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Pricing Bivariate Option under GARCH-GH Model with Dynamic Copula: Application for Chinese Market

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Abstract. This paper develops the method for pricing bivariate contingent claims under General Autoregressive Conditionally Heteroskedastic (GARCH) process. In order to provide a general framework being able to accommodate skewness, leptokurtosis, fat tails as well as the time varying volatility that are often found in financial data, generalized hyperbolic (GH) distribution is used for innovations. As the association between the underlying assets may vary over time, the dynamic copula approach is considered. Therefore, the proposed method proves to play an important role in pricing bivariate option. The approach is illustrated for Chinese market with one type of better-of-two-markets claims: call option on the better performer of Shanghai Stock Composite Index and Shenzhen Stock Composite Index. Results show that the option prices obtained by the GARCH-GH model with time-varying copula differ substantially from the prices implied by the GARCH-Gaussian dynamic copula model. Moreover, the empirical work displays the advantage of the suggested method.
Keywords: call-on-max option; GARCH process; generalized hyperbolic (GH) distribution; normal inverse Gaussian (NIG) distribution; copula; dynamic copula

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1 Introduction

Following the great work of Black and Scholes (1973) and Merton (1973), the option literature has been developed a lot. Over the years, various generalizations of the Brownian motion framework due to Black and Scholes (1973) have been used to model multivariate option prices. Examples include Margrabe (1978), Stulz (1982),

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Johnson (1987), Reiner (1992), and Shimko (1994). In all these papers, correlation was used to measure the dependence between assets. However, Embrechts et al. (2002) and Forbes and Rigobon (2002) have pointed out that, correlation may cause some confusion and misunderstanding. Indeed, it is a financial stylized fact that correlations observed under ordinary market differ substantially from correlations observed in hectic periods.

On the other hand, to take into account the heteroskedasticity of assets returns, a lot of models have been put forward, such as the constant-elasticity-of-variance model of Cox (1975), the jump-diffusion model in Merton (1976), the compound-option model in Geske (1979) and the displaced-diffusion model in Rubinstein (1983). Opposed to the aforementioned models, a bivariate diffusion model for pricing option on assets with stochastic volatilities was introduced by Hull and White (1987). Unfortunately, the bivariate diffusion option model requires the conditions stronger than no arbitrage and it faces the difficulty in empirical study that the variance rate is unobservable.

Through an equilibrium argument, Duan (1995) showed that options can be priced when the dynamics for the price of the underlying asset follows a GARCH process. This GARCH option pricing model has so far experimented some empirical successes in Heynen et al. (1994), Duan (1996) and Heston and Nandi (2000). In order to extend the risk neutralization developed in Rubinstein (1976) and Brennan (1979), Duan (1999) developed the GARCH option pricing model by providing a relatively easy transformation to risk-neutral distributions.

Now the distribution of the error term in GARCH process attracts a lot of attention. In Engle (1982) the normal distribution is used but alternative distributions such as the t distribution or the GED distribution have been considered to capture the excess kurtosis and fat tail. Unfortunately, as explained in Duan (1999), using t distribution to model continuously compounded asset returns is inappropriate, since the moment generating function of t distribution with any finite degree of freedom does not exist, and because of the symmetry of the GED distribution, a more flexible appropriate distribution is called for.

In Jensen and Lunde (2001), it was found that the normal inverse Gaussian (NIG) models, the special case of generalized hyperbolic (GH) distribution, are able to outperform some of the most praised GARCH models when considering daily U.S. stock return data. In particular, a big gain is found in modelling the skewness of equity returns as in Eberlein and Keller (1995) and Eberlein and Prause (2002). It is concluded that allowing conditional skewness leads to more accurate predictions of conditional variance and excess return. Moreover, GH distribution has the moment generating function, which gains an advantage over the t distribution.

As multivariate options are regarded as excellent tool for hedging the risk in today's finance, a more appropriate measure for dependence structure is required, here we concentrate on the copula. Copulas are functions that join or "couple" multivariate distribution functions to their one-dimensional marginal distribution functions, Joe (1997) and Nelsen (1999). It has been known since the work of Sklar (1959) that any multivariate continuous distribution function can be uniquely factored into its marginals and a copula. In a word, copula has proven to be an interesting tool to take into account all the dependence structure and even to capture the nonlinear dependence of data set.

Copulas have also been introduced to price bivariate options as shown in Rosenberg (1999), Cherubini and Luciano (2002). In these papers, all the appropriate preliminary copulas are supposed to remain static during the considered time period. However, most of data sets often cover a reasonably long time period and economic factors induce changes in dependence structure. Thus the basic properties of financial products change in different periods (the stable period and the crisis period). Therefore, to price the bivariate option in a robust way, a dynamic copula approach should be adopted.

In the present paper, a new dynamic approach to price the bivariate option under GARCH-GH process using time-varying copula is proposed. Through fitting two GARCH-GH models on two underlying assets, the return innovations are obtained. Observing that the dependence structure for the two series of innovations changes over time, we analyze the changes in copulas through moving windows. Then a series of copulas are selected on different subsamples according to AIC criterion (Akaike, 1974). Through this method, the changes of the copula can be observed and the change trend displays more and more clearly. Conditioning on the result of the moving window process, the dynamic copula with time-varying parameter is expressed similarly as in Dias and Embrechts (2003), Jondeau and Rockinger (2004), Granger et al. (2006), and Guégan and Zhang (2006) for instance. An innovating feature of the present paper is investigating the dynamic evolution of the copula's parameter as a time-varying function of predetermined variables, which gives a considerably dynamic expression to the changes of the copula and makes the changes of parameters more tractable.

In the empirical study, call option on the better performer based on two important Chinese equity index returns (Shanghai Stock Composite Index and Shenzhen Stock Composite Index) is used to illustrate the innovated method described previously. The Student t copula is the best fitting copula and time-varying parameter is considered. We provide the option prices implied by GARCH-NIG model with time-varying copula and these prices are compared with those obtained by GARCH-Gaussian model. It can be observed that the prices implied by the GARCH-Gaussian are generally underestimated.

The remainder of this paper is organized as follows. In Section 2, the basic framework of option pricing and the notations are introduced. Section 3 introduce the new model for pricing bivariate option based on GARCH-GH process with time-varying copula. In section 4, empirical study is described and results are provided. Section 5 concludes.

2 Preliminaries and Related Work

We specify the framework for option pricing that we choose, then we introduce the model with which we work and the innovation distribution that we use.

2.1 Option valuation

This paper concentrates on European option on the better performer of two assets, but the technique is sufficiently general to be applied for other alternative multivariate options as well. The call option on the better performer belongs to one type of better-of-two markets and can be referred to as call-on-max option. The payoff of a unit amount call-on-max option is

$$\max\{\max(S_1(T), S_2(T)) - K, 0\},$$

where S_i is the price at maturity T of the i-th asset $(i = 1, 2)$, and K is the strike price. In the following, $R_{i,t}$ is used to denote the return on i-th index $(i = 1, 2)$ from time $t-1$ to time t, and the corresponding log-return is denoted as $r_{i,t} = \log(R_{i,t})$.

