# Predicate of existence and predictability for a theoretical object in physics

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# Predicate of existence and predictability for a theoretical object in physics *

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*Summary*

We consider the question of the predicate of existence for a physical object from both points of view of the evolution of theories and concepts and of the theoretical structure, considering that they are not mutually exclusive but that, taken together, they can bring to light evidences that would remain otherwise unnoticed. A first aspect of our inquiry is related with "novelty", i.e., that which was *unthought* before springing up and being established. Then we deal with the question of the *physical meaning* of an experimental result and its relation with the theoretical system - network of concepts and principles shaping a structure - that is proposed to provide intelligibility, considering the variations of meaning which it suffers through the evolution of theoretical explanations. Mathematization of physics is essentially related to its characteristic predictability. We then analyze of what predictability is : of existence or property ? and take as a clarifying example that of "indistinguishibility" of quantum particles. Dealing with mathematization, we have to consider the questions of truth in mathematics and of "mathematical reality" as they stand after the deontologization of mathematics, and to investigate how they are reflected in the problems rised by the use of mathematical notions in physical theoretical constructions. We are then able to confront ourselves with the question of the nature of predictability for physical objects, considering various forms of predictions, and the difference between prediction and prevision. Finally, any mathematization is a formalization, and physics sets in its own genuine and fundamental manner the problem of the relationship between the formal and the

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content, whether this last one is considered as pointing at the real or at the empirical.

In the invitation to participate the Colloquium on "The status of existence for hidden physical entities", several aspects of the problem were mentioned to spur our reflexion. It was suggested, first, to consider the question of experimental *evidence* for a particular physical entity (what I will call a "physical object", be it a quantum particle, or a neutral current, etc.) from either the point of view of *historical study* or that of *logical reconstruction*.

This opposition or duality seems to me to correspond somewhat to a dichotomy between philosophy and history of science that has been - and still often is - current with analytic philosophy as well as in a reduction-to-socialized-paradigms conception of history. If one believes in a more dynamical idea of the relations between philosophy and history of science[1], the duality as it has been stated can be understood in a less constraining way, which could be formulated as: the existence of a physical object can be considered from the viewpoint of the evolution of theories and concepts or from the structural one. Nothing a priori makes two such points of view mutually exclusive, because nothing indicates that, considering knowledge, its structures are ahistorical and do not evolve, when, on the contrary, "logical reconstructions" deliberately deny any historical dimension. Then mutual ignorance or incompatibility would not be unavoidable, a comparison between both standpoints would not be senseless anymore - and it will be, indeed, most instructive.

## 1
## EVOLUTION

The point of view of historical study helps itself, indeed, with the thing as it has been judged (*la chose jugée*), the existence of which has been admitted; the "object" standing here, the question is to know in which way its coming to light has happened, in the space of scientific ideas and practices. Historical accuracy - in the sense of a history of concepts - would induce to pick it - or to try to do so - in the very moment of the passage from the state when it did not yet exist for the mind to the state when the possibility, and then the effectiveness of its existence, is stated and eventually established.

[1] Paty (1990), chapter 4 and (1993), chapter 1.

History would make us grasp, so to speak, the states of the respective conditions which prevailed before, then after, the standing up in its existence of the object: it should also grasp, between the two, its state when it rises and comes into view, that is, the *novelty*, the physical object previously unthought. History - conceptual history - would emphasize the nature of this *unthought* that prepares - or opposes to - what will spring up and be established. The difficulty will be to characterize the breaking point, if any, that (conceptual) novelty represents when it sets up, even before it has been set (in the sense of its evidence for anybody), in the very moment where it is being set. A brief evocation of some examples will help to portray what I am trying to define.

Take Newton's *Principia*'s instantaneous time, considered in its singularity, that allows the expression of the law of motion (these providing the equation of the trajectory as a function of time): this law that is commonly called the law of causality because it connects, as a necessary consequence, what exists at a given instant to what is at the instant immediately next to the first, thanks to the concepts of differential and integral calculus. Perhaps the greatest innovation of the *Principia* is this thought of an instantaneous time magnitude, singular and nevertheless relational, when everybody by then - including Newton himself - conceived time as being, fundamentally, duration in a continuous flow [2]. The trace of the character of radical novelty of this founding concept of mechanics and of its analytic treatment is to be found in the difficulties met by the Bernoulli brothers (Jacob and Johann), by Varignon, Euler, Clairaut, d'Alembert, to conceive, from a physical point of view, what is an instant in itself and when compared to the instant that comes immediately after, and their difference, $dt$ [3]. As a matter of fact, such difficulties were to persist for more than a half century afterwards, and would not vanish before Lagrange's *Analytical mechanics*.

Another example would be the introduction, by Michael Faraday, of the field concept with contiguous propagation with a finite velocity, within a conceptual universe where attraction at a distance was considered instantaneous. One must say that this introduction had been prepared by the use of the notion of propagation function of a potential (such as the continuous pressure of a fluid), whose origin is directly linked to the views on partial differential equations developed by d'Alembert, then by Euler and by Lagrange. Partial differential equations had been introduced by d'Alembert precisely with this physical problem in mind[4].

The transformation that a physics which was traditionally thought

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[2] Paty (1994a).

[3] Paty (in press b).

[4] Around 1746-1749 (resolution of the equation of vibrating strings, treatise on the winds, first dissertation - in latin - on the resistance of the fluids). Cf. d'Alembert (1746, 1747, 1749, 1752). See Paty [1994a], and [in press, b]. A first study of this aspect of d'Alembert achievements has been performed by Grinberg [in press]. On Euler's works on the subject - more elegantly presented than d'Alembert's, but posterior -, see Euler [1752], Truesdell [1954].

through the concepts of mechanics underwent with the introduction of an alien entity such as the field concept - which it was by then, and it was, indeed, a “physical entity” -, results from an evolution that goes through Maxwell’s electromagnetic theory to Einstein’s relativity: special relativity, with regard to the modification of the concepts of space and time, general relativity for the bond between the latter and that of matter. At which time can we consider that such a physical entity was existent and proven ? Is it at Faraday’s time, when the concept of electromagnetic field is still only phenomenological ? or at Maxwell’s time, when the concept is crystallizing a whole theory and unifies itself ? or with Hertz’s experiments detecting the propagated waves that make it manifest and verify the theoretical prediction derived from it ? or with Lorentz’ stationary ether, conceived as independent from movable bodies and as being the proper place of electromagnetic field, by then distinct from matter ? or not before Einstein’s theory which, by making useless the mechanical support of the field, provides to the concept its foundational basis ?

By studying the historical evolution of the field concept and of field theory - as well as many other similar cases - we would undoubtedly be taught that any decision is an approximation which depends on conditions of intelligibility that are linked to a context : the status of the electromagnetic field is not the same at the times and under the respective conceptions of Faraday, Maxwell, Hertz, Lorentz and Einstein. Its corresponding physical contents being different in each case, we are led, by considering this lapse of time in the history of scientific ideas, to state that the justification for a predicate of existence can only be a relative one. At each of these moments, it was legitimate to enunciate the existence of such a thing as the field. But this existence had not the same meaning in these different situations : it was, at the beginning, that of shakings or vibrations of a mechanical ether, and, at the end, that of a self-sufficient physical entity defined as propagating through space in the course of time.

The largest sense content got in the end seems well to ought to the central place that the concept - I mean, the physical entity, or “object” - holds in the network of theoretical propositions. From mechanics to relativistic covariance the field concept passes from a marginal situation and an auxiliary role to a foundational function: from the first to the latter, the weight, so to speak, of the predicate of existence has increased. Let us not forget, however, that thought about physical theory itself has evolved during this time, and that its requirements with covariance of field equations differ henceforth from those of theoretical and analytic physics as they were conceived aat the beginning of the nineteenth century. The very nature of concepts and of “physical entities” underwent, meanwhile, modifications in the scientists’ minds, together with what was meant by “predicate of existence”.

My intention is not to go further in a detailed way through the analysis of this example, however rich and instructive. It has given us, at least, some idea of the extent of the perspectives that history can open on the question at stake : it dissuades us, in any case, from giving to it an overly simplistic answer. A question related to it is embedded in a series of other ones, or reflected in others as if in an infinite succession of mirrors.

