Hedging tranches index products : illustration of model dependency

Dominique Guegan, Julien Houdain

Hedging Tranched Index Products: Illustration of the Model Dependency

J. HOUDAIN* **D. GUEGAN****

April 2006

Abstract

Synthetic CDOs have been the principal growth engine for the credit derivatives market over the last few years. The appearance of credit indices has helped the development of a more transparent and efficient market in correlation. This increase in volumes makes it necessary to use models of increasing diversity and complexity in order to model credit variables. Tranched index products are exposed to spread movements, defaults, correlation and recovery uncertainties. Hedging these risks requires an understanding of the sensitivities of the different tranches in the capital structure to these sources of risk. The dynamic hedging of index tranches presents dealers with two main challenges. First, the dealer must calculate the hedge positions (delta or hedge ratio) of the index or individual CDS or other index tranches. These deltas or hedge ratios are model-dependent, which leaves dealers with model risk. Second, the value of an index tranche depends on the correlation assumption used to price and hedge it. Since default correlation is unobservable, a dealer is exposed to the risk that his correlation assumption is wrong (correlation risk). In this paper, index tranches' properties and several hedging strategies are discussed. Next, the model risk and correlation risk are analyzed through the study of the efficiency of several factor-based copula models (like the Gaussian, the double-t and the double-NIG using implied correlation and a particular NIG one-factor model using historical correlation) versus historical data in terms of hedging capabilities. We comment on each model's underlying theoretical approach and then describe and analyze their computational complexity. We show that there is significant model and correlation risk in the credit derivatives market due to the discrepancies between models in terms of hedging results and to the frequent change in the tranches' behavior.

JEL *classification: G12, G13.*

Keywords: CDO, Hedging, Index tranches, Delta, Hedge ratio, Model dependency, Correlation, Correlation smile, Factor models, NIG.

*Corresponding author, **Fortis Investments**, Structured Finance Quantitative Research and Risk Manager, 23, rue de l'Amiral d'Estaing 75016 Paris. **École Normale Supérieure**, Cachan, CES Antenne Cachan, 61, avenue du Président Wilson, 94230 Cachan. Tél.: + 33 (1) 53 67 27 24, julien.houdain@fortisinvestments.com.

****École Normale Supérieure**, Cachan, Senior Academic Fellow de l'IEF, Full Professor, Head of the Economics and Management Department, 61, avenue du Président Wilson, 94230 Cachan - France. Tel: + 33 (1) 47 40 55 75 - Fax: + 33 (1) 47 40 24 60, guegan@ecogest.ens-cachan.fr.

"Slowly it began to dawn on me that what we faced was not so much risk as uncertainty. Risk is what you bear when you own, for example, 100 shares of Microsoft, you know exactly what those shares are worth because you can sell them in a second at something very close to the last traded price. There is no uncertainty about their current value, only the risk that their value will change in the next instant. But when you own an exotic illiquid option, uncertainty precedes its risk, you don't even know exactly what the option is currently worth because you don't know whether the model you are using is right or wrong. Or, more accurately, you know that the model you are using is both naive and wrong, the only question is how naive and how wrong." **Emanuel Derman**, 2004.

1 Introduction

Index tranches offer the opportunity to trade correlation products through their standardized form and liquidity. Tranched index products are similar in many respects to synthetic CDO tranches. These products are exposed to defaults, spread[1] movements, correlation and recovery uncertainties. Pricing and hedging[2] index tranches require advanced modeling and an understanding of the sensitivities of different tranches in the capital structure to these sources of risk.

Synthetic CDOs have been the principal growth engine for the credit derivatives market over the last few years. They create new, customized asset classes by allowing various investors to share the risk and return of an underlying portfolio of credit default swaps[3](CDS). Multiple tranches of the underlying portfolio are issued, offering investors various maturity and credit risk characteristics. Thus, the attractiveness to investors is determined by the underlying portfolio of CDS and the rules for sharing the risk and return. A synthetic CDO is often called "a correlation product" because, in simple words, it is a contract that references the default of more than one obligor. Investors in this product are buying correlation risk, or more exactly, joint default risk between several obligors. The underlying portfolio loss distribution directly determines the tranche cash flows and thus the tranche valuation.

The appearance of credit indices has helped the development of a transparent and efficient market in correlation. Typical examples of standardized credit indices are the DJ iTraxx Europe and the DJ CDX North American indices. Given that the underlying portfolio is agreed to and has a fixed maturity date, market-makers of these indices have also agreed to quote standard tranches on these portfolios from an equity or first loss tranche to the most senior tranche.

With a transparent, liquid market in iTraxx and CDX indices, tranche market participants can now optimize their credit views, either gaining exposure or hedging existing positions across different seniorities. Index tranches have grown in popularity because, on the surface, they offer several advantages. First, an investor can quickly gain exposure to 125 issuers. Second, general market spread risk can be separated from idiosyncratic risk, since the senior tranches are exposed to general spread widening while the junior tranches are exposed to specific company default risk. Finally, these products offer the prospect of going long credit risk (sell protection) or short credit risk (buy protection) at different points in the capital structure.

Equity tranches have become a very popular way of taking credit risk. Hedge funds[4] typically sell protection on an equity tranche and delta hedge the resulting exposure to the spreads and correlation of underlying names. Effective hedging begins with a clear specification of the hedge goals. There exist several different hedging strategies in the tranched index products market. For example, if we sell protection on a given tranche, we can hedge the position using the underlying index or some underlying individual credits, or even using another tranche. A most common trade over the past two years has been to take a long position in the equity tranche, and hedge or short the first mezzanine tranche. As long as the mezzanine tranche moves in line with the pricing model expectations, or as long as correlation

[1] With respect to credit derivative products, credit spread refers simply to the effective premium an investor would need to pay to receive CDS protection, or the effective premium an investor would receive to provide CDS protection.

[2] Hedging is the taking of offsetting risks.

[3] In its basic form, a credit default swap (CDS) is essentially a contract that transfers default risk from one party to another; the risk protection buyer pays the protection seller a premium (spread), usually in the form of quarterly payments.

[4] A largely unregulated investment fund that specializes in taking leveraged speculative positions.

between the equity and mezzanine tranches remains constant, the hedging strategy works fine, and the investor pockets the income differential. If, however, mezzanine and equity tranches move differently than the theoritical hedge ratio, the hedge fund can lose on both sides of the trade. A large market shock reveals that the assumptions underlying the models may have been inaccurate not only in size, but in sign.

A hedging strategy is model dependent, in the sense that the calculated deltas or hedge ratios between tranches are different from one model to the next. The dynamic hedging of index tranches presents dealers with two main challenges. First, the dealer must calculate the hedge positions (delta or hedge ratio) of the index or individual CDS or other index tranches. These deltas or hedge ratios are model-dependent, which leaves dealers with model risk. Second, the value of an index tranche depends on the correlation assumption used to price and hedge it. Since default correlation is unobservable, a dealer is exposed to the risk that its correlation assumption is wrong (correlation risk).

The Gaussian one-factor model has become the established way of pricing correlation products. The new availability of relatively liquid market levels has led to the price quotes of the tranches in terms of base correlation. Nevertheless, it is a well known fact that the pricing is not entirely driven with the Gaussian one-factor model as it does not provide an adequate solution for pricing simultaneously various tranches of an index, nor for adjusting correlation against the level of market spreads. In reality, the tranche pricing is also done using proprietary and often more complex and detailed models. We can find in the literature models that reproduce very well, or even perfectly, selected market prices for the different tranches of the same reference index (see Andersen and Sidenius (2003, 2005), Guégan and Houdain (2005), Hull and White (2004, 2005), Walker (2005, 2006)) and Burtschell *et al.* (2005). All these models are quite different and offer different advantages in term of hedging and risk managing capabilities.

In this paper, we analyze the performances of some models versus historical data in terms of hedging capabilities for index tranches and we show that there is significant model and correlation risk in the credit derivatives market due to the discrepancies between models in terms of hedging results and to the frequent change in the tranches' behavior.The remainder of the paper is organized as follows. Section 2 presents the typical structure of standard tranched index products. In Section 3, we introduce the most common index tranches hedging strategies. We highlight the main differences between the more popular pricing models in Section 4. Finally, in Section 5, we analyze the accuracy of each model versus historical data.

