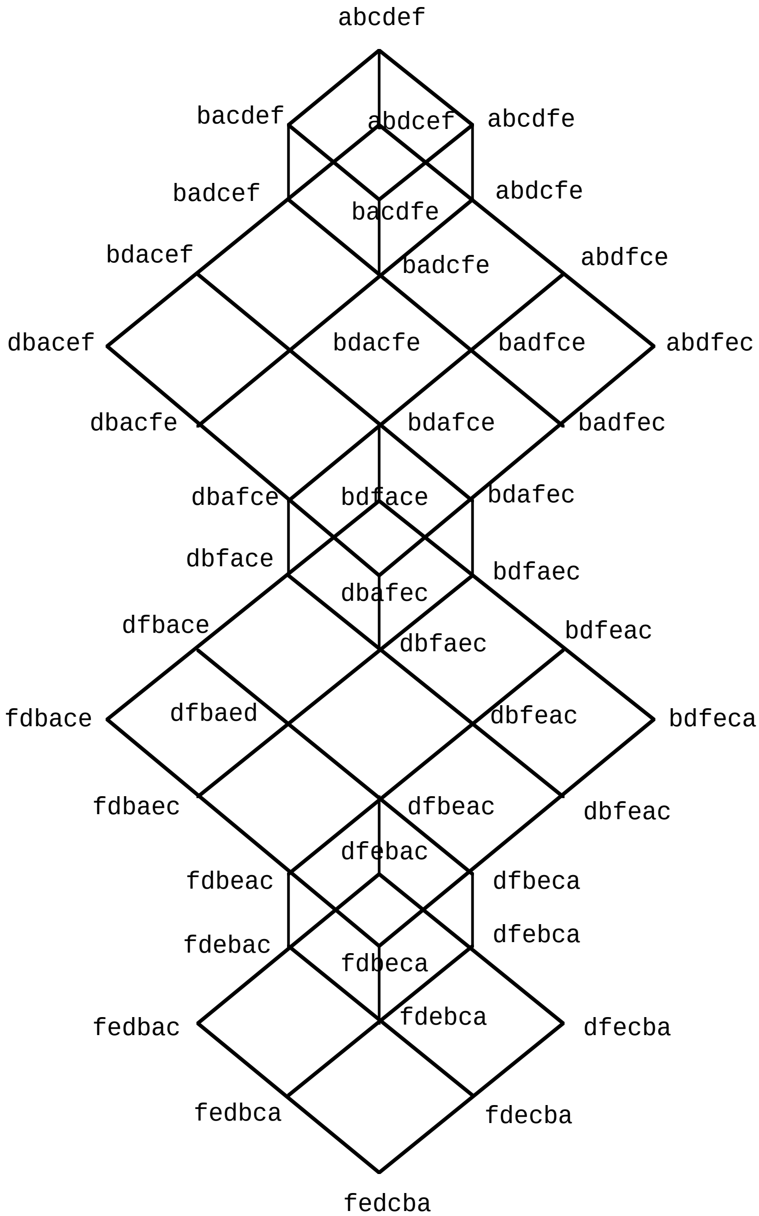

Figure 3

A Condorcet domain sublattice of the permutoèdre lattice on 6 elements.