One could, however, think that the two examples above might be too exceptional, depending on entities or magnitudes unusually fundamental, and argue that objects more common or familiar to our nowadays physics, more "concrete" to us and apparently more "intuitive" as, for example, atoms or elementary particles of a given kind - other things being equal, despite their specific theoretical characterization - should also be considered. We should not forget, nevertheless, that more peculiar and specific objects imply at their basis other ones that are more general. Those which I just evoked would not fail to be dependent on the concept of field - be it quantum field - as well as of that of time, and replies to questions about the former would not fail to depend on the nature of questions about the latter.

We shall, therefore, now leave these perspectives, keeping in mind the character of *novelty* of a physical object when it appears, predicted or ascertained, as well as the variety of implications that this character entails on our manner to conceive the status of propositions at the respective times before, during and after this springing up.

## 2
## STRUCTURE

As for the structural aspect - I prefer to call it so, instead of “logical reconstruction”, we shall see why -, it is usually considered from the recognition of the object, once its existence has been established, or at least firmly supposed. The logical structuration which is then looked for holds with the “objects” and their related concepts in the state they are afforded, these being ordered to each other in such a manner that their meanings - i.e. the description of their physical contents and the meaning of them - appear as the result of their structural relationships. It is, indeed, structure that manages meaning - and perhaps, until a certain point, that grants it.

Looking into this structural aspect is often claimed to perform a “logical reconstruction”. But there is, indeed, nothing more here than a mere analysis of a state of things which is existing in itself. True, by doing it, the historical point of view is set aside, no attention being given to the way in which theses concepts and relations of concepts that constitute physical objects have been obtained. But nothing is added to the relations which have been afforded by the physical procedures - experimental and theoretical. The theoretical representation is in no way modified, it is not reconstructed, but given as it stands, and what one then does is only to try to understand the logical connexions which make the meaning.

There is a great inconvenience in speaking of “logical” or “rational” “reconstruction”, in that it tends to give credit to the idea that physical thought - which operates through creative processes - is not by itself logical or rational enough - due probably to its intuitive aspects -, and that only through an extra-interpretation are we able to find in it the logics and rationality it was lacking. Beneath this question, actually, stands the problem of what the status of theory is, in relation to phenomena and experiments as well as with the use of mathematics. We shall meet, in the following, various aspects of this problem. For now, we shall content ourselves in keeping the most neutral formulation in this respect, and reject for that reason the expression “logical reconstruction”, when for example “logical” or “sructural” “analysis” is adequate to express exactly what is intended.

In such an analysis, one is probably better aware of what makes the deep meaning of the character, which we have recognized as a “novelty”, of the object under designation - to stay with our problem -, but its novelty itself, as such, has for a long time got wind. Such is eventually the limitation of the mere structural viewpoint : when we adhere unilaterally to it, we are not able to solve the question of the moment when any “existence” for a particular physical object has been evidenced.

We shall, at most, be able to utter more statements about this existence and about the admitted characters and properties of this object, in such a way that the evidence, in connexion to experiment, for these properties and this object gets more weight and is endowed with a particular meaning. Here we meet with an aspect of the question of the physical meaning of an experimental result : this meaning is a direct function of the theoretical system - network of concepts and principles shaping a structure - that is proposed to provide intelligibility. We are aware of it from (historical) examination of the variations of meaning for a given experimental result, when we follow the evolution of its theoretical explanations which go along with the elaboration of theories. See, for example, the interpretations of the negative result of Michelson and Morley's experiment showing the absence of any displacement of interference fringes for the interferometer in motion, respectively at the three following stages of the theory of light in relation with the motion of bodies: Fresnel's stationary optical ether versus Stokes' moving one, Lorentz' electrodynamic theory with length contraction -, and Einstein's special theory of relativity - for this last one, it was nothing more than another piece of evidence in favour of the statement of the relativity principle[5]. The weight afforded to the experimental result in statements about the validity of the theory under consideration is directly dependent on the genuine character of these "structures".

When speaking in that way of "structures" (what other call "logical reconstructions") and of physical content or meaning of propositions, we find, in filigree, the question of the evolution (of concepts, of theories), and of the relationship between these structures. As these can be incommensurable, or at least embedded in a discontinuous series, that of the succession of our representations, the question we find is that of the relationships between varied meanings, for the same concept or set of concepts, from a structure to a different one. Generally speaking, a proven experimental fact persists through such an evolution : what varies is its meaning[6]. Concepts are gradually elaborated on it (or, more exactly, on such results, the plural being relevant here) : through this or these results, and through their permanence from a theoretical structure to another one, concepts or scientific statements in succession feed from one to another: contents transform themselves, while maintaining from one state to another some terms of comparison.

Let us only evoke some examples: the principle of relativity, the constancy of the speed of light in vacuum independently of its source's motion, the energy discontinuity in radiation, wave - corpuscle duality, etc. We notice that the results which correspond to these statements are not only empirical or observational, as they are designated by terms which have something theoretical,

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[5] Paty (1992b).

[6] Paty (1992a).

although in a "minimal" way, that is to say that they call only upon elementary theoretical notions, that are supposed to be definitively acquired, and whose direct meaning transcends particular theories.

To the structure is referred the question of consistence, which is of a rational nature, and that of the greatest consistence which is provided by the statement of existence for a physical entity of the type we consider here : we shall return further to this point. One must observe that the "duality" of the possible viewpoints of the evolution of concepts and theories, on the one hand, and of their logical structuration, on the other hand, would not allow us, if we were to consider it as an exclusive one and as a dichotomy, to apprehend the question at stake : to some degree, it is necessary to hold together these two points of view, or at least to take into account indications from each of them that seem to be essential.

The nature of the argumentation that leads to the statement of existence, the effectiveness of this statement (that is to say, what supplement it brings to acquired knowledge), are others aspects of the "question to the contests". They concern predictability, degree of rational coherence including the rise of coherence just mentionned above and, indeed, precisions on the function of experiment(s), be it in the singular or in the plural, that serve to control this coherence and predictability. These aspects we are going to consider now: will shall see, on our way, that they are directely connected with the preceding ones, which dealt with evolution and structure.

We will examine, first of all, what the theoretical prediction of an object or a new physical phenomenon is, in relation with the theoretical statements on which this prediction is supported: of which nature are the mathematical relationships at stake (simple deduction or creation of a formalism), is there any addition of physical statements (for example, hypotheses or principles), what is the exact nature of the dragging induced by the theoretical form which has led to make a prediction ? These questions carry in filigree a conception of "truth" concerning mathematical statements as well as physical ones - both being distinct in nature but non deprived of a link -, and refer more generally to the problem of physical interpretation of a theoretical formalism. The nature of the final legitimization of a predicate of existence depends on the solution of such problems : is the legitimization already obtain *a priori*, and does it merely ratify a feature that was already contained implicitly in the theory, or is it acquired only *a posteriori*, through the experimental result, being effective only then in the modification or the construction of the theory ?

## 3
## THEORETICAL PREDICTIONS, MATHEMATICAL RELATIONSHIPS AND PHYSICAL STATEMENTS

We may propose from the start that predictability is the mark *par excellence*, among the characters of scientific knowledge, of mathematization. This is, as least, what is observed. The least predictive knowledges are the farthest from utilization of mathematics : for example, history, which belongs to humanities, is not able to predict the future. This judgement begs however for some nuance, and we must take into account the possibility of local predictions : with regard to the past, in history, the establishment of connexions between facts may help to predict a missing link of events and causes. Consider also paleontology, in which science one is able, by considering a set of data, to infer from it the existence of some unknown element, that appears some later time to be effective (consider, for instance, a morphological feature of a fossil species that can be predicted from the form of a tooth). Similar situations occur with naturalists' classifications, or with geologists' hypotheses (cf. the emergence of plate tectonics as a science or volcanology), as well as in other sciences where mathematics are not directly concerned in the expression of the main concepts.

Notice however that, in such cases, it is the tightness - or high focalization - of the network of reasonings and of coherence between facts that makes prediction possible - sometimes with quasi-certainty. Such is also the case of series of reasonings in day-to-day life: this is perhaps due to the fact that reason is, in the sequence of its statements, structured like mathematics.