The fair value of the option is determined by taking the discounted expected value of the option's payoff under the risk-neutral distribution. As the call-on-max is typically traded over the counter, price data are not available. Therefore, valuation models cannot be tested empirically. However, comparing models with different assumptions can be implemented.

The Black Scholes approach for option pricing assumes the efficiency of the financial market and all the pricing theory developed after their seminal work lies on the existence of the risk neutral measure. The measure verifies the martingale property for the theory of contingent claim pricing. Recently, some works have proposed new approaches for pricing, based on historical measure. These new works are really interesting because they are close to the reality, see for instance Barone-Adesi et al. (2004). Nevertheless, the present work keeps a historical approach for pricing options using the risk-neutral environment, providing a new strategy in the bivariate context.

2.2 Generalized Hyperbolic (GH) distributions

In order to take into account specific stylized fact of the assets (skewness and kurtosis mainly), we will work with the generalized hyperbolic (GH) distribution that we present briefly now, we refer to Eberlein and Keller (1995) for more details.

The one dimensional generalized hyperbolic distribution admits the following density function

$$f_{GH}(x; \lambda, \alpha, \beta, \delta, \mu) = \kappa(\lambda, \alpha, \beta, \delta)\tau^{(\lambda-1/2)} K_{\lambda-1/2}(\alpha\tau) \exp(\beta(x-\mu)), \tag{1}$$

where K_λ is the modified Bessel function of the third kind and

$$\kappa(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})},$$

$$\tau = \sqrt{\delta^2 + (x-\mu)^2},$$

and $x \in \mathbb{R}$.

The parameters $\lambda, \alpha, \beta, \delta, \mu \in \mathbb{R}$ are interpreted as follows: $\mu \in \mathbb{R}$ is the location parameter and $\delta > 0$ is the scale parameter. The parameter $0 \leq |\beta| < \alpha$ describes the skewness and $\alpha > 0$ gives the kurtosis. Particularly, if $\beta = 0$, the distribution is symmetric, and if $\alpha \to \infty$, the Gaussian distribution is obtained in the limit. The parameter $\lambda \in \mathbb{R}$ characterizes certain subclasses of the distribution and considerably influences the size of the probability mass contained in the tails of the distribution. If the random variable x is characterized by a generalized hyperbolic distribution, we denote it $x \sim GH(\lambda, \alpha, \beta, \delta, \mu)$.

Generally we will use in applications the parameters $\bar{\alpha} = \alpha\delta$ and $\bar{\beta} = \beta\delta$ corresponding to the scale and location invariant parameters. Then, the density function of the generalized hyperbolic distribution expressed in terms of the invariant parameters becomes:

$$f_{GH}(x; \lambda, \bar{\alpha}, \bar{\beta}, \delta, \mu) = \kappa(\lambda, \bar{\alpha}, \bar{\beta}, \delta)\chi^{(\lambda-1/2)} K_{\lambda-1/2}(\bar{\alpha}\chi) \exp(\bar{\beta}(\frac{x-\mu}{\delta})), \tag{2}$$

where

$$\kappa(\lambda, \bar{\alpha}, \bar{\beta}, \delta) = \frac{(\bar{\alpha}^2 - \bar{\beta}^2)^{\lambda/2}}{\sqrt{2\pi}\bar{\alpha}^{\lambda-1/2}\delta^{\lambda}K_{\lambda}(\sqrt{\bar{\alpha}^2 - \bar{\beta}^2})},$$

$$\chi = \sqrt{1 + (\frac{x-\mu}{\delta})^2},$$

and $x \in \mathbb{R}$. In that case, $GH(\lambda, \bar{\alpha}, \bar{\beta}, \delta, \mu)$ is a location-scale distribution family, and we have

$$x \sim GH(\lambda, \bar{\alpha}, \bar{\beta}, \delta, \mu) \Leftrightarrow \frac{x-\mu}{\delta} \sim GH(\lambda, \bar{\alpha}, \bar{\beta}, 1, 0). \tag{3}$$

In the following, we will use the relationship in Equation (3).

A special case of the GH distribution is the Normal Inverse Gaussian (NIG) distribution obtained by assuming that $\lambda = -1/2$ in Equation (1). The density function of the NIG distribution expressed in terms of the invariant parameters $\bar{\alpha} = \delta\alpha$ and $\bar{\beta} = \delta\beta$ is equal to:

$$f_{NIG}(x; \bar{\alpha}, \bar{\beta}, \delta, \mu) = \frac{\bar{\alpha}}{\pi\delta} \exp[\sqrt{\bar{\alpha}^2 - \bar{\beta}^2} + \bar{\beta}(\frac{x-\mu}{\delta})] \frac{K_1(\bar{\alpha}\sqrt{1 + (\frac{x-\mu}{\delta})^2})}{\sqrt{1 + (\frac{x-\mu}{\delta})^2}}, \tag{4}$$

where $x, \mu \in \mathbb{R}$, $\delta > 0$ and $0 < |\bar{\beta}| < \bar{\alpha}$. If the random variable x has a NIG distribution, we denote it as $x \sim NIG(\bar{\alpha}, \bar{\beta}, \delta, \mu)$. In the application, we will use this particular case of the generalized hyperbolic distribution.

2.3 GARCH process transformation

Here we are interested in pricing options, thus we need to derive the joint risk-neutral return process from the objective bivariate distribution. Instead of deriving the bivariate risk-neutral distribution directly, the proposed way is to transform each of the marginal process separately. First of all, we assume that the one-period log-return for every index, under probability measure P, follows a GARCH process, that is, for $i = 1, 2$:

$$\begin{aligned} r_{i,t} &= m_{i,t} + \sqrt{h_{i,t}}\varepsilon_{i,t}, \\ h_{i,t} &= \alpha_{i,0} + \textstyle\sum_{j=1}^{q} \alpha_{i,j}\varepsilon_{i,t-j}^2 + \sum_{j=1}^{p} \beta_{i,j}h_{i,t-j}, \\ \varepsilon_{i,t}|\varphi_{i,t-1} &\sim D(0,1) \quad \text{under measure P.} \end{aligned} \tag{5}$$

where the conditional mean $m_{i,t}$ is any predictable process, measurable with respect to the information set $\varphi_{i,t-1}$ of all information up to and including time $t-1$. Under historical measure P, $\varepsilon_{i,t}$ follows some distribution D, whose distribution function is denoted as F_D, with zero mean and variance 1. Other restrictions are $p \geq 0, q \geq 0$; $\alpha_{i,0} > 0$; $\alpha_{i,j} \geq 0 (j = 1, \ldots, q)$; $\beta_j \geq 0 (j = 1, \ldots, p)$. To ensure covariance stationarity of the GARCH (p, q) process, $\sum_{j=1}^{q} \alpha_{i,j} + \sum_{j=1}^{p} \beta_j$ is assumed to be less than 1.