2 Tranched Index Products

The iTraxx Europe Main and the CDX North America Main are the most liquid CDS indices, trading in large size and at bid/offer spreads currently under 1 basis point per annum. Each Main index includes 125 issuers from their respective region. These issuers are investment grade[5] at the time an index series[6] is launched, with a new series launched every six months (roll[7]). In practice, "on the run"[8] Main indices are mostly composed of A-rated and BBB-rated issuers, about evenly split between BBB and higher quality ratings. One of the most significant developments in financial markets in recent years has been the creation of CDS index tranches.

Tranched index products differ from other synthetic CDOs in one key respect. The main advantage of index tranches is that they are standardized. Standardization applies to both the composition of reference pool and the structure of the tranches. Index tranches are quoted in the broker market on a daily basis. While synthetic CDOs are generally private buy-and-hold transaction for which information and liquidity are limited, investors can buy or sell index tranches. A transaction can be initiated with

[5]An asset whose credit rating is BBB or better.

[6]New index series are created semi-annually. Currently, the most recent CDX indices are Series 6, while the most recent iTraxx indices are Series 5. A new index series will have a maturity date slightly longer than the previous index series to allow the market to have a more-or-less constant maturity. A new series will typically have a slightly different collection of issuers than the previous series. For example, CDX Main Series 5 does not have General Motors Acceptance Company (GMAC), unlike CDX Main Series 4, as the entity was no longer investment grade at the time Series 5 was created.

[7]The roll refers to the process whereby investors and dealers trade out of the previous "on-the-run" index and into the new on-the-run index. Both indices roll every six months around 20^{th} September and 20^{th} March.

[8]The most recently issued series, usually the most liquid.

one dealer, and unwound with either the original dealer or with another one. For an introduction to standardized CDS indices and index tranches see Amato and Gyntelberg (2005).

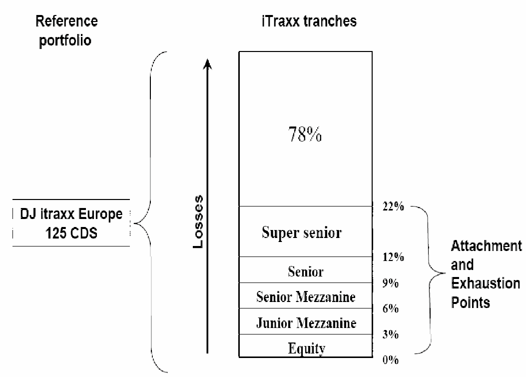

Figure 1: DJ iTraxx Europe tranches structure.

Figure 1 illustrates the iTraxx standard[9] tranches structure. Each tranche is defined by its attachment point which defines the level of subordination and its exhaustion (or detachment) point which defines the maximum loss of the underlying portfolio that would result in a full loss of tranche notional. The attachment and exhaustion points of the standard index tranches evolved to create instruments with distinct risk profiles. The first-loss 0-3% equity tranche is exposed to the first several defaults in the underlying portfolio. This tranche is the riskiest as there is no benefit of subordination but it also offers high returns if no defaults occur. The 3-6% and the 6-9% tranches, the junior and senior mezzanine, are levered in the underlying portfolio spread, but are less immediately exposed to the portfolio defaults. The 9-12% tranche is the senior tranche, while the 12-22% tranche is the low-risk super senior piece. The tranching of the indices in Europe and North America is different. In North America, the CDX index is tranched into standard classes representing equity 0-3%, junior mezzanine 3-7%, senior mezzanine 7-10%, senior 10-15% and super senior 15-30% tranche.

In return for bearing the risk of losses, sellers of protection receive a quarterly payment from buyers of protection equal to a premium times the effective outstanding amount of a given tranche[10]. The premiums on the mezzanine and senior tranches are running spread with no upfront payment. By contrast, the equity tranche is quoted in terms of upfront payment. Buyers of protection on an equity tranche make an upfront payment that is a percentage of the original notional of the contract, in addition to paying a running spread premium of 500 basis points. Figure 2 illustrates the evolution of the tranches' spread or upfront quotes for the iTraxx and the CDX since March 2004.

In order to evaluate the fair spread of a tranche, we need to determine a tranche loss function and relate it to the portfolio loss experienced within a generic interval. In other words, we have to calculate the loss of the tranche conditional on the loss of the underlying portfolio. If a certain percentage portfolio loss l occurs, the impact on the tranche holder's position will be driven by the attachment L^- and detachment L^+ point of the tranche.

The tranche loss function TL is defined as:

$$TL_t^{L^-,L^+} = \frac{max[min(l_t, L^+) - L^-, 0]}{L^+ - L^-}. \tag{1}$$

At any time t, this function, given any portfolio loss l, provides the corresponding loss suffered by the tranche holder.

[9] Trading index tranches is not limited to standard tranches currently on offer. Investors can create their own tranches in a negotiated transaction.

[10] The effective notional is the original notional less any losses incurred due to defaults that have impacted on the tranche

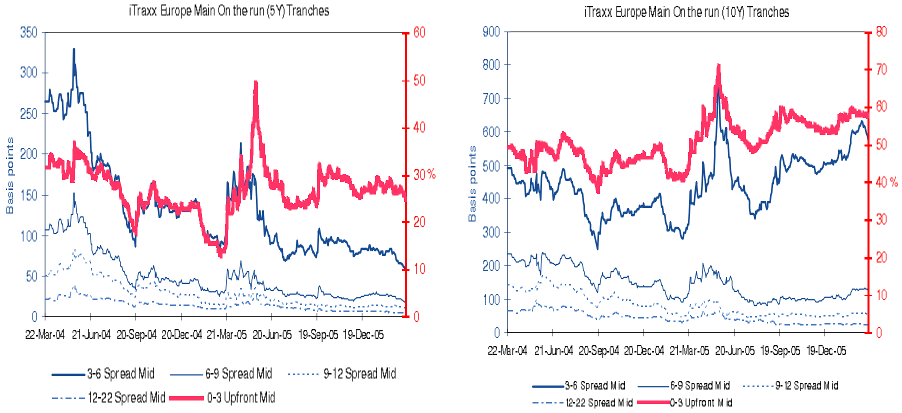

Figure 2: Evolution of the tranches' spread or upfront since March 2004 on the iTraxx market.

The next step consists of determining the expected tranche loss for any given time. This step is crucial in computing the tranche breakeven spread, which depends on the outstanding notional at each payment date. Thus the tranche expected loss at a given time t is defined as fallow:

$$E[TL_t^{L^-,L^+}] = \sum_{l_t} TL_t^{L^-,L^+} \times P[Loss = l_t]. \tag{2}$$

Given the expected term structure of the tranche loss distribution, it is then possible to proceed with the estimation of the breakeven spread. Similar to the more traditional single name CDS contracts, an index single tranche can be represented by two legs:

- *Premium Leg*: the premium leg of a CDO tranche consists of payments that are proportional to the difference between the tranche notional and the cumulative default losses.

- *Default Leg*: the default leg is a stream of payments that cover portfolio losses as they occur, given that the cumulative losses are larger than L^- but do not exceed L^+.

With regard to the premium leg PL, we can formally express it as:

$$PL^{L^-,L^+} = \sum_{i=1}^{N} Spread \times (1 + r_{t_i})^{-t_i} \times \left[Notional - E[TL_{t_i}^{L^-,L^+}]\right] \times \triangle T_i, \tag{3}$$

where N is the total number of premium payments, which depends on the maturity of the tranche and the payment frequency, t is the time in years[11], $\triangle T_i = t_i - t_{i-1}$, r is the swap rate, and $Notional$ is the original notional of the tranche. Equation 3 shows that the premium leg is computed by discounting back the tranche premium adjusted for the notional outstanding at each payment date.

As far as the default leg is concerned, the computation effort is heavier, since theoretically the default leg should be computed by averaging the present value of payments over all the set of possible default times. In practice a discrete set of default dates is chosen (daily, monthly or yearly spaced) and it is assumed that a default can occur only upon the specified set of default dates. If we impose that a default can occur only upon the set of payment dates, we can then write down the expression of the default as follows:

$$DL^{L^-,L^+} = \sum_{i=1}^{N} (1 + r_{t_i})^{-t_i} \times \left[E[TL_{t_i}^{L^-,L^+}] - E[TL_{t_{i-1}}^{L^-,L^+}]\right], \tag{4}$$

[11] Actual/360.