No one, in any case, will deny that sciences whose object is described by quantitative magnitudes, and whose utilization of mathematics is, for this reason, constitutive of themselves, are sciences where predictability stands as a constant property, resulting in its institution as a criterion of method and of judgement. Their predictions are quantitative and precise, and the correspondence between this precision and that of observations and measurements on physical "objects" (a general term that designates phenomena as well) allows endowment of statements on predictability with qualifications of accuracy and even of truth (or, on the contrary, inaccuracy and falseness). We will have however to specify the nature of these qualifications (whether they are absolute or relative), and the source to which which we have to refer them (whether they deal with mathematics or with fundamental physical statements).

What is, therefore, the nature of this predictability ? The word "prediction" itself, used since the highest antiquity, is sufficent neither to indicate the bond of prediction to mathematization, nor to the procedure of scientific knowledge. "If prevision and prediction equate to science", Alexandre Koyré noticed, "nothing is more scientific than babylonian astronomy"[7]. But, he observed, "it is Greeks that, for the first time, have conceived and shaped the intellectual exigency of theoretical knowledge: *to save phenomena*, that is to say to formulate an explanatory theory of the given that is observable ; a thing that Babylonians never did". According to Koyré, incidentally, "to save phenomena" is not only to connect them by calculation, and to give the means to anticipate[8], but "to reveal the underlying reality", that is to say, to reveal "under the apparent disorder of the immediately given, a real unity, ordered and intelligible".

In mathematized physics, one can distinguish two kinds of "predictions": the one that results from a "causal law" (i. e. a law expressed by a differential equation in function of the time that relates, to an event produced at the given time *t*, the event produced at any time, whose archetype is Newton' s law of gravitation), and which is pure mathematical deduction of the equation; and the one that corresponds to a predicate of existence, which is of a "qualitative" nature, of a physical object which is different from the starting data (for example, the existence of an unknown perturbating planet, Neptune, inferred from a phenomenal property relative to a law of motion - irregularities of the path of a given planet, Uranus, that differ from the calculated law -, or again the existence of a new particle such as neutrino, or such as quarks inside the proton, or the supposed existence of a black hole in a given region of the sky).

It is obviously the second type of prediction that is of concern to us here. One can, however, wonder where is a difference between the two situations. In examples of the second case, the path is given by those "ingredients" which have been considered (sources of the field of gravitation, Lagrangian, etc.). For a truly covering and deductive theory, these "ingredients" have to be included at least in an implicit way (that is to say through indirect properties that lead back to them), in such a way that the predicate of existence about one of them derives from the very statement of the theory. Then we are faced with the problem of knowing whether this predicate corresponds to a real term: the prediction has to be tested. Consider such examples as the bending of light rays in the vicinity of great masses or the slowing down of clocks in a gravitation field, implied by the general theory of relativity. It is so unless this prediction does not correspond precisely to a previously known property as in the case of the secular advance of Mercury's perihelion, which is a direct consequence of the equations of general relativity. One will wonder then why it is however necessary to test, when the admitted theory included already, implicitly or not, the considered property, the predicate of existence. If the theory is truly new, the test is an element that guarantees its validity - or, rather, its non-falsehood - in this respect. If it is already admitted, it is rather a test of coherence on a verifiable property. Its possibility, or

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[7] Koyré (1973), p. 89.

[8] According to the positivist acception given by Pierre Duhem, for instance Duhem (1906, 1908). Remind Auguste Comte's aphorism concerning science: "To know for prevision, to foresee for action".

its necessity, has to do with the interpretation of the theory and the physical character of its statements.

"Local time" in Lorentz's electrodynamics theory, considered by Lorentz as a mere auxiliary quantity and not a physical one[9], appears in Einstein's special theory of relativity as being as physical as the time taken in a system of reference at rest [10]. The slowing down of clocks in a system in relative motion with respect to a system considered at rest is a permanent prediction of the theory, that is not obtained as a reinterpretation of Lorentz' theory[11], but by construction in Einstein's theory which is endowed with a different architecture[12]. The "qualitative" character of this prediction is such only subjectively, for those who stand in front of the theory considered as a new one, at the moment of its presentation, and who had been familiarized with a different idea of physical time. For those who have admitted the theory, on the contrary, the prediction is not anymore now than mere routine: it does not add anything to the statements of the theory such as it has been constituted. The interpretation of this physical character of local time is, indeed, contained in the formulation of the theory itself, as it is shown in Einstein's article of 1905, in its sections which deal with the "Deduction of the formulae of transformation", and with the "physical interpretation" of the same formulae which derives from the procedure of construction of physical quantities (space and time) which are the object of the transformation[13].

The second case to which we referred above corresponds, on the contrary, to a prediction that is not directly deduced, and that needs a supplementary statement. What does such an addition correspond to, in the theoretical procedure? Is it a mathematical proposition, superimposed to the existent formalisation, or a physical one, and, in this case, of what nature ? Model ? Principl ? Mere demand for coherence? For example, the prediction of existence of the neutrino according to Pauli and to Fermi is implied by the energy and spin conservations, to the extent that is only necessary to secure there, for a new physical phenomenon ($\beta$ decay), the validity of this principle[14].

## 4
## EXISTENCE AND PROPERTY
### *Remarks on the "indistinguishibility" of quantum particles*

[9] Lorentz (1904); cf. Paty (1993a), chap. 2 to 4.
[10] Einstein (1905).
[11] As opposed to what Heisenberg had told (Heisenberg (1955), p. 15).
[12] Paty (1993a).
[13] Einstein (1905). Voir Paty (1993a), chapter 4.
[14] Paty (1988), chapter 9.

"The word to exist", wrote Henri Poincaré speaking of existence in mathematics, "can have only one meaning, that of being exempt from contradiction". Some would judge that such a position is a formalistic one, with Emile Borel, who considered a formalist to be one who identifies a truth judgement with a non contradictory one and speaks of existence without asking any correspondence with some concrete meaning. To this Borel opposed the empiricist (in logic and in mathematics), who demands "a clarification of the concrete meaning of purely existential statements such as 'it exist' or 'it does not exist'". The empiricist, he estimated, will eventually go, in this direction to free himself from the formal rules of logic[15]. The opposition of the formalist and the empiricist is not conceived, even in mathematics, in the same manner by everybody : indeed, Poincare was far from considering himself a formalist in mathematics, and as a matter of fact he was not, being by his intuitionism opposed to formalism in the strict sense (one can consider, furthermore, that he was an empiricist with respect to physics).

This variety of interpretations is strikingly evidenced by the even more complex interplay of formalism and empiricism in physics. Consider a case of "prediction of properties" (as it is, the existence of a class of objects endowed with a given property) that gives rise to another point of view, beyond the mere formal or the empirist in the sense of logic as evoked by Borel, about the multiple meanings that could *a priori* be possibly attached to a predicate of existence. This case, a well known one, is that of indistiguibility of identical particles.

One knows that indistiguibility of identical quantum particles stands, as a matter of fact, at the very basis of quantum theory since its beginnings (it is the deep reason for Planck's counting of a resonator's cells, at variance from Boltzmann's in gas physics), and afterwards when it transformed into quantum mechanics (I refer here to Bose-Einstein statistics, which appears linked to the notion of matter wave), and also as the theoretical explanation of Pauli's exclusion principle (reduced to the properties of Fermi-Dirac statistics)[16].

Indistiguishibility, revealed in that way, is not a new particle concept, but a concept of "type" on particles, bearing on a property of a set of particles, sufficiently general to cover all the quantum domain. This concept characterizes any particle, known or unknown, considered at the quantum level, and whose nature ("genidentity", after a denomination proposed by Reichenbach) is determined by physical quantities of mass, spin, charge, etc. These quantities are sufficent to characterize particles by their state, and it is not possible to distinguish these otherwise (as, for example, in allocating them an identification number that they would keep in the course of their evolution : several electrons or photons in a

[15] Borel (1950).

[16] Paty (1988), chap 6, (1993b) and (to be publ.); Darrigol (1990).

given state, each having a self identity). It is the state that defines identity, whatever the number of individuals, which can be thought of as existing individually (and so we count them). The concept of indistiguishibility makes all the difference between quantum particles and particles that classical physics and mechanics succeed in describing.