In order to develop the GARCH option pricing model and finally obtain the risk-neutral price, Duan (1999) has generalized the conventional risk-neutral valuation relationship to accommodate heteroskedasticity of the asset return process with non Gaussian innovations by a probability measure Q being risk-neutral in some sense. The generalized principle is described below.

Assumption 1 *The equilibrium pricing measure Q, defined over the interval $[t_l, t_u]$ is said to satisfy the generalized locally risk-neutral valuation relationship if, for $\forall t$ such that $t_l \leq t \leq t_u - 1$, the following conditions are all satisfied:*

1. *the measure Q is mutually absolutely continuous with respect to the objective measure P;*
2. *there exists a predictable process $\lambda_{i,t}$ such that $\Phi^{-1}[F_D(\varepsilon_{i,t})] + \lambda_{i,t}$, conditionally on $\varphi_{i,t-1}$, is a standard normal random variable with respect to the measure Q;*
3. *$E^Q(R_{i,t}|\varphi_{i,t-1}) = \exp(r_t)$,*

where r_t denotes the one period risk free interest rate at time t, and $\Phi(\cdot)$ denotes the standard normal distribution function.

Assuming that the Assumption 1 holds, the asset return process which follows a GARCH model under measure P can be characterized by a simple risk-neutral dynamic GARCH model described in the following theorem:

Theorem 1. *Under the pricing measure Q defined by Assumption 1, the one-period log-return $r_{i,t}$, $i = 1, 2$, follows the model:*

$$\begin{gathered} r_{i,t} = m_{i,t} + \sqrt{h_{i,t}}\varepsilon_{i,t}, \\ h_{i,t} = \alpha_{i,0} + \sum_{j=1}^{q} \alpha_{i,j}\varepsilon_{i,t-j}^2 + \sum_{j=1}^{p} \beta_{i,j}h_{i,t-j}, \\ \varepsilon_{i,t} = F_D^{-1}[\Phi(Z_{i,t} - \lambda_{i,t})], \end{gathered} \tag{6}$$

where $Z_{i,t}$, conditional on $\varphi_{i,t-1}$, is a Q-standard normal random variable. Moreover, $\lambda_{i,s}$ is the solution to

$$E^Q[\exp(m_{i,s} + \sqrt{h_{i,s}}F_D^{-1}[\Phi(Z_{i,s} - \lambda_{i,s})])|\varphi_{i,s-1}] = \exp(r_s). \tag{7}$$

It can be easily seen that when the distribution of the innovations is a Normal distribution, the innovations $\varepsilon_{i,t}$ become $Z_{i,t} - \lambda_{i,t}$, that means, the innovations just translate a quantity $\lambda_{i,t}$.

Theorem 1 implies that the log-return $r_{i,t}$ follows a process close to a GARCH (p, q) under the risk-neutral measure. It provides a relatively easy transformation to generalize local risk-neutral distributions that is skewed and leptokurtic. According to this theorem, the terminal asset price is derived in the following corollary with the same notation:

Corollary 1. *Under the Assumption 1, the terminal price $S_{i,T}$ for the i-th $(i = 1, 2)$ asset is equal to:*

$$S_{i,T} = S_{i,t}\exp\{\sum_{s=t+1}^{T}[m_{i,s} + \sqrt{h_{i,s}}F_D^{-1}[\Phi(Z_{i,s} - \lambda_{i,s})]]\}, \tag{8}$$

where $h_{i,s}$, $Z_{i,s}$ and $\lambda_{i,s}$ are given in Equation (6).

Considering the importance of the martingale property for the theory of contingent claim pricing, it is necessary to note that the discount asset price process $e^{-r_t t}S_{i,T}$ is a Q-martingale. Therefore, under the GARCH specification, the call-on-max option, with exercise price K at maturity T, has the time-t value given by

$$COM_t = e^{-\sum_{s=t+1}^{T} r_s}E^Q[\max\{\max(S_{1,T}, S_{2,T}) - K, 0\}]. \tag{9}$$

This equation provides the faire value for call-on-max option. Now we are interested to get the multivariate distribution for this bivariate option.

2.4 An interesting tool: copula

In the present paper, the dependence structure of the log-return for the underlying assets is based on copula theory. Some notations are specified now.

Let $\mathbf{X} = (\mathbf{X}_n)_{n\in\mathbb{Z}} = \{(X_{i1}, X_{i2}, \ldots, X_{id}) : i = 1, 2, \ldots, n\}$ be a d-dimension random sample of n multivariate observations from the unknown multivariate distribution function $F(x_1, x_2, \ldots, x_d)$ with continuous marginal distributions $F_1, F_2, \ldots, F_d$. The characterization theorem of Sklar (1959) implies that there exists a unique copula C_θ such that

$$F(x_1, x_2, \ldots, x_d) = C_\theta(F_1(x_1), F_2(x_2), \cdots, F_d(x_d))$$

for all $x_1, x_2, \ldots, x_d \in \mathbb{R}$. Conversely, for any marginal distributions $F_1, F_2, \ldots, F_d$ and any copula function C_θ, it is said that the function $C_\theta(F_1(x_1), F_2(x_2), \ldots, F_d(x_d))$ is a multivariate distribution function with given marginal distributions $F_1, F_2, \ldots, F_d$. This theorem provides the theoretical foundation for the widespread use of the copula approach in generating multivariate distributions from univariate distributions, Joe (1997) and Nelsen (1999).

In order to adjust a copula C_θ on a set of process, we will use maximum likelihood method and AIC criterion (Akaike, 1974). This means that we will adjust the copula on the innovations of the returns, to assure independence, without losing the dependence specification. We specify now the approach that we follow here.

3 Methodology: option pricing under GARCH-GH process with dynamic copula

In the proposed scheme for valuating the bivariate option, the objective bivariate distribution of the log-returns $(r_{1,t}, r_{2,t})$ is specified conditionally on $\varphi_{t-1} = \sigma((r_{1,s}, r_{2,s}) : s \leq t-1)$, the information set of all information up to and including time $t-1$. In order to derive the joint risk-neutral log-return process from this objective bivariate conditional distribution in a convenient transformation way, it is proposed to transform each of the marginal process and the copula instead of deriving the bivariate risk-neutral distribution directly.

The objective marginals are specified by the model with GARCH-GH process introduced as in Equation (5) with the consideration that the distribution D is a GH distribution.