The default leg is therefore expressed as the discounted value of the marginal tranche loss over each payment interval.

The breakeven spread is then computed by exploiting the initial equivalence relationship between the premium and default leg. Thus we can express the current fair breakeven spread as:

$$Spread_{Current} = \frac{\sum_{i=1}^{N}(1+r_{t_i})^{-t_i} \times \left[E[TL_{t_i}^{L^-,L^+}] - E[TL_{t_{i-1}}^{L^-,L^+}]\right]}{\sum_{i=1}^{N}(1+r_{t_i})^{-t_i} \times \left[Notional - E[TL_{t_i}^{L^-,L^+}]\right] \times \triangle T_i}. \tag{5}$$

The theory and calculation method behind standard tranches valuation are similar to those of a CDS, see O'Kane and Turnbull (2003) for a complete description.

In order to determine the *Mark-to-Market* (MTM) of a given tranche we need to introduce the *Risky Duration* (DV01). We define the Risky Duration of a tranche as the expected present value of 1 basis point paid on the premium leg until default or maturity, whichever is sooner.

$$DV01_{Current} = \sum_{i=1}^{N}(1+r_{t_i})^{-t_i} \times \left[Notional - E[TL_{t_i}^{L^-,L^+}]\right] \times \triangle T_i. \tag{6}$$

The $DV01$ essentially reflects the market's expectation of time to default. For example, a wider spread implies a lower risky duration and vice-versa. Thus, the *Mark-to-Market* of a tranche is expressed as:

$$MTM = (Spread_{Current} - Spread_{Initial}) \times DV01_{Current}. \tag{7}$$

Having introduced standard index tranches, we now move on to presenting the most common hedging strategies that are used for these products.

3 Hedging Tranched Index Products

Tranched products are exposed to spread movements, defaults, correlation and recovery uncertainties. Hedging these exposures requires an understanding of the sensitivities of different tranches in the capital structure to these sources of risk. Equity tranches have become a popular way of taking levered credit risk. Hedge funds typically sell protection on an equity tranche and delta hedge the resulting exposure. Thus, in this section, we mainly focus on the hedging of a long position on an equity tranche. For a detailed analysis of hedging strategies see Calamaro *et al.* (2004).

The simplest trading strategy involves a naked position[12] in an index tranche. As with all unhedged trading strategies, an outright position implicitly expresses a market view. For instance, a long protection position in the equity tranche expresses a bullish[13] view on spreads and defaults.

There are three common approaches to hedge an index tranche. First, many investors use the underlying index to delta-hedge. In fact, index tranches are often quoted with delta exchange, which means the prices are based on the assumption that the transaction has two legs: the tranche and the delta amount of the index. The theoretical delta of a tranche versus an index is the sensitivity of its price or mark-to-market to a 1 basis point parallel shift[14] in the underlying spread curve of the index. Thus the theoretical delta of tranche versus an index is expressed as:

$$Delta_{index} = \frac{\Delta MTM_{tranche}}{1bp_{index} \times DV01_{index}}. \tag{8}$$

When hedging a tranche with the index, the net position has residual first-order exposures. For example, equity is underhedged against movements in high spreads names with mostly idiosyncratic risk and

[12] An outright short or long position in a tranche.

[13] Believing that a particular financial asset, a sector, or the overall market is about to rise. Opposite of bearish.

[14] With reference to spread curve movements, a parallel shift is an equal shift of the whole curve either upwards or downwards.

overhedged against low spread names with mostly systemic risk.

Second, investors can use single name CDS to hedge their position. The theoretical delta of a tranche versus an individual credit spread is the sensitivity of its price or mark-to-market to a 1 basis point parallel shift in the underlying spread curve of the single name CDS. Thus the theoretical delta of a tranche versus a single name CDS is expressed as:

$$Delta_{CDS} = \frac{\Delta MTM_{tranche}}{1bp_{CDS} \times DV01_{CDS}}. \tag{9}$$

There are two major determinants of single name deltas: spread and correlation. Higher spread names have a higher implied default probability, and thus are likely to have more impact on the junior tranches. The opposite is true for low spread names, which are more important to senior tranches. The second determinant of delta is correlation. On the first hand, credits that are more correlated to the market are less important to the equity tranche. On the other hand, credits that are uncorrelated to the market as a whole represent an idiosyncratic risk and have the potential to hurt the equity tranche because they can widen materially or even default without a corresponding effect across the market. The delta of the equity tranche to these credits is high.

Third, the most common trade over the past two years has been to take a long position in the equity tranches, and hedge or short the first mezzanine tranche. Investors delta hedge with other tranches rather than an index to increase the carry. The theoretical hedge ratio between two tranches can be expressed as:

$$HedgeRatio_{1,2} = \frac{\Delta MTM_{tranche_1}}{1bp_{index} \times DV01_{index}} / \frac{\Delta MTM_{tranche2}}{1bp_{index} \times DV01_{index}}. \tag{10}$$

This is the ratio between the sensitivity of each tranche mark-to-market to a 1 basis point parallel shift in the underlying spread curve of the index. In reality, the hedge ratio is commonly calculated using a proportional shift of 5 or 10 basis points in the underlying spread curve of the index. The investor receives more premium income by hedging with a leveraged product in exchange for assuming the risk that the market may perform differently from model expectations. As long as the mezzanine tranche moves in line with the model expectations, or as long as correlations between the equity and mezzanine tranches remain constant, the delta hedge works fine, and the investor pockets the income differential. If, however, mezzanine and equity tranches move differently than the assumed hedge ratio, the hedge fund can lose on both sides of the trade. In fact, correlation has changed and tranche levels are not moving as models had predicted. A large market shock reveals that the assumptions underlying the models may have been inaccurate not only in size, but in sign.

As mentioned in Gibson (2004), the dynamic hedging of index tranches presents dealers with four challenges. First, the dealer must calculate the hedge positions (deltas or hedge ratios) of the index or given CDS in the reference index portfolio or other tranches. These deltas or hedge ratios are model-dependent, thus index tranches leave dealers with model risk. Second, the value of a index tranche depends on the correlation assumption that is used to price and hedge it. Since default correlation is unobservable, a dealer is exposed to the risk that his correlation assumption is wrong ("correlation risk"). Third, as deltas change over time and the dealer dynamically adjusts his hedges, he is exposed to liquidity risk. The credit default swap market may not have enough liquidity for the dealer to adjust its hedge as desired without incurring high trading costs. Fourth, a dealer prefers to be hedged against both small moves in credit spreads (spread risk) and unexpected defaults (jump-to-default risk). Hedging against both risks adds complexity.

In the two next Sections we focus on the first two risks previously mentioned, the model risk and the correlation risk. We exhibit their implications by comparing empirical results versus historical data.

4 The pricing models under study

In this Section we briefly expose the main properties of four factor models[15] used to price CDO tranches: the Gaussian, the double-t, the double-NIG and a particular NIG one-factor model.

Factor models represent an useful and efficient framework to model the dependency structure for portfolios underlying most synthetic CDOs. They are used to derive portfolio loss distributions. The idea behind factor models is to break down the firms' asset values into a risk component that is idiosyncratic to the asset, plus one or a number of factors that are systematic to all assets in the portfolio. Thus, defaults of different firms in the credit portfolio are independent conditional on a common market factor.

A well-known approach used to derive loss distribution is the large homogenous portfolio approximation (LHP). It is assumed that it is possible to approximate the real reference credit portfolio with a portfolio consisting of a large number of equally weighted identical instruments (having the same term structure of default probabilities, recovery rates, and correlations to the common factor). This LHP limit approximation employing the Law of Large Numbers was first proposed by Vasicek (1987) . Because spreads and correlation with the common factor should be different between obligors, we prefer to use another approach (non homogeneous) than the LHP model which seems to be too restrictive regarding the importance of the spreads, recovery rate and correlation dispersion for the calculation of prices, deltas and hedge ratios of tranched index products. We refer to Beinstein *et al.* (2005) for a complete study of the LHP versus the non homogeneous methodology. In the following approach we use each obligor's spread curve and sensitivity to the common factor.