Among the solutions to which one can *a priori* think of to give an account of such a property, we shall consider two of them, one of a purely logical and formal nature, the other of a conceptual nature. Consider the first. One can contemplate viewing the question of indistiguishibility as a purely logico-mathematical one, and determine consequently the framework of formal reasoning in which physics undertakes its conceptual and theoretical designations. One would claim the suppression of the (logical) relationship that means identity between thought objects, by proposing, for instance, an extension of set theory such that the elements of a set would be, henceforth, no longer submitted to the usual definition of identity[17]. One would then be led, however, to the following alternative: either to make use of a particular mathematical logic for this case while keeping classical logic for the rest - this would need justification and refer, in the end, to the physical nature of the problem; either to reconstruct the whole of logic and mathematics on this new basis, which is obviously not a very economic solution.

The second direction of thought is that which takes the point of view of the concepts of physics. One can question the notion of individual identity that usually characterizes physical systems and states, and of which one could think, until 1924[18], that it was universal and characterized elementary quantum particles as well as classical ones. With the final elucidation by Dirac of the general character of indistiguishibility for identical quantum states, this property appeared as a feature of the laws of nature and therefore to be phenomenal, properly described by the theory, quantum mechanics. The absence of self identity for indistiguishible particles could from then on be stated as a property of existence, the nature of which is both formal and empirical. It is in this sense that Paul Langevin called to a reconsideration of our common manner at conceiving the notion of a particle[19], by suppressing the notion of individual identity at the quantum level. From his side, Hans Reichenbach spoke, as we have quoted, of “genidentity” to designate this restriction of quantum particles identity to their generic characters alone, when described in the temporal order; he conceived, however, this “genidentity” as being functional and not material, and the property of existence that it expresses would have then to be considered as being of a formal nature[20].

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[17] Krause, French (1993).

[18] The year when Bose and Einstein’s works were published, and also Schrödinger’s first paper that prepared his wave mechanics. Cf. Paty (1993b).

[19] Langevin (1933). Cf. Paty (1988), chapter 6.

[20] Reichenbach (1956).

In front of that state of things relative to the indistiguishibility of identical particles, common sense, that is to say the adoption of a minimalist position from the theoretical and metatheoretical viewpoint, would make us observe, before any other consideration or attempt of interpretation, that the conceptual elements which are at our disposal theoretically to describe such "objects" are reduced to the only quantities that point at them effectively (mass, spin, charge, other quantum numbers, etc.). Considered within a given system (for instance, localized in an atom), objects that would be described by identical values of these quantities (for instance, two electrons) would be strictly equivalent - nothing would distinguish them from one another -, and therefore permutable. (This is why the property of antisymmetry of their state function entails the impossibility that two fermions - for example, two electrons, or two nucleons, protons or neutrons - be found in the same quantum state, within a same system: otherwise the state function would be null. Such is the explanation of exclusion principle). This kind of objects are therefore equivalent, according to the quantum description, even if one can count them (for example, so many photons in a definite state): such is the exactl meaning of indistiguishibility, revealed by phenomena and whose theory (quantum mechanics) furthermore gives account perfectly, providing the explanations of the properties of black body radiation and of monoatomic gases through Bose-Einstein's statistic and of Pauli's exclusion principle through Fermi-Dirac's statistic .

We can interpret this mapping of the property (equivalence of particles with similar characteristics, being hold in the same state within a system, and which can be counted but that nothing distinguishes) by the theoretical description as a kind of narrow encompassing of the concept (of quantum particle) by the theory. Instead of considering it as a shortfall, as we are naturally inclined by the common intuition that we have of the notion of "particle", taken from the immediate experience of bodies in our surroundings as well as from our habits of classic physics, would not have we to consider that nothing authorizes us to think, about such objects, of properties that are not refered to by theory ?

In the description of particles by classical mechanics and physics[21], on the contrary, one mentally adds, to the characters that point out a particle, an identity - as one numbers marbles - that distinguishes it among its likes. But it is because classical physics (and mechanics) deals with such objects by taking them as given. It provides, for example, the equation of motion of a particle endowed with mass and charged; and one supposes always - explicitly or not - that in addition to its characteristics namely represented by physical quantities (of a theoretical nature), this object possesses an aseity, that is not designated by a theoretical quantity, but that is self-evident (identifiable projectile, stone that one has held in hand before launching it, or particular celestial body that has got a

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[21] Mechanics is not enough by itself to characterize a particle : electromagnetism, for exemple, is implied in the définition of electric charge.

name, as the Moon, or the comet of Halley). Indeed, classical physical theory does not present itself as a theory of this particle, but as a theory of what happens to it and of some of its properties: the very notion of particle as an identifiable individual is external to the classical theory. This theory takes it as given, and is conceived to describe the behaviour of objects that possess this characteristic. On the whole, this notion, being endowed with a non explicit and not theorized content is, in relation to classic physics, a kind of substance. Moreover quantum physics was born from the exploration of the "intimate nature" of material bodies; "substance" has been lost on the way, namely the identification of indistiguishibles. The deprivation of this substantial property reminds of the elimination of ether in electromagnetic field theory when modified by relativistic kinematics[22].

If, moreover, we restrict ourselves to consider what quantum theory does point out, the object that it describes is not exactly a particle: it has got only a part of the properties that are attributed to particles, and it has got as well only a part of those attributed to a wave, exhibiting them only under some conditions. It is therefore, properly speaking, neither a particle nor a wave : it has become usual to name this object endowed with so unprecedented properties a "quanton"[23]. There is no need to invoke a made-to-measure philosophy (as Bohr's complementarity) to consider that quantum mechanics (and quantum field theory) describes physical objects that are quantons, which can be, under some approximations, compared with particles and waves properties in the classical sense or in the common intuition. (Especially, when one wants to catch them with the help of a corpuscle detecting device, or of a wave detecting device, they let themselves be taken as such, while losing a part of their genuine characteristics: and is it not in this sense that one would have to understand the expression "reduction" ?).

The theoretical entity that describes the quanton (or, generally, the quantum system under consideration) is the $\psi$ function of quantum mechanics (or the quantum field in field quantum theory). If it defines no other *identification* of the system than those quantities that it implies (and, consequently, does not authorize us to conceive that this system be distinguishable from others that are in all respects identical), it defines however, in the limits of this very identity, an individuality, since it is conceived as a description of individual systems and not only of ensembles of systems. It is important to distinguish, as quantum physics obliges us, two notions that are merged by current life and classical physics, that of individuality and distinguishibility for identical systems (particles, quantons). Quantum systems are individuated (therefore enumerable) but not distinguishable (or numberable).

Let us conclude by taking again the parallel which we considered

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[22] Einstein (1905). Cf. Paty (1993a).

[23] Bunge (1973).

above. In the same way as the concept of field (in the classical sense) has proven to be self-sufficient, without a medium of the type of a mechanical ether, similarly the quanton (or the quantum system, or the quantum field) is also self-sufficient, without underlying distinguishable, ondulatory or corpuscular, substance, without projection or reduction on concepts that would remain external to the theory and that nothing justifies physically, and that are, therefore, useless.

Undistinguishibility of identical individual systems is a predicate of a general property that conditions the specific predicates of existence. One could ask, concerning this predicate *of property*, questions analogous to those asked above on the predicate *of existence*; the possible questions or answers would not be fundamentally different in nature.

## 5
## TRUTH IN MATHEMATICS AND PHYSICS

In his book *Mathematics in western culture*, Morris Kline considers that, mathematics being today freed from any ontological reach (for instance, relatively to physical world), and no geometry neither algebra being able to pretend to be true (in this sense), mathematics is transformed into a pure creation ofthought, and freed “from the bondage of producing truths”)[24].

This expression appears to me disputable because, in its proper domain, mathematics produce, indeed, truths : but these are simply “mathematical truths”. They are relative to axioms and to definitions chosen as starting point : these have been are simply posited, and have not in themselves any truth value other that their coherence between them. These axioms constitute the reference of the truth of propositions : propositions have to be consistent with axioms, they will be true in consideration of axioms (truth *modulo* the axioms).