The system above is in principle self-contained. However, problems may occur when it comes to actually implementing. In particular, the requirement that $\lambda_{i,s}$ is the solution to Equation (7) may be extremely difficult to deal with. It is noted that there exists an one to one correspondence between $\lambda_{i,s}$ and the mean specification $m_{i,s}$. In particular, if $m_{i,s}$ is assumed to be measurable with respect to the information set $\varphi_{i,t-1}$, Equation (7) may be rewritten as

$$m_{i,s} = r_s - \ln E^Q[\exp(\sqrt{h_{i,s}} F_D^{-1}[\Phi(Z_{i,s} - \lambda_{i,s})])|\varphi_{i,s-1}]. \tag{10}$$

Therefore, Equation (8) is displayed as

$$\begin{aligned} S_{i,T} = S_{i,t} \exp\{ & \textstyle\sum_{s=t+1}^{T} [r_s - \ln E^Q[\exp(\sqrt{h_{i,s}} F_D^{-1}[\Phi(Z_{i,s} - \lambda_{i,s})])|\varphi_{i,s-1}] \\ & + \sqrt{h_{i,s}} F_D^{-1}[\Phi(Z_{i,s} - \lambda_{i,s})]]\}. \end{aligned} \tag{11}$$

Moreover, if $\lambda_{i,s}$ is assumed to be constant as λ_i, the Equation (6) simplifies to the case where λ_i is the unit risk premium. Therefore for technical reason, we will assume that $\lambda_{i,s} = \lambda_i$ for all s in the empirical study, generalization will be discussed in another paper.

Furthermore, the objective copula and risk-neutral copula are assumed to be the same:

Proposition 1. *The objective copula describing the dependence between the two assets' log-returns in Equation (5) under the objective background and the risk-neutral copula for the dependence of the log-returns in Equation (6) under the risk-neutral environment are the same, if $\lambda_{i,s} = \lambda_i$ for all s.*

Under Proposition 1, the objective joint log-return process can be transformed easily into its risk-neutral counterpart. In a word, the transformation from Equation (5) to Equation (6), in conjunction with the assumption that the objective and local risk-neutral conditional copulas remain the same, allows a particularly convenient pricing method for bivariate options.

After specifying the objective marginals according to Equation (5), the joint distribution of the indexes will be described by fixing the conditional copula. With the aim of choosing the best fitting copula, AIC criterion is used.

Since most of data often cover a reasonably long time period, the economic factors induce some changes in the dependence structure. Therefore, a dynamic copula approach is adopted. After determining the change type of the copula as introduced in Dias and Embrechts (2003), Guégan and Zhang (2006) and Guégan and Cyril (2007), the corresponding dynamic copula approach is applied. Here we are mainly interested in the case that copula parameters change with static copula family.

Considering firstly the period as a whole, one copula is chosen to best fit the log-return innovations after the GARCH filter defined in Equation (5). As the dependence structure of the underlying assets is treated dynamically, with moving window, the best copulas are chosen according to AIC criterion. Assuming that the results of the series of best copulas on subsamples show that the copula family remains changeless while the copula parameters change, the innovating method is to define a time-varying parameter function permitting to take into account kinds of correlations inside the parameters.

Thus with the standardized innovations $(\varepsilon_{1,t}, \varepsilon_{2,t})$ of the log-return GARCH model, the dynamic copula C is assumed to have the time dependent parameter vector $\theta_t = (\theta_{1,t}, \theta_{1,t}, \ldots, \theta_{m,t})$, such that

$$\theta_{l,t} = \theta_0 + \sum_{i=1}^{g} \eta_i \prod_{j=1}^{2} \varepsilon_{j,t-1} + \sum_{k=1}^{s} \zeta_k \theta_{l,t-k} \tag{12}$$

for $l = 1, 2, \ldots, m$ and $\eta_i (i = 1, 2, \ldots, g)$, $\zeta_k (k = 1, 2, \ldots, s)$ are scalar model parameters. Equation (12) defines a dynamic structure motivated by GARCH process for the dependence parameters. Other specification can be considered and this will be done in another paper.

To estimate the parameters in Equation (12), the maximum likelihood method is needed. Recalling that the standardized innovations are assumed to be distributed conditionally as the generalized hyperbolic distribution (GH), the bivariate conditional distribution function is such that

$$F(\varepsilon_{1,t}, \varepsilon_{2,t}; \theta_t) = C(GH_1(\varepsilon_{1,t}), GH_2(\varepsilon_{2,t}); \theta_t),$$

where C is the copula function, GH_i $(i = 1, 2)$ is the GH distribution function.

The corresponding conditional density function is then

$$f(\varepsilon_{1,t}, \varepsilon_{2,t}; \theta_t) = c(GH_1(\varepsilon_{1,t}), GH_2(\varepsilon_{2,t}); \theta_t) \prod_{i=1}^{2} gh_i(\varepsilon_{i,t}),$$

where the copula density c is given by

$$c(u_1, u_2; \theta) = \frac{\partial^2 C(u_1, u_2; \theta)}{\partial u_1 \partial u_2},$$

with $(u_1, u_2) \in [0, 1]^2$ and gh_i $i = 1, 2$ represents the generalized hyperbolic distribution density.

The conditional log-likelihood function can be finally evaluated as

$$\sum_{t=b+1}^{n} (\log c(GH_1(\varepsilon_{1,t}), GH_2(\varepsilon_{2,t}); \theta_t) + \sum_{i=1}^{2} \log gh_i(\varepsilon_{i,t})) \tag{13}$$

where $b = \max(p, r)$.

Numerical maximization of Equation (13) gives the maximum likelihood estimates of the model. However, the optimization of the likelihood function with several parameters is numerically difficult and time consuming. It is more tractable to estimate firstly the marginal model parameters and then the dependence model parameters using the estimates from the first step. In order to do so, the two marginal likelihood functions

$$\sum_{t=p+1}^{n} \log gh_i(\varepsilon_{i,t}) \quad \text{for} \quad i = 1, 2, \ldots, d,$$

are independently maximized. From here the marginal parameters estimates are obtained and then are plugged in Equation (13). So the final function to maximize becomes

$$\sum_{t=b+1}^{n} (\log c(GH_1(\varepsilon_{1,t}), GH_2(\varepsilon_{2,t}); \theta_t)). \tag{14}$$

From this dependence estimates, $\hat{\theta}_t$ are obtained and the model is fitted.

Specifically, for one-parameter copulas, the time-varying parameter function can be presented directly for this alone parameter; but for multi-parameter copulas, the complexity of estimating parameters results in the choice of the one most important parameter, letting the other static.

Benefiting from the identification assumption for the objective and local risk-neutral conditional copulas, pairs of standard normal random variables $Z_{i,t}$ $(i = 1, 2)$ in the transformed GARCH-GH model in Equation (6) can be drawn from the dynamic copula acting as the estimated conditional risk-neutral measure of association. This procedure is accomplished with the aid of Monte Carlo simulations. These generated random variables are then applied to obtain the transformed innovations as shown in Equation (6). Eventually, according to Corollary 1, the payoffs implied by these innovations are averaged and discounted at the risk-free rate, and the fair value of the call-on-max option can be expressed as in Equation (11).

4 Empirical work

4.1 Models for each data set

For the empirical work, the valuation scheme for the bivariate option under GARCH-GH model with dynamic copula outlined in Section 3 is applied to call-on-max option on the Shanghai Stock Composite Index and the Shenzhen Stock Composite Index. The sample contains 1857 daily observations from 4 January 2000 to 29 May 2007. The log-returns of Shanghai Stock Composite Index and Shenzhen Stock Composite Index are shown in Figure 1, it is noted that the outliers typically occur simultaneously and almost in the same direction.