For $i = 1, ..., n$ ($n \in \mathbf{N}$), we define V_i the i^{th} firm's asset value as:

$$V_i = \rho_i Z + \varepsilon_i \sqrt{1 - \rho_i^2}. \tag{11}$$

where:

- Z is the common factor of the model,
- ε_i is the idiosyncratic risk of the i^{th} firm,
- Z and ε_i are independent random variables with zero-mean and unit-variance absolutely continuous distribution function with respect to Lebesgue measure,
- ρ_i[16] represents the sensitivity of V_i to Z with respect to $-1 \leq \rho_i \leq 1$,
- and we denote respectively φ and ϕ as the probability density functions of the random variables V_i and Z.

In order to determinate the conditional default probability of the i^{th} firm we assume that a firm i defaults when its asset value hits the barrier k_i ($k_i \in R$). Thus, looking at the expression of the i^{th} firm's asset value given in equation (11), we argue that for specific realizations of the common factor Z, the i^{th} firm defaults as soon as its asset value hits the default barrier k_i. We denote $P_i(Z)$ the conditional default probability of the i^{th} firm, then for $Z = z$:

$$P_i(z) = Prob[V_i \leq k_i | Z = z]. \tag{12}$$

Using equation (11), the expression (12) becomes:

$$P_i(z) = Prob\left[\varepsilon_i \leq \frac{k_i - \rho_i z}{\sqrt{1 - \rho_i^2}} | Z = z\right]. \tag{13}$$

[15] The use of factor models in credit risk management is reportedly due to Vasicek (1987). This approach is also used in Belkin et al. (1998), Finger (1999), Schonbucher (2000), and Frey, McNeil and Nyfeler (2001). The pricing of CDOs using factor models has been also studied by Andersen, Sidenius and Basu (2003), Laurent and Gregory (2003) and similar techniques were later proposed by Hull and White (2004) and Guégan and Houdain (2005).

[16] The correlation between the two random variables V_f and V_j defined previously is given, for $1 \leq f \leq n$ and $1 \leq j \leq n$, by: $Corr[V_f, V_j] = E[V_f V_j] = \rho_f \rho_j E[Z^2] = \rho_f \rho_j$.

We denote $\mathbf{C}_{\varepsilon_i}$ the cumulative distribution for the random variable ε_i, then the probability given in (13) for $Z = z$ and for $i = 1, ..., n$ is equal to:

$$P_i(z) = \mathbf{C}_{\varepsilon_i}\left[\frac{k_i - \rho_i z}{\sqrt{1-\rho_i^2}}\right]. \tag{14}$$

In order to determine the real value of the default barrier k_i for each firm i, we denote Q_i the unconditional default probability of the i^{th} firm. Q_i is recovered from CDS market data using an intensity-based model[17] or by bootstrapping the CDS spread curve of the underlying firm. The default barrier k_i is equal to $\varphi^{-1}(Q_i)$. When it is impossible to determine the distribution of the random variable V_i then we need to use the following approach. By definition we have:

$$Q_i = Prob[V_i \leq k_i] = E[P_i(Z)]. \tag{15}$$

If ϕ represents the density function of the random variable Z, then:

$$E[P_i(Z)] = \int_{-\infty}^{+\infty} P_i(z)\phi(z)dz, \tag{16}$$

which implies that:

$$Q_i = \int_{-\infty}^{+\infty} \mathbf{C}_{\varepsilon_i}\left[\frac{k_i - \rho_i z}{\sqrt{1-\rho_i^2}}\right]\phi(z)dz. \tag{17}$$

Now, to determine the default barrier of the i^{th} firm we need to solve in k_i the Equation 17.

Under such modeling assumptions, it is possible to determine the underlying portfolio loss distribution by numerically computing the inverse Fast Fourier Transform[18] or some other inversion method. Because of computation complexity of the Fast Fourier Transform, we present in this paper an alternative based on a recursive methodology introduced by Galiani *et al.* (2004).

We define $\Omega_{n,z}^k$ as the conditional aggregate probability of having k defaults in an n-credit portfolio. For a one-credit portfolio and a specific realisation of Z, the probability $\Omega_{1,z}^1$ of having one default equals $P_1(z)$ whereas the probability $\Omega_{1,z}^0$ of having no default is equal to $1 - P_1(z)$. Regarding a two-credit portfolio and given the conditional independence of the defaults, the probability $\Omega_{2,z}^0$ of having no default is equal to $(1 - P_1(z))(1 - P_2(z))$ and the probability of having two defaults $\Omega_{2,z}^2$ is equal to $P_1(z)P_2(z)$. The probability $\Omega_{2,z}^1$ of having only one default is less intuitive and is equal to $\Omega_{1,z}^1(1-P_1(z))+\Omega_{1,z}^0 P_2(z)$. This recursive approach can be extend to any n-credit portfolio using the following recursive algorithm:

$$\Omega_{n+1,z}^k = \Omega_{n,z}^k(1 - P_{n+1}(z)) + \Omega_{n,z}^{k-1}P_{n+1}(z). \tag{18}$$

The unconditional distribution is given by:

$$\Omega_{n+1}^k = \int_{-\infty}^{+\infty} \Omega_{n+1,z}^k \phi(z)dz. \tag{19}$$

Using the previously described properties we can derived a loss distribution for each payment date. Now we are going to describe different approaches that have been generated to fit the observed market prices of index tranches using factor models.

4.1 The Gaussian one-factor model

The Gaussian one-factor model has become the established way of pricing tranched index products. The new availability of relatively liquid market levels has led to the price quotes of the tranches in terms

[17] The first published intensity model appears to be Jarrow and Turnbull (1995). Subsequent research includes Duffie and Huang (1996), Jarrow, Lando and Turnbull (1997) and Duffie, Singleton (1997a, 1997b) and Schonbucher and Schubert (2001). The fundamental idea of the intensity-based model framework is to model the default probability as the first jump of a Poisson process.

[18] See Press et al. (1992).

of implied correlation. In this approach, regarding Equation (11), the common factor Z and the idiosyncratic risks ε_i are assumed to be Gaussian independent random variables, the recovery rate of each firm is assumed to be 40%, and ρ_i is a constant parameter. Thus, the correlation between underlying the obligors is assumed to be the same. This correlation is the model variable and is implied by the market prices. The Gaussian one-factor model does not provide an adequate solution for pricing simultaneously various tranches of an index, nor for adjusting correlation against the level of market spreads.

The first way to calculate an implied tranche correlation is to calculate the flat correlation that reprices each tranche to fit market prices. This method computes what is known as compound correlation. Calculating compound correlation is straightforward, requiring a simple one-dimensional root searching algorithm. This works fine in almost all cases. However, there is sometimes a problem in that either we cannot find a solution or that we get two solutions. The shape of the compound correlation has become known as the correlation "smile". This is because the compound correlation is higher for the equity and senior tranches than it is for the mezzanine tranches. Another disadvantage of this method is that it is not possible to extend compound correlation to the pricing of non-standard tranches.

Initially the market chose compound correlation as its quotation convention. Since one year and a half, base correlation[19] has become more widely used in the market. The fundamental idea behind the concept of base correlation is that all tranches are decomposed into combinations of base tranches, where a base tranche is simply another name for an equity tranche. Consider a first mezzanine tranche with lower and upper attachment points L^- and L^+. The same risk position can be obtained by being short the equity tranche with upper attachment point L^- and long the equity tranche with upper attachment point L^+. While for compound correlation we calculate the flat correlation required for each tranche to match the market spreads, for base correlation we value any tranche as the difference between two base tranches. We then calculate the flat correlation required for each base tranche so that we match the observed market spreads. Base correlation produces a correlation "skew". The way that base correlation is defined means that the base correlation calculated for each tranche is linked to the base correlation of the tranche below. There is always either one solution or no solution. The situation of having two solutions never arises. See O'Kane and Livesey (2004) for a complete description of compound and base correlation.

One of the drawbacks of this Gaussian approach is, as we mentioned previously, that it cannot fit the prices of the different tranches with one single correlation coefficient. As a result, the method generates a correlation smile or skew. Thus, researchers and practitioners proposed the use of heavy tail distributions such as the Student-t distribution or even heavy tail and skewed distributions like Normal Inverse Gaussian distributions to correct this problem.

