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0.1 Introduction

The theory of asset pricing, which takes its roots in the Arrow-Debreu model (Theory of value [1959, chap. 7]), the Black and Sholes formula (1973) and Cox and Ross (1976 a and b), has been formalized in a general framework by Harrison and Kreps (1979), Harrison and Pliska (1979) and Kreps (1981). In these models, securities markets are assumed to be frictionless. The main result is that a price process is arbitrage free (or, equivalently, compatible with some equilibrium) if and only if it is, when appropriately renormalized, a martingale for some equivalent probability measure. The theory of pricing by arbitrage follows from there. Contingent claims can be priced by taking their expected value with respect to an equivalent martingale measure. If this value is unique, the claim is said to be priced by arbitrage. The new probabilities can be interpreted as state prices (the prices of 1 dollar tomorrow in each state of the world) or as the intertemporal marginal rates of substitution of an agent maximizing his expected utility.

In this work, we will propose a general model that takes frictions into account. There is an important body of literature on this subject concerned with optimal portfolio selection problems. Among others we can cite: Magill and Constantinides (1976), Constantinides (1986), Taksar and al. (1988), Duffie and Sun (1990), Grossman and Laroque (1990), Fleming and al. (1990), Davis and Norman (1990), and Dumas and Luciano (1991). In these studies the bid and ask price processes of a risky asset are exogenously given: they are usually diffusions of constant ratio (i.e. transaction costs are proportional). Typically, it is then found that portfolios are not rebalanced by maximizing agents as long as their share of wealth invested in the risky asset remains in a certain interval.

Some authors have also studied hedging strategies in the presence of transaction costs. Early investigations are Gilster and Lee (1984) and Leland (1985). In a continuous time model they find hedging strategies, for call options, that are revised at a finite number of times only and that are asymptotically exact (when the number of revisions goes to infinity). The strategies consist in following Black and Sholes ratios with an adjusted (upward) volatility. Dybvig and Ross (1986), Prisman (1986) and Ross (1987) have studied the case of a two-period economy with taxes. Figlewski (1989) has performed numerical simulations to evaluate the importance of transaction fees in hedging strategies. More recently, Bensaid and al. (1992) have developped an algorithm to hedge any contingent claim in the presence of proportional transaction costs in a binomial model. Their method elaborates on the idea that perfect duplication may be a suboptimal way of hedging if transaction costs are large enough.

In the spirit of Harrison and Kreps (1979) we start by characterizing bid and ask securities price processes that are arbitrage free (or, equivalently, viable, i.e. compatible with some equilibrium for a certain class of maximizing agents) in a fairly general finite horizon model of securities market. It turns out that all such processes can be obtained as a perturbation of a price process that is arbitrage free in a frictionless economy (the bid price lying above and the ask price lying below this process). Indeed, we find that a bid-ask price process is arbitrage free if and only if there exists an equivalent probability measure that transforms some process between the bid and the ask price processes into a martingale or a super/sub-martingale(after a normalization). Such a probability measure will be called a martingale measure by analogy with the frictionless case. In particular, we find that the bid and the ask price processes do not have to be arbitrage free in an economy without frictions in order to be viable. As in the perfect markets case, the martingale probabilities can be interpreted as possible intertemporal marginal rates of substitution of an agent maximizing his expected utility. They can also be interpreted as state prices and we show how to compute arbitrage bounds on the bid-ask prices of any contingent claim using the set of martingale measures. These bounds are respectively the minimum cost necessary to hedge the contingent claim, and the maximum amount one could borrow against it, using traded securities. They define a possible range for the bid and the ask price at which a new security could be traded if it were to be introduced in the market. Indeed, nobody would buy the security for more than the upper bound since there would be a way of getting at least the same payoff for cheaper, and nobody would sell it for less than the lower bound since there would be a way of borrowing a larger amount against it, using traded securities (see Bensaid and al. [1992] for an extensive discussion of this point). We find that the interval defined by these bounds is equal (modulo the boundary) to the set of expectation of the claim with respect to all the martingale measures. There is also a sense, made explicit in the work, in which these are the tightest bounds one can find without knowledge about agents' preferences. Other potential applications of our analysis are those of the martingale approach in the frictionless case.

Two simple examples can illustrate intuitively our results. Consider, to start with, a deterministic world where agents consume at date 0 and T and can trade two securities (1 and 2). Security 1 (the numeraire) is assumed to be always worth 1. At any date t, security 2 can be bought at its ask price $Z(t)$, and can be sold at its bid price $Z'(t)$. Note that in this deterministic world a martingale is merely a constant process. Now suppose that there is no constant process lying between the bid price Z' and the ask price Z of security 2. It is easy to see that this means that there exist two dates t and t' such that $Z'(t') > Z(t)$. In this case, buying an arbitrarily large amount A of security 2 at date t and selling [1] it at date t' one ends up with an arbitrarily large net profit $A(Z'(t') - Z(t))$ at the final date, without spending anything at date 0. Therefore, the bid-ask price processes Z and Z' are not arbitrage free in this case. Conversely, if there is a constant process between Z and Z', security 2 can never be bought at a lower price than it can ever be sold at, and there are no opportunities of arbitrage regardless of the behavior of the bid-ask price processes. In particular, the bid and the ask prices do not have to be constant, as in a frictionless economy. In order to make sense of the more general results recall that a martingale is the stochastic analog of a constant.

Another simple example with uncertainty can also illustrate the link between the martingale measures and the arbitrage bounds on the price of a contingent claim: the minimum cost to hedge it and the maximum amount that can be borrowed against it through securities trading. Consider an economy where there are two dates, 0 and 1, and two possible states of the world at date 1 : "up" and "down". Two securities can be traded: a bond that is always worth 1 (i.e. the riskless rate is equal to zero), and a stock that is worth $S_u = 110$ in state "up" and $S_d = 90$ in state "down". We assume that there is a bid-ask spread in trading the stock at date 0 : it can be bought for S_0

[1]If $t < t'$ the purchase is financed by going short in security 1 and if $t > t'$ the proceeds from the sale are invested in security 1.

and it can be sold for S_0'. It is easy to see that there are no arbitrage opportunities in this economy as long as $S_0 \geq S_0'$, $S_0 > S_d$, and $S_u > S_0'$. It is also easy to see that these three inequalities are satisfied if and only if there exists a probability measure that puts a strictly positive weight p^* on state "up" and a strictly positive weight $1-p^*$ on state "down", and for which we have $S_0 \geq p^* S_u + (1-p^*)S_d \geq S_0'$. Consider now a call option on the stock with exercise price equal to 100, that pays 10 in state "up" and 0 in state "down". If there is no bid-ask spread on the stock and its price is $S_0 = 100$, it is easy to see that there is only one risk-neutral (martingale) probability, 0.5 for the "up" and the "down" state. The call is then worth its expected payoff with respect to this probability: 5. The same result could be obtained by duplicating the call, i.e. buying 0.5 shares of stock and selling 45 in bonds, since this portfolio generates the same payoff as the call and its value is precisely 5. Now suppose that there is a bid-ask spread on the stock: the stock can still be sold for $S_0' = 100$ but can be bought for $S_0 > 100$. It is then easy to see that the minimum cost to hedge the call is $0.5S_0 - 45$ if $S_0 \leq 110$ and 10 otherwise. Indeed, in the first case the optimal strategy is to buy 0.5 shares of stock and to sell 45 in bonds. In the second case however the transaction costs on the stock are too large and it is optimal to buy 10 in bonds, although this strategy does better than duplicate the call since it yields a payoff of 10 in every state of the world. Moreover, the maximum amount that can be borrowed against the call is 5, since the optimal strategy consists in selling 0.5 shares of stock and buying 45 in bonds. On the other hand, the processes that lie between the bid and the ask price take the values $S_u = 110$ in state "up", $S_d = 90$ in state "down" and any value between $S_0' = 100$ and S_0 at date 0. The set of positive probabilities that transform some of these processes into a martingale are of the form $1 > p^* > 0$ for state "up", with $\frac{S_0 - 90}{20} \geq p^* \geq 0.5$. It is then easy to check that the interval defined by the expected values of the payoff of the call with respect to these probabilities is equal to the interval defined by the bounds computed using optimal hedging strategies.[2]

There is no doubt that shortselling and borrowing costs are another salient feature of financial markets. For instance, investors usually do not have full use of the proceeds from short sales of stocks (see Cox and Rubinstein [1985] p. 98-103), and this effectively represents a shortselling cost that is proportional to the holding period. In the Treasury bond market, short sales are performed through reverse repurchase agreements (reverse repos) in which the shortseller lends money at the reverse repo rate and takes the bond as a collateral. The shortselling cost is then the spread between the repo rate (at which an owner of the bond can borrow money collateralized by the bond through a repurchase agreement) and the reverse repo rate. If the bond is "on special", i.e. if it is particularly difficult to borrow, its repo rate is lower than the repo rate on general collateral. In this case, the shortselling cost is the sum of the spread between the repo rate and the reverse repo rate and of the spread between the repo rate and the repo rate on general collateral. Stigum (1983) reports typical costs between 0.25% and 0.65%, but they can be much larger for specific bonds. Amihud and Mendelson (1991) show that costs of this magnitude are substantial enough to wipe out the profits from the arbitrage between Treasury bills and Treasury notes with less than six months to maturity (with identical cashflows) that are relatively cheap, even taking trading costs into account. On the other hand, borrowing costs

[2] Note that when $S_0 > 110$ they coincide up to the boundary only.

vary substantially with the size and the credit rating of the market participant, and since they can be quite large, their significance is even less questionable.

Our framework permits to study a market with short sales constraints and different borrowing and lending rates. More precisely we can consider two sorts of securities. Shortselling the first type of securities is not allowed, i.e. they can only be held in nonnegative amounts, whereas the second type of securities can only be held in nonpositive amounts. In particular, we do not assume that the borrowing rate is equal to the lending rate. However this model includes the case where some (or all) securities are not subject to any constraints; we include these securities twice: in the first type and in the second type so that they can be held in nonnegative and nonpositive amounts. We show that this type of economy is arbitrage-free if and only if there exists a numeraire and an equivalent probability measure that transforms the normalized (by the numeraire) price processes of traded securities that cannot be sold short into a supermartingale, and the normalized price processes of the securities that can only be held short into a submartingale.[3] In such an economy, even if a contingent claim can be duplicated by dynamic trading, it is not necessarily possible to price such a contingent claim by arbitrage. This comes from the fact that the underlying securities cannot be sold short. However, arbitrage bounds can be computed for arbitrary contingent claims: they are the minimum amount it costs to hedge the claim and the maximum amount that can be borrowed against it using dynamic securities trading (see Bensaid et al. [1992] for an extensive discussion of this point). These are the tightest bounds that can be inferred on the price of a claim without knowing preferences. We find that these arbitrage bounds on the bid-ask prices of a contingent claim are respectively equal to the smallest and the largest expectations of its future normalized cashflows - with respect to all the numeraire processes and supermartingale probability measures. This model includes situations where holding negative amounts of a security is possible but costly (in the form of a lower expected return per unit of time), and where the riskless borrowing and lending rates differ. In this case, and when the underlying security price follows a diffusion process, we use the previous results to characterize arbitrage-free economies and we determine the set of super/submartingale probability measures.

To interpret these results, note that a martingale is a process that is constant on average: the expectation of its future value at any time is equal to its current value. Therefore, if securities prices are martingales investors cannot enjoy the possibility of a gain without the risk of a loss, and cannot suffer the risk of a loss without enjoying the possibility of a gain. This prevents arbitrage opportunities, i.e. sure gains, that could be generated by buying securities or by shortselling them. On the other hand, if short sales are prohibited a security (or a portfolio of securities) may provide the risk of a loss without providing the possibility of a gain; this does not constitute an arbitrage opportunity since the arbitrage would consist in shortselling the security (or the portfolio of securities) and this is not permitted. As a result, price processes only need to be nonincreasing on average (i.e. supermartingales) to prevent arbitrage opportunities.

Another possible interpretation is that a probability measure that transforms price processes into martingales corrects for the risk aversion of the agents involved in

[3] A martingale is a process that is constant on average, a supermartingale is a process that is nonincreasing on average, and a submartingale is a process that is nondecreasing on average.

trading: it puts more weight on "bad" (low consumption) states than the subjective probability measure and as a result the price of an asset can be computed by taking the expectation, weighted by the new probabilities, of its future cashflows. In other words, these new probabilities make the securities price processes compatible with expected utility maximization by risk-neutral agents (hence the often used name of "risk-neutral probabilities"). Indeed, with these probabilities, securities appear to be fair bets to risk-neutral agents. On the other hand, a supermartingale is a process that is nonincreasing on average: the expectation of its future value at any time is less than or equal to its current value. Hence a security that has a supermartingale as a price process does no longer appear as a fair bet to a risk-neutral agent. He would like to sell the security short in unlimited amounts. The short sales constraint, however, prevents him from doing so.

In Section 2 we analyze a two period economy where agents trade in the first period contingent claims to consumption in the second period, belonging to a convex cone of a larger space of payoffs. The prices of these claims are assumed to be given by a sublinear functional (positively homogeneous and subadditive, i.e. such that the price of the sum of two claims is at most equal to the sum of their prices). Enlarging the set of possible price functionals from the linear to the sublinear functionals allows us to take frictions into account: in particular, a long position in a claim costs more than one gets by going short in the same claim. We find that a model is arbitrage free (or equivalently, viable, i.e. compatible with some equilibrium) if and only if there exists a strictly positive linear functional that lies below the sublinear price functional on the set of marketed claims. We show how to compute bounds on the bid and the ask prices of contingent claims: the minimum amount it costs to hedge the claim and the maximum that can be borrowed against it, using securities trading. These bounds can be related to equilibrium analysis and there is a sense in which they are the tightest bounds that can be derived without knowledge about preferences. We show that for any contingent claim the interval defined by these "arbitrage" bounds is equal (modulo its boundary) to the set of expectations of the payoff of the claim with respect to all the martingale measures.

Our choice of a sublinear price functional is justified by the multiperiod model introduced in Section 3, where consumption takes place at dates 0 and T, and where consumers can buy a finite number of securities at their ask price and sell them at their bid price at any time under some constraints. In this section, we apply the results of section 2 to characterize arbitrage free (or equivalently, viable, i.e. compatible with some equilibrium) securities bid-ask price processes. We find that a securities bid-ask price process is arbitrage free if and only if, after a normalization, there exists an equivalent probability measure and a martingale or a super/sub-martingale, with respect to this probability, that lies between the bid and the ask price processes. This result allows us to associate to each economy with frictions a family of possible underlying frictionless economies, and every economy with frictions can be seen as the perturbation of a frictionless one, and conversely. Note that we do not impose any particular form on the price processes (diffusion or others) and then the martingale property is not contained in our assumptions; Furthermore, we do not impose proportionality between the bid and the ask price and the spread can evolve arbitrarily.

In Section 4, we apply the general framework introduced in section 3 to transaction costs, incomplete markets and shortselling costs including different borrowing and lending rate. We establish in these contexts valuation formulas for derivative assets. In the shortselling costs case, this allows us to derive a (nonlinear) partial differential equation that must be satisfied by the arbitrage bounds on the prices of derivative securities and to determine the optimal hedging strategies. This partial differential equation is similar to that obtained by Black and Scholes (1973), with two additional nonlinear terms proportional to the spread between the borrowing and lending rates, and to the shortselling cost. Numerical results suggest that the arbitrage bounds can be quite sharp; and substantially sharper than those obtained by using the Black and Scholes hedge ratios.

In section 5, we characterize efficient consumption bundles in dynamic economies with uncertainty, taking market frictions into account. We define an efficient consumption bundle as one that is an optimal choice of at least a consumer with increasing, state-independent and risk-averse Von Neumann-Morgenstern preferences. In an economy with frictions, a consumption bundle (i.e. a contingent claim to consumption) is available, through securities trading, for a minimum cost equal to its largest price in the underlying linear economies defined by the underlying pricing rules. The linear pricing rule for which this value is attained is said to "price" the contingent claim. We show that a consumption bundle is efficient if and only if it does not lead to lower consumption in cheaper - for the linear pricing rule that "prices" it - states of the world. We also characterize the size of the inefficiency of a consumption bundle, i.e. the difference between the investment it requires and the smallest investment needed to make every maximizing agent at least as well off.

These results allow us to define a measure of portfolio performance that does not rely on mean-variance analysis (and avoids the problems associated with it : see Dybvig [1988 a] and Dybvig and Ross [1985 a and b]), taking market frictions into account. These results can also be used to evaluate the efficiency of a hedging strategy. In the perfect market case hedging and investment decisions can be separated into two distinct stages : duplicate the contingent claim to be hedged and invest optimally the remaining funds. Hence the efficiency of a hedging strategy is not really an issue since hedging amounts to duplication (or perfect hedging). In the presence of market frictions, however, hedging and investment decisions are intimately related and cannot be separated. Hence the efficiency of hedging strategies becomes an issue that can be handled with the previous results. We also evaluate the inefficiency of investment strategies followed by practitioners (such as stop-loss strategies) as in Dybvig (1988 b), and we find that the presence of market frictions may rationalize trading strategies otherwise inefficient, or at least reduce substantially their inefficiency.