The notion of mathematical truth appears clear, thus conceived, if one considers the propositions of mathematics as relative to “objects” - mathematical objects -, that are of a formal nature, these objects being represented by the symbols themselves. This consideration does not oblige us, for all that, to restrain to the formalist idea (Hilbert) that, as we know it since Gödel, does not exhaust the question of the nature of these “formal contents”[25]. Mathematical truth is maintained again if the notion of mathematical object is dissolved in pure

[24] Kline (1964),1964 ed., p. 466.

[25] The expression "formal contents" is due to G. Granger (1982).

operationality, with a conception of an intuitionist, operative type, according to which there would be no mathematical object, but only operations of the mathematical reasoning (see, with differences, Brouwer, Poincare, Hermann Weyl), what does not prevent to conceive a kind of "mathematical reality" as a basis for the consistency of mathematics propositions.

We will try to justify it - summarily - in both cases. As for the logicist conception, that bases mathematics on pure logic, it makes in fact mathematics a pure form, without content. But we shall not retain it here, its interest being (with Russell's and Whitehead's *Principia mathematica*) to have allowed important developments of the mathematical logic, but not of mathematics themselves.

The denial of the notion of "mathematical object", motivated by the refusal of a platonician position, leads to see in mathematics a "science of relationships and structures" by opposition to a "science of ideal entities"[26]. This idea would invalidate henceforth - since seventeenh century - an ideal reality of mathematical objects.

One will notice nevertheless that, from a historical point of view, the conception of an ideal reality is still present in Newton, with a meaning that is not very far from the traditional one (which is not surprising, if one considers his neo-platonism, shared with Henry Moore and Isaac Barrow) : see his conception of "*true, absolute, and mathematical* magnitudes" by opposition to "*apparent, relative, sensible*" ones[27]. This conception goes along with his mathematizing the science of motion and his construction of the theory of gravitation, together with the specific form of geometry he used (and edified) to this end in the *Principia*, a geometry of ratios and limits of ratios, impregnated with the ideas of his calculus of fluxions[28]. It is true that the genetic conception of the origin of knowledge (opposed to Cartesian innateness as well as to the Platonician idealism) issued by Locke has contributed to transform the way in which the nature of the relationship between mathematics and physics is considered.

But someone like d'Alembert, for example, who located himself in this lineage (his philosophy of knowledge is tributary of Locke's and Condillac's ideas), conceived that mathematical quantities have, at least some of them, despite their idea-like (*idéelle*) (and ideal) character, some degree of "reality". The type of reality at stake is a reality conceived for a mathematical being, i.e. endowed with a particular meaning indeed, but that holds to this author's proper exigency for intelligibility. The notion of limit is, for d'Alembert, a mathematical reality to the extent that one can make a direct ad clear idea of it : it is operative (by the procedure of passage to the limit in geometry) and objectal (in the sense that it can be put in correspondence with a definite quantity)[29]. Such "real" notions are those

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[26] Funkenstein (1986), p. 297; Tuchanska (1993).

[27] Newton (1687).

[28] Paty (1994a).

[29] For definitions of "operational" and "objectal", see Granger (1992).

to which we refer other mathematical notions that are only operative (and in fact, for d'Alembert, non objectal), such as differential magnitudes, considered as a mere convenient symbolism : these do hold their meaning, according to him, from the limit that is the ratio of two of them.

This meaning of "mathematical reality" is more limited than what today mathematicians undertand by this, admitting that mathematics are free to create notions and rules that notions have to follow, and that underlying the game there is as well, in their eyes, some kind of "reality". (One admits, for example, that differential magnitudes are not, so to speak, only operative but objectal, and the notion of infinity itself has for this reason its citizenship in mathematics). But the basic idea remains the same : it is that of reference. Mathematical propositions are referred to this particular "reality" that underlies them and that is only of a mathematical nature: at variance with d'Alembert, would we say, and for this nearer to a kind of ontology, they hold to mathematics themselves, to what determines in an internal way mathematical concepts and propositions rather than to the idea that we make for ourselves of what is more or less real (which someone like d'Alembert identified to the character of being concrete : for example, numbers, conceived as solutions of the equations of curves intersections, which remains a geometrical conception).

The existence of a "mathematical reality"[30] is conceived as well by formalist mathematicians as by intuitionist ones (as Poincaré[31]), that are not in any way Platonist. What these mathematicians mean when they speak of "mathematical reality", is that mathematical propositions that result from a demonstration, as are theorems, are not evident from the start, but discovered through some process of reasoning that is truly a research : confronted with statements on "beings" initially posited by itself, thought experiences something hard that resists, and thus in fact escapes it, independent or exterior to its process, and endowed with a kind of inner consistency. Demonstration, in mathematics, is not simply deduction of the syllogistic type, and theorems are not sequences of tautologies. The situation presents an analogy with the confrontation with a material reality, which explains that the same word is used in the two cases. "Externality" that "resists" - it is besides made of a tight network or stuff of properties that are enunciated as propositions - corresponds to what mathematicians call "mathematical reality". It is not thought, of course, on the same mode as "physical reality", as the world of phenomena of the physical world that is revealed to us through experience, through sensibility. It is an other kind of reality that it at stake, an idea-like reality (*une réalité idéelle*).

To admit the existence of mathematical objects belonging to a mathematical reality understood in this sense opposes the dissolution of mathematics in pure relationships : because, at least, these relationships, what

[30] See, for ex., Lautmann (1977).

[31] See, for. inst., Poincaré (1902).

mathematical propositions are, weave a network that is so tight and opaque to direct evidence, and eventually rebelling against rational resolution, that it does not differ, by this feature, from what one generally means by "object" as being hold by the understanding.

We know that this object is not confused with material or physical object - and one can speak, in this sense, of a deontologization of mathematics, albeit what is meant by "ontology" is far from being clear, even speaking of material objects themselves (one admits an ontology of the real world, even if one has dropped the notion of substance). If, by ontology one means something relative to the reality of the material world, one will claim, indeed, the absence of an ontology for mathematics, and one will reserve the corresponding philosophical or metaphysical question for the underlying basis of the sciences of nature. The "content" of mathematical propositions is empty with respect to possible physical contents.[32]

It is in this sense that the deontologization of mathematics has reached, in the nineteenth century, geometry itself that, since Descartes and Newton at least, was directly linked to spatial representations of the physical world. Science of figures in space, then science of space, geometry could be only conceived as unique until the discovery, in the nineteenth century, of non Euclidean geometries. From then on, space was endowed with several possible geometries, and geometry, as the science of space, was henceforth the science of these various theories that would be in principle (eminently with Riemann and Clifford) decidable by experience, as in physics[33].

Axiomatisation of geometry had nevertheless to free it from this "ontology" that made its difference with the other branches of mathematics. "Geometrization" of physics such as the one proposed by the general theory of relativity is not connected to an ontological idea of geometry. This last enters in theoretical constructions of physics in a manner that is not intrinsically different from the utilization of mathematical magnitudes in general. "Geometrization" here means only that the conceptual substratum of physical phenomena is a four dimensional space-time continuum. It has moreover to do, since Minkowski's four-dimensional space, with an extension, of a formal nature, of a geometrization conceived in the usual meaning, that would deal with ordinary space, to a fourth imaginary coordinate representing time (the imaginary character reminds us that time is not space, the space-time metric being no longer definite positive as that of Euclidean geometry). On the other hand, the geometry of interest here is not the mathematical science known under this name, that is to say pure and axiomatic geometry : it is, actually, a "physical geometry", an elementary physics of space, an interpreted transcription of magnitudes and of relationships of geometry in the proper sense (expressed in the language of differential magnitudes) to be used for

[32] See, for. inst., Einstein (1921).

[33] For a discussion, cf. Paty (1992c) and (1993a), chapters 6 and 7.

the spatial and temporal properties of bodies[34].

Anyhow, "deontologization" (understood with the above meaning) of mathematics has made possible a new conception of the relations of the former to nature. Mathematics, for example geometry, does not correspond any more to an ideal and global representation, but is used in (or applied to) establishing laws and physical theories : such was, already, the geometrization of motion by Galileo through the expression of proportions between spaces covered, speeds and durations. The overall conception of mathematics as a form of intelligibility and constructive utilization had to go for some time together. But one sees clearly with differential and integral calculus - or, at least, after decantation during the time of its assimilation and of the exploration of its possibilities of application - that the nature of mathematical magnitudes is not to be confused, due to the construction's conditions, with that of physical magnitudes, even idealized ones.