4.2 The double-t one-factor model

This model is an extension of the Gaussian one-factor model using the LHP model. It has been considered for the pricing of CDOs by Hull and White (2005). The underlying idea is to correct the correlation smile or skew by using a distribution with fat tails, hoping that a single distribution reprices all tranches silmutaneously. In this approach, regarding Equation (11), the common factor Z and the idiosyncratic risks ε_i are Student-t independent random variables and ρ is a constant parameter such as the correlation is the same between each firm. Equation (11) needs to be adjusted in order to respect the fact that Z and ε_i are assumed to be independent random variables with zero-mean and unit-variance. Thus using this approach, for $i = 1, ..., n$ ($n \in \mathbf{N}$), we assume that V_i the i^{th} firm's asset value is explained by the following model:

$$V_i = \rho_i \left(\frac{\nu_z - 2}{\nu_z}\right)^{1/2} Z + \sqrt{1-\rho_i^2} \left(\frac{\nu_{\varepsilon_i} - 2}{\nu_{\varepsilon_i}}\right)^{1/2} \varepsilon_i, \tag{20}$$

where ν_z and ν_{ε_i} are the degrees of freedom of the respective Student-t distributions.

It has been demonstrated in Hull and White (2005) and in Burtshell *et al.* (2005) that this model can give a good fit of the market prices with almost the same compound correlation level for each tranche. But if we want to perfectly fit the quotes, as in the Gaussian case, this approach produces implied correlation

[19] The base correlation has been introduced by JP Morgan.

because the correlation parameter ρ becomes one of the three variables of the model. The two other variables are the degrees of freedom of the Student-t distributions of Z and of the ε_i. In this paper, we have implemented this model using the non homogeneous methodology previously described in Section 4 and constant 40% recovery rate.

4.3 The NIG one-factor model with historical correlations

This model has been introduced by Guégan and Houdain (2005). The underlying idea of this approach is to use the price quotes of the tranches available in the market to determine the implied NIG-distribution of the common factor for a given input correlation level. In the market standard methodology, the distributions used in the factor model are fixed and the implied correlation of a tranche is calculated under these assumptions. We have used an opposite approach: we fix the correlation of the portfolio and the distribution of the idiosyncratic risk of each firm and then we determine the implied NIG-distribution of the common factor in the model.

The Normal Inverse Gaussian[20] distribution has the remarkable property of being able to represent stochastic phenomena that have heavy tails and/or are strongly skewed. This distribution is characterized by 4 parameters $(\alpha, \beta, \mu, \delta)$. The parameter α is related to steepness, β to symmetry, and μ and δ respectively to location and scale. The $NIG(\alpha, \beta, \mu, \delta)$ Probability Density Function is given by:

$$NIG(x; \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta) q \left(\frac{x-\mu}{\delta} \right)^{-1} K_1 \left(\delta \alpha q \left(\frac{x-\mu}{\delta} \right) \right) e^{\beta x}, \tag{21}$$

with:

$$q(x) = \sqrt{1+x^2}, \tag{22}$$

and,

$$a(\alpha, \beta, \mu, \delta) = \pi^{-1} \alpha \exp(\delta \sqrt{\alpha^2 - \beta^2 - \beta\mu}). \tag{23}$$

Here K_1 is a modified Bessel function of the third kind with index 1 defined as:

$$K_1(x) = x \int_1^{\infty} \exp(-xt) \sqrt{t^2-1} dt. \tag{24}$$

The necessary conditions for a non-degenerated density are $\delta > 0$, $\alpha > 0$ and $\frac{|\beta|}{\alpha} < 1$. The Moment Generating Function of a NIG-distributed random variable X is equal to:

$$M_X(u) = \exp(u\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+u)^2})). \tag{25}$$

From equation (25) we can derive:

$$E[X] = \mu + \delta \left(\frac{\beta}{\gamma} \right), \tag{26}$$

$$Var[X] = \delta \left(\frac{\alpha^2}{\gamma^3} \right), \tag{27}$$

$$Skew[X] = 3 \left(\frac{\beta}{\alpha} \right) \left(\frac{1}{(\delta\gamma)^{1/2}} \right), \tag{28}$$

$$Kurt[X] = 3 \left(1 + 4 \left(\frac{\beta}{\alpha} \right)^2 \right) \left(\frac{1}{(\delta\gamma)} \right), \tag{29}$$

[20] The NIG distribution was introduced to investigate the properties of the returns from financial markets by Barndorff-Nielsen (1997). Since then, applications in finance have been reported in several papers, both for the conditional distribution of a GARCH model (Jensen and Lunde (2001); Forsberg and Bollerslev (2002); Venter and de Jongh (2002)) and for the unconditional distribution (Prause (1997); Rydberg (1997); Bolviken and Benth (2000); Lillestol (1998-2000-2001-2002); Venter and de Jongh (2002); Guegan and Houdain (2005)).

with $\gamma = \sqrt{\alpha^2 - \beta^2}$. From equations (28) and (29), and since $\frac{|\beta|}{\alpha} < 1$, we obtain some properties of the kurtosis and the skewness:

$$Kurt[X] > 3, \tag{30}$$

and,

$$3 + \frac{5}{3}(Skew[X])^2 < Kurt[X]. \tag{31}$$

To respect the fact that Z and ε_i are independent random variables with zero-mean and unit-variance the parameters α, β, μ and δ need to be scaled.

One of the main properties of the NIG distribution class are the scaling property:

$$X \sim NIG(\alpha, \beta, \mu, \delta) \Rightarrow \varsigma X \sim NIG(\frac{\alpha}{\varsigma}, \frac{\beta}{\varsigma}, \varsigma\mu, \varsigma\delta). \tag{32}$$

Another important property of the Normal Inverse Gaussian distribution is its behavior under convolutions. Indeed, if X_1 and X_2 are independent and distributed respectively as $NIG(\alpha, \beta, \mu_1, \delta_1)$ and $NIG(\alpha, \beta, \mu_2, \delta_2)$. Then $X_1 + X_2$ is a $NIG(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2)$.

Using the previously described properties of the NIG distribution it is possible to fit the market quote of each tranche with the same correlation structure. In the real world the asset value of each firm would have a different correlation with the asset value of each other firm in the portfolio. In this methodology the correlation between the underlying assets of the portfolio becomes an input. In this paper, we estimate the pairwise correlations using the methodology proposed by Andersen, Sidinius and Basu (2003) and the correction proposed by Jackel (2005) based on equity returns correlation [21] and Frobenius norm to determine the factor loadings. We also use the recovery rates provided by the rating agencies. For more details please refer to Guégan and Houdain (2005).

It has been demonstrated that this model seems to be the best alternative if we want to price a non standard or bespoke tranche CDO. Thus it would be interesting to see if the model should be a good alternative to hedge standard tranche index products.

4.4 The double-NIG one-factor model

The double-NIG one-factor model is also an extension of the Gaussian one-factor model using the LHP model. It has been considered for the pricing of CDOs by Kalemanova *et al.* (2005). The underlying idea is to correct the correlation smile or skew by using a heavy tail and skewed distribution. In this approach, regarding Equation (11), the common factor Z and the idiosyncratic risks ε_i are Normal Inverse Gaussian independent random variables and ρ is a constant parameter such as the correlation is the same between each firm. As in the Double-t model, if we want to perfectly fit the quotes, this approach produces implied correlation.

The underlying idea of the double-NIG one-factor model is to benefit from the two last described properties of the NIG distribution and then to easily compute the default barrier of each firm $k_i = \varphi^{-1}(Q_i)$. In this paper, we have implemented this model using the non homogeneous methodology previously described in Section 4 and constant 40% recovery rate.

[21] On a one year period.

4.5 Summary of the studied models

We can summarize the specificities of each model as fallow:

Assumption	LHP (Gaussian, Student-t, NIG)	Non Homogeneous (Gaussian, Student-t, NIG) with Base correlation	Non Homogeneous (NIG) with Historical correlation
Index portfolio	Infinite obligors	Actual number of obligors	Actual number of obligors
Spread curves shape	Flat curves	Full curves	Full curves
Underlying spreads	Single Index spread	Individual obligor spreads	Individual obligor spreads
Correlation structure	Identical pairwise correlation	Identical pairwise correlation	Historical pairwise correlation
Recovery rate	40%	40%	Rating agencies recovery rates

Table 1: Summary of the different models.

In the next section we compare the hedging capabilities of these different non homogeneous factor models.