In section 6, we consider a model in which agents face investments opportunities (or investments) described by their cash flows as in Gale (1965), Cantor and Lippman (1983,1995), Adler and Gale (1993) and Dermody and Rockafellar (1991,1995). These cash flows can be at each time positive as well as negative. It is easy to show that such a model is a generalization of the classical one with financial assets. As in Cantor and Lippman (1983,1995) and Adler and Gale (1993), we will show that the absence of arbitrage opportunities is equivalent to the existence of a discount rate such that the net present value of all projects is nonpositive. We will extend this result in three directions : allowing our model to contain an infinite number of investments, allowing the cash flows to be continuous as well as discrete and finally, considering risky cash flows which is never the case for all the mentioned references. We impose that the opportunities can not be sold short and that the opportunities available today are those that are available tomorrow and the days after...(stationarity). In fact our framework permits to encompasses short sales constraints and transaction costs as well. In such a model we prove that the set of arbitrage prices is smaller than the set obtained without stationarity. Under some assumptions, we prove that there is a unique price for an option compatible with the no arbitrage condition : the Black and Scholes price even if there is transaction costs.

0.2 Arbitrage and equilibrium in a two period model

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X = L^2(\Omega, \mathcal{F}, P)$ the space of square integrable random variables[4] on $(\Omega, \mathcal{F}, P)$, that we assume to be separable, and X_+ the set of random variables $x \in X$ such that $P(x \geq 0) = 1$ and $P(x > 0) > 0$. If B is an element of $\mathcal{F}$, we denote by 1_B the element of X which is equal to 1 on B and to 0 elsewhere. We also denote by R the real line and by $\bar{R}$ the extended real line $R \cup \{-\infty, +\infty\}$, and if $a = (a_i)$ and $b = (b_i)$ are vectors of R^N we denote by $a \cdot b$ the dot product in R^N of a and b, by a^+ the vector $(max(0, a_i))$ and by a^- the vector $(-min(0, a_i))$. If M is a subset of X, we denote by $cl(M)$ the closure of M, and we say that M is a convex cone if for all x, y in M and all λ in R_+ we have $x + y \in M$ and $\lambda x \in M$. For instance, a linear subspace of X is a convex cone. If $\pi : M \rightarrow R$ is a functional defined on M, π is sublinear if for all x, y in M and all λ in R_+ we have $\pi(x + y) \leq \pi(x) + \pi(y)$ and $\pi(\lambda x) = \lambda \pi(x)$. Note that a sublinear functional is convex. A functional $f : X \rightarrow R$ is positive if for all $x \in X_+$ we have $f(x) > 0$. We denote the set of positive linear functionals on X by Ψ.

Consider a two period economy where consumers can purchase, at date zero, claims to consumption at date one denoted $m \in M$, where M is a convex cone of $X = L^2(\Omega, \mathcal{F}, P)$. The claim m is available to consumers at a price $\pi(m)$, in terms of today's consumption, where π is a sublinear functional defined on M. We consider this class of functionals, that includes the linear functionals, in order to take frictions and more precisely bid-ask spreads into account. For instance, consumers pay, at date 0, $\pi(m)$ to take a long position in m and receive $-\pi(-m)$ when taking a short position in m (if it is available). Since π is sublinear, we have $\pi(m) + \pi(-m) \geq 0$ and consumers pay more to buy the claim m than they receive when selling it. A more detailed justification of this choice will be given in the context of the multiperiod model section 3, where π is the result of a dynamic hedging process and appears naturally to be sublinear. In the next we shall characterize the price systems (M, π) that are compatible with equilibrium for the class of consumers that have continuous, convex and strictly increasing preferences, and show that they coincide with the price systems that are arbitrage free.

More precisely, the consumption space of our consumers is supposed to be $R \times X$ and we assume that every consumer is defined by a preorder of preferences $\preceq$ satisfying the following assumption [5]:

[4] We shall identify random variables that are equal everywhere except on a set of probability zero, and consider X as a space of classes of random variables.

[5] Although this assumption looks quite general, it actually excludes typical examples. It might be possible to extend the results below to a more general class of preferences, but we have not investigated this issue yet, as we shall emphasize the no arbitrage condition in this work.

Assumption (C) : *(i) for all* $(r^*, x^*) \in R \times X$, $\{(r, x) \in R \times X : (r^*, x^*) \preceq (r, x)\}$ *is convex,*
(ii) for all $(r^*, x^*) \in R \times X$, $\{(r, x) \in R \times X : (r^*, x^*) \preceq (r, x)\}$ *and* $\{(r, x) \in R \times X : (r, x) \preceq (r^*, x^*)\}$ *are closed.*
(iii) for all $(r^*, x^*) \in R \times X$, *for all* $r > 0$ *and all* $x \in X_+$ *we have* $(r^*, x^*) \prec (r^* + r, x^*)$ *and* $(r^*, x^*) \prec (r^*, x^* + x)$.

The class of such preferences is denoted by $\mathcal{C}$. We can now define the following notion of viability for a price system (M, π) which is identical to the definition in Harrison and Kreps (1979), except that because of the frictions induced by the sublinearity of the price functional, our consumers have budget sets that are convex cones instead of being half spaces.

Definition 0.2.1 *A price system* (M, π), *where* M *is a convex cone of* X *and* π *is a sublinear functional on* M, *is said to be viable if there exist a preorder* $\preceq$ *in the class* $\mathcal{C}$ *and a pair* $(r^*, m^*) \in R \times M$ *such that:*

$$r^* + \pi(m^*) \leq 0$$

and $(r, m) \preceq (r^*, m^*)$ *for all*$(r, m) \in R \times M$ *satisfying* $r + \pi(m) \leq 0$.

This says that a price system (M, π) is viable if some consumer in the class $\mathcal{C}$ can find an optimal net trade (r^*, m^*) in $R \times M$, subject to his budget constraint $r^* + \pi(m^*) \leq 0$. In fact, a viable pair (M, π) is one for which we can find equilibrium plans for some consumers belonging to the class $\mathcal{C}$ who can trade claims in $R \times M$, and this is the interpretation that we shall keep in mind.

Indeed, it is obvious that if some consumers are in equilibrium, they all have found an optimal net trade. Conversely, assume that a pair (M, π) is viable in the sense of Definition 2.1 and note $\preceq \in \mathcal{C}$ a preorder of preferences for which an optimal net trade (r^*, m^*) can be found. Define a new preorder $\preceq^*$ on $R \times X$ by: $(r, x) \preceq^* (r', x')$ if $(r + r^*, x + m^*) \preceq (r' + r^*, x' + m^*)$. It is then easy to show that $\preceq^*$ belongs to $\mathcal{C}$. Moreover $(0, 0)$ is an optimal net trade for a consumer with preorder of preferences $\preceq^*$. If otherwise let $(r', m') \in R \times M$ such that $(0, 0) \prec^* (r', m')$, i.e. $(r^*, m^*) \prec (r' + r^*, m' + m^*)$, and $r' + \pi(m') \leq 0$. We then have $r' + r^* + \pi(m' + m^*) \leq r' + r^* + \pi(m') + \pi(m^*)$ by sublinearity of π and hence we must have $r' + r^* + \pi(m' + m^*) \leq 0$, in addition to $(r^*, m^*) \prec (r' + r^*, m' + m^*)$, which contradicts the optimality of (r^*, m^*) in the budget set of our consumer with preorder of preferences $\preceq$. Therefore, in an economy populated by agents with preferences given by the preorder $\preceq^* \in \mathcal{C}$ and who can trade claims in $R \times M$, the prices given by the functional π are equilibrium prices since all agents weakly prefer the net trade $(0, 0)$ to any other trade available to them in their budget set.

The following theorem characterizes the viability of a price system (M, π) in terms of comparisons of the price functional π with positive and continuous[6] linear functionals. We find that (M, π) is viable if and only if there exists a (frictionless) linear price functional that is viable and lies below π on M.

[6]Note that continuity is redundant since positive linear functionals on a Banach lattice (and hence on X) are continuous (see Jameson [1974, prop. 33.14]).

Theorem 0.2.1 *A price system* (M, π)*, where* M *is a convex cone of* X *and* π *is a sublinear functional on* M*, is viable if and only if there exists a positive and continuous linear functional* ψ *defined on* X *such that* $\psi \mid_M \leq \pi$*. In particular, a necessary condition of viability is that* π *is positive.*

The proof of this Theorem is adapted from the proof of the similar one in Harrison and Kreps (1979) and is given in Jouini and Kallal (1996b).

The price system (M, ψ) can be interpreted as a price system of a frictionless economy where the price of every claim lies between its bid and ask price in the economy with frictions. Such a price system will be called an underlying frictionless price system of (M, π). This terminology can be justified if we think of our transaction costs economy as a perturbation of a frictionless one. The underlying frictionless price systems are then candidates for such initial frictionless economies.

Also, if M is a linear subspace of X and π is a linear functional as in the frictionless case, we find the classical result of Harrison and Kreps (1979): (M, π) is viable if and only if there exists a strictly positive linear extension of π to the whole space X.

We are now going to study the link between viability and the absence of opportunities of arbitrage in our model. Strictly speaking, an opportunity of arbitrage is a positive claim to consumption tomorrow available for nothing (or less) today. Although a viable system obviously cannot admit such opportunities, the converse does not hold in general: we also need to eliminate the possibity of getting arbitrarily close to an arbitrage opportunity in order to obtain the viability of the price system (see Kreps [1981] for an example). Hence, the following

Definition 0.2.2 *A free lunch is a sequence of real numbers* r_n *that converges to some* $r^* \geq 0$*, a sequence of contingent claims* x_n *in* X *that converges to some* $x^* \geq 0$ *such that* $r^* + x^* \in X_+$*, and a sequence of claims* m_n *in* M *such that* $m_n \geq x_n$ *and* $r_n + \pi(m_n) \leq 0$ *for all* n[7].

Hence a free lunch is a way to get a payoff arbitrarily close to a given positive claim at no cost, or to get a payoff arbitrary close to a nonnegative claim for a negative cost. [8] This definition is fairly natural, given that we have assumed that agents have continuous preferences.

Theorem 0.2.2 *A price system* (M, π)*, where* M *is a convex cone of* X *and* π *is a sublinear functional on* M*, admits no free lunch if and only if it is viable.*

In Jouini and kallal (1995a) we prove that the no free-lunch assumption is equivalent to the existence of an underlying frictionless economy. Theorem 2.2 is then a direct consequence of this last result with Theorem 2.1. A direct proof of Theorem 2.2 is given in Jouini and Kallal (1996b). In the following, the terms viable and arbitrage free will be synonymous.

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[7] I.e. (r_n, m_n) is in the budget set for all n.

[8] Note that if $M \cap X_+ \neq \emptyset$ one implies the other. Hence in this case we could define a free lunch as a sequence of claims $x_n \in X$ that converges to some claim $x^* \in X_+$, and a sequence of claims $m_n \in M$ such that $m_n \geq x_n$ and $\pi(m_n) \leq 0$, and get the same results in what follows. If M is closed and is such that $M = M - X_+$, and if π is nondecreasing lower semicontinuous the condition: $\pi(x) > 0$ for all $x \in M \cap X_+$, is equivalent to the absence of our free lunches.

If we now consider a claim x that does not necessarily belong to M we can, as in Harrison and Kreps (1979), ask what prices would be reasonable (in the sense of viability) for x. We introduce the notion of bid-ask price consistent with the economy (M, π). Possible interpretations are discussed later on. We shall assume in the whole section that the price functional π is l.s.c..

Definition 0.2.3 *Let (M, π) be a viable price system. Let $x \in X$ and $(q, p) \in \bar{R}^2$, we say that (q, p) is a bid-ask price system of x consistent with (M, π) if there exists a l.s.c. sublinear functional π' defined on the convex cone $M' = \{m + \lambda x : (m, \lambda) \in M \times R\}$ such that $(-\pi'(-x), \pi'(x)) = (q, p)$, $\pi' |_M = \pi$ and (M', π') is viable.*[9]

Note that the sublinearity of π implies that if (q, p) is a bid-ask price system of x consistent with (M, π) then $p \geq q$. In Jouini and Kallal (1996b) we prove that in a viable economies, any claim admits a consistent bid-ask price system.

Let us also define for all $x \in X$ the set

$$\Pi(x) = \{\psi(x) : \psi \in \Psi \text{ and} \psi |_M \leq \pi\}.$$

This set is precisely the set of prices of x in all underlying frictionless economies and it is easy to see that $\Pi(x) = -\Pi(-x)$. Note that if (M, π) is viable then $\Pi(x)$ is a nonempty interval of R, since the set $\{\psi \in \Psi : \psi |_M \leq \pi\}$ is nonempty and convex.

As it is shown in Jouini and Kallal (1996b), in the multiperiod securities price model a pleasant fact is that the set of consistent prices $C(x)$ is equal, modulo its boundary, to $\Pi(x)$, the set of prices of x in all the underlying frictionless economies.

As pointed out in Kreps (1981), one can take the view that $x \notin M$ is marketed somewhere and that π prices claims in M that are also marketed at the same time. A consistent price is one for which this situation is viable. Another view, perhaps more useful in practice, is to consider a market where claims in M are traded, before a market for a claim $x \notin M$ opens. A price of x consistent with (M, π) is then a price for which x might sell in this new market. However, as the new claim x is introduced, new opportunities are available to the agents and the old equilibrium might collapse. In a frictionless economy, this does not happen if x is priced by arbitrage (see Kreps [1981, Theorem 5]): as you introduce a new claim x that is priced by arbitrage, an equilibrium remains so.

In our case, as one introduces a new claim $x \notin M$, an extended price system (M', π') does not necessarily admit $(0, 0)$ as an equilibrium for the preorder $\preceq$ if (M, π) does. This is nonetheless true for some extended price system, namely the supremum of all the underlying frictionless price functionals.

Proposition 0.2.3 *Let (M, π) be a viable price system such that $-X_+ \subset M$ and $\pi(-y) \leq 0$ for all $y \in X_+$, and let $\preceq$ be a preoder of preferences in the class $\mathcal{C}$ such that: $(r, m) \preceq (0, 0)$ for all $(r, m) \in R \times M$ satisfying $r + \pi(m) \leq 0$. Let $x \in X$, and let the price system (M', π'), defined by $M' = \{m + \lambda x : (m, \lambda) \in M \times R\}$, and $\pi'(m') = \sup_{\psi' \in \Psi : \psi' |_M \leq \pi} \{\psi'(m')\}$, for all $m' \in M'$. Then (M', π') is viable and we have $(r', m') \preceq (0, 0)$ for all $(r', m') \in R \times M'$ satisfying $r' + \pi'(m') \leq 0$.*

[9] In fact, to be consistent with our framework we should not allow for infinite prices. The extension to this case, however, is straightforwood. In particular, the viability of (M', π') implies that $p > -\infty$ and $q < +\infty$. If $p = +\infty$ this means that the claim x is not marketed and if $q = -\infty$ it means that a short position in the claim x is not marketed.

Proof : Separate strictly the convex sets J and K as in the proof of Theorem 2.1.∎ This supremum gives in fact, as it is shown in Jouini and Kallal (1996b) the only prices that could lead to the same equilibrium if the claim were to be introduced in the market, for every possible preferences in the general class $\mathcal{C}$.

Therefore, without knowledge about preferences, it is not possible to infer tightest bounds than those given by $C(x) = cl(\Pi(x))$ for the newly introduced claim x. In our multiperiod securities price model these as tight as possible bounds are in fact the minimum amount it costs to hedge the claim x and the maximum that can be borrowed against it using securities trading.

0.3 The multiperiod model and the martingale approach

We consider a multiperiod economy where consumers can trade a finite number of securities at all dates $t \in \mathcal{T}$, with $\mathcal{T} \subset [0, T]$. Although we impose a finite horizon there is no other restriction on market timing: our framework includes discrete as well as continuous time models. Without loss of generality we shall assume that $\{0, T\} \subset \mathcal{T}$. These securities have a bid and an ask price at any time $t < T$, and deliver a contingent amount of consumption units at date T. Moreover, we assume that there are two sorts of securities: those which cannot be held short (i.e. in negative amounts) and those that can only be held short. The first set of securities might include a riskless asset in which consumers can invest, and the second set of securities might include a riskless asset that they can use to borrow funds. Of course, this model encompasses the case where a security can be held both in positive and negative amounts : when this is the case we include its price process twice in the model, once in each set of securities.