Among the conceptual difficulties in the beginnings of mathematization of mechanics by differential calculus, one was related, it seems, to the apparent opposition between, on the one side, the continuous character of mathematical space as well as of time understood as duration and, on the other side, physical magnitudes expressing properties of the motion of bodies that were, as these, discontinuous and singular - even when bodies were reduced to the abstraction of material points. The variations of these magnitudes suppose the concept of time. Now, precisely, the notion of instantaneous time and the precise meaning of its differential, the discontinuity of the instant in the causal equation[35] of the path of a material point submitted to a force - from its invention by Newton - came up against the definition of physical time as a continuous flow, and made itself manifest in the geometrical transcription of these magnitudes that made their comprehension intuitive[36]. Difficulties of application and ambiguities of interpretation, notably concerning the calculation of accelerations and of corresponding forces, would go along the development of the mechanics of solid body during some fifty years.

Mathematics is introduced in a physical representation, conceived as a construction, as a tool that first transcribes the admitted notions in a more or less intuitive manner, and it progressively becomes the very condition for the thought of magnitudes of physics. The mentioned difficulties about the differential notions in mechanics were absorbed when this last became totally analytic, with Lagrange.

Geometrical notions admitted in physics on an intuitive basis, such as infinitesimal segments of curve and straight lines relative to trajectories, represented by differentials, yielded their place to these same differentials conceived as mathematical magnitudes in themselves, implemented in physical theory without referring any more to geometrical representation as intuitive

[34] See, f. inst. Einstein (1921). For a discussion, cf. Paty (1993a), chapter 7.

[35] I am using here the present terminology and not that of the time.

[36] Paty (1994a).

translation. Analysis, algebraically conceived, became the very symbolic language in which physical concepts are constituted and put in relationships. A new form of physical intuition developed thus, in correlation with another intelligibility. From then on, we think of physical space and time according to the intelligibility that their definition allows, through mathematics, as magnitudes on a continuum, and according to the meaning that we attach, for such magnitudes, to operations of differentiation and integration.

Time and space have then been reconstructed in reference to physical phenomena, as in the theory of special relativity, where they are submitted to the condition of obeying the principle of relativity and that of constancy of the speed of light independently of the motion of the source: this condition made their physical character, being henceforth no more “absolute and mathematical”. This reconstruction of space and time, more adequate to their real physical character, has been undertaken from their mathematical form, and this last has been consequently appreciably modified under the species of transformation formulae for referentials in relative motion. It is again as physical concepts thought according to their mathematical form that they have suffered again another modification with the general theory of relativity, being by their new definition submitted to an even more general property of physical phenomena (general covariance).

This taking place - a movement already sketched in Galileo and rather well visible in Huygens and Newton -, the relationship between mathematics and physics was no longer only conceived on the mode of ideal intelligibility (that of the analogy with pure forms), but on a mode so to speak “practical” of the intelligible[37], that constitutes mathematically the instrument of thought. Newton buildd up instantaneous time by the “first and ultimate ratios of vanishing magnitudes”, that is to say differential quantities conceived according to their ratios and to the limits of these ratios[38], and established the properties of physical quantities and of laws or of the principles that govern them on this basis. Since then, physics has been continuously constituting itself mathematically.

The narrow and constituting relationship that physics maintains with mathematics leads one inevitably to think of the eventuality of a possible bond between the question of “mathematical truth” (“do mathematical truths exist ?) and that of the nature of the “certainty” , the “evidence” and also of the “truth” of the conclusions drawn from a theoretical reasoning in physics. It might appear that, from the mathematical viewpoint, only consistency is required, for, in any case, “ontology” there would be no longer that of mathematics, but that of physics.

Nevertheless, the physical content of the propositions of a theory is not empty from the properly mathematical point of view if, precisely, mathematical

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[37] But that is not, as it were, merely pragmatical, for the construction of an explanatory system is at stake.

[38] Newton (1687), Book I, scholie of lemma XI.

notions are taking part in the very thought of physical construction, as we have meant with the examples of differential notions for space and time, and as one would see also with absolute differential calculus in the case of the general theory of relativity, or again with vectors in Hilbert space for quantum mechanics.

This question requires nevertheless an examination of the various aspects of "interpretation". The "mathematical content" in its wholeness is not perhaps strictly needed - and "consistency" would be eventually enough - when the covering of a physical magnitude by the mathematical magnitude is only partial: such is perhaps the case of the properties of Hilbert space in quantum mechanics. We must also take into account the approximation character of a physical quantity as compared to the ideal (or rather only "idea-like") situation of mathematical magnitudes: the continuum of space and the material point can be instruments of physical theory, but they might well be unsatisfactory with respect to some of its demands. (They are, for example, the cause of the presence of infinite quantities, in calculations of quantum field theory, that are eliminated only by complex and somewhat artificial procedures).

It seems nevertheless that what mathematics confers in certainty to theoretical calculations of physics depends directly on their truth value. This certainty is dependent, indeed, on the reference to axioms and to the magnitudes that have been chosen as fundamental ; although mathematized, the latter are chosen for physical reasons, which makes, in any event, the truth of the corresponding physical theory a truth relative to a given state of conditions, and that is subject to revision.

The notion of mathematical truth appears as founding the choice of ideal concepts, without substituting itself in any way to the demand for the latter to be, in their very approximate character, in conformity to phenomena (or, better, appropriate to their description).

"Physical truth" does not cover over "mathematical truth", but implies it. To what should we attribute otherwise the success of differential conceptions in mechanics, or that of Riemanian geometry in General Relativity ? This success holds to the dragging effect, on physical conceptualisation, of the mathematical form, or formalism, that constitutes in itself a thought without equivalent (for example, tensors, necessary to express general covariance, according to Einstein himself[39]). Theorems of tensor analysis, expressing a mathematical truth, constitute the way of specifying the physical demands that characterize General Relativity, and the equations of the latter are a direct consequence of them.



[39] Paty 1993a, chapter 5.

## ON THE NATURE OF PREDICTABILITY FOR PHYSICAL OBJECTS

With modern science, that made its entry in the seventeenth century, the conception of the relationship between mathematics and physics (at least in those geometrized parts of physics that were then mechanics, astronomy, optics) underwent a mutation. An imposition of mathematics onto physics was no longer at stake, being either analogical or essential, nor even "technical" (when geometry stepped in from material construction of objects such as parabolic or spherical mirrors in optics[40], or again, according to Newton himself, arising from problems and constructions of mechanics[41]).

Then appeared, first of all, a new conception of "mathesis", that of mathematics as the language in which the book of the Universe was written according to Galileo, or, more fundamentally, as a (or even the only) model of intelligibility in Descartes, a conception adopted by Newton.

But it is also, taking a more precise qualification, the embedding of mathematics in physics by the constitution of the latter, that relates to the idea of theoretical construction, as well as to that of quantified knowledge linked to the ideas of certainty, accuracy and precision. Take, therefore, the two aspects of mathematization: on the one hand, evidence and certainty (d'Alembert saw, as for him, this last as a delayed series of evidences linked in a chain), and, on the other hand, quantity (through deontologization, here heard as suppression of qualities and essences), that gives access to measurement, and to the numerical definition of precision.

Mathematized knowledge was going to equate absolute knowledge with a divine origin : in the facts for Galileo (who asserted the equal legitimacy of the two Books, that of Revelation and that of Nature), as a matter of principle for Descartes, according to a new conception of the function and the foundation of reason. It was enough for the latter to be founded in God; this being admitted, its aspirations and its operations then had no other reference than itself, reason, unique source of an evidence that is a mark of higher intelligibility. Reason, the mediator between absolute - divine - knowledge and the world.

Indeed, with Newton, physics (the mathematizable, rational part of it) was still thought on the mode of an absolute mathematization, that of geometry; but the latter, that can express the physics of motion and of the system of the world, has been transformed to this very end. The geometry of the *Principia* is a local and temporal geometry, where the point and the instant can be seized in their continuous variation and described by going to the limits of ratios of magnitudes:

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[40] Cf. Rashed (to be published).