5 Models outputs versus historical data

It has been demonstrated in the literature that the previously described models can fit the tranches' market quotes. Thus, in this section we are going to study their differences in terms of hedging capabilities. In order to compare our results with historical data, we will study the behavior of the iTraxx series 3 and 4 and its $0-3\%$ and $3-6\%$ underlying tranches from the 21^{th} of March 2005 to the 20^{th} of March 2006. The idea is to offset the Mark-to-Market risk of a long position in the equity tranche with a well chosen short position in the first mezzanine tranche. For simplicity we assume that tranches are traded without the index delta exchange and that the equity piece is traded in running spread without upfront payment. For the equity tranche, running spread and upfront are linked by the following relation:

$$Spread_{running} = \frac{Upfront}{DV01_{tranche}} \times 100 + 500. \tag{33}$$

In Figure 3, we illustrate the daily MTM variations of the two less subordinated iTraxx underlying tranches from the 20^{th} of March 2005 to the 20^{th} of March 2006. The index tranche market experienced a great deal of volatility in the second week of May 2005, after S&P downgraded General Motors and Ford to junk status which lead to an extensive spread widening in the CDS indices. These two North American companies were not part of the iTraxx so we have been, during this period, the witnesses of a very important contagion effect between the North American tranche market and the European one. Some practitioners and researchers have tried and are currently trying to explain this phenomena or more exactly this major crisis. One of the principal explanations is that just after these downgrades, hedge funds began to reduce leverage in the index tranche market and reversed the existing behavior between the tranches' returns. By the way, for the purpose of this paper we just have to keep in mind that during May 2005 the behavior of the European tranches has changed but not as much as the North American tranches.

Regarding the studied strategy, the underlying idea of hedging the equity tranche with the first mezzanine one is to find the best hedge ratio between these two tranches. Figures 3 illustrates the fact that the sensitivity of the tranches is quite different regarding the maturity and the well known fact that for a 10 year maturity the first mezzanine tranche behaves more like an equity piece. This behavior should be explained by the fact that the expected loss of the iTraxx 5Y is arround 1.7% on the studied period but arround 5.7% for the iTraxx 10Y which means that the first mezzanine tranche with an attachment point at 3% is more likely to be eaten for a 10 years maturity.

Figures 4 and 5 present the scatter plots of the daily MTM variations of the $0-3\%$ and $3-6\%$ tranches for a 5 and 10 years maturity from the 21^{th} of March 2005 to the 20^{th} of September 2005 for the iTraxx Series 3 and from the 21^{th} of September 2005 to the 20^{th} of March 2006 for the iTraxx Series 4. We can see some extreme events in the iTraxx Series 3 due to the May 2005 crisis. We can see also that the historical hedge ratios are easily observable but should be quite volatile in time. Thus, in Table 2 and 3 we present the historical hedge ratios between the equity and the first mezzanine tranches of the iTraxx for a 5 and 10 years maturity on monthly, quarterly and semestrial time intervals which should

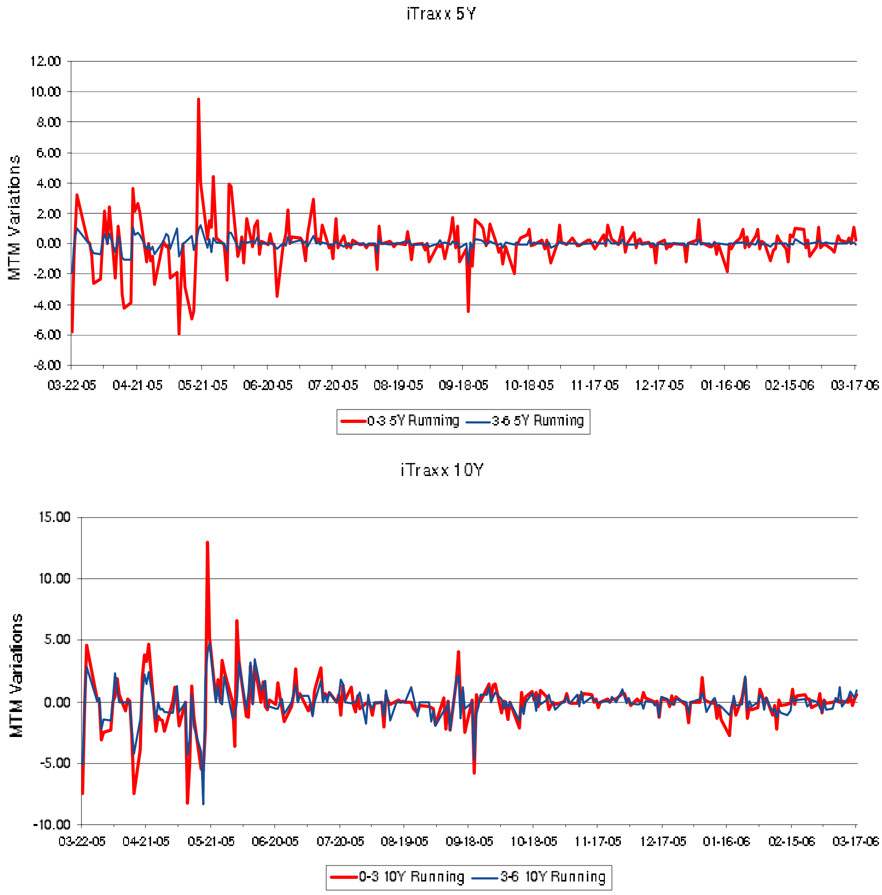

Figure 3: Evolution of the tranches' Mark-to-Market since March 2005.

be respectively analyzed as short term, mean term and long term hedging strategy. These hedge ratios are the slope in a regression of daily MTM variations in the tranches. We compare this historical results with the ratios given by the several models previously described in Section 4. In columns 4, 6, 8 and 10 we have calculated the absolute difference between historical hedge ratios and model dependent hedge ratios. For each time interval we present the average of these absolute differences. Note that for the calculation of the hedge ratios we used Equation 10 with a 5 basis points proportional shift in the underlying spread curve of the iTraxx. We chose a 5 basis points proportional shift because it gives better results than a parallel or another proportional shift for all the models comparing to historical hedge ratios. It is in line with the market that the curves may be perturbed proportionally, with wider spread names moving more than tight spread ones, to reflect better typical spread moves.

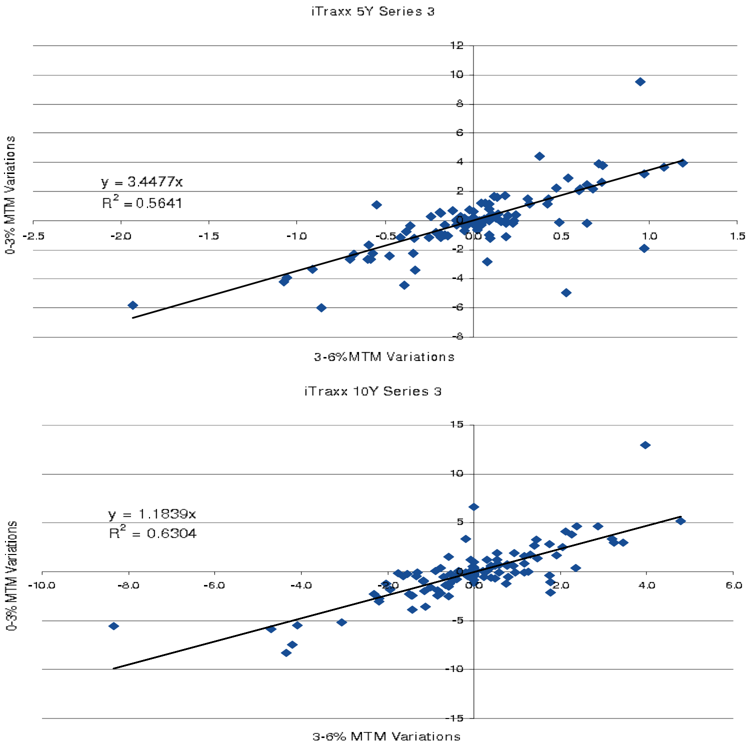

Figure 4: Slope of MTM variations of the $0-3\%$ and $3-6\%$ tranches for the iTraxx Series 3 5Y and 10Y.