More precisely, let a filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ model the information structure of our economy. Let $Z'(t) = (Z'_1(t), \ldots, Z'_K(t))$ and $Z(t) = (Z_1(t), \ldots, Z_K(t))$ be two K dimensional positive processes that denote the bid and the ask prices of the K securities traded in the market. Consumers can buy security k at a price $Z_k(t)$ and sell it at a price $Z'_k(t)$ at any date $t \in \mathcal{T}$. Moreover, we assume that Z and Z' are adapted to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$, i.e. that bid-ask prices at any date depend on past and current information only, and that $E((Z_k)^2(t)) < \infty$ and $E((Z'_k)^2(t)) < \infty$ for all $t \in \mathcal{T}$ and $k = 1, \ldots, K$, which means that bid and ask prices have finite second moments. We also assume that the bid-ask price processes Z and Z' and the filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ are right-continuous.[10]

For the same reasons as in Harrison and Kreps (1979), consumers are only allowed to use simple strategies, i.e. they are only allowed to trade at a finite (but arbitrarily large) number of (arbitrary) dates, that do not depend on the path followed by prices. This restriction eliminates in particular doubling strategies.[11] Furthermore since the bid and the ask prices are possibly different, we separate trading strategies

[10] A filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is right-continuous if for all $t \in [0, T]$, $\mathcal{F}_t$ is the intersection of the σ-algebras $\mathcal{F}_s$, where $s > t$.

[11] Harrison and Pliska (1979) and Dybvig and Huang (1988) suggest that imposing a lower bound on wealth instead would lead to similar results even allowing continuous trading. However, these papers assume the absence of free lunch, and one of the aims of this section is to characterize the absence of free lunch.

into a cumulative long position and a cumulative short position (which are consequently nondecreasing). The difference between these two gives the net position of our consumer in the K securities at any date. Formally, a simple trading strategy is a pair (θ, θ') of nonnegative and nondecreasing K dimensional processes such that there exists a set of trading dates $0 = t_0 \leq \ldots \leq t_N = T$ for which $(\theta(t,\omega), \theta'(t,\omega))$ is constant over the intervals of time $[t_{n-1}, t_n)$, for $n = 1, \ldots, N$. We also assume that (θ, θ') is adapted to $\{\mathcal{F}_t\}_{t \in \mathcal{T}}$, which means that investors can only use current and past information to trade, and we impose the integrability conditions $E((\theta_k Z_k)^2(t)) < \infty$, $E((\theta'_k Z'_k)^2(t)) < \infty$, $E((\theta_k Z'_k)^2(t)) < \infty$, and $E((\theta'_k Z_k)^2(t)) < \infty$ for all $t \in \mathcal{T}$ and $k = 1, \ldots, K$, which guarantee that the payoffs of the strategies have finite second moments.

We incorporate short sales constraints into the model by restricting investors to hold nonnegative net positions in securities $k = 1, \ldots, S$, and nonpositive net positions in securities $k = S+1, \ldots, K$. We say that a trading strategy is admissible if it satisfies these constraints.

Definition 0.3.1 *A simple strategy is said to be admissible if it does not require selling short securities $k = 1, \ldots, S$, i.e. if $\theta_k(t,\omega) \geq \theta'_k(t,\omega)$ for all $k = 1, \ldots, S$, and if it does not require holding positive quantities of securities $k = S+1, \ldots, K$, i.e. if $\theta_k(t,\omega) \leq \theta'_k(t,\omega)$ for all $k = S+1, \ldots, K$.*

Also, we normalize the price of one of the securities that cannot be sold short to 1, i.e. we assume that $Z_1(t) = Z'_1(t) = 1$, for all $t \in \mathcal{T}$. This essentially amounts to express securities prices in terms of a numeraire (or unit of account) that can be stored. On the other hand, we assume that it is possible to have negative balances in the unit of account (i.e. to borrow) but that it can be costly to do so. In order to model this point, we assume that one of the securities that can be held in nonpositive quantities only has a price process that is nondecreasing and bounded, i.e. we assume that the price process $Z_{S+1}(t) = Z'_{S+1}(t)$ is positive, nondecreasing[12] and bounded. These assumptions amount to set a zero interest rate and to allow for a spread between the borrowing and lending rate. If an interest rate is actually earned[13] on positive balances of the numeraire, we can normalize all securities prices by the value of the accumulation process, i.e. the value of a portfolio where one unit of numeraire is invested at the initial date, and we are back to our model. Therefore, this assumption is made without much loss of generality.

Consumers are assumed not to have external sources of financing, they consume only at dates 0 and T, and hence sell (or short) some securities in order to purchase others. This is formalized by the notion of self-financing strategy which characterizes the feasible (in the sense of the budget constraints) trading strategies.

Definition 0.3.2 *A simple strategy (θ, θ') is said to be self-financing if for $n = 1, \ldots, N$ we have*

$$(\theta(t_n) - \theta(t_{n-1})) \cdot Z(t_n) \leq (\theta'(t_n) - \theta'(t_{n-1})) \cdot Z'(t_n).$$

[12] We could actually only assume that it has bounded variation and derive the fact that it has to be nondecreasing from the absence of arbitrage.

[13] The interest rate can be negative in real terms.

The set of simple self-financing admissible strategies is denoted by Θ. It is easy to see that Θ is a convex cone of the set of trading strategies. In fact, a self-financing trading strategy (θ, θ') can be seen as a way of transferring wealth from the initial date 0 to the final date T : it guarantees, at date T,

$$(\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T)$$

units of consumption contingent on the state of the world, and costs $\theta(0) \cdot Z(0) - \theta'(0) \cdot Z'(0)$ units of date 0 consumption.

In the subsequent analysis we shall characterize economies where there are no opportunities of arbitrage profits, i.e. where it is not possible to obtain a payoff that is nonnegative and strictly positive in some states of the world, at no cost. To be more precise, we shall examine economies in which there are no free lunches, which include arbitrage opportunities : roughly speaking, a free lunch is the possibility of getting payoffs arbitrarily close to a given positive contingent claim at an arbitrarily small cost. Back and Pliska (1990) provide an example of an economy where there are no arbitrage opportunities and yet the "fundamental theorem of asset pricing" fails : prices do not have the "martingale property" and there is no linear pricing rule. Since our aim is to find the counterpart of the "fundamental theorem of asset pricing" in an economy with market frictions, we shall study the implications of the absence of free lunches in our model.

Definition 0.3.3 *A free lunch is a sequence of contingent claims x_n in X converging to some x^* in X_+ and such that there exists a sequence of strategies (θ^n, θ'^n) in Θ satisfying $(\theta^n - \theta'^n)^+(T) \cdot Z'(T) - (\theta^n - \theta'^n)^-(T) \cdot Z(T) \geq x_n$ for all n, and*

$$\lim_n \{\theta^n(0) \cdot Z(0) - \theta'^n(0) \cdot Z'(0)\} \leq 0.$$

An immediate consequence of the absence of free lunch is that the ask price of any security must lie above its bid price: $P(\{\omega : Z'(t, \omega) \leq Z(t, \omega)\}) = 1$, for all $t \in \mathcal{T}$ (see Jouini and Kallal [1995a]).

In order to describe the investment opportunities in our economy, we consider the set M of marketed claims, i.e. of payoffs that can be hedged by an admissible self-financing simple trading strategy. Formally, a claim $x \in X$ belongs to M if there exists an admissible self-financing simple trading strategy (θ, θ') in Θ satisfying $(\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T) \geq x$. It is easy to see that M is a convex cone.

When there are shortsale constraints and/or transaction costs it is not true that the cheapest way to achieve at least a given contingent payoff at date T is to duplicate it.[14] However, the set of available contingent rates of return in our economy can be represented by the price functional π defined, for every $x \in X$, by

$$\begin{aligned}\pi(x) = \inf\{ \; & \liminf_n \{\theta^n(0) \cdot Z(0) - \theta'^n(0) \cdot Z'(0)\} : (\theta^n, \theta'^n) \in \Theta, \\ & (\theta^n - \theta'^n)^+(T) \cdot Z'(T) - (\theta^n - \theta'^n)^-(T) \cdot Z(T) \geq x_n \\ & \text{and } (x_n) \subset X \text{ converges to } x\}.\end{aligned}$$

In words, $\pi(x)$ represents the infimum cost necessary to get at least a payoff arbitrarily close to x at date T. This means that π indeed summarizes the investment opportunities in our dynamic economy. It turns out that this price functional is sublinear,

[14] This fact has been underlined by Bensaid et al. (1992) in a discrete time and states framework.

which means that it is less expensive to hedge the sum $x+y$ of two claims than to hedge x and y separately. It is easy to see why: the sum of the trading strategies that hedge x and y hedges $x+y$ but some orders to buy and sell the same security at the same date might cancel out, generating some savings on transaction costs and/or making the short sales constraints nonbinding.

We are now able to appreciate the main differences between our model and the frictionless case: in a frictionless economy the set of marketed claims M is a *linear subspace* and the price functional π that characterizes the opportunity set of returns is *linear.* Instead we find that in securities market models with frictions, M is a convex cone and π is a sublinear functional.

At this point, we are in a position to prove our main result that characterizes arbitrage free[15] bid and ask price processes. We find that the absence of free lunch [16] is equivalent to the existence of a numeraire process Z_0 and an equivalent probability measure that transforms some processes lying between the normalized bid-ask price processes of the securities that cannot be sold short into *supermartingales,* and some processes lying between the normalized bid-ask price processes of the securities that can only be held short into *submartingales.* [17]

Let us first define the set $\Theta'(t,k) = \{(\theta,\theta') \in \Theta : (\theta,\theta')(s) = 0$ for all $s < t, \theta'_k(T) = 1, \theta_j(T) = 0$ for $j \neq 0$, $\theta'_j(T) = 0$ for $j \notin \{0,k\}$ and $\theta(t)\cdot Z(t) = \theta'(t)\cdot Z'(t)\}$, and the set $\Theta(t,k) = \{(\theta,\theta') \in \Theta : (\theta,\theta')(s) = 0$ for all $s < t, \theta_k(T) = 1, \theta'_j(T) = 0$ for $j \neq 0$, $\theta_j(T) = 0$ for $j \notin \{0,k\}$, and $\theta(t)\cdot Z(t) = \theta'(t)\cdot Z'(t)\}$.

In words, $\Theta'(t,k)$ is the set of the strategies that consist in going short in one security k (and investing the proceeds in security 0) between time t and the final date T (but not necessarily at the same time in different events). The set $\Theta(t,k)$ is defined symmetrically.

The following theorem characterizes arbitrage free bid-ask price processes with shortsales constraints and generalizes the main result of Jouini and Kallal (1995a). This last one will be derived as a consequence of the following Theorem in section 4.

Theorem 0.3.1 *Assume that the borrowing and the lending rates are equal (i.e. $Z_1 = Z_{K+1} = 1$) then :*
(i) The securities price model admits no multiperiod free lunch if and only if there exist at least a probability measure P^* equivalent[18] *to P with $E((\frac{dP^*}{dP})^2) < \infty$ and a process Z^* satisfying*[19] *$Z' \leq Z^* \leq Z$ such that Z^*_k is a supermartingale for $k = 1,\ldots,K$ and a submartingale for $k = K+1,\ldots,S$ with respect to the filtration $\{\mathcal{F}_t\}$ and the probability measure P^*.*
(ii) Moreover, if we denote by E^* the expectation operator associated to the "(super/sub)-martingale measure" P^*, there is a one-to-one correspondence between the set of such expectation operators and the set of linear functionals $\psi \in \Psi$ such that $\psi\mid_M \leq \pi$. This

[15] We shall use the terms "arbitrage free" and "absent of free lunches" interchangeably.

[16] We could also relate the absence of free lunch to the viability as in section 2.

[17] A stochastic process Y is a martingale with respect to the probability measure P^* and the filtration $\{\mathcal{F}_t\}$ if $E^*(Y(s)\mid\mathcal{F}_t) = Y(t)$ for all $s \geq t$, where E^* is the expectation operator associated to P^*. A process Y is a supermartingale with respect to P^* and $\{\mathcal{F}_t\}$ if $E^*(Y(s)\mid\mathcal{F}_t) \leq Y(t)$ for all $s \geq t$. A process Y is a submartingale with respect to P^* and $\{\mathcal{F}_t\}$ if $E^*(Y(s)\mid\mathcal{F}_t) \geq Y(t)$ for all $s \geq t$.

[18] I.e. P and P^* have exactly the same zero measure sets.

[19] This means that for all t, $P(\{\omega : Z'(t,\omega) \leq Z^*(t,\omega) \leq Z(t,\omega)\}) = 1$.

correspondence is given by the following formulas:

$$P^*(B) = \psi(1_B),\ \textit{for all } B \in \mathcal{F} \textit{ and } \psi(x) = E^*(x),\ \textit{for all } x \in X.$$

Furthermore for such an expectation operator we can take

$$Z_k^*(t,\omega) = \sup_{(\theta,\theta')\in\Theta'(t,k)} \{E^*((\theta_0 - \theta_0')(T) \mid \mathcal{F}_t)(\omega)\},\ \textit{for } k = 1,\dots,K$$

$$\textit{and } Z_k^*(t,\omega) = \inf_{(\theta,\theta')\in\Theta(t,k)} \{E^*(-(\theta_0 - \theta_0')(T) \mid \mathcal{F}_t)(\omega)\},\ \textit{for } k = K+1,\dots,S.$$

(iii) Furthermore, for all $m \in M$ we have

$$[-\pi(-m), \pi(m)] = cl\{E^*(m) : P^* \textit{ is a (super/sub)-martingalemeasure }\}.$$

The proof of this Theorem is at the end of this section. One consequence of this result is that *any* arbitrage-free (or equivalently, viable) bid ask price process (Z', Z) can be seen as a perturbation of an arbitrage-free frictionless economy price process Z^*, with $Z' \leq Z^* \leq Z$. Conversely, it is obvious that if Z^* is an arbitrage-free frictionless price process, then any perturbed price system process (Z', Z), with $Z' \leq Z^* \leq Z$, is arbitrage-free and Z^* defines one of the underlying frictionless economies.

In particular, neither Z' nor Z needs to be a martingale (i.e. arbitrage-free by themselves in a frictionless model) for the securities price model to be arbitrage-free. This has been illustrated in the introduction by the example of a deterministic economy. Let us now briefly examine two stochastic examples. Consider, to start with, an economy where a stock bid and ask price processes follow diffusions with a singularity (zero volatility) at a point where their drift is not equal to the riskless rate. In a frictionless model this would lead to arbitrage opportunities. However, if the bid price is strictly lower than the ask price, according to our result the model with transaction costs is arbitrage-free. Now let us turn to a two-period economy with two possible states of the world ("up" and "down") at date 1. Suppose that the interest rate is equal to zero, and that a stock is available for trading at date 0 and pays off (at date 1) $S_u = \$110$ in state "up" and $S_d = \$90$ in state "down". We denote by S_0 its ask price, and by S_0' its bid price at date 0. According to Theorem 3.1 (part (i)), the model is arbitrage-free if and only if $S_0 \geq S_0'$, $S_0 > 90$, and $S_0' < 110$ (which do not imply that S or S' is a martingale). Also note that if the model is arbitrage-free, there are infinitely many equivalent martingale measures: they are of the form $p^* = \frac{S_0^* - 90}{110-90}$ where S^* is any process satisfying $S_u^* = S_u$, $S_d^* = S_d$, and $S_0^* \in [S_0', S_0] \cap [90, 110]$. Nonetheless, it is easy to see that in this model any contingent claim can be obtained by trading in the bond and in the stock, i.e. that markets are *complete*.[20]

Another consequence of this Theorem (part (iii)) is that it allows a representation, in terms of the set of equivalent martingale measures, of the set of investment opportunities available in this economy. A contingent claim x is available at a cost (in terms of today's consumption) $\pi(x)$ which is equal to the *largest* expected value with respect to the martingale measures. Similarly, the maximum amount $-\pi(-x)$

[20]It would be possible to have infinitely many processes S^* between the bid and the ask price processes, with each of the processes S^* having infinitely many martingale measures. To see this, add a third state of the world (e.g. "middle") to this example, with the stock paying $S_m = 100$ in that state. In this case, markets would be incomplete.

that can be borrowed (for today's consumption) against the claim x is equal to its *smallest* expected value with respect to the martingale measures. This means that the martingale measures can be interpreted as possible stochastic discount factors, i.e. today's prices of \$1 to be paid in a given state of the world. This also means that the interval $[-\pi(-x), \pi(x)]$ defines arbitrage bounds on the price of the contingent claim x. Indeed, x would not be traded for a price outside this interval since this would either lead to an arbitrage opportunity or be dominated by trading in the underlying securities. On the other hand, it is possible to show that any bid-ask price pair in this interval can be supported by some strictly increasing continuous linear (and hence convex) preferences. Therefore, without any further knowledge about preferences, $-\pi(-x)$ and $\pi(x)$ are the tightest bounds that can be inferred on the price of the claim x.