In this empirical work, we restrict the GH distribution to the NIG distribution that is more tractable and has a lot of nice features such as it is closed under convolution. The NIG fitting results are shown in Figure 2 and Table 1. The fitted NIG distributions are asymmetric. But simulation provides skewness parameter $\bar{\beta}$ close to 0 and location parameter μ nearly equal to 0, thus in order to make the GARCH-NIG fitting more tractable, an assigned symmetric NIG distribution with 0 location is refitted and the results are shown in Figure 3, Figure 4 and Table 2.

The parameter estimates for the GARCH (1,1) with symmetric NIG innovation models (see Equation (5)) for the underlying assets log-returns are listed in Table 3, and in order to compare, the results for GARCH-Gaussian model are also provided. From the AIC and BIC values of the two types of model, GARCH-NIG models appear better for both Shanghai Stock Composite Index and Shenzhen Stock Composite Index.

4.2 Dynamic copula method

Here, we consider the bivariate vector composed with the two assets. Several kinds of copulas are considered to describe the dependence structure between these assets on the whole period, including Gaussian, Frank, Gumbel, Clayton, Student t copulas (Joe, 1997). All the copulas mentioned above are fitted to the support set of the standardized innovation pairs from GARCH-NIG and GARCH-Gaussian models respectively. The fitting results are listed in Table 4. AIC criterion is used to choose the best fitting copula. From the models fitted to the standardized innovations for Shanghai and Shenzhen stock composite indexes, the one which has the smallest AIC value is the Student t copula both for GARCH-NIG and GARCH-Gaussian models. Therefore, Student t copula is considered as the best fitting copula in case of static dependence for both models.

Using moving window allows to observe the change trend in a direct way, and makes the dynamics specification more reasonable corresponding to the real setting. Therefore, the whole sample is divided into subsamples separated by the moving window. 16 windows in which each consists of 300 observations are moved by 100 observations. Along with the moving of the window, series of best fitting copulas on different subsamples are decided by AIC criterion. The results for the best fitting copulas on all subsamples for GARCH-NIG and GARCH-Gaussian model are shown in Table 5. Results listed in Table 5 show that on almost all subsamples, Student t copula turns out to be the best fitting copula for the GARCH-NIG model. So it is rather reasonable to assume that for the GARCH-NIG model, the copula family remains static as Student t, while the parameter changes along the time. As far as the GARCH-Gaussian model is concerned, the copula changes a lot. For the 2^{nd}, 3^{rd} and 5^{th} windows, although the Gaussian copula seems as the best fitting, the Student t copula offers the very close AIC value (with the difference not bigger than 2). And for the 7^{th}, 8^{th}, 9^{th}, 10^{th}, 11^{th} windows, the Frank copula provides the best fitting, and the Student t copula is the secondly best fitting. Thus we still assume that the copula family is static as the Student t but the parameters vary. In addition, it can be observed that the correlation does not change a lot for both GARCH-NIG and GARCH-Gaussian models while the degree of freedom varies obviously for both of the two models. Therefore, it seems reasonable to assume that the degree of freedom varies along time while the correlation remains static.

The time-varying function for the degree of freedom of the Student t copula is put forward as:

$$\nu_t = l^{-1}(s_0 + s_1\varepsilon_{1,t-1}\varepsilon_{2,t-1} + s_1 l(\nu_{t-1})), \tag{15}$$

where s_0, s_1, s_2 are real parameters and $l(\cdot)$ is a function defined by

$$l(\nu) = \log(\frac{1}{\nu - 2}),$$

to ensure that the degree of freedom is not smaller than 2.

The corresponding estimate results for the dynamic copula parameter described in Equation (15) are listed in Table 6.

4.3 Pricing bivariate option

Standard normal random variables can then be generated from this conditional Student t copula with time-varying parameter, and according to two NIG margin distributions, log-return innovations can be sampled to compute the price of the option. Considering that the initial asset prices need to be close for the option to make sense, it is assumed here that they are normalized to unity. The Monte Carlo study is based on 100,000 replications, resulting in simulation errors in the order of magnitude of 1 basis point for 1 month maturity claims. Different maturities can be considered, and 1 month (20 trading days) are displayed here just devoting itself to illustrating the approach. Moreover, the strike price is set at levels between 0.5 and 2.7. The risk-free rate is assumed to be 6% per annum. And λ_i is considered as 5%, thus $\lambda_1 = \lambda_2$. Using the proposed dynamic copula method with time-varying parameter, the option prices are represented in Figure 5. Compared with the option prices implied by the GARCH-Gaussian dynamic model in Figure 6, it can be observed that the GARCH-Gaussian model generally underestimates the price.

5 Conclusion

In this paper, a systematic new approach for bivariate option pricing under GARCH-GH model with dynamic copula has been introduced. The introduction of GARCH-GH model on each asset permits to take into account most of the stylized facts observed on the data set. The risk neutral model permits to get an analytical expression for the fair value of the call-on-max option. The bivariate option pricing approach lies on Assumption 1 which permits to use the same copula under historical and risk neutral measures. This work can be extended almost in two ways. The first one concerns an extension of the pricing modelling using BL-GARCH (Storti and Vitale, 2003). Indeed this class of models permits to take into account explosion and clusters as stylized facts. The second one concerns the weakness of Assumption 1.

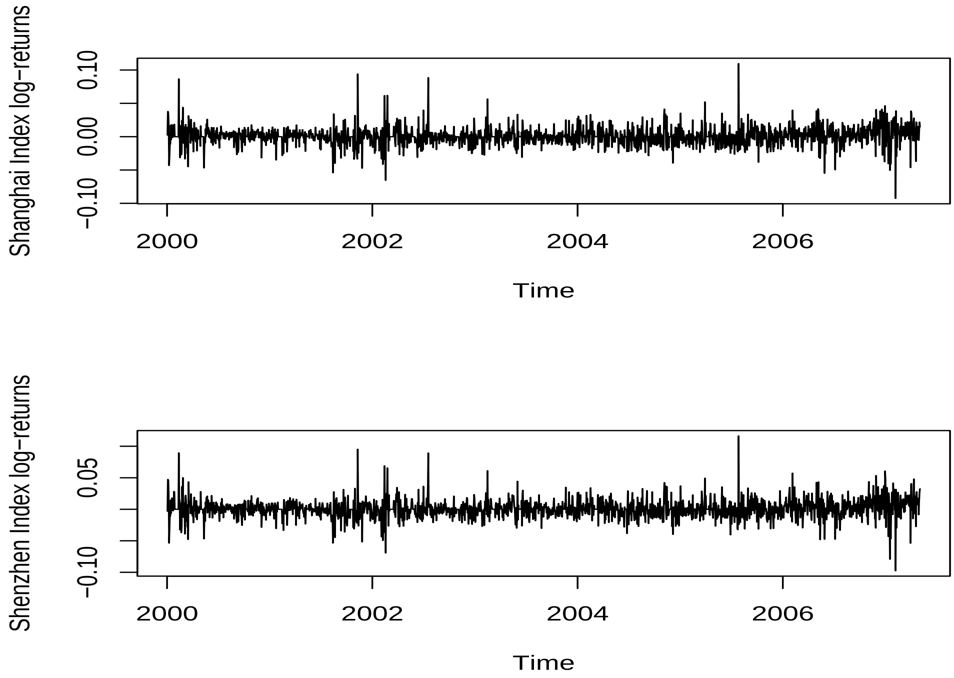

Fig. 1. Log-returns for Shanghai Stock Composite Index and Shenzhen Stock Composite Index from 4 January 2000 to 29 May 2007