[41] Newton (1687), Preface to the first edition of the *Principia* .

"deontologization" of mathematical or physical magnitudes was achieved when considering only their inter-relationships in the expression of laws. For Newton, mathematical magnitudes were ideal, but by this, precisely, they could express the real properties of the real world, beyond appearance.

Gravitational force was a mathematical entity, whose physical nature was unknown, maybe unknowable; but Newton granted it a physical content, that he expressed by the law of attraction (through its mathematical form, that was relational). The physical content corresponded to phenomena of falling bodies, of the attraction of the Sun or of the Moon. In a general way, for this force as for all others, the mathematical form realizes itself in the physical contents through the expression of the law of motion, transcribed in the equation that connects the state of the material system at a given time and in a given place of its trajectory to the immediately following time and place (this has been, afterwards, been commonly called differential causality law).

Thus mathematization of physics (as it is, mechanics) led, from its first realizations, to the most elementary and the most universal form of prediction. Consider again the remark that we made previously, concerning the distinction between prediction and prevision (in the latter nothing qualitatively new steps in), that is perhaps byzantine, for the establishment of the law of motion, i. e. of Newtonian causality, in some problem, makes use of a number of considerations whose nature is akin to that of more qualitative predictions.

At the moment when they were newly issued, results relative to the motions of such planet, or on tides, or on the form of the Earth, or, for example, on the return of Halley's comet (through Clairaut's calculations), corresponded to predictions of Newtonian theory. The word prediction here is to be taken with its most undeniable meaning, contrary to immediate predictions or previsions, such as that of the state of a system at any instant $t$ knowing the state at another given instant: one does not add anything, from a qualitative point of view, to the acquired knowledge, at least once the theory has been established and generally accepted.

In these relatively "simple" examples, the law of motion asks for specifications that were not included in the general form of the equation of the Newtonian law of gravitation: data on the great axes of orbits, hypotheses on the constitution of the Earth as a fluid mass, methods of calculation for the three-body problem, etc. Now these predictions become as elementary as the equation of motion, and reduce to simple previsions as soon as the corresponding transformations of the theory are universally adopted.

There are few cases of simple "prevision", where no specification of the particular problem that one wants to calculate is to be added to the admitted theory. In the strict meaning, a "prevision" would be expressed in the form a table as resulting from standard calculations (tables of tides, of the Moon, etc.). Such tables call upon specifications of the indicated type, but that one admits as ratified. The difference between prediction and prevision would be therefore ultimately subjective, regarding not the nature of the theory and of the calculations but their degree of acceptance, their consideration as hypothetical or not, their extrinsic legitimation.

It remains true that some problems exist that, when "put into equations", result in predictions that are not simply numerical values of the type of tables in which all the ingredients necessary for calculation would be known, but that correspond to a statement on the existence of new phenomena. Clearly, these "predictions" do not result only from the game of mathematical deduction, and it is not mathematics alone that produce some physical reality (a new planet in the case of Adam's and Le Verrier's calculations predicting the existence of Neptune from the irregularities of Uranus' motion[42], a new particle in the case of the neutrino, from Pauli to Reines and Cowan and to the intense beam production of neutral leptons, "fundamental bricks of the material universe"[43]).

The physical object predicted by the theory is actually homogeneous in nature, not to mathematical entities that enter the theory, but to physical magnitudes that these entities express, that are themselves fastened to concepts and to categories that constitute them as physical entities and give them the vocation to represent phenomena.

The new objects that might then eventually appear, the predicted objects, can be traced step by step through the process of theoretical reasoning and mathematical deduction. The origin of the prediction is actually to be found in the principles or physical concepts from which one has started, that may include supplementary hypotheses - but always of a physical nature - that eventually might simply be reinterpretations of the initial physical contents.

An example of such a reinterpretation is Dirac's relativist equation of the electron, whose negative energy solutions (purely mathematical ones, apparently, being devoid of any physical counterpart), when reinterpreted, led to the prediction of antiparticles. This reinterpretation of terms that are issued from a mere calculation calls on physical principles (exclusion principle, etc.) that allow one to render the mathematical solution compatible with the general physical constraints that are homogeneous to other physical solutions (with positive energy) of the equation.

Afterwards, once quantum field theory had operated the "logical reconstruction" of the electron relativist equation, the antiparticule was found to be contained mathematically and physically in the theory. But it was contained, henceforth, for the same reason as the other ingredients of the theory.

Prediction requires one always to set or to make explicit physical interpretation, this being, moreover, most often contained in filigree inside initial statements. Mathematics provide the relationships, unseen in general, that derive from these statements and that, when interpreted, entail prediction. Its role, as

[42] Cf., f. inst., Roseweare (1982).
[43] Paty (1988), chapter 9.

Poincaré noticed, is to express the hidden unity that happens to stand in phenomena.Through the play of its relationships, statements and reasonings, it makes elements of this unity explicit when, without it, they would remain overshadowed. Prediction holds essentially to that. It begins as a necessity from the viewpoint of the relationship between the mathematical magnitudes that are used to represent physical quantities. This mathematically necessary conclusion is thereafter evaluated from the viewpoint of the corresponding physical content.

If the physical meaning of quantities goes directly along with their mathematical expression, then indeed mathematical relationships generate directly something that is of the order of a physical property statement (such are the equations of transformation of space and time coordinates in Einstein's elaboration of the special theory of relativity). If formalization precedes physical thought and guides it (as it is the case with the general theory of relativity, with quantum mechanics, and with quantum field theory), it is necessary to interpret the statements obtained in terms of their physical content according to criteria of coherence and likelihood that will be gauged in relation with the possibilities of getting evidence for phenomena. But the net meaning is not fundamentally different in the two cases.

A characteristic example of prediction for a physical theory, in quantum field theory, is that of the unified electroweak gauge theory of Glashow, Salam and Weinberg. This theory makes use of a specific parameter (the $\theta_{W-S}$ "mixing angle", connected to the ratio of the masses of $W^{\pm}$ and $Z^0$ intermediate bosons), that stands, through the play of mathematical relationships between the physical quantities of the theory, in the various terms of the Lagrangian, and is henceforth incorporated into the predictions. The prediction of the existence of weak neutral current is directly homogeneous with the entities that enter the initial formulation of the theory, on the basis of the previous knowledge about weak charged current : the Lagrangian has been reformulated merely to satisfy a larger symmetry group that includes electromagnetic interaction together with the weak one. The intensity of these weak currents is given by this parameter (or rather by the square of its sine). The same holds for the prediction of intermediate bosons with masses in the above indicated ratio - the two results proving furthermore consistent, by providing a unique value for the parameter[44].

Analogously, the prediction by Einstein's general theory of relativity of the secular advance of Mercury's perihelion results not from a particular hypothesis, but from the very form of the physical theory as it is constituted, without in any way presupposing this implication. More simply, and in an almost similar way, the construction of the concept of time and the resulting deduction, in the special theory of relativity, of Lorentz' transformation formulae for space and time coordinates, ends at the prediction - without reinterpretation, but by simply

44 See, for inst., Glashow (1980), Weinberg (1980), Salam (1980), and, for recent considerations, Trân Thanh Vân (1992).

following the thread of the consequences of the mathematized definition of physical time - that local time in the system in motion is indeed the physical time that clocks placed in this system would indicate[45].

In all these cases, one speaks of prediction to the extent that the physical phenomenon (or effect) appears qualitatively new and that the verification of the prediction still remains to be ascertained. When the validity of the former is acknowledged, its novelty melts into the theory, and the prediction, that is henceforth nothing more than routine, is transformed into prevision, and its interest is only a practical one.

## 7
## PREDICTION AND FORMAL MEANING

Mathematization is a *sine qua non* condition for any theory in contemporary physics, and in this physics no statement or predicate, be it a predicate of existence or of property, can be expressed in other terms than mathematized quantities. Theoretical principles themselves, that carry with them an essential part of the physical content, since they inform and govern concepts and their relationships, are stated and even thought in terms of mathematical notions (from minimum principles in the classical period to nowadays symmetry principles).