Let us illustrate this point by going back to our two-periods-and-states-of-the-world example. Suppose that $S_0' = 100$ and $S_0 \geq 100$. It is easy to see that the minimum cost to hedge the call is equal to $0.5S_0 - 45$ (i.e. the cost of duplication) if $S_0 \leq 110$, and is equal to 10 otherwise. Indeed, if $S_0 > 110$ the transaction costs on the stock are too large and duplication is no longer optimal. Moreover, the maximum amount that can be borrowed against the call is \$5. On the other hand, the martingale measures are of the form $(p^*, 1-p^*)$, with $1 > p^* > 0$ and $\frac{S_0-90}{20} \geq p^* \geq 0.5$. It is easy to check that the interval defined by the expected values of the payoff of the call with respect to these probabilities is equal to the interval defined by the bounds computed using the optimal hedging strategies (note that when $S_0 > 110$ they coincide up to the boundary only).

Also, in our securities price model a maximizing agent will equate his marginal utility to one of the martingale measures (one that "prices" his optimal consumption bundle). Indeed, it is easy to see that if (r^*, m^*) is an optimal consumption bundle satisfying the budget constraint $r^* + \pi(m^*) \leq 0$, then some hyperplane separates feasible bundles from bundles that are strictly preferred to (r^*, m^*). This hyperplane is given by one of the positive linear functionals ψ that lie below π on M and is tangent to the indifference surface that goes through (r^*, m^*). Moreover, it is easy to see that we must have $\psi(m^*) = \pi(m^*)$, which means that ψ "prices" m^*. And by Theorem 3.1 (part (ii)) such a linear pricing rule ψ corresponds to a martingale measure. Conversely, any equivalent martingale measure E^* could be the vector of intertemporal marginal rates of substitution of a maximizing agent in our economy. Indeed, by Theorem 3.2 (part (ii)) the linear pricing rule ψ defined by $\psi(x) = E^*(x)$ for all $x \in X$ lies below π on M. Moreover it is easy to see that an agent with a utility function defined by $u(r, x) = r + \psi(x)$ for all $(r, x) \in R \times X$ and with an endowment $(0, 0)$ will be happy not to trade.

In the following Theorem we will focus our attention to the shortselling constraints and for this purpose we will assume that there is no bid-ask spreads.

Theorem 0.3.2 *Assume that there is no bid-ask spreads on the marketed securities (i.e. $Z = Z'$) and that the lending rate can be written on the form $Z_{K+1}(t, \omega) = exp(\int_0^t r(t, \omega)dt)$ where r is some nonnegative bounded (by some $\bar{r}$) and right continuous adapted instantaneous rate process then :*
(i) The securities price model admits no free lunches if and only if there exist a probability measure P^* equivalent to P with $E((\frac{dP^*}{dP})^2) < \infty$, and a positive numeraire

process $Z_0^* = exp(\int_0^t r_0(s,\omega)ds)$ *with* $0 \leq r_0(t,\omega) \leq r(t,\omega)$*, such that* $(\frac{Z_k}{Z_0^*}$ *is a supermartingale for* $k = 1,\ldots,K$ *and a submartingale for* $k = K+1,\ldots,S$ *with respect to the filtration* $\{\mathcal{F}_t\}$ *and the probability measure* P^**.*
(ii) If we denote by E^* *the expectation operator associated to* P^**, there is a one-to-one correspondence between the set of such expectation operators and the set of positive linear functionals* $\psi \in \Psi$ *such that* $\psi \mid_M \leq \pi$*. This correspondence is given by :*

$$P^*(B) = \psi(1_B), \textit{ for all } B \in \mathcal{F} \textit{ and } \psi(x) = Z_0^*(0)E^*(\frac{x}{Z_0^*(T)}), \textit{ for all } x \in X.$$

(iii) For all $x \in M$ *we have,*
$$\begin{aligned}[-\pi(-x), \pi(x)] &= cl\{\psi(x) : \psi \mid_M \leq \pi \textit{ and } \psi \in \Psi\} \\ &= cl\{Z_0^*(0)E^*(\tfrac{x}{Z_0^*(T)}) : Z_0^* \textit{ and } P^* \textit{ are as in } (i)\}.\end{aligned}$$

The proof is at the end of this section. In words, we find that the absence of free lunch is equivalent to the existence of (1) a numeraire process Z_0^* that increases at a rate smaller than or equal to the borrowing rate and larger than or equal to the lending rate, and (2) an equivalent probability measure that transforms the (normalized by the numeraire) price processes of the securities that cannot be sold short into *supermartingales,* and the (normalized by the numeraire) price processes of the securities that can only be held in nonpositive quantities into *submartingales.* This super/submartingale measure is in fact what is sometimes called a "risk-neutral" measure. In particular, if a security is not subject to any constraint, it follows that its (normalized) price process $\frac{Z^*}{Z_0^*}$ must be a supermartingale and a submartingale, i.e. a martingale, under the "risk-neutral" measure. Hence, if there are no constraints at all in the economy, and if there is no spread between the borrowing and the lending rates, the numeraire must be equal to 1 and there must exist an equivalent probability measure for which securities prices are martingales (as shown by Harrison and Kreps [1979]).

While part (i) of the Theorem characterizes the absence of arbitrage in terms of a numeraire process and of supermartingale/submartingale measures, part (ii) states that these probability measures and numeraire define all the positive linear functionals that lie below the sublinear functional π (that describes the random returns available in our dynamic economy). These positive linear functionals can be interpreted as representing the price functionals of underlying frictionless economies. This turns out to be useful to compute arbitrage bounds on the price of any contingent claim. Indeed, part (iii) states that for every contingent claim x, the interval $[-\pi(-x), \pi(x)]$ is equal to the closure of the set of prices of the claim x in all the underlying frictionless economies. Recall that $\pi(x)$ is the smallest amount necessary to reach (or dominate) the payoff x by trading in the underlying securities. Hence, if the claim x were to be introduced in the market at date 0 it would not be bought for more than $\pi(x)$ or sold for a price under $-\pi(-x)$, since a better deal could be achieved through securities trading in both cases. Therefore the bid and the ask prices of x must fall in the interval $C(x) = [-\pi(-x), \pi(x)]$. Note that this does not mean that if the bid price of x is below $-\pi(-x)$ or if the ask price of x is above $\pi(x)$ then there are opportunities of arbitrage; it only means that there are no transactions at these prices[21]. However,

[21]Note that we are assuming the opening of a spot market for the contingent claim x at date 0 only.

if the bid price of x were to be above $\pi(x)$ or its ask price were to be below $-\pi(-x)$ then selling the claim x at the bid price or buying it at the ask price and hedging the position through securities trading would constitute a free lunch. Also, Theorem 3.2 characterizes the set $\Pi(x)$ using the set of martingale measures. Hence, the interval $C(x) = [-\pi(-x), \pi(x)]$ defined by the arbitrage bounds on the bid-ask prices of the claim x is equal to the closure of the set of expectations of x with respect to all the equivalent probability measures that transform some process between the bid and the ask price processes of each traded security into a martingale.

We conclude this section by the proofs of Theorem 3.1 and 3.2. **Proof of Theorem 3.1 :** First, let P^* be a probability measure equivalent to P and let Z^*, with $Z' \leq Z^* \leq Z$, be a super/submartingale with respect to P^* and $\{\mathcal{F}_t\}$ (as in the theorem). Define the linear functional ψ by $\psi(x) = E^*(x)$ for all $x \in X$. Since $\rho = \frac{dP^*}{dP} \in X$ we have by the Riesz representation Theorem that $\psi(x) = E^*(x) = E(\rho x)$ is continuous. Since P and P^* are equivalent, we have $\psi \in \Psi$.

Let $m \in M$ and let $(\theta, \theta') \in \Theta$ with trading dates $0 = t_0 \leq t_1 \leq \ldots \leq t_N = T$, such that $(\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T) \geq m$. Since $Z' \leq Z^* \leq Z$ and (θ, θ') is nondecreasing and self-financing, we have, for $n = 1, \ldots, N$,

$$\begin{aligned}&E^*((\theta(t_n) - \theta(t_{n-1}))^*(t_n) - (\theta'(t_n) - \theta'(t_{n-1})) \cdot Z^*(t_n) \mid \mathcal{F}_{t_{n-1}})\\ \leq &E^*((\theta(t_n) - \theta(t_{n-1}))(t_n) - (\theta'(t_n) - \theta'(t_{n-1})) \cdot Z'(t_n) \mid \mathcal{F}_{t_{n-1}}) \leq 0.\end{aligned}$$

and then

$$E^*((\theta - \theta')(t_n) \cdot Z^*(t_n) \mid \mathcal{F}_{t_{n-1}}) \leq E^*((\theta - \theta')(t_{n-1}) \cdot Z^*(t_n) \mid \mathcal{F}_{t_{n-1}}).$$

Using the fact that Z^*_k is a supermartingale when the security k can not be sold short (i.e. $\theta - \theta' \geq 0$) and a submartingale when security k can only be held in nonpositive amount (i.e. $\theta - \theta' \leq 0$) we obtain

$$E^*((\theta - \theta')(t_n) \cdot Z^*(t_n) \mid \mathcal{F}_{t_{n-1}}) \leq (\theta - \theta')(t_{n-1} \cdot Z^*(t_{n-1}.$$

By iteration, $E^*((\theta - \theta')(T) \cdot Z^*(T)) \leq (\theta - \theta')(0) \cdot Z^*(0) \leq \theta(0) \cdot Z(0) - \theta'(0) \cdot Z'(0)$. Since $(\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T) \leq \{(\theta - \theta')^+(T) - (\theta - \theta')^-(T)\} \cdot Z^*(T)$ we have $E^*((\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T)) \leq \theta(0) \cdot Z(0) - \theta'(0) \cdot Z'(0)$. It is easy to see that this implies that there cannot be any multiperiod free lunch.

Moreover, taking the infimum over the strategies $\theta \in \Theta$ such that $(\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T) \geq m$, we obtain that $\psi(m) = E^*(m) \leq \tilde{\pi}(m)$, where $\tilde{\pi}(m) = \inf\{\theta(0) \cdot Z(0) - \theta'(0) \cdot Z'(0) : (\theta, \theta') \in \Theta \text{ and } (\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T) \geq m\}$, for all $m \in M$. Using the fact that ψ is continuous, we have by Lemma 1 below that $\psi \mid_M \leq \pi$.

Lemma 1 (Jouini and Kallal (1995a)) *π is the largest l.s.c. functional that lies below $\tilde{\pi}$.*

Conversely, assume that the securities price model, or equivalently (M, π), admits no free lunch and let $\psi \in \Psi$ such that $\psi \mid_M \leq \pi$, as guaranteed by Theorem 2.1. Define P^* from ψ by $P^*(B) = \psi(1_B)$ for all $B \in \mathcal{F}$. By linearity and strict positivity of ψ it is clear that P^* is a measure equivalent to P. Using the fact that $Z_0 = Z'_0 = 1$ it is easy to show that $P^*(1_\Omega) = 1$.

Since ψ is continuous, by the Riesz representation Theorem there exists a random variable $\rho \in X$ such that $\psi(x) = E(\rho x)$, for all $x \in X$. Thus, $P^*(B) = E(\rho 1_B)$ for all $B \in \mathcal{F}$ and $\frac{dP^*}{dP} = \rho$ is square integrable. It remains to show that there exists a process Z^*, with $Z'_k \leq Z^*_k \leq Z_k$, and such that Z^*_k is a supermartingale for $k = 1, \dots, K$ and a submartingale for $k = K+1, \dots, S$ with respect to P^* and $\{\mathcal{F}_t\}$.

Consider the processes Z^* defined in (ii). In words, when the security k can be sold short, Z^*_k is the supremum of the conditional expected value of the proceeds from the strategies that consist in going short in one security k (and investing the proceeds in security 0) between time t and the final date T (but not necessarily at the same time in different events). When the security k can only be held in nonpositive quantities, the process Z^* is defined symmetrically. We obviously must have $Z^*_k(t) \leq Z_k(t)$ for $k = 1, \dots, K$ and for all t. Indeed, one of the strategies in $\Theta(t,k)$ consists in going long at once at date t in one unit of security k. Moreover we also must have $Z'(t) \leq Z^*(t)$ for $k = 1, \dots, K$ and all t. Indeed, assume instead that there exists $t \in [0, T]$ and $B \in \mathcal{F}_t$ such that $P(B) > 0$ and $Z'(t,\omega) > Z^*(t,\omega)$ for all $\omega \in B$. Then there would exists a strategy $(\alpha, \alpha') \in \Theta(t,k)$ satisfying $E^*(Z'_k(t)1_B) > E^*(-(\alpha_0 - \alpha'_0)(T)1_B)$. At the end of this strategy we have one unit of security k and we can sale it at a price $Z'_k(T)$. This operation defines a new trategy (θ, θ') with $\theta_0(T) = \alpha_0(T) + Z'_k(T)$, $\theta_k(T) = 0$ and $\theta = \alpha$ in all other cases. We then have $E^*((\theta_0 - \theta'_0)(T)1_B) > 0$, and hence $E^*([(\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T)]1_B) > 0$ which contradicts the fact that $E^*([(\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T)]1_B) \leq \pi([(\theta - \theta')^+(T) \cdot Z'(T) - (\theta - \theta')^-(T) \cdot Z(T)]1_B) \leq 0$. Hence $Z'_k(t) \leq Z^*_k(t)$ for all t and Z^*_k lies between Z_k and Z'_k for $k = 1, \dots, K$. A symetric argument gives the result for $k = K+1, \dots, S$.

Moreover we must have $E^*(Z^*_k(s) \mid \mathcal{F}_t) \leq Z^*_k(t)$ for $k = K+1, \dots, S$ and $E^*(Z^*_k(s) \mid \mathcal{F}_t) \geq Z^*_k(t)$ for $k = 1, \dots, K$ for all $s \geq t$, since $\Theta'(s,k) \subset \Theta'(t,k)$ and $\Theta(s,k) \subset \Theta(t,k)$ for all $s \geq t$ and all $\omega \in \Omega$, and by the law of iterated expectations.

Part (iii) then follows from Theorem 2.4. ■

Proof of Theorem 3.2: First note that if there is no multiperiod free lunch, then the set M is a convex cone and the price functional π is a lower semicontinuous[22] sublinear functional on M which takes value in R. It is then easy to see that our multiperiod model admits no free lunches if and only if the induced two-period model (M, π) has the same property.

Then let P^* be a probability measure equivalent to P such that $\frac{dP^*}{dP} \in X$ and let Z^*_0 be a bounded (and bounded away from zero) absolutely continuous positive numeraire process such that $\frac{Z_k}{Z^*_0}$ is a supermartingale for $k = 1, \dots, K$ and a submartingale for $k = K+1, \dots, S$. Without loss of generality, we shall assume that $Z^*_0(0) = 1$. Define the linear functional ψ by $\psi(x) = E^*(\frac{x}{Z^*_0(T)})$ for all $x \in X$. Since $\rho = \frac{dP^*}{dP} \in X$ we have that (by the Riesz representation Theorem) ψ is continuous. Since P and P^* are equivalent, ρ is strictly positive. Thus, ψ is also strictly positive and we have $\psi \in \Psi$.