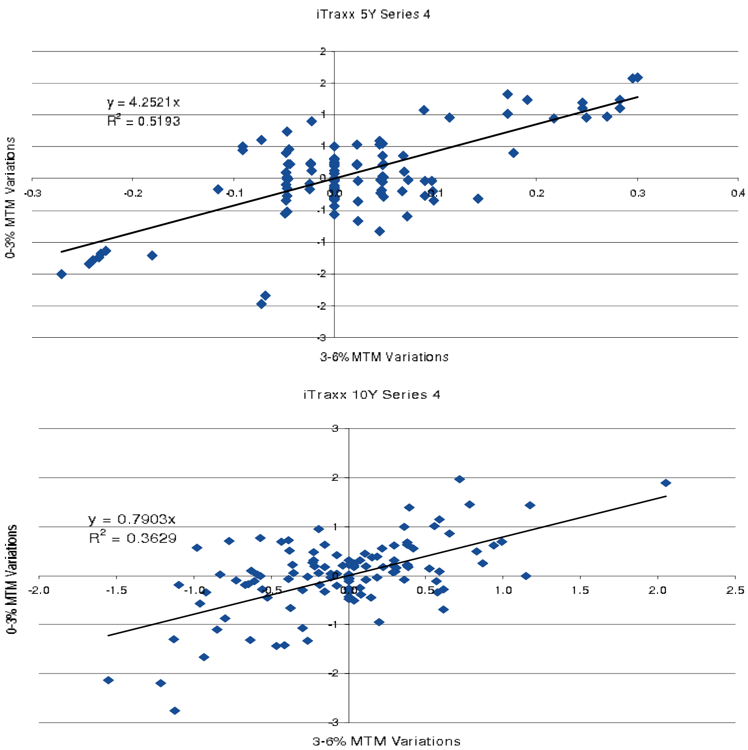

Figure 5: Slope of MTM variations of the $0-3\%$ and $3-6\%$ tranches for the iTraxx Series 4 5Y and 10Y.

	Market	Base Correlation						Historical Correlation	
Period	Slope	Gaussian	Abs Diff	Double-t	Abs Diff	Double-NIG	Abs Diff	NIG	Abs Diff
Mar-Apr05	3.38	4.38	1.00	4.09	0.71	4.19	0.81	3.89	**0.51**
Apr-May05	3.62	4.45	0.83	4.16	0.54	4.26	0.64	3.95	**0.33**
May-Jun05	3.77	4.38	0.61	4.10	0.33	4.20	0.43	3.89	**0.12**
Jun-Jul05	4.76	4.59	**0.17**	4.29	0.47	4.40	0.36	4.08	0.68
Jul-Aug05	2.29	4.49	2.20	4.20	1.91	4.30	2.01	3.99	**1.70**
Aug-Sep05	3.47	4.63	1.16	4.33	0.86	4.44	0.97	4.11	**0.64**
Sep-Oct05	5.07	4.77	**0.30**	4.46	0.82	4.57	0.5	4.24	0.83
Oct-Nov05	3.54	4.75	1.21	4.44	0.90	4.55	1.01	4.22	**0.68**
Nov-Dec05	4.06	4.80	0.74	4.48	0.42	4.60	0.54	4.26	**0.20**
Dec-Jan06	5.26	4.82	**0.44**	4.51	0.75	4.62	0.64	4.28	0.98
Jan-Feb06	3.80	4.73	0.93	4.42	0.62	4.53	0.73	4.20	**0.40**
Feb-Mar06	3.80	4.80	1.65	4.40	1.25	4.50	1.35	4.18	**1.03**
Average Abs Diff			0.94		0.78		0.83		**0.68**
Mar-Jun05	3.56	4.38	0.82	4.09	0.53	4.19	0.63	3.89	**0.33**
Jun-Sep05	3.69	4.59	0.90	4.29	0.60	4.40	0.71	4.08	**0.39**
Sep-Dec05	4.48	4.77	0.29	4.46	**0.02**	4.57	0.09	4.24	0.24
Dec-Mar06	4.08	4.82	0.74	4.51	0.43	4.62	0.54	4.28	**0.20**
Average Abs Diff			0.69		0.40		0.49		**0.29**
iTraxx3	3.56	4.38	0.82	4.09	0.53	4.19	0.63	3.89	**0.33**
iTraxx4	4.32	4.77	0.45	4.46	0.14	4.57	0.25	4.24	**0.08**
Average Abs Diff			0.64		0.34		0.44		**0.21**

Table 2: Hedge Ratio calculations vs historical Hedge Ratio for the iTraxx 5Y.

	Market	Base Correlation						Historical Correlation	
Period	Slope	Gaussian	Abs Diff	Double-t	Abs Diff	Double-NIG	Abs Diff	NIG	Abs Diff
Mar-Apr05	1.51	1.14	**0.37**	1.05	0.46	0.99	0.52	0.81	0.70
Apr-May05	1.26	1.06	**0.20**	0.97	0.29	0.92	0.34	0.79	0.47
May-Jun05	0.73	1.02	0.29	0.94	0.21	0.88	0.15	0.83	**0.10**
Jun-Jul05	0.91	0.94	**0.03**	0.87	0.04	0.82	0.09	0.80	0.11
Jul-Aug05	0.33	0.92	0.59	0.85	0.52	0.80	**0.47**	0.82	0.49
Aug-Sep05	1.03	0.95	**0.08**	0.87	0.16	0.82	0.21	0.83	0.20
Sep-Oct05	0.86	0.90	**0.04**	1.05	0.19	0.98	0.12	0.74	0.12
Oct-Nov05	-0.18	1.10	1.28	1.01	1.19	0.95	1.13	0.69	**0.87**
Nov-Dec05	0.65	1.08	0.43	0.99	0.34	0.93	0.28	0.68	**0.03**
Dec-Jan06	1.29	1.12	**0.17**	1.03	0.26	0.97	0.32	0.71	0.58
Jan-Feb06	1.02	1.11	0.09	1.02	**0.00**	0.96	0.06	0.68	0.34
Feb-Mar06	0.22	0.91	0.69	1.10	0.88	0.98	0.76	0.70	**0.48**
Average Abs Diff			**0.36**		0.38		0.37		0.37
Mar-Jun05	1.26	1.14	**0.12**	1.05	0.21	0.99	0.27	0.81	0.45
Jun-Sep05	0.85	0.94	0.09	0.87	**0.02**	0.82	0.03	0.80	0.05
Sep-Dec05	0.61	0.98	**0.29**	1.05	0.44	0.98	0.37	0.74	0.13
Dec-Mar06	0.91	1.12	0.21	1.03	**0.12**	0.97	0.06	0.71	0.20
Average Abs Diff			**0.18**		0.20		**0.18**		0.21
iTraxx3	1.18	1.14	**0.04**	1.05	0.13	0.99	0.19	0.81	0.37
iTraxx4	0.80	0.90	0.10	1.05	0.25	0.98	0.18	0.74	**0.06**
Average Abs Diff			**0.07**		0.19		0.19		0.22

Table 3: Hedge Ratio calculations vs historical Hedge Ratio for the iTraxx 10Y.

Regarding Table 2, the hedge ratios given by the different models on the same period are as expected not identical. Nevertheless, the difference with historical hedge ratios are non-negligible for all the models. These historical hedge ratios are quite volatile comparing to the model dependent ones. We can say that for each term the NIG factor model using historical correlation gives the best results especially for the iTraxx Series 4 but it seems to be very hazardous to hedge on a short term basis using model dependent hedge ratios. The same kind of analysis should be done regarding Table 3, except that the calculated hedge ratios are closer between the different models. This time the Gaussian factor model using base correlation seems to be the more appropriate model especially for the iTraxx Series 3. Note that the NIG factor model using historical correlation gives again the best result for a long term hedge on the iTraxx Series 4.

We can also see that sometimes the difference between the historical hedge ratio and the model dependent ones is huge. This is the case for example in Table 3 for the $Oct - Nov05$ time period. During this period the historical hedge ratio is negative which means that the tranches' behavior has changed. This is the typical case when a large market shock reveals that the assumptions underlying the models may have been inaccurate not only in size, but in sign.

Those results illustrate the fact that there is a real model risk and correlation risk in the tranched index products market. We can not argue that this market is driven by a specific model nor that there is a full agreement on the correlation structure that has to be used. The only thing that we can say is that nowadays more and more dealers seems to use a historical correlation structure for their trading and hedging activities. That should explain the good results obtained on the iTraxx Series 4 with the NIG factor model using historical correlation. For the moment, the best alternative before entering in this kind of hedging strategy would be to follow the evolution of historical hedge ratios and to compare with the evolution of the maximum of different model dependent hedge ratios.