It is, furthermore, this mathematical expression that entails prediction, at the different levels that we have considered. Predictability in physics rests on proper mathematical deductibility, but it is effective only as much as it expresses - beyond the consideration of mathematics as pure language - the relationships of a law or a physical theory. Thus physics does not reduce itself to mathematics, although it could not state its laws, propositions and predicates without having recourse to appropriate mathematical notions and symbols. It is therefore necessary to conceive of physics as a thought in itself, bearing on specific objects, while making a "constitutive" usage of mathematical thought. Physical thought borrows, in sum, from mathematical thought its framework and many particular tools (notions, theories, relationships).

Physical thought is being built, by means of mathematics, towards an object that is regularly redefined from perception and exploration of phenomena, and gradually focalized and specified through the elaboration and refinement of

[45] Paty (1993a), chapter 4.

concepts and procedures : at every step of this construction process that encompasses the "physical object", mathematical notions and reasonings are implemented. This is why this "science of nature" is also an "exact science": even if its statements are in principle always open to rectification, the language that expresses them is always that of mathematical accuracy.

Any mathematization is a formalisation, and physics sets in its own genuine and fundamental manner the problem of the relationship between the formal and the real, the formal and the empirical, the formal and the content[46] (each of these instancies, the real, the empirical, the content, being situated at distinct epistemological levels[47]). Predicates of existence and of properties, predictive statements and their meaning, are eminent manifestations of this relationship. One must however notice, to conclude, that all the mathematical formalizations, with respect to physics, are not of a unique nature, and we can make a distinction between various degrees among them.

Let us sketch three of them. To the first degree, mathematization goes immediately along physical elaboration: statements on physical quantities and properties thought "intuitively" and so to speak qualitatively (distance, speed[48], weight, force, etc.) are transcribed directly in mathematical terms, magnitudes being represented by symbols, and their physically defined properties as relationships between these symbols. The control of the physical meaning of the statements thus obtained goes along at every step, so to speak, with the mathematical processing. Examples are numerous and familiar: composition of forces or of impulsions in classical mechanics (one finds on this question, especially, significant remarks in Newton, d'Alembert, Lagrange), their representation in space "in magnitude and direction" (i.e. vectors); vector calculation for forces; elaboration of the concept of time in the theory of special relativity, etc.

A higher degree of abstraction is reached when one endeavours to translate a previous mathematization into more formalized terms: for example, Lagrange's invention of variational calculus, stimulated by the desire to formulate mathematically Maupertuis' principle of least action. But application is not always guaranteed, and formalism may well escape physical thought, momentarily or not : this is what happened with Lagrange's variational calculus and least action principle in their application to problems of mechanics. There is no identification between the (mathematical) formal and the physical, but only partial overlaps, that

[46] We mean here, by "content", the physical signification of what is contained in the formal expression of a quantity.

[47] See, for inst., Paty (1992a).

[48] It was, then, a mean speed, as, for instance, in Galileo's work. Instantaneous speed, considered by Newton, required, as we have seen, a mathematical conceptual mediation, that of differential (or fluxion) calculus. Similarly, the notion of force, that was criticized in the eighteenth century (in particular by d'Alembert) for having metaphysical connotations, would be fully "rehabilitated" by Lagrange considering that Analysis provided henceforth an unambiguous expression for it.

are most often refined through modifications of the physical approach (Lagrange withdrawed the principle of least action in favour of the principle of virtual work, and the first would find again fruitfulness only in the hands of Hamilton[49]).

The use of Hilbert space formalism in quantum mechanics might be of this nature, just as the construction of the quantum mechanics itself, performed in a relatively empirical way, by juxtaposing a formal apparatus onto experimental data. This state, or this phase, of the relationship between the formal and the empirical content, seems to be characterized by the importance of the "interpretation" conceived as connecting two instances otherwise rather loosely linked.

In cases of this kind, the power of formal relationships has exceeded the physical interpretation that went along their elaboration : such is the wave function of Schödinger's wave mechanics, initially conceived by its author according to a direct physical interpretation, an ondulatory one, that of a wave amplitude propagating in coordinate space. The relationships obtained, always in agreement with phenomena, obliged Schrödinger and other physicists to a quite different reinterpretation of $\psi$. The one that was to overcome was Born's probabilist interpretation, in terms of "probability amplitude" or "probability wave" in the *3n* dimensioned configuration space, whose physical counterpart was not obvious (what kind of a hybrid entity was it, mathematical and nevertheless carrier of physical content ?[50]).

Heisenberg, for his part, introduced the state vector of a Hilbert space to represent the physical state of a system, and this mathematical magnitude was as much abstract to his eyes, but not more, fundamentally, than Lorentz's "mathematical" local time, with which he saw a parallel[51]. The work of theoretical physics, at this stage, was for him to physically interpret this formal (purely mathematical) magnitude, and from there issued all the particularity and novelty of the status of quantum mechanics with respect to the formalism and interpretation. This is a viewpoint on the theory, at a certain stage of its formalization. It is followed by effects : the interpreted formalism entails the prediction.

Finally, the third stage would be that where physical thought and formal (mathematical) thought overlap each other nearly exactly - at least momentarily, because the overlapping cannot be complete and will have to be fitted again -, so that formal thought is, at this stage, the only means of expression for physical thought and drags this last down, allowing it to be established and firmly asserted.

One can distinguish there several degrees. One of them would be in the immediate continuation of the preceding stage: physical thought, powerless in

49 Martin-Viot (1994).

50 Paty (1993b).

51 See above.

its present formulation, states a problem that can be expressed in a general and formal manner, and then has at its disposal a formalism, foreign to physics until then, by means of which it is possible to translate the statements of the problem, which thus can be solved. A example of it, an historical one, is provided by Einstein's elaboration of his general theory of relativity : the physicist has been able to state the (physical) condition of general covariance with respect to the theory of gravitation, but this condition can only be expressed exactly through the formalism of tensors and of absolute differential calculus, thanks to which the physical problem initially set can subsequently be solved[52].

Quantum mechanics pertains to this situation in many aspects, if one looks it as a theoretical construction rather than as an interpreted formalism: specific properties of quantum phenomena such as wave-corpuscle duality, indetermination of conjugated quantities, indistinguishibility of identical particles, non-local separability, are all contained in its fundamental formalism in terms of state vectors obeying the principle of superposition[53].

Another degree of the coincidence between the formal and the physical is reached when the preliminary phase of the elaboration is passed, and the formalism holds so to speak a function of substitution (one leaves physical thought for a formal thought as one advances in unknown lands). One then settles with full legitimacy inside this thought that is indissociably physical and formal, reasoning physically by the very means of the formal, up to the point where one says, with Einstein, that "it is in mathematics that the creative principle is to be found"[54].

The object built up or still in construction identifies then with the instrument by the very conditions of the conceptual construction, always submitted to the verdict of verification. One of the earliest examples of this new physical method of thoughtis found in Poincare's paper "The dynamics of the electron", in which he plans to formulate a relativistic theory (in the restricted or "special" meaning) of gravitation by constructing a Lagrangian from the condition that it be Lorentz invariant[55]. This procedure is henceforth a current one in theoretical physics, when it is wanted to construct the form of interaction fields. Einstein's works on General Relativity, then his and Weyl's and others' researches on the unified field, performed afterwards large advances in this way[56].

What justifies from then on the overlapping of the formal and the physical, of which present physics gives us many examples in several areas, is not the adoption of a panmathematical, metaphysical and arbitrary view *à la*

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[52] Paty (1993a), chapter 5.
[53] Paty (1986), 1992a).
[54] Einstein (1933).
[55] Poincaré (1905).
[56] Cf., for instance, Paty (1993a), chapter 5.

Minkowski[57]; it is not either a renunciation at the concern for the physical object in favour of purely formal problems as in "mathematical physics"; but it is indeed the encounter of "physical sense" with its privileged way of expression, that seems to go increasingly in the direction of the mathematical-formal, notwithstanding that the two remain always somewhat different, the formal and the physical never identify exactly.

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[57] Despite the fact that physicists contemporary to Minkowski considered his theory as more physical than Einstein's one, that seemed to them a more abstract one. But this was due to the dynamical versus kinematical aspect of the theories, which I have discussed elsewhere (Paty (1993a), and (in press c). Minkowski's space-time theory was clearly depending, in his mind, on the electromagnetic world picture.

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