Since there is no bid-ask spreads on the traded securities, the cost and the result of a strategy (θ, θ') only depend from the net position $\theta - \theta'$. In the next, we will denote by θ this net position and we will consider that the strategy is entirely described by θ. Let $m \in M$ and let θ be a simple self-financing strategy that is admissible (i.e. that

[22]I.e. Such that $\{(m, \lambda) \in M \times R : \lambda \geq \pi(m)\}$ is closed in $M \times R$, or equivalently such that $\{m \in M : \lambda \geq \pi(m)\}$ is closed in M for all $\lambda \in R$, or equivalently such that $\liminf_n \{\pi(m_n)\} \geq \pi(m)$ whenever the sequence $(m_n) \subset M$ converges to $m \in M$.

satisfies the short sales constraints), with trading dates $0 = t_0 \leq t_1 \leq \ldots \leq t_N = T$, such that $\theta(T) \cdot Z(T) \geq m$. Since the strategy θ is self-financing, we must have

$$E^*((\theta(t_n) - \theta(t_{n-1})) \cdot \frac{Z(t_n)}{Z_0^*(t_n)} \mid \mathcal{F}_{t_{n-1}}) \leq 0,$$

which can also be written

$$E^*(\theta(t_n) \cdot \frac{Z(t_n)}{Z_0^*(t_n)} \mid \mathcal{F}_{t_{n-1}}) \leq E^*(\theta(t_{n-1}) \cdot \frac{Z(t_n)}{Z_0^*(t_n)} \mid \mathcal{F}_{t_{n-1}}).$$

Since θ is adapted to the filtration $\{\mathcal{F}_t\}$ we must have

$$E^*(\theta(t_{n-1}) \cdot \frac{Z(t_n)}{Z_0^*(t_n)} \mid \mathcal{F}_{t_{n-1}}) = \theta(t_{n-1}) \cdot E^*(\frac{Z(t_n)}{Z_0^*(t_n)} \mid \mathcal{F}_{t_{n-1}})$$

Moreover using the supermartingale and the submartingale properties with respect to P^*, together with the fact that $\theta_i \geq 0$ for all $i = 1, \ldots, K$, $\theta_j \leq 0$ for all $j = K+1, \ldots, S$, we obtain

$\theta(t_{n-1}) \cdot E^*(\frac{Z(t_n)}{Z_0^*(t_n)} \mid \mathcal{F}_{t_{n-1}})\theta(t_{n-1}) \cdot \frac{Z(t_{n-1})}{Z_0^*(t_{n-1})}$,
which can be rewritten

$$E^*(\theta(t_n) \cdot \frac{Z(t_n)}{Z_0^*(t_n)} \mid \mathcal{F}_{t_{n-1}}) \leq \theta(t_{n-1}) \cdot \frac{Z(t_{n-1})}{Z_0^*(t_{n-1})}.$$

Iterating this reasoning, and using the law of iterated expectations, we obtain that

$$E^*(\theta(T) \cdot \frac{Z(T)}{Z_0^*(T)} \leq \theta(0) \cdot Z(0).$$

This inequality precludes the existence of a free lunch. Taking the infimum over the strategies $\theta \in \Theta$ it is then easy to show that $\psi\mid_M \leq \pi$.

Conversely, assume that the multiperiod model (or equivalently the induced two-period model (M, π)) admits no free lunch, and let $\psi \in \Psi$ such that $\psi\mid_M \leq \pi$, as guaranteed by Theorems 2.1 and 2.2. Since ψ is continuous, by the Riesz representation Theorem there exists some $\rho \in X$ such that $\psi(x) = E(\rho x)$, for all $x \in X$.

Suppose, to start with, that $Z_{K+1} = Z_1 = 1$. Define P^* from ψ by $P^*(B) = \psi(1_B)$ for all $B \in \mathcal{F}$. By linearity and strict positivity of ψ it is clear that P^* is a measure equivalent to P. As in Jouini and Kallal (1995a) it is easy to show that $\pi(1_\Omega) = 1$, $\pi(-1_\Omega) = -1$ and hence that $\psi(1_\Omega) = P^*(1_\Omega) = 1$. It follows that P^* is a probability measure. Moreover, $P^*(B) = E(\rho 1_B)$ for all $B \in \mathcal{F}$ and $\frac{dP^*}{dP} = \rho$ is square integrable. It remains to show that Z_k is a supermartingale for $k = 1, \ldots, K$ and a submartingale for $k = K+1, \ldots, S$ with respect to P^*. In order to do this, consider the strategy $\theta \in \Theta$ that consists in buying security Z_i, for some $i = 1, \ldots, K$ at time t in event $B_t \in \mathcal{F}_t$, and borrowing the funds by selling short security Z_{K+1}. This strategy does not cost anything and its payoff at the final date T is $1_{B_t}(Z_i(T) - Z_i(t))$, therefore we must have $\pi(1_{B_t}(Z_i(T) - Z_i(t)) \leq 0$, and hence $\psi(1_{B_t}(Z_i(T) - Z_i(t)) \leq 0$ which means $E^*(1_{B_t}(Z_i(T) - Z_i(t)) \leq 0$. This implies that $E^*(Z_i(T) \mid \mathcal{F}_t) \leq Z_i(t)$, which means that Z_i is a supermartingale with respect to P^*. Similarly, in order to show that Z_j is a submartingale with respect to P^* consider the startegy that consists in selling short security Z_j, for some $j = K+1, \dots, S$ at time t in event $B_t \in \mathcal{F}_t$, and in investing the proceeds in security Z_1.

Let us now turn to the general case where $Z_{K+1}(t) = exp(\int_0^t \tilde{r}(s)ds)$ for some nonnegative, bounded (by some constant $\bar{r}$), and right-continuous adapted process $\tilde{r}$. Consider the set of dates $0 = t_0 < t_1 < \dots < t_N = T$ and the sequence of real numbers $\bar{R}_n = exp(\int_{t_n}^{t_{n+1}} \bar{r})$ for $n = 0, \dots, N-1$. Let K be the set of sequences $R = (R_0, R_1, \dots, R_{N-1})$ of essentially bounded random variables such that R_n is $\mathcal{F}_{t_n}$ measurable and takes a.e. values in $\prod_0^{N-1}[1, \bar{R}_n]$ for all $n = 0, \dots, N-1$. For any sequence of random variables $R = (R_0, R_1, \dots, R_{N-1}) \in K$, and with the convention $R_N = 1$, let us define

$$a_n(R) = \sup_{i=1,\dots,K} E[\tfrac{Z_i(t_{n+1})}{Z_i(t_n)} \tfrac{R_{n+1}E(\rho R_{n+2}\dots R_N|\mathcal{F}_{t_{n+1}})}{E(\rho R_{n+1}\dots R_N|\mathcal{F}_{t_n})} \mid \mathcal{F}_{t_n}],$$

$$b_n(R) = \inf_{j=K+1,\dots,S} E[\tfrac{Z_j(t_{n+1})}{Z_j(t_n)} \tfrac{R_{n+1}E(\rho R_{n+2}\dots R_N|\mathcal{F}_{t_{n+1}})}{E(\rho R_{n+1}\dots R_N|\mathcal{F}_{t_n})} \mid \mathcal{F}_{t_n}].$$

Note that by choosing $i = 1$ and $j = K+1$, it is easy to see that $a_n(R) \geq 1$ and $b_n(R) \leq \bar{R}_n$. This implies that $(a_n(R), b_n(R)) \cap [1, \bar{R}_n] \neq \emptyset$, where for any real numbers a and b the set (a, b) denotes the closed interval $[a, b]$ if $a \leq b$, and the closed interval $[b, a]$ otherwise.

Let us denote $[\alpha_n(R), \beta_n(R)] = (a_n(R), b_n(R)) \cap [1, \bar{R}_n]$ and let us define the set-valued map ϕ for all $R \in K$ by

$$\phi(R) = (\phi_0(R), \dots, \phi_{N-1}(R)) = \prod_0^{N-1} [\alpha_n(R), \beta_n(R)].$$

It is easy to see that K is convex and compact for the weak-star topology $\sigma(L^\infty, L^1)$ of L^∞, that $\phi(R) \subset K$ for every $R \in K$, that ϕ takes convex, nonempty and compact values for this topology, and that its graph is closed (which implies that it is upper-semicontinuous).

Let us now modify ϕ in the following way:

(1) If there exists $n \geq 0$ such that $b_n(R)$ is not larger than or equal to $a_n(R)$, let us denote by n_0 the smallest index for which this happens. This means that there exist a nonnull event $B_{n_0} \in \mathcal{F}_{t_{n_0}}$, and i_0 and j_0 such that $b_{n_0}(R) < a_{n_0}(R)$ on B_{n_0}. For every $n > n_0$, let

$\phi_n^*(R) = \{\inf(a_n(R), \beta_n(R))\}$ if $\frac{Z_{i_0}(t_{n_0+1})}{Z_{i_0}(t_{n_0})} - \frac{Z_{j_0}(t_{n_0+1})}{Z_{j_0}(t_{n_0})} \geq 0$, and

$\phi_n^*(R) = \{\sup(b_n(R), \alpha_n(R))\}$ otherwise.

(2) Let $\gamma_0(R) = \frac{1}{E(\rho R_1 \dots R_N)}$. Then let

$\phi_0^*(R) = \{\gamma_0(R)\}$ if $\gamma_0(R) \in [\alpha_0(R), \beta_0(R)]$,

$\phi_0^*(R) = \{\beta_0(R)\}$ if $\gamma_0(R) > \beta_0(R)$, and

$\phi_0^*(R) = \{\alpha_0(R)\}$ if $\gamma_0(R) < \alpha_0(R)$.

(3) Everywhere else we let

$\phi_n^*(R) = \phi_n(R)$.

It is easy to show that the graph of ϕ^* is included in the graph of ϕ, and that ϕ^* is also convex, nonempty and compact valued and that it is upper-semicontinuous.

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By the Kakutani-Fan Theorem (see Fan [1952]), this implies that ϕ^* admits a fixed point, that we shall denote by R^*. We shall now proceed in three steps.

Step 1: For all $n = 0, \ldots, N$ we have $a_n(R^*) \leq b_n(R^*)$.

Indeed, suppose that this property is not satisfied, and consider the smallest index n_0 for which it fails to be true. This means that there exist a nonnull event $B_{n_0} \in \mathcal{F}_{t_{n_0}}$, and i_0 and j_0 such that, with the convention $R^*_N = 1$, $E((\frac{Z_{i_0}(t_{n_0+1})}{Z_{i_0}(t_{n_0})} - \frac{Z_{j_0}(t_{n_0+1})}{Z_{j_0}(t_{n_0})})\frac{\rho R^*_{n_0+1} \ldots R^*_N}{E(\rho R^*_{n_0+1} \ldots R^*_N | \mathcal{F}_{t_{n_0}})} \mid \mathcal{F}_{t_{n_0}}) > 0$ on B_{n_0}. For every $n > n_0$, let $i(n)$ be the ($\mathcal{F}_{t_n}$ measurable) index that realizes the supremum in $a_n(R^*)$ and $j(n)$ be the ($\mathcal{F}_{t_n}$ measurable) index that realizes the supremum in $b_n(R^*)$. Consider the self-financing trading strategy that is triggered at time t_{n_0} in the event B_{n_0}, and consists in buying one dollar of security i_0 and selling short one dollar of security j_0, liquidating the position at time t_{n_0+1} and investing the proceeds in security $i(n_0+1)$ or financing the losses by selling security $j(n_0+1)$ depending on the state of the world, and rolling over the position at each date n by buying security $i(n)$ or selling short security $j(n)$. The payoff of such a strategy is equal to

$$1_{B_{n_0}}(\frac{Z_{i_0}(t_{n_0+1})}{Z_{i_0}(t_{n_0})} - \frac{Z_{j_0}(t_{n_0+1})}{Z_{j_0}(t_{n_0})})\frac{Z_{i(n_0+1)}(t_{n_0+2})}{Z_{i(n_0+1)}(t_{n_0+1})} \cdots \frac{Z_{i(N-1)}(t_N)}{Z_{i(N-1)}(t_{N-1})}$$

if $\frac{Z_{i_0}(t_{n_0+1})}{Z_{i_0}(t_{n_0})} - \frac{Z_{j_0}(t_{n_0+1})}{Z_{j_0}(t_{n_0})} \geq 0$, and to

$$1_{B_{n_0}}(\frac{Z_{i_0}(t_{n_0+1})}{Z_{i_0}(t_{n_0})} - \frac{Z_{j_0}(t_{n_0+1})}{Z_{j_0}(t_{n_0})})\frac{Z_{j(n_0+1)}(t_{n_0+2})}{Z_{j(n_0+1)}(t_{n_0+1})} \cdots \frac{Z_{j(N-1)}(t_N)}{Z_{j(N-1)}(t_{N-1})}$$

otherwise.
Denote by Δ this payoff. Since this strategy is self-financing, satisfies the short sales constraints, and does not require any investment, we must have $E(\rho\Delta) \leq 0$. Note that

$$E(\rho\Delta) = E(1_{B_{n_0}} E((\frac{Z_{i_0}(t_{n_0+1})}{Z_{i_0}(t_{n_0})} - \frac{Z_{j_0}(t_{n_0+1})}{\tilde{Z}_{j_0}(t_{n_0})})A_{n_0+1} \mid \mathcal{F}_{t_{n_0}})),$$

where k is equal to i or j depending on the state of the world and

$$A_{n_0+1} = E(\rho\frac{Z_{k(n_0+1)}(t_{n_0+2})}{Z_{k(n_0+1)}(t_{n_0+1})} \cdots \frac{Z_{k(N-1)}(t_N)}{Z_{k(N-1)}(t_{N-1})} \mid \mathcal{F}_{t_{n_0+1}}).$$

Let us focus on A_{n_0+1}. At date t_{n_0+1} we know what values are taken by k. Let us suppose at first that we have $k = i$ with $i \leq K$. We then have $E(\rho\frac{Z_{k(n_0+1)}(t_{n_0+2})}{Z_{k(n_0+1)}(t_{n_0+1})} \cdots \frac{Z_{k(N-1)}(t_N)}{Z_{k(N-1)}(t_{N-1})} \mid \mathcal{F}_{t_{n_0+1}}) = E(\frac{Z_{i(n_0+1)}(t_{n_0+2})}{Z_{i(n_0+1)}(t_{n_0+1})} \cdots \frac{Z_{i(N-2)}(t_{N-1})}{Z_{i(N-2)}(t_{N-2})} E(\rho\frac{Z_{i(N-1)}(t_N)}{Z_{i(N-1)}(t_{N-1})} \mid \mathcal{F}_{t_{N-1}}) \mid \mathcal{F}_{t_{n_0+1}})$. Note that by construction we have $E(\rho\frac{Z_{i(N-1)}(t_N)}{Z_{i(N-1)}(t_{N-1})} \mid \mathcal{F}_{t_{N-1}}) = a_{N-1}(R^*)E(\rho \mid \mathcal{F}_{t_{N-1}})$. Moreover, since $i \leq K$ we must have $R^*_{N-1} = \inf(a_{N-1}(R^*), \beta_{N-1}(R^*))$ which implies $R^*_{N-1} \leq a_{N-1}(R^*)$ and hence we must have
$A_{n_0+1} \geq$
$E(\frac{Z_{i(n_0+1)}(t_{n_0+2})}{Z_{i(n_0+1)}(t_{n_0+1})} \cdots \frac{Z_{i(N-3)}(t_{N-2})}{Z_{i(N-3)}(t_{N-3})} E(\frac{Z_{i(N-2)}(t_{N-1})}{Z_{i(N-2)}(t_{N-2})} R^*_{N-1} E(\rho \mid \mathcal{F}_{t_{N-1}}) \mid \mathcal{F}_{t_{N-2}}) \mid \mathcal{F}_{t_{n_0+1}})$.

By construction, we have

$$E(\frac{Z_{i(N-2)}(t_{N-1})}{Z_{i(N-2)}(t_{N-2})} R^*_{N-1} E(\rho \mid \mathcal{F}_{t_{N-1}}) \mid \mathcal{F}_{t_{N-2}}) = a_{N-2}(R^*)E(\rho R^*_{N-1} \mid \mathcal{F}_{t_{N-2}}),$$

and we have $R^*_{N-2} \leq a_{N-2}(R^*)$, which after a sufficiently large number of iterations leads to

$$A_{n_0+1} \geq R^*_{n_0+1} E(\rho R^*_{n_0+2} \dots R^*_{N-1} \mid \mathcal{F}_{t_{n_0+1}}).$$

Similarly, in the other case where $k = j$ with $j \geq K+1$ we find that

$$A_{n_0+1} \leq R^*_{n_0+1} E(\rho R^*_{n_0+2} \dots R^*_{N-1} \mid \mathcal{F}_{t_{n_0+1}}).$$

Altogether, this implies that

$$E(\rho\Delta) \geq E(1_{B_{n_0}} E((\frac{Z_{i_0}(t_{n_0+1})}{Z_{i_0}(t_{n_0})} - \frac{Z_{j_0}(t_{n_0+1})}{Z_{j_0}(t_{n_0})}) E(\rho R^*_{n_0+1} \dots R^*_{N-1} \mid \mathcal{F}_{t_{n_0+1}}) \mid \mathcal{F}_{t_{n_0}})).$$

Given the hypothesis made this implies that $E(\rho\Delta) > 0$, which is a contradiction.

Step 2: $\gamma_0(R^*) \in [a_0(R^*), b_0(R^*)]$.

Consider the payoff (which clearly defines the strategy that generates it)

$$\Delta' = \frac{Z_{i(0)}(t_1)}{Z_{i(0)}(t_0)} \cdots \frac{Z_{i(N-1)}(t_N)}{Z_{i(N-1)}(t_{N-1})}.$$

Since this strategy consists in rolling over an initial investment of one dollar at date 0, we must have $E(\rho\Delta') \leq 1$. As in step 1 we can show that $E(\rho\Delta') \geq E(\frac{Z_{i(0)}(t_1)}{Z_{i(0)}(t_0)} E(\rho R^*_1 \dots R^*_{N-1} \mid \mathcal{F}_{t_1}))$. Multiplying both sides by $\gamma_0(R^*)$ we obtain $E(\frac{Z_{i(0)}(t_1)}{Z_{i(0)}(t_0)} \frac{E(\rho R^*_1 \dots R^*_{N-1} \mid \mathcal{F}_{t_1})}{E(\rho R^*_1 \dots R^*_{N-1})}) \leq \gamma_0(R^*)$, which means that $a_0(R^*) \leq \gamma_0(R^*)$. In a similar way, we can show that $b_0(R^*) \geq \gamma_0(R^*)$.