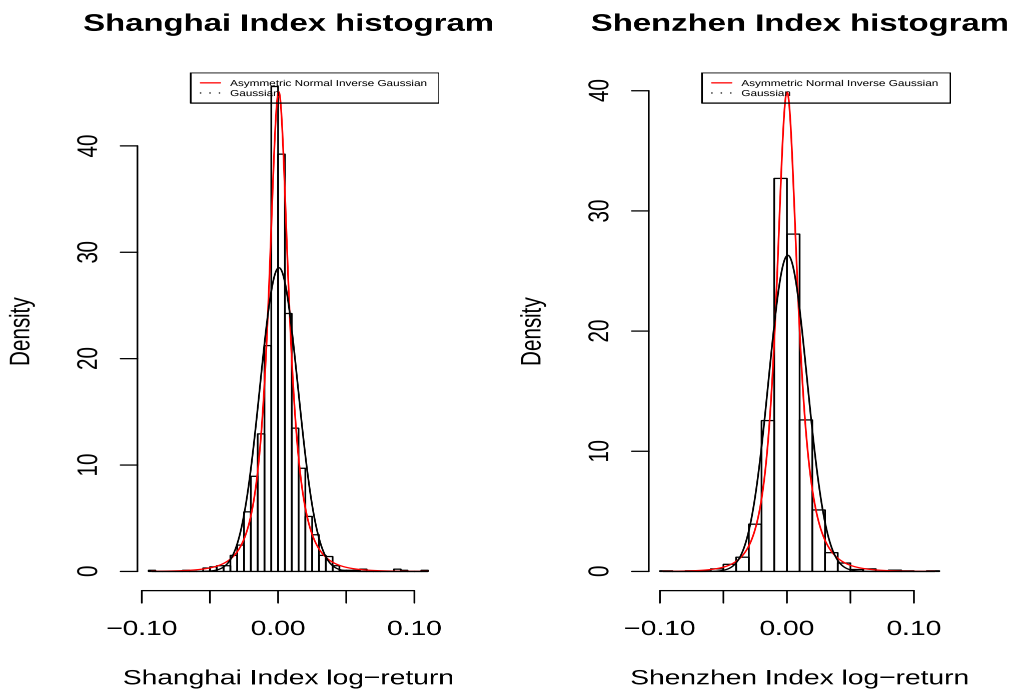

Fig. 2. Asymmetric NIG fitting for log-returns of Shanghai Stock Composite Index and Shenzhen Stock Composite Index

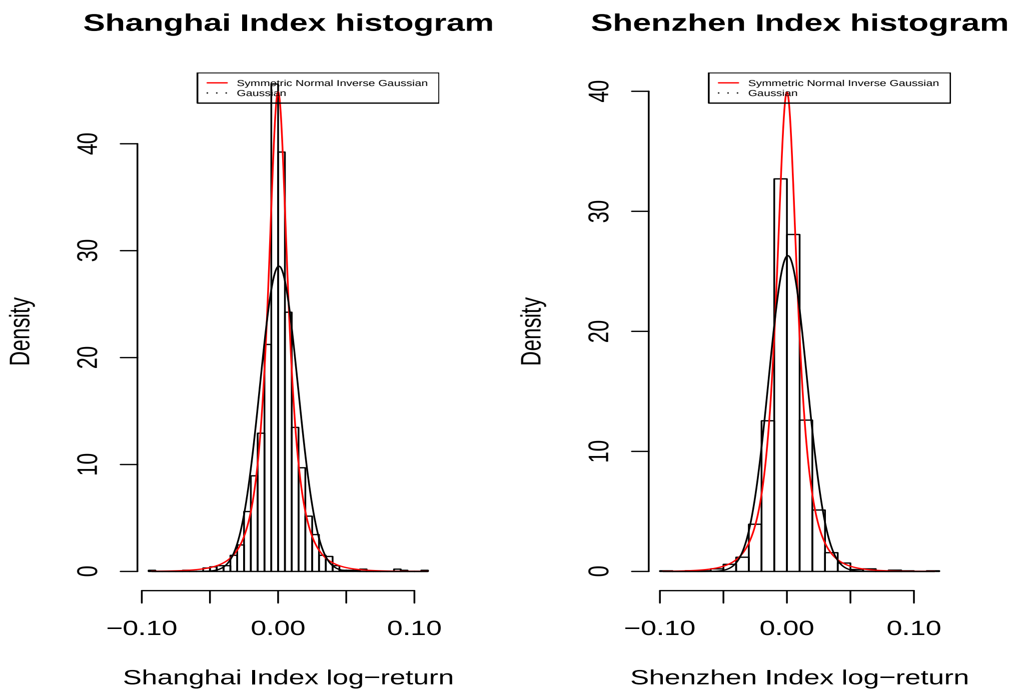

Fig. 3. Symmetric NIG fitting for log-returns of Shanghai Stock Composite Index and Shenzhen Stock Composite Index

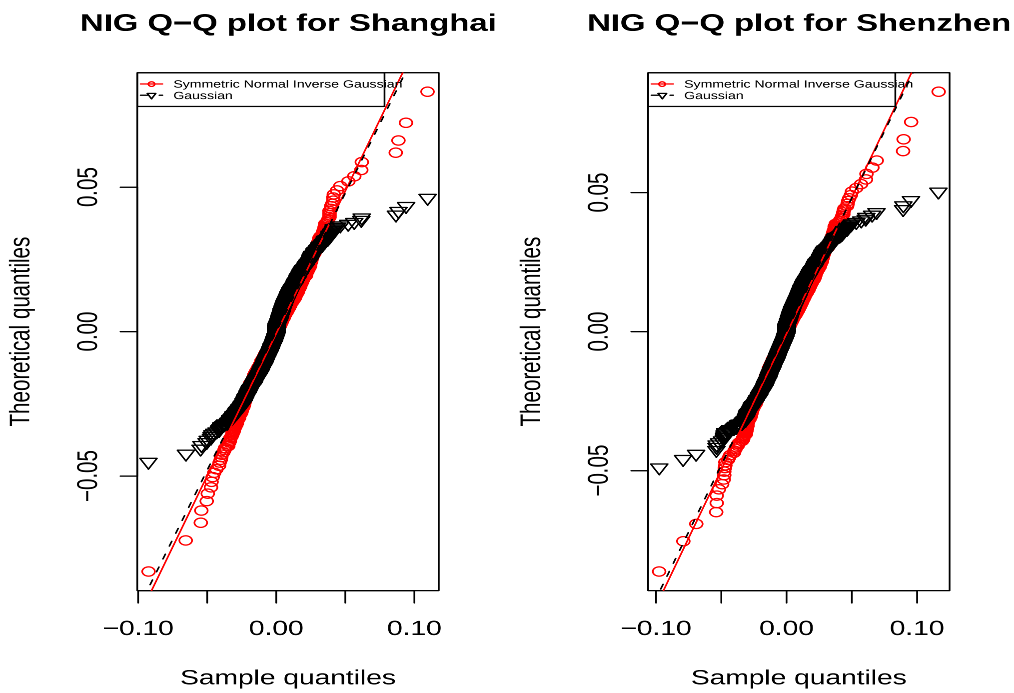

Fig. 4. Q-Q plots of symmetric NIG fitting for log-returns of Shanghai Stock Composite Index and Shenzhen Stock Composite Index