6 Conclusion

In this paper we have presented tranched index products and different hedging strategies. We also described a computational methodology in order to implement several different factor models that take into account that a credit portfolio is not an homogeneous portfolio. Finally we have compared historical hedge ratios versus calculated and model dependent hedge ratios for a specific hedging strategy. Regarding our results it seems that hedging index tranches will stay a very hard task for the moment. The historical hedge ratios are quite volatile and the calculated ones are quite sensitive to the model that we used. Nevertheless on a long term basis it seems possible to build an acceptable hedging strategy. This study is an illustration that the tranched index products market is a difficult market which is not yet standardized in terms of modeling assumptions. If we want to better control the model risk and the correlation risk we have to continue this study in the future and to extend it to other models and hedging strategies.

ACKNOWLEDGMENTS: The corresponding author would like to express his thanks to the following people: Sébastien de Kort for his contributions and programming support, Cyril Caillault and Mitchell Reznick for stimulating discussions.

References

[1] Amato, J.D., and J. Gyntelberg, 2005, "CDS Index Tranches and the Pricing of Credit Risk Correlations", BIS Quarterly Review, March.

[2] Andersen, L., and J. Sidinius, 2005, "Extensions to the Gaussian Copula: Random Recovery and Random Factor Loadings", Journal of Credit Risk 1, 1 (Winter), 29-70.

[3] Andersen, L., J. Sidinius, and S. Basu, 2003, "All Your Hedges in One Basket", RISK, November, 67-72.

[4] Barndorff-Nielsen, O.E., 1997, "Normal Inverse Gaussian processes and stochastic volatility modelling", Scandinavian Journal of Statistics, vol. 24, 1-13.

[5] Belkin, B., S. Suchover, and L. Forest, 1998, "A one-parameter representation of credit risk and transition matrices", Credit Metrics Monitor, 1(3), 46-56.

[6] Beinstein, E., A. Sbityakov, P. Allen, and D. Dirk, 2005, "Enhancing our framework for index tranche analysis", Credit Derivatives Research, JPMorgan.

[7] Burtschell, X., J. Gregory, and J-P. Laurent, 2005, "A comparative analysis of CDO pricing models", www.defaultrisk.com.

[8] Burtschell, X., J. Gregory, and J-P. Laurent, 2005, "Beyond the Gaussian Copula: Stochastic and Local Correlation", www.defaultrisk.com.

[9] Calamaro, J-P., T. Nassar, K. Thakkar, and J. Tierney, 2004, "Trading Index Tranche Products: The First Steps", Quantitative Credit Strategy, Deutsche Bank.

[10] Derman, E., 2004, "My Life As A Quant: Reflections on Physics and Finance", John Wiley & Sons.

[11] Duffie, D., and M. Huang, 1996, "Swap Rates and Credit Quality", Journal of Finance, 51(2), 921-949.

[12] Duffie, D. and K. Singleton, 1997, "Modeling term structures of defaultable bonds", Review of Financial Studies, 12(4), 687-720.

[13] Duffie, D. and K. Singleton, 1997, "An Econometric Model of the Term Structure of Interest-Rate Swap Yields", Journal of Finance, 52(4), 1287-1321.

[14] Finger, C.C., 1999, "Conditional approaches for credit metrics portfolio distributions", Credit Metrics Monitor, 2(1), 14-33.

[15] Forsberg, L., and T. Bollerslev, 2002, "Bridging the Gap Between the Distribution of Realized (ECU) Volatility and ARCH modeling (of the EURO): The GARCH-NIG model", Journal of Applied Econometrics, Vol. 17, No. 5, 535-548.

[16] Frey, R., A. McNeil, and N. Nyfeler, 2001, "Copulas and Credit Models", RISK, October, 111-114.

[17] Jensen, M.B., and A. Lunde, 2001, "The NIG-S&ARCH model: a fat-tailed, stochastic, and autoregressive conditional heteroskedastic volatility model", Econometrics Journal, Royal Economic Society, vol. 4(2), 10.

[18] Galiani, S., 2004, "Understanding the Risk of Synthetic CDOs", http://www.federalreserve.gov/Pubs/FEDS/2004/200436/200436pap.pdf.

[19] Gibson, M.S., M. Shchetkovskiy and A. Kakodkar, 2004, "Basket Default Swap Valuation", Credit Derivatives, Merrill Lynch.

[20] Guégan, D., and J. Houdain, 2005, "Collateralized Debt Obligations Pricing and Factor Models: A New Methodology Using Normal Inverse Gaussian Distributions", www.defaultrisk.com/pp crdrv 93.htm.

[21] Hull, J., and A. White, 2004, "Valution of a CDO and an n^{th}-to-default CDS without Monte Carlo simulation", Working Paper, University of Toronto.

[22] Hull, J., and A. White, 2005, "The Perfect Copula", Working Paper, University of Toronto.

[23] Jackel, P., 2005, "Splitting the core", Working paper, http://www.jaeckel.org.

[24] Jarrow, R.A., and S.M. Turnbull, 1995, "Pricing Derivatives on Financial Securities Subject to Credit Risk", Journal of Finance, 50(1), 53-85.

[25] Jarrow, R., D. Lando, and S. Turnbull, 1997, "A Markov model for the term structure of credit spreads", Review of Financial Studies, 10(2), 481-523.

[26] Kalemanova, A., B. Schmid and R. Werner, 2005, "The Normal inverse Gaussian distribution for synthetic CDO", working paper.

[27] Laurent, J-P. and J., Gregory, 2003, "Basket Default Swaps, CDO's and Factor Copulas", Working Paper ISFA Actuarial School, University of Lyon and BNP-Paribas.

[28] Lillestol, J., 1998, "Fat and skew? Can NIG cure? On the prospects of using the Normal inverse Gaussian distribution in finance", Discussion paper 1998/11, Department of Finance and Management Science, The Norwegian School of Economics and Business Administration.

[29] Lillestol, J., 2000, "Risk analysis and the NIG distribution", The Journal of Risk, 2, 41-56.

[30] Lillestol, J., 2001, "Bayesian Estimation of NIG-parameters by Markov chain Monte Carlo Methods", Discussion paper 2001/3, Department of Finance and Management Science, The Norwegian School of Economics and Business Administration.

[31] Lillestol, J., 2002, "Some crude approximation, calibration and estimation procedures for NIG-variates", Department of Finance and Management Science, The Norwegian School of Economics and Business Administration.

[32] O'Kane, D. and M. Livesey, 2004, "Base Correlation Explained", Fixed Income Quantitative Credit Research, Lehman Brothers.

[33] Prause, K., 1997, "Modelling financial data using generalized hyperbolic distributions", FDM Preprint 48. University of Freiburg.

[34] Press, W.H., B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, 1992, "Fast Fourier Transform", Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed. Cambridge, England: Cambridge University Press, Ch.12, 490-529.

[35] Rydberg, T., 1997, "The normal inverse Gaussian Lévy processes: simulation and approximation", Commun. Statist.-Stochastic Models., 13(4), 887-910.

[36] Schonbucher, P.J., 2000, "Factor models for portfolio credit risk", Working Paper.

[37] Schonbucher, P.J., and D. Schubert, 2001, "Copula-dependent default risk in inten- sity models", Department of Statistics, Bonn University, Working Paper.

[38] Vasicek, O., 1987, "The loan loss distribution", Working Paper, KMV Corporation.

[39] Venter , J.H. and de Jongh, P.J., 2002, "The effects on risk and volatility estimation when using different innovation distributions in GARCH models", Research Report FABWI-N-RA: 2002-98, Potchefstroom University for CHE.

[40] Venter , J.H. and de Jongh, P.J., 2002, "Risk Estimation using the Normal Inverse Gaussian Distribution", The Journal of Risk, Vol. 4, No. 2, 1-23.

[41] Walker, M.B., 2005, "An Incomplete-Market Model for Collateralized Debt Obligations", http://www.physics.utoronto.ca/qocmp/incompleteCDO.2005.10.05.pdf.

[42] Walker, M.B., 2005, "CDO Models Towards the Next Generation: Incomplete Markets and Term Structure", http://www.physics.utoronto.ca/ qocmp/nextGenDefaultrisk.pdf.