Step 3:

By the fixed-point property, we must have
(i) $R^*_0 = \gamma_0(R^*)$
Indeed, since $\gamma_0(R^*) \in [a_0(R^*), b_0(R^*)]$ by Step 2, and since $a_0(R^*) \geq \alpha_0(R^*)$ and $b_0(R^*) \leq \beta_0(R^*)$ we must have $\gamma_0(R^*) \in [\alpha_0(R^*), \beta_0(R^*)]$.
(ii) $a_n(R^*) \leq R^*_n \leq b_n(R^*)$ for all $n = 0, \dots, N-1$.
Indeed, we have shown that $a_n(R^*) \leq b_n(R^*)$ for all $n = 0, \dots, N-1$ in Step 1.

Let $Z_0(t_0) = 1$ and $Z_0(t_n) = R^*_0 \dots R^*_{n-1}$, for $n = 1, \dots, N$. Since by the fixed-point property we must have $R^*_0 = \gamma_0(R^*)$, we have $E(\rho R^*_0 \dots R^*_{N-1}) = 1$ and hence $E(\rho Z_0(t_N)) = 1$. Moreover since $R^*_n \geq a_n(R^*)$ we must have

$$E(\frac{Z_i(t_{n+1})/Z_0(t_{n+1})}{Z_i(t_n)/Z_0(t_n)} \frac{\rho Z_0(t_N)}{E(\rho Z_0(t_N) \mid \mathcal{F}_{t_n})} \mid \mathcal{F}_{t_n}) \leq 1, \text{ for } n = 0, \dots, N-1, \text{ and } i = 1, \dots, K.$$

And since $R^*_n \leq b_n(R^*)$ we must have $E(\frac{Z_j(t_{n+1})/Z_0(t_{n+1})}{Z_j(t_n)/Z_0(t_n)} \frac{\rho Z_0(t_N)}{E(\rho Z_0(t_N) \mid \mathcal{F}_{t_n})} \mid \mathcal{F}_{t_n}) \geq 1$, for all $n = 0, \dots, N-1$, and all $j = K+1, \dots, S$.

By letting $E^*(x) = E(\rho Z_0(T) x)$, E^* defines an equivalent probability measure (with $\frac{dP^*}{dP} = \rho Z_0(T) \in X$) and satisfies $E^*(\frac{Z_i(t_{n+1})}{Z_0(t_{n+1})} \mid \mathcal{F}_{t_n}) \leq \frac{Z_i(t_n)}{Z_0(t_n)}$ and $E^*(\frac{Z_j(t_{n+1})}{Z_0(t_{n+1})} \mid \mathcal{F}_{t_n}) \geq \frac{Z_j(t_n)}{Z_0(t_n)}$, for all $n = 0, \dots, N-1$, all $i = 1, \dots, L$, and all $j = 1, \dots, S$, which are the desired super/submartingale properties.

To conclude, we extend the numeraire process Z_0 found this way for each time grid $t_0, \dots, t_N$ into a continuous time process by defining $Z_0(t) = exp(\int_0^t r_N(s, \omega) ds)$, with $r(s) = y_n(s)$ for every $t_n \leq s < t_{n+1}$, where $R_n^* = exp(\int_{t_n}^{t_{n+1}} y_n(s)ds)$ for all $n = 0, 1, \ldots, N-1$ (which implies $0 \leq r \leq \bar{r}$). Note that Z_0 is absolutely continuous (see Billingsley [1986, chap. 6]) and can be written $Z_0(t) = \int_0^t g_N(t, \omega)$, where $g_N(t, \omega) = exp(\int_0^t r_N(s, \omega)ds) r_N(t, \omega)$ belongs to $L^\infty([0, T] \times \Omega)$, and satisfies $0 \leq g_N \leq exp(\int_0^T \bar{r})\bar{r}$. We then do this for finer and finer grids $t_0 \leq \ldots \leq t_N$ of union equal to some dense subset D of $[0, T]$. Since the sequence of slopes $g_N(t, \omega)$ constructed this way is bounded in $L^\infty([0, T] \times \Omega)$, it admits a convergent (for the weak-star topology $\sigma(L^\infty, L^1)$) subsequence. Let us denote by $g^*(t, \omega)$ its limit, and consider the numeraire process $Z_0^*(t) = \int_0^t g^*(s, \omega)ds$. It is easy to see that $Log(Z_0^*(t))$ is also absolutely continuous, which allows us to write $Z_0^*(t) = exp(\int_0^t r_0(s, \omega)ds)$, for some bounded process r_0. By letting $E^*(x) = E(\rho Z_0^*(T)x)$, it is easy to see that E^* defines an equivalent probability measure (with $\frac{dP^*}{dP} = \rho Z_0^*(T) \in X$) and satisfies (using the right-continuity of the filtration and the price processes) $E^*(\frac{Z_i(t)}{Z_0(t)} \mid \mathcal{F}_s) \leq \frac{Z_i(s)}{Z_0(s)}$ and $E^*(\frac{Z_j(t)}{Z_0(t)} \mid \mathcal{F}_s) \geq \frac{Z_j(s)}{Z_0(s)}$ for all $T \geq t \geq s \geq 0$, all $i = 1, \ldots, K$, and all $j = K+1, \ldots, S$, which are the desired super/submartingale properties. The property $0 \leq r_0 \leq \tilde{r}$ then follows.

Since M is a convex cone and π is a lower semi-continuous sublinear functional, statement (iii) is a consequence of section 2. ■

Note that by assuming in our multiperiod securities price model that contingent claims that can be hedged at a finite cost are marketed, we implicitly treat $-\pi(-x)$ and $\pi(x)$ as the bid and the ask price of x in the economy. Another possible interpretation is to consider claims that can be hedged as nonmarketed derivative securities and ask what could be equilibrium prices for them. Following the same reasoning as in section 2, it is easy to see that any bid-ask price system (q, p), with $p \geq q$, in the interval $[-\pi(-x), \pi(x)]$ is compatible with the securities price process. However, without knowledge about preferences it not possible to infer tightest bounds than $-\pi(-x)$ and $\pi(x)$ on the price of the claim x.

0.4 Valuation of derivatives

In this section we apply the theoretical results of section 2 and 3 to some particular cases of economies with frictions.

0.4.1 Pricing derivatives with transaction costs

In this subsection we assume that there is no short sales constaints and that the borrowing and the lending rate are equal (to 1 after normalization). In this case, following Theorem 3.1, there exists a probability measure P^* and two processes between Z and Z' such that the first one is a supermartingale and the second one a submartingale with respect to P^*. In fact we can prove the following stronger result.

Theorem 0.4.1 (Jouini and Kallal, 1995 a) *(i) The securities price model admits no multiperiod free lunch if and only if there exist at least a probability measure P^* equivalent to P with $E((\frac{dP^*}{dP})^2) < \infty$ and a process Z^* satisfying $Z' \leq Z^* \leq Z$ such that Z^* is a martingale with respect to the filtration $\{\mathcal{F}_t\}$ and the probability measure P^*.*

(ii) Moreover, if we denote by E^* the expectation operator associated to P^*, there is a one-to-one correspondence between the set of such expectation operators and the set of linear functionals $\psi \in \Psi$ such that $\psi\mid_M \leq \pi$. This correspondence is given by the following formulas:

$$P^*(B) = \psi(1_B), \text{ for all } B \in \mathcal{F} \text{ and } \psi(x) = E^*(x), \text{ for all } x \in X.$$

(iii) Furthermore, for all $m \in M$ we have

$$[-\pi(-m), \pi(m)] = cl\{E^*(m) : P^* \text{ is a martingale measure }\}.$$

Proof of Theorem 4.1.1 :

Using Theorem 3.1, we only have to prove that there exists a martingale relatively to P^* between Z and Z'. Since there is no constraints, Theorem 3.1 (ii) permits us to construct two processes $\overline{Z}^*$ and $\underline{Z}^*$ between Z and Z' such that $\overline{Z}^*$ is a supermartingale relatively to P^* and $\underline{Z}^*$ a submartingale.

Note that we must have $\overline{Z}^*(t) \leq \underline{Z}^*(t)$ for all t. Indeed, assume instead that there exists $t \in [0,T]$ and $B \in \mathcal{F}_t$ such that $P(B) > 0$ and $\overline{Z}^*(t,\omega) > \underline{Z}^*(t,\omega)$ for all $\omega \in B$. Then there would exist strategies $(\theta,\theta') \in \Theta'(t,k)$ and $(\alpha,\alpha') \in \Theta(t,k)$ satisfying $E^*((\theta_0 - \theta'_0)(T)1_B) > E^*(-(\alpha_0 - \alpha'_0)(T)1_B)$, i.e. $E^*(((\theta_0+\alpha_0)-(\theta'_0+\alpha'_0))(T)1_B) > 0$, and hence $E^*([\{(\theta+\alpha)-(\theta'+\alpha')\}^+(T)\cdot Z'(T) - \{(\theta+\alpha)-(\theta'+\alpha')\}^-(T)\cdot Z(T)]1_B) > 0$ which contradicts the fact that $E^*([\{(\theta+\alpha)-(\theta'+\alpha')\}^+(T)\cdot Z'(T) - \{(\theta+\alpha)-(\theta'+\alpha')\}^-(T)\cdot Z(T)]1_B) \leq \pi([\{(\theta+\alpha)-(\theta'+\alpha')\}^+(T)\cdot Z'(T) - \{(\theta+\alpha)-(\theta'+\alpha')\}^-(T)\cdot Z(T)]1_B) \leq 0$. Hence $\overline{Z}^*(t) \leq \underline{Z}^*(t)$ for all t. To conclude, we use the following

Lemma 2 *If $Z'(t) \leq \overline{Z}^*(t) \leq \underline{Z}^*(t) \leq Z(t)$, where $\overline{Z}^*(t)$ is a supermartingale and $\underline{Z}^*(t)$ is a submartingale with respect to $\{\mathcal{F}_t\}$ and a probability $P^* = \rho P$, with $\rho \in X$, then there exists a martingale Z^* with respect to $\{\mathcal{F}_t\}$ and P^* such that $Z' \leq Z^* \leq Z$.*

The proof of this Lemma is given in Jouini and Kallal (1995a).

■

Note that Theorem 3.1 is not in this last reference and Theorem 4.1.1 is then proved directly.

In the next, we will show how we can use this approach to price contingent portfolios. The main difference with the previous approach is that, in the next, we will not compare liquidation values for different strategies but the final quantity of each security for each strategy. We replace then the main definitions of previous sections as follows.

Definition 0.4.1 *A contingent claim C is defined by $(C_0, \ldots, C_K) \in X^{K+1}$ the contingent portfolio guaranted by C*

Note that a contingent claim C is not necessarily attainable by a strategy belonging to Θ. Thus, we consider the following set M of marketed claims, i.e., claims that can be dominated (or hedged) by the payoff of a simple self-financing trading strategy.

Definition 0.4.2 *A claim C is said to be marketed if there exists a self-financing simple strategy (θ,θ') in Θ such that $(\theta_k - \theta'_k)(T,\omega) \geq C_k(\omega)$. for $k = 0,\ldots,K$ and for almost every ω.*

As we have already seen, when there are transaction costs it is not true that the cheapest way to obtain a given minimal contingent payoff at date T is to duplicate it by dynamic trading. A simple example can illustrate it. Assume that a call option on a stock is to be hedged using a riskless bond and the underlying stock only. Also suppose that there are transaction costs in trading the stock at intermediate dates (between now and maturity). It is then easy to see that if transaction costs are prohibitively high it is cheaper to buy the stock and hold it until maturity (which leads to a payoff that is strictly larger than the payoff of the call) than to try to duplicate the call. Hence, we consider the price functional π defined for every marketed claim $m \in M$ by

$$\begin{aligned}\pi(C) = \inf\{ \ &\liminf_n \{\theta^n(0)\cdot Z(0) - \theta'^n(0)\cdot Z'(0)\} : (\theta^n, \theta'^n) \in \Theta,\\ &(\theta^n - \theta'^n)(T) \geq C^n\\ &\text{and } (C^n) \subset M \text{ converges to } C\}.\end{aligned}$$

In words, $\pi(C)$ represents the infimum cost necessary to get at least a final contingent portfolio arbitrarily close to C at date T. In the subsequent analysis we shall characterize the situations in which there are no possibilities of arbitrage profits. In fact, as in the previous sections we shall rule out a weaker form of arbitrage opportunities: (portfolio) free lunches. In words, a (portfolio) free lunch is the possibility of getting a contingent portfolio (at date T) with a value arbitrarily close to a given element of X_+ using a self-financing simple strategy at a cost (in terms of date 0 consumption) arbitrarily small. More precisely

Definition 0.4.3 *A (portfolio) free lunch is a sequence of contingent claims C^n in M, converging to some C and a sequence (θ^n, θ'^n) in Θ such that $C^+Z'(T) - C^-Z(T) \in X_+$, $(\theta^n - \theta'^n)(T) \geq C^n$ for all n and $\lim_n \{\theta^n(0)\cdot Z(0) - \theta'^n(0)\cdot Z'(0)\} \leq 0$.*

An obvious consequence of the absence of (portfolio) free lunches is that the bid price of any security must lie above the ask price (i.e. for all $t \in \mathcal{T}$, $P(\{\omega : Z'(t,\omega) \leq Z(t,\omega)\}) = 1$).

We shall now see that the price functional π is sublinear: it is less expensive to hedge the sum $x + y$ of two contingent claims than to hedge the claims x and y separately and add up the costs. It is easy to see why: the sum of two strategies that hedge the claims x and y hedges the claim $x + y$ but some orders to buy and sell the same security at the same date might cancel out. Some of the transaction costs might be saved this way.

Proposition 0.4.2 *If there is no (portfolio) free lunch, then the set of marketed claims M is a convex cone and the price functional π is a lower semicontinuous sublinear functional which takes value in R.*

Proof of Proposition 4.1.2 : Using the fact that Θ is a convex cone and the fact that $Z \geq Z'$, it is relatively easy to show that M is a convex cone. Since the null strategy permits to dominate the null portfolio at a zero cost we have $\pi(0) \leq 0$. If $\pi(0) < 0$, it is easy to construct a (portfolio) free lunch and then $\pi(0) = 0$. Furthermore we can easily see that $\pi(\lambda C) = \lambda\pi(C)$ for all $C \in M$ and $\lambda \in R_+$. Using a limit argument it is also easy to obtain that for all $C, C' \in M$ $\pi(C+C') \leq \pi(C)+\pi(C')$.

Let $\lambda \in R$ and C^n be a sequence in M converging to $C \in M$ such that $\pi(C^n) \leq \lambda$, for all n. Then, by a diagonal extraction process, there exist a sequence $\tilde{C}^n \in M$ and a sequence $(\theta^n, \theta'^n) \in \Theta$ such that $\|\tilde{C}^n - C^n\| \leq \frac{1}{n}$, $(\theta^n - \theta'^n)(T) \geq \tilde{C}^n$ and $\theta^n(0) \cdot Z(0) - \theta'^n(0) \cdot Z'(0) \leq \lambda + \frac{1}{n}$. Since $\tilde{C}^n$ converges to C we must then have $\pi(C) \leq \lambda$. Hence, the set $\{C \in M : \pi(C) \leq \lambda\}$ is closed and π is l.s.c.

Let now $C \in M$ such that $\pi(C) = -\infty$, for all $\lambda > 0$, we have $\pi(\lambda C) = -\infty$ and if λ converges to 0 we obtain by the lower semi-continuity property that $\pi(0) = -\infty$. This contradiction implies that π takes its values in R. ∎

We are now in a position to prove our main result that characterizes arbitrage free bid and ask price processes. We find that the securities prices model is arbitrage free if and only if there exists a process, lying between the bid and ask price processes, that is a martingale under some equivalent measure. Moreover, we obtain that the arbitrage price of a contingent portfolio is equal to the supremum of the expectations of its terminal value evaluated with some security price process Z^* in the bid-ask interval and relatively to a probability measure P^* such that Z^* is a P^* martingale. Recall that Theorem 3.1 claims that when we only compare terminal liquidation values, the arbitrage price is equal to the supremum of the expectations of the terminal liquidation value relatively to a probability measure P^* such that there exists some martingale Z^* relatively to P^* in the bid-ask interval. The process Z^* does not appear in the valuation formula it only has an effect on the choice of P^*. In the following Theorem Z^* imposes restrictions on P^* and appears directly in the valuation formula.