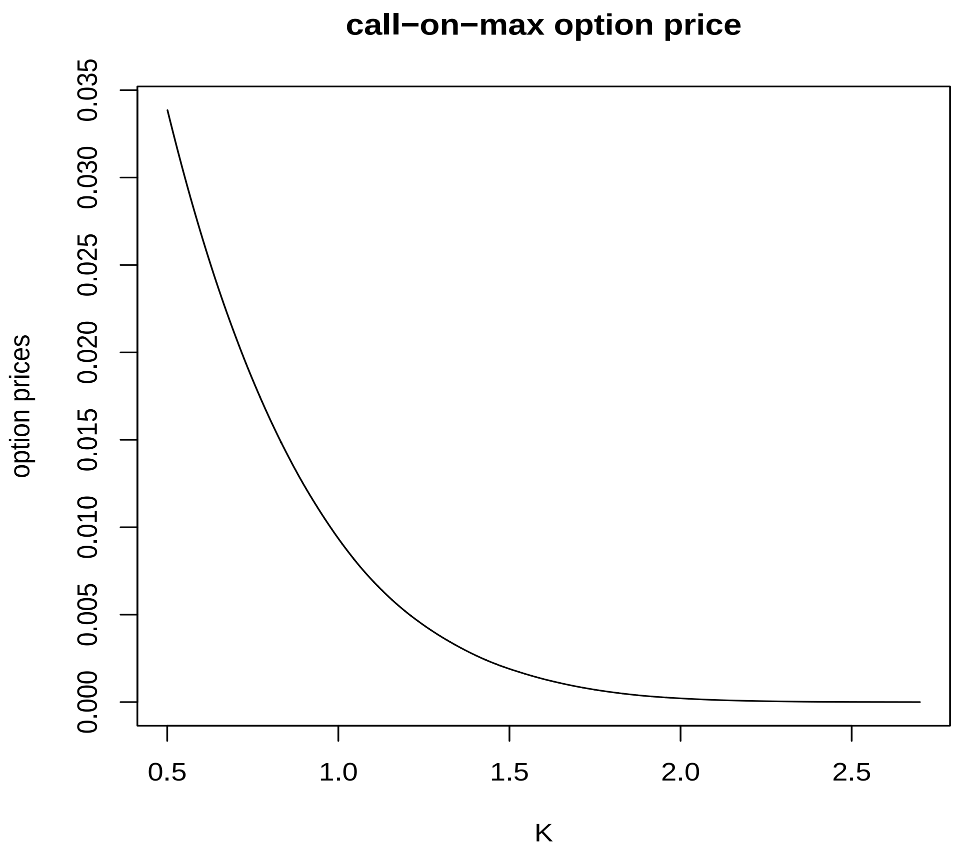

Fig. 5. 1 month maturity call-on-max option prices as a function of the strike using the method of dynamic Student t copula with time-varying parameter

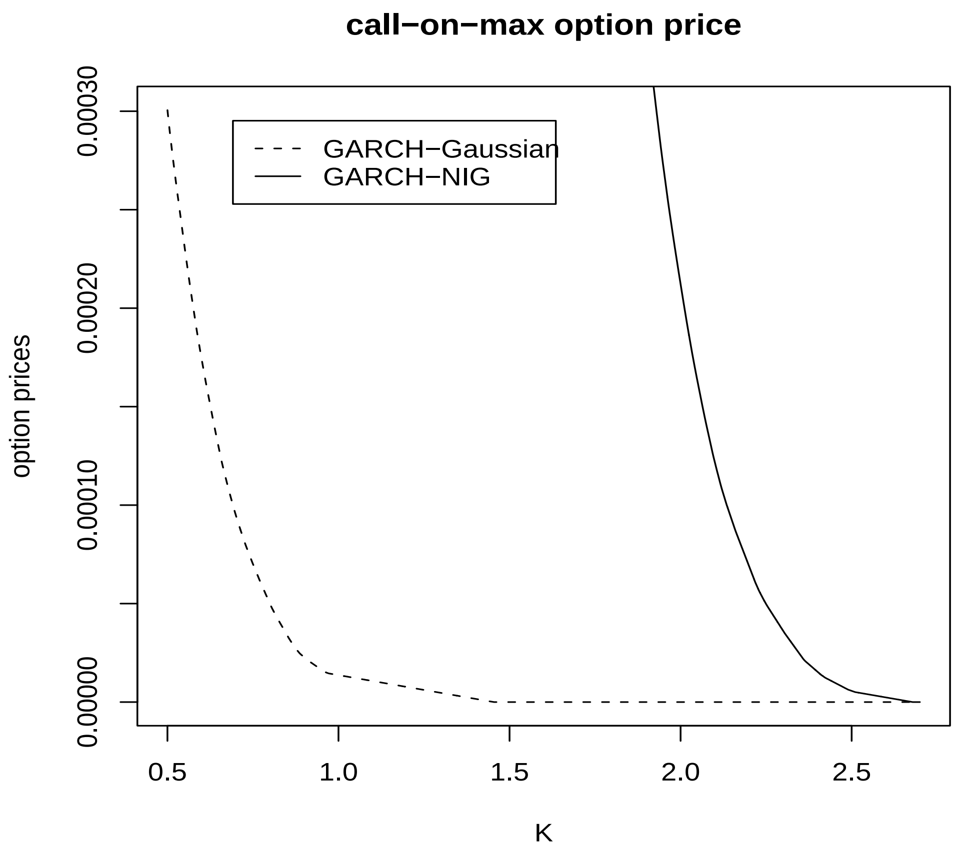

Fig. 6. 1 month maturity call-on-max option prices as a function of the strike from GARCH-NIG and GARCH-Gaussian models with dynamic copula

Table 1. Estimates of asymmetric NIG fitting parameters for marginal log-returns

Parameter	Shanghai Index		Shenzhen Index	
$\bar{\alpha}$	4.460e-01	(4.407e-03)	5.364e-01	(6.396e-03)
$\bar{\beta}$	1.607e-04	(2.256e-07)	8.440e-04	(2.981e-07)
μ	4.428e-04	(1.160e-07)	-1.171e-04	(1.719e-07)
σ	1.409e-02	(2.067e-07)	1.510e-02	(2.116e-07)
AIC	-10972.56		-10649.73	
BIC	-10950.46		-10627.62	

$\sigma = \delta\bar{\alpha}/\sqrt{\bar{\alpha}^2 - \bar{\beta}^2}$ is reparameterized as a dispersion parameter that can be seen as the volatility. Figures in brackets are standard errors.