Theorem 0.4.3 *(i) The securities price model admits no (portfolio) free lunch if and only if there exist at least a probability measure P^* equivalent to P with $E((\frac{dP^*}{dP})^2) < \infty$ and a process Z^* satisfying*[23] *$Z' \leq Z^* \leq Z$ such that Z^* is a martingale with respect to the filtration $\{\mathcal{F}_t\}$ and the probability measure P^*.*
(ii) For all contingent claim C in M we have $\pi(C) = supE^*(C \cdot Z^*(T))$ where the supremum is taken over all the expectation operators E^* associated to a probability measure P^* and all the processes Z^* such that (P^*, Z^*) satisfy the conditions of (i).

Proof of Theorem 4.1.3 : First, let P^* be a probability measure equivalent to P and let Z^*, with $Z' \leq Z^* \leq Z$, be a martingale with respect to P^* and $\{\mathcal{F}_t\}$. Define the linear functional ψ by $\psi(C) = E^*(Z^*(T) \cdot C)$ for all $C \in X^{K+1}$. Since $\rho = \frac{dP^*}{dP} \in X$ we have by the Riesz representation Theorem that $\psi(C) = E^*(Z^*(T) \cdot C) = E(\rho Z^*(T) \cdot C)$ is continuous.

Let $\tilde{\Omega}$ be the space equal to $\Omega \times \{0, \ldots, K\}$ endowed with $(\tilde{\mathcal{F}}, \tilde{P})$ the natural probability structure defined by $(\mathcal{F}, P)$. Let $\tilde{X}$ be the set defined by $\tilde{X} = L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. The set $\tilde{X}$ can be identified with X^{K+1} and π can be seen as a sublinear functional on a subset M of $\tilde{X}$. In the next $\tilde{\Psi}$ is defined for $\tilde{X}$ as Ψ for X

Since P and P^* are equivalent, we have $\psi \in \tilde{\Psi}$.

Let $C \in M$ and let $(\theta, \theta') \in \Theta$ with trading dates $0 = t_0 \leq t_1 \leq \ldots \leq t_N = T$, such that $(\theta - \theta')(T) \geq C$. Since $Z' \leq Z^* \leq Z$ and (θ, θ') is nondecreasing and self-financing, we have, for $n = 1, \ldots, N$,

$$\begin{aligned} &E^*((\theta(t_n) - \theta(t_{n-1}))^*(t_n) - (\theta'(t_n) - \theta'(t_{n-1})) \cdot Z^*(t_n) \mid \mathcal{F}_{t_{n-1}}) \\ &\leq E^*((\theta(t_n) - \theta(t_{n-1}))(t_n) - (\theta'(t_n) - \theta'(t_{n-1})) \cdot Z'(t_n) \mid \mathcal{F}_{t_{n-1}}) \leq 0. \end{aligned}$$

[23] This means that for all t, $P(\{\omega : Z'(t, \omega) \leq Z^*(t, \omega) \leq Z(t, \omega)\}) = 1$.

Using the fact that Z^* is a martingale with respect to $\{\mathcal{F}_t\}$ and P^*, we have

$$E^*((\theta-\theta')(t_n)\cdot Z^*(t_n) \mid \mathcal{F}_{t_{n-1}}) \leq E^*((\theta-\theta')(t_{n-1})\cdot Z^*(t_n) \mid \mathcal{F}_{t_{n-1}}) \leq (\theta-\theta')(t_{n-1})\cdot Z^*(t_{n-1}).$$

By iteration, $E^*((\theta-\theta')(T)\cdot Z^*(T)) \leq (\theta-\theta')(0)\cdot Z^*(0) \leq \theta(0)\cdot Z(0) - \theta'(0)\cdot Z'(0)$. Since $(\theta-\theta')^+(T)\cdot Z'(T) - (\theta-\theta')^-(T)\cdot Z(T) \leq \{(\theta-\theta')^+(T) - (\theta-\theta')^-(T)\}\cdot Z^*(T)$ and, by positivity of the price processes, $C^+Z'(T) - C^-Z(T) \leq (\theta-\theta')^+(T)\cdot Z'(T) - (\theta-\theta')^-(T)\cdot Z(T)$ we have $E^*(C^+Z'(T) - C^-Z(T)) \leq \theta(0)\cdot Z(0) - \theta'(0)\cdot Z'(0)$. It is easy to see that this implies that there cannot be any (portfolio) free lunch.

Furthermore, $\psi(C) = E^*(Z^*(T)\cdot C) \leq E^*((\theta-\theta')(T)\cdot Z^*(T))\theta(0)\cdot Z(0) - \theta'(0)\cdot Z'(0)$. Taking the infimum over the strategies $(\theta,\theta')\in\Theta$ such that $(\theta-\theta')(T)\geq C$, we obtain that $\psi(C) = E^*(C) \leq \tilde{\pi}(C)$, where $\tilde{\pi}(m) = \inf\{\theta(0)\cdot Z(0) - \theta'(0)\cdot Z'(0) : (\theta,\theta')\in\Theta \text{ and } (\theta-\theta')(T)\geq C\}$, for all $C\in M$. We can prove as in Lemma 1 that π is the largest l.s.c. functional that lies below $\tilde{\pi}$. Using the fact that ψ is continuous, we have then $\psi\mid_M \leq \pi$.

Lemma 3 *The absence of (portfolio) free lunch is equivalent to the absence of free lunch for (M,π).*

Proof of the Lemma : Consider a (portfolio) free lunch as in definition 4.1.3. We have $\theta^n(0)\cdot Z(0) - \theta'^n(0)\cdot Z'(0) \geq \pi((\theta^n - \theta'^n)(T)) \geq \pi(C^n)$ and taking the lim inf on both sides we see that for a subsequence of the sequence C^n, we have $\lim_n\{\theta^n(0)\cdot Z(0) - \theta'^n(0)\cdot Z'(0)\} \leq 0$. If (M,π) admits no free lunch, by Theorems 2.1 and 2.2 there exists $\psi\in\tilde{\Psi}$ such that $\psi\mid_M\leq\pi$ and we have then $\psi(C)\leq 0$ which is impossible.

Conversely, consider a free lunch for (M,π). We have $x_n\in M$ and $r_n+\pi(x_n)\leq 0$ for all n. Using the definition of π we can find, by a diagonal extraction process, sequences of strategies $(\theta^n,\theta'^n)\in\Theta$ and of claims $y_n\in M$ such that $\|x_n - y_n\|\leq 1/n$, $(\theta^n-\theta'^n)(T)\geq y_n$ and $r^* + \lim_n\{\theta^n(0)\cdot Z(0) - \theta'^n(0)\cdot Z'(0)\}\leq 0$. If $r^*=0$, we then have a multiperiod free lunch since y_n converges to $x^*\in\tilde{X}_+$ and then $C^+Z'(T) - C^-Z(T)\in X_+$. If $r^*>0$, it suffices to modify the strategies such as the amount r^* is invested in some security to obtain a (portfolio) free lunch. This ends the proof of the Lemma.

Assume now that there is no (portfolio) free lunch or equivalently that there is no free lunch for (M,π) and let $\psi\in\tilde{\Psi}$ such that $\psi\mid_M\leq\pi$, as guaranteed by Theorems 2.1 and 2.2. Since ψ is continuous, by the Riesz representation Theorem there exists a random variable $\rho\in\tilde{X}$ such that $\psi(x) = E(\rho x)$, for all $x\in\tilde{X}$ or equivalently there exists $(\rho_0,\ldots,\rho_K)$ in X^{K+1} such that $\psi(C) = E(\rho\cdot C)$, for all $C\in X^{K+1}$. Define P^* from ψ by $P^*(B) = E(\rho_0 1_B)$ for all $B\in\mathcal{F}$. By linearity and strict positivity of ψ it is clear that P^* is a measure equivalent to P. Using the fact that $Z_0 = Z'_0 = 1$ it is easy to show that $P^*(1_\Omega) = 1$ and $\frac{dP^*}{dP} = \rho_0$ is square integrable.

It remains to show that there exists a process Z^*, with $Z'_k\leq Z^*_k\leq Z_k$, and such that Z^*_k is a martingale with respect to P^* and $\{\mathcal{F}_t\}$, for $k=1,\ldots,K$. In fact, we will prove that the martingale relatively to P^* and $\{\mathcal{F}_t\}$ defined by $Z^*_k(t) = E^*(\rho_k/\rho_0\mid\mathcal{F}_t)$ lies between Z'_k and Z_k.

Let $k\in\{1,\ldots,K\}$, $t\in\mathcal{T}$ and $B\in\mathcal{F}_t$. Let C the contingent claim defined by $C_k = 1_B$, $C_0 = -Z_k(t)1_B$ and $C_h = 0$ for $h\neq 0,k$. The contingent claim C is duplicable. It suffices to buy at t, if $\omega\in B$, one unit of the security k

and to pay with security 0 units. This strategy costs nothing and we have then $E^*((-C_0 Z_k(t) + \rho_k/\rho_0)1_B) = E((-\rho_0 Z_k(t) + \rho_k)1_B) = \psi(C) \leq \pi(C) \leq 0$. Then, we have $E^*(\rho_k/\rho_0 1_B) \leq E^*(Z_k(t)1_B)$, for all t and all $B \in \mathcal{F}_t$. This implies that $Z_k^* \leq Z_k$. By a symetric argument we obtain $Z_k^* \geq Z'_k$ which achieves to prove the point (i) of the theorem.

In fact, we have also proved that every $\psi \in \tilde{\Psi}$ such that $\psi \mid_M \leq \pi$ is equal to $E^*(C \cdot Z^*)$ for some process Z^* between Z' and Z and some probability measure P^* such that Z^* is a martingale relatively to P^* and conversely.

Following section 2, $\pi(C) = \sup\ \psi(C)$ where the supremum is taken over all the functionals $\psi \in \tilde{\Psi}$ such that $\psi \mid_M \leq \pi$. This result permits to achieve the proof of (ii). ■

0.4.2 Shortselling costs in the diffusion case

We now turn to an illustration of our results, in an economy where traded securities prices follow a continuous time diffusion process.

In this section, we consider the continuous time case where the set of trading dates $\mathcal{T}$ is equal to $[0, T]$ and securities prices follow diffusion processes. To keep things simple we shall treat the case where there are only two securities, a bond and a stock that follows a diffusion process. We model the spread between the borrowing and lending rate by having a (locally) riskless security in which investors can only go short and a riskless security in which they can only go long. Similarly, we model the cost of shortselling the underlying stock by having a risky security in which investors can only go short, and a risky security in which they can only go long, with possibly different returns. We shall not consider bid-ask spreads in the traded securities here. Again to simplify, we shall assume that there is a single source of uncertainty in the economy, modeled by a Brownian Motion.[24] Finally, we shall focus our attention on the pricing of derivative securities that pay a given function of the underlying stock price at maturity.

Formally, let $W = \{W(t) : 0 \leq t \leq T\}$ be a one-dimensional standard Brownian Motion defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_t\}$ be the augmented[25] filtration generated by W, with $\mathcal{F}_T = \mathcal{F}$. Consider the risky securities price processes Z_1 and Z_2 satisfying the stochastic integral equations:

$$Z_1(t) = Z_1(0) + \int_0^t \mu_1(Z_1(s), s) Z_1(s) ds + \int_0^t \sigma_1(Z_1(s), s) Z_1(s) dW(s)$$

$$Z_2(t) = Z_2(0) + \int_0^t \mu_2(Z_1(s), s) Z_2(s) dt + \int_0^t \sigma_2(Z_1(s), s) Z_2(s) dW(s)$$

and the locally riskless securities price processes

$$B_1(t) = exp(\int_0^t r_1(Z_1(s), s) ds),$$

$$B_2(t) = exp(\int_0^t r_2(Z_1(s), s) ds),$$

[24] There is no real difficulty in introducing more sources of risk and more traded securities.

[25] This implies in particular that it is right-continuous.

for all $t \in [0,T]$, where the drifts $\mu_i(x,t)$, the volatilities $\sigma_i(x,t) : R \times [0,T] \rightarrow R$ and the riskless rates $r_i(x,t)$ are given continuous functions of the state variable x (the price of the stock) and time t, such that $\sigma_i(x,t) > 0$, $r_i(x,t) \geq 0$ and $r_i(x,t)$ is bounded for $i = 1,2$. Initial values $Z_1(0)$ and $Z_2(0)$ are assumed to be positive so that the stock price processes remain positive.

Investors are assumed to be able to hold only long positions in security Z_1 and only short positions in security Z_2. In fact, the processes Z_1 and Z_2 can model the returns on the same stock, respectively for a long and a short position. In this case, the difference in return between Z_2 and Z_1 models the cost of shortselling the stock[26]. Also, r_1 models the riskless lending rate and r_2 models the riskless borrowing rate, as we assume that investors can only hold nonnegative quantities of B_1 and nonpositive quantities of B_2.

We shall assume that the drift and volatility are sufficiently regular to guarantee the existence and uniqueness of continuous solutions Z_1 and Z_2 with bounded moments to the stochastic integral equations above. We refer to Gihman and Skorohod (1972, Theorem 1, p. 40) for sufficient Lipschitz and growth conditions on $\mu_i(x,t)$ and $\sigma_i(x,t)$.

Absence of arbitrage and (super/sub)martingale measures

Following Theorem 3.1 consider P^*, a candidate probability measure consistent with the absence of arbitrage, i.e. an equivalent probability measure with $dP^* = \rho dP$ where ρ is strictly positive and square integrable. Using Kunita and Watanabe (1967) Representation Theorem it is easy to show that $E(\rho \mid \mathcal{F}_t)$ can be written as $1 + \int_0^T \xi(s)dW(s)$, where ξ is a measurable adapted process satisfying $\int_0^T E(\xi^2(s))ds < \infty$, and it follows from Ito's Lemma that $E(\rho \mid \mathcal{F}_t) = exp\{\int_0^t \gamma(s)dW(s) - \frac{1}{2}\int_0^t \gamma^2(s)ds\}$ and in particular

$$\rho = exp\{\int_0^T \gamma(s)dW(s) - \frac{1}{2}\int_0^T \gamma^2(s)ds\}$$

where $\gamma(t) = \frac{\xi(t)}{E(\rho|\mathcal{F}_t)}$. Let the process $W^*(t)$ be defined by

$$W^*(t) = W(t) - \int_0^t \gamma(s)ds, 0 \leq t \leq T.$$

Then according to Girsanov's (1960) Fundamental Theorem 1, W^* is a Brownian Motion on $(\Omega, \mathcal{F}, P^*)$, and the processes Z_i satisfy, for $i = 1,2$, the stochastic integral equation

$$Z_i(t) = \int_0^t \{\mu_i(Z_1,s) + \sigma_i(Z_1,s)\gamma(s)\}Z_i(s)ds + \int_0^t \sigma_i(Z_1,s)Z_i(s)dW^*(s).$$

We shall further assume that for each price process Z_i, and every numeraire process $B_0(t) = exp(\int_0^t r_0(s)ds)$ with $r_1(s) \leq r_0(s) \leq r_2(s)$, there exists an equivalent probability measure P_i, with $E((\frac{dP_i}{dP})^2) < \infty$, that transforms $\frac{Z_i}{B_0}$ into a martingale. A sufficient condition is $E[exp(8\int_0^T \gamma_i^2(s)ds)] < \infty$, for $i = 1,2$, where $\gamma_i(t) = -\frac{\mu_i(t)-r_0(t)}{\sigma_i(t)}$ is the instantaneous price of risk associated to the normalized price processes (see the Appendix). Note that if $\gamma_i(t)$ is bounded this condition is satisfied.

[26]Although we let the volatilities of the long and short positions in the stock be possibly different, the cost of shortselling a security is more likely to be paid in terms of expected return.

We are now in a position to characterize the absence of arbitrage in our model. According to Theorem 3.2, this model is arbitrage free if and only if we can find a numeraire $B_0(t) = exp(\int_0^t r_0(s)ds)$ with a rate of accumulation r_0 that is between the lending rate r_1 and the borrowing rate r_2, and an equivalent probability measure for which $\frac{Z_1}{B_0}$ is a supermartingale and $\frac{Z_2}{B_0}$ is a submartingale. This leads to the following result established in Jouini and Kallal (1995b).