Table 2. Estimates of symmetric NIG fitting parameters for marginal log-returns

marginal	Shanghai Index		Shenzhen Index	
$\bar{\alpha}$	4.536e-01	(4.539e-03)	5.275e-01	(6.117e-03)
$\bar{\beta}$	0.000		0.000	
μ	0.000		0.000	
σ	1.409e-02	(2.044e-07)	1.516e-02	(2.146e-07)
AIC	-10971.40		-10649.37	
BIC	-10960.34		-10638.32	

Figures in brackets are standard errors.

Table 3. Estimates of GARCH-NIG and GARCH-Gaussian parameters for marginal log-returns

GARCH-NIG		
	Shanghai Index	Shenzhen Index
m	6.065e-04 (1.952e-04)	7.260e-04 (2.300e-04)
$\bar{\alpha}$	8.103e-01 (2.807e-03)	7.959e-01 (5.837e-03)
α_0	3.597e-05 (2.032e-01)	3.532e-05 (1.989e-01)
α_1	2.758e-01 (3.865e-01)	3.015e-01 (5.477e-01)
β_1	5.651e-01 (2.988e-01)	5.558e-01 (7.747e-01)
AIC	-11037.14	-10708.29
BIC	-11009.51	-10680.66
GARCH-Gaussian		
	Shanghai Index	Shenzhen Index
m	3.833e-04 (2.419e-04)	3.761e-04 (2.882e-04)
α_0	5.136e-06 (7.682e-07)	5.529e-06 (9.011e-07)
α_1	8.115e-02 (4.726e-03)	8.721e-02 (5.496e-03)
β_1	8.966e-01 (5.034e-03)	8.950e-01 (5.249e-03)
AIC	-10793.11	-10518.3
BIC	-10771	-10496.2

Figures in brackets are standard errors.

Table 4. Copula Fitting Results

	GARCH-NIG	
Copula	Parameter	AIC value
Gaussian	9.191e-01 (4.205e-02)	-3517.934
Gumbel	3.732 (7.285e-02)	-3460.264
Clayton	3.905 (1.051e-01)	-2915.304
Frank	14.070 (3.242e-01)	-3255.558
Student t	9.221e-01 (3.914e-02); 3.675 (2.119)	-3683.532
	GARCH-Gaussian	
Copula	Parameter	AIC value
Gaussian	9.314e-01 (4.402e-02)	-3757.412
Gumbel	3.971 (7.845e-02)	-3528.21
Clayton	4.081 (1.114e-01)	-2728.87
Frank	16.593 (3.611e-01)	-3591.436
Student t	9.349e-01 (5.095e-02); 5.807 (1.601)	-3797.926

Figures in brackets are standard errors and for Student t copula, the first parameter is the correlation, the second parameter is the degree of freedom.

Table 5. Dynamic Copula Analysis using Moving Window

	GARCH-NIG		GARCH-Gaussian	
i^{th}	Co	Parameter	Co	Parameter
1	t	9.109e-1(1.032e-1); 2.444(1.302)	Ga	9.308e-1(1.034e-1)
2	t	9.064e-1(9.990e-2); 3.084(9.590e-1)	Ga	9.247e-1(1.036e-1)
3	t	9.308e-1(9.505e-2); 5.594(9.384e-1)	Ga	9.381e-1(1.103e-1)
4	t	9.451e-1(1.157e-1) 3.903(1.017)	t	9.539e-1(6.831e-2) 14.784(2.675)
5	t	9.602e-1(2.804e-1); 7.919(4.215)	Ga	9.646e-1(1.437e-1)
6	t	9.697e-1(1.142e-1) 6.826(3.352)	t	9.730e-1(1.164e-1) 15.262(5.337)
7	t	9.654e-1(1.442e-1); 8.098(3.385)	Fr	25.157(1.293)
8	t	9.598e-1(9.931e-2); 6.005(2.104)	Fr	22.866(1.193)
9	t	9.444e-1(2.117e-1); 7.087(3.630)	Fr	18.971(1.003)
10	t	9.385e-1(1.594e-1); 7.675(2.000)	Fr	18.115(9.655e-1)
11	t	9.419e-1(1.612e-1); 9.947(1.352)	Fr	18.914(1.000)
12	Gu	4.450(2.167e-1)	Gu	4.800(2.347e-1)
13	t	9.228e-1(1.533e-1); 5.682(2.388)	Ga	9.371e-1(1.143e-1)
14	t	8.831e-1(2.594e-1); 3.574(10.030)	Ga	9.062e-1(9.686e-2)
15	t	8.727e-1(7.247e-2) 3.300(8.206)	t	9.009e-1(1.092e-1) 5.012(3.371e-1)
16	t	8.493e-1(1.030e-1) 4.937(2.129)	t	8.765e-1(1.043e-1) 10.513(1.601)

Figures in brackets are standard errors. "Co" represents "Copula type", the short notes "t", "Gu", "Ga" and "Fr" represent respectively "Student t", "Gumbel", "Gaussian" and "Frank" copulas. And for the Student t copula, the first parameter is the correlation, the second parameter is the degree of freedom.

Table 6. Parameter estimates for dynamic Student t copula with time-varying parameter

	GARCH-NIG	GARCH-Gaussian
p	9.176e-01 (2.361e-02)	9.267e-01 (2.483e-02)
s_0	4.384e-01 (1.497)	1.065 (5.452e-03)
s_1	-6.055e-02 (7.407e-01)	1.676e-01 (1.190e-03)
s_2	-9.414e-01 (4.165e-01)	-6.971e-01 (3.289e-02)

Figures in brackets are standard errors and p represent the correlation estimate.

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Appendix 1. Proof of Proposition 1

Proof. It should be noted firstly that as presented in Duan (1999), $\Phi^{-1}[F_D(\varepsilon_{i,t})]$ is a standard normal random variable conditional on $\varphi_{i,t-1}$ with respect to measure P. In fact, from the second condition in Assumption 1, the Q-standard normal variable $Z_{i,t}$ $(i = 1, 2)$ in Equation (6) can be represented as $\Phi^{-1}[F_D(\varepsilon_{i,t})] + \lambda_{i,t}$. According to the one-to-one relationship of the innovation and the standard normal random variable described in Duan (1999), this means that the risk-neutralization has an invariance property indicating that the nature of the distribution for the transformed innovation remains unchanged. In fact, the risk neutralization merely causes the transformed innovation to undergo a shift in mean with the magnitude of λ_i determined by the third condition in Assumption 1. So the dependence structure between the two underlying assets remains unchanged after the transformation of the two marginals, that is, the objective copula is the same as the risk-neutral one. □