Theorem 0.4.4 *(i) The model is arbitrage free if and only if there exists an accumulation rate process* $r_0(t)$ *such that* $r_1(t) \leq r_0(t) \leq r_2(t)$ *and* $\frac{\mu_1(t)-r_0(t)}{\sigma_1(t)} \leq \frac{\mu_2(t)-r_0(t)}{\sigma_2(t)}$ *for all* $t \in [0,T]$.
(ii) In this case, for each such accumulation rate process r_0*, the set of the (square integrable) Radon-Nikodym derivatives of the equivalent probability measures for which* $\frac{Z_1}{exp(\int_0^t r_0(s)ds)}$ *is a supermartingale and* $\frac{Z_2}{exp(\int_0^t r_0(s)ds)}$ *is a submartingale is*

$$\Pi = \{\rho : \rho = exp[\int_0^T \gamma(s)dW(s) - \tfrac{1}{2}\int_0^T \gamma^2(s)ds] \text{ for some adapted process } \gamma(t) \text{ such that } -\frac{\mu_2(t)-r_0(t)}{\sigma_2(t)} \leq \gamma(t) \leq -\frac{\mu_1(t)-r_0(t)}{\sigma_1(t)}\}.$$

If the (borrowing and lending) riskless rate is equal to r_0, then $-\gamma_i(t) = \frac{\mu_i(t)-r_0(t)}{\sigma_i(t)}$ represents the instantaneous risk-premium per unit of risk, i.e. the expected return of the security in excess of the riskless rate per unit of volatility. The theorem then says that the price processes are arbitrage free if and only if the price of risk is higher for a short position than for a long position in the stock, i.e. if investors, when long in the stock, get a smaller expected return per unit of risk, than they have to pay per unit of risk when short in the stock. Moreover, the instantaneous price of risk is known to be useful to compute the price of derivative securities in frictionless securities market models. Adding this return to the Brownian Motion we obtain a security that has a unit constant volatility, and an expected return equal to the price of risk. In a world where this basic security is a martingale, i.e. where the adjusted price of risk is equal to zero, all derivative securities should be martingales as well, and their prices should be the expected value of their payoffs (with respect to the new probability). In our case there are two instantaneous prices of risk: the price for a short position and the price for a long position in the stock. As we shall see, bounds can be found on any derivative security price by computing all the expected values of its normalized payoff with respect to the probabilities that make the adjusted price of risk equal to zero, starting from any price of risk between the prices of risk for a long and a short position in the stock. As we would expect, the relevant price of risk is equal to the price of risk for a long position in the stock if the optimal hedging strategy consists in being long in the stock, and it is equal to the price of risk for a short position in the stock if the optimal hedging strategy consists in being short in the stock. If the borrowing and the lending rates differ, the previous steps need to be followed using all the riskless rates that lie between the borrowing and the lending rates, and for which the risk-premium on a long position in the stock is lower than the risk-premium on a short position.

Arbitrage bounds on contingent claims

Let us consider a contingent claim to consumption at the final date T that is a given function of the stock price at that date, i.e. that is of the form $h(Z_1(T))$. Denote by $\pi(h)$ and $-\pi(-h)$ the arbitrage bounds on the bid and the ask price of the claim

$h(Z_1(T))$, where $\pi(h)$ is the minimum amount it costs to hedge it and $-\pi(-h)$ the maximum that can be borrowed against it through dynamic securities trading. We have already seen that no investor would agree to pay more than $\pi(h)$ for the claim $h(Z_1(T))$, since there is a way of obtaining (at least) this payoff by trading in the underlying securities. Also, no investor would sell the claim $h(Z_1(T))$ for less than $-\pi(-h)$ since there is a way of obtaining $-\pi(-h)$ against this payoff by trading in the underlying securities.

According to Theorem 3.1, the interval $[-\pi(-h), \pi(h)]$ is equal (modulo its boundary) to the set of expectations $\{E(\rho\frac{h(Z_1(T))}{exp(\int_0^T r_0(t)dt)}) : r_1(t) \leq r_0(t) \leq r_2(t)$ and $\rho \in \Pi(r_0)\}$, where $\Pi(r_0)$ is the set of probability measures for which $\frac{Z_1(t)}{exp(\int_0^t r_0(s)ds)}$ is a supermartingale and $\frac{Z_2(t)}{exp(\int_0^t r_0(s)ds)}$ is a submartingale. This allows us to express these bounds as the solutions of optimal control problems and to derive a partial differential equation with a boundary condition that they satisfy. These partial differential equations are in fact similar to the partial differential equation derived by Black and Sholes in the frictionless case, with two additional nonlinear terms. One term is proportional to the spread between the borrowing and the lending rates, whereas the other term is proportional to the spread between the risk-premia on long and short positions in the stock.

To derive these results, we consider payoffs that are sufficiently regular and do not grow too fast. More precisely, we assume that the function $h : R \to R$ is continuous and satisfies the polynomial growth condition $|h(x)| \leq C(1 + |x|)^n$ for some constant C and some nonnegative integer n. We denote by $C_p^{2,1}(E)$ the class of functions from a subset E of $R \times [0, T]$ into R that are twice continuously differentiable with respect to the first variable and continuously differentiable with respect to the second variable. We then have

Theorem 0.4.5 (Jouini and Kallal (1995b)) *(i) If the securities price model is arbitrage free, and if there exists a function $V(x, t) \in C_p^{2,1}(R \times (0, T))$, continuous on $R \times [0, T]$, and satisfying the partial differential equation*

$$r_1 V = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 V}{\partial x^2} + r_1 x \frac{\partial V}{\partial x}$$

$$+\sigma_1 x(\frac{\mu_2 - r_1}{\sigma_2} - \frac{\mu_1 - r_1}{\sigma_1})[\frac{\partial V}{\partial x}]^- + (r_2 - r_1)[V - x(\frac{\partial V}{\partial x})^+ + x\frac{\sigma_1}{\sigma_2}(\frac{\partial V}{\partial x})^-]^-$$

with boundary condition

$$V(x, T) = h(x), \text{ for all } x \in R$$

then $\pi(h) = V(Z_1(0), 0)$.
(ii) In this case, we have

$$h(Z_1(T)) = V(Z_1(0), 0) + \int_0^T (\frac{\partial V}{\partial x})^+ dZ_1 - \int_0^T \frac{\sigma_1 Z_1}{\sigma_2 Z_2}(\frac{\partial V}{\partial x})^- dZ_2 + \int_0^T \frac{[\Delta_0]^+}{B_1} dB_1 + \int_0^T \frac{[\Delta_0]^-}{B_2} dB_2$$

where $\Delta_0 = V - Z_1[(\frac{\partial V}{\partial x})^+ - \frac{\sigma_1}{\sigma_2}(\frac{\partial V}{\partial x})^-]$.

This means that the solution to this partial differential equation V, that gives the arbitrage bound as a function of the stock price and calendar time, can be used to find the optimal hedging strategy in terms of the hedge ratio $\frac{\partial V}{\partial x}$, as in the frictionless case. The optimal hedging strategy, that duplicates the payoff $h(Z_1(T))$ for the lowest possible cost, consists in holding, at any time, $\Delta_1 = (\frac{\partial V}{\partial x})^+$ units of security Z_1, i.e. of stock, and $\Delta_2 = -\frac{\sigma_1 Z_1}{\sigma_2 Z_2}(\frac{\partial V}{\partial x})^-$ units of security Z_2, i.e. going short in $(\frac{\sigma_1}{\sigma_2})(\frac{\partial V}{\partial x})^-$ units of stock, and investing the surplus $(\Delta_0)^+ = [V - \Delta_1 Z_1 - \Delta_2 Z_2]^+ = [V - Z_1(\frac{\partial V}{\partial x})^+ + Z_1\frac{\sigma_1}{\sigma_2}(\frac{\partial V}{\partial x})^-]^+$ in the riskless bond B_1 or financing the deficit $(\Delta_0)^- = [V - \Delta_1 Z_1 - \Delta_2 Z_2]^- = [V - Z_1(\frac{\partial V}{\partial x})^+ + Z_1\frac{\sigma_1}{\sigma_2}(\frac{\partial V}{\partial x})^-]^-$ by selling short the riskless bond B_2. This strategy only requires an initial investment $\pi(h) = V(Z_1(0), 0)$.

It is also possible to determine the equivalent probability measure P^* and the riskless rate that "price" the contingent claim $h(Z_1(T)$, i.e. that are such that $\pi(h) = E^*[\frac{h(Z_1(T))}{exp(\int_0^T r_0(t)dt)}]$. It follows from Theorem 3.2 that the riskless rate is given by

$$r_0 = r_1 1_{\{V - Z_1(\frac{\partial V}{\partial x})^+ + Z_1\frac{\sigma_1}{\sigma_2}(\frac{\partial V}{\partial x})^- \geq 0\}} + r_2 1_{\{V - Z_1(\frac{\partial V}{\partial x})^+ + Z_1\frac{\sigma_1}{\sigma_2}(\frac{\partial V}{\partial x})^- < 0\}}.$$

The equivalent probability measure P^* is given by $\frac{dP^*}{dP} = exp\{\int_0^T \gamma(s)dW(s) - \frac{1}{2}\int_0^T \gamma^2(s)ds\}$, where the risk-premium γ is given by

$$\gamma(t) = -\frac{\mu_1 - r_0}{\sigma_1} 1_{\{\frac{\partial V}{\partial x} \geq 0\}} - \frac{\mu_2 - r_0}{\sigma_2} 1_{\{\frac{\partial V}{\partial x} < 0\}}.$$

In other words, r_0 and γ are respectively the riskless rate and the risk-premium faced by an investor hedging the contingent claim $h(Z_1(T))$.

It is easy to see that these results apply to the lower bound $-\pi(-h)$ on the price of the claim $h(Z_1(T))$. Indeed, we have $-\pi(-h) = -V(Z_1(0), 0)$ where V satisfies the same partial differential equation as in Theorem 4.2.1 but with the opposite boundary condition $V(x, T) = -h(x)$. This partial differential equation is nonlinear whenever $r_2 > r_1$ and/or $\frac{\mu_2 - r_1}{\sigma_2} > \frac{\mu_1 - r_1}{\sigma_1}$, i.e. whenever there actually is a cost of borrowing and/or a cost of shortselling the stock. Because of the nonlinearity, the upper and lower bounds $\pi(h)$ and $-\pi(-h)$ do not necessarily collapse to a single arbitrage price as they would in the absence of frictions.

Note that a heuristic reasoning could have led us to these partial differential equations. Indeed, from the sign of $\frac{\partial V}{\partial x}(x, t)$ we can infer whether a hedging strategy needs to be long or short in the stock: if $\frac{\partial V}{\partial x}(x, t) > 0$ it needs to be long since it must rise in value as the stock price moves up and if $\frac{\partial V}{\partial x}(x, t) < 0$ it needs to be short since it must fall in value as the stock price moves up. In the first case, assuming that the riskless rate is r_0, the price of risk for a long position in the stock is $-\frac{\mu_1 - r_0}{\sigma_1}$ and the risk-neutral adjusted expected return on the stock is equal to the riskless rate. In the second case, the price of risk for a short position in the stock is $-\frac{\mu_2 - r_0}{\sigma_2}$ and the risk-neutral adjusted expected return on the stock is then equal to the riskless rate plus the cost (in terms of excess expected return) $\sigma_1(\frac{\mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1})$ of holding a short position in the stock with the same volatility. The riskless rate r_0 is then set equal to the lending rate r_1 if the hedging portfolio involves lending, and to the borrowing rate r_2 otherwise. Writing Black and Sholes (1973) equation with these risk-neutral adjusted expected returns and riskless rates, we then obtain our partial differential equation.

So far we only have a Verification Theorem, that is to say that if we can find a sufficiently regular solution to the partial differential equations above we can compute the arbitrage bounds on the prices of derivative securities, as well as determine the optimal hedging strategies. Under some additional restrictions on the model, it is possible to show that these partial differential equations indeed have a solution. An existence Theorem is included in the Jouini and Kallal (1995 b).

An example of a model that would satisfy these assumptions is an economy where the stock price Z_1 follows a geometric Brownian motion as in the Black-Sholes economy, with a constant cost (in terms of expected return) of shortselling the stock, and where the riskless borrowing and lending rates are constant. More precisely, let

$$\frac{dZ_1(t)}{Z_1(t)} = \mu dt + \sigma dW(t) \text{ and } \frac{dZ_2(t)}{Z_2(t)} = (\mu + c)dt + \sigma dW(t)$$

$$B_1(t) = exp(rt) \text{ and } B_2(t) = exp((r+s)t)$$

where the drift μ, the volatility σ, the cost of shortselling c, the riskless rate r and the spread s between the borrowing and the lending rates are constant. In this case, the stock price processes can also be written $Z_1(t) = Z_1(0)exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\}$ and $Z_2(t) = exp\{-c(T-t)\}Z_1(t)$. The portfolio of derivative securities to be hedged can be chosen with a payoff of the form

$$h(x) = Max(0, x - K_1) + Max(0, K_2 - x)$$

which corresponds to a call of strike price K_1 plus a put of strike price K_2. It follows from Theorem 4.2.2 that there are no opportunities of arbitrage if and only if $s \geq 0$ and $c \geq 0$, i.e. if the spread in the riskless rate and the shortselling cost are nonnegative. Moreover, the arbitrage bounds on the claim $h(Z_1(T))$ can then be found by solving the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} + cx[\frac{\partial V}{\partial x}]^- + s[V - x\frac{\partial V}{\partial x}]^- = rV$$

with boundary condition

$$V(x,T) = h(x).$$

Although a closed form solution is not available at this point, this equation can be solved by classical numerical methods, such as the finite difference method. Another possibility is to approximate the model by a binomial model. We have computed some numerical values of the arbitrage bounds of a portfolio formed of a put and a call (both at-the-money, i.e. of strike price equal to the stock price), for reasonable parameter values, and a cost c of the order 1%. We find that the arbitrage bounds are quite sharp, and substantially sharper than the arbitrage bounds obtained by implementing the hedging strategy that consists in adding up the hedge ratios given by the Black and Sholes formula. For instance, for $\sigma = 30\%$, $r = 5\%$, $s = 0$, $T = 6$ months, $Z_1(0) = K_1 = K_2 = 100$, and $c = 1\%$ we find that the arbitrage bounds are approximatively 16.90 and 17.17. Hence, in this case, the spread represents roughly 1.6% of the value of the portfolio. The Black and Sholes hedge ratios would give instead arbitrage bounds equal to 16.78 and 17.29, i.e. an interval twice as wide. These qualitative results are robust to changes in parameter values. More numerical results are given in the next subsection.

4.3 Shortselling costs in the finite periods and states case

In this section we consider a model where the set of states of the world Ω and the set of possible trading dates $\mathcal{T}$ are finite. More specifically, we shall assume that the stock price follows a multiplicative binomial process and that the riskless rate is constant, although it will be clear that this is by no means essential to our analysis.

Suppose that there are two securities in the economy: a stock and a riskless bond with a constant return $R = exp(r)$ over each period. The return on a long position in the stock is assumed to take two possible values $u_1 > d_1$ in states "up" and "down" at each date. In order to model the cost in going short in the stock, we assume that a short position in the stock yields different returns $u_2 > d_2$ in states "up" and "down" at each date. [27] As in the continuous time diffusions model, we shall consider two securities price processes Z_1 and Z_2 with these returns, and assume that security 1 cannot be sold short whereas security 2 cannot be held long. Also, we shall restrict our attention to derivative securities with payoffs that are functions of the stock price Z_1, so that the actual values taken by the process Z_2 are irrelevant, and its return matters only. Therefore, we can fix $Z_2(0)$ at an arbitrary (positive) level.

In this economy, a state of the world is the realization of a sequence of T "ups" and "downs" where T is the (finite) horizon of the economy. At any date t we can observe the sequence of "ups" and "downs" that have occured up to that date and there is a certain probability that the next element in the sequence is an "up" or a "down." Therefore, a probability measure on the set of states of the world Ω can be described by an array of conditional probabilities of going "up," at any date t, given each possible sequence of "ups" and "downs" up to time t. The set of probabilities for which the normalized process $\frac{Z_1(t)}{R^t}$ is a supermartingale (if there are any) are of the form α for state "up" tomorrow (and $1-\alpha$ for state "down" tomorrow), where $\alpha \leq \frac{R-d_1}{u_1-d_1}$, at each date in every event. The probabilities for which the normalized process $\frac{Z_2(t)}{R^t}$ is a submartingale are of the form α' for state "up" tomorrow (and $1-\alpha'$ for state "down" tomorrow), where $\alpha' \geq \frac{R-d_2}{u_2-d_2}$, at each date in every state. Therefore, according to Theorem 3.1, the model is arbitrage free if and only if $\alpha_2 \leq \alpha_1$, where $\alpha_i = \frac{R-d_i}{u_i-d_i}$ is the probability (for state "up" tomorrow) at each date in every state that transforms $\frac{Z_i(t)}{R^t}$ into a martingale.

The arbitrage bounds on the price of a derivative security that pays a certain function $h(Z_1(T))$ of the stock price at maturity, [28] i.e the minimum cost of hedging it and the maximum amount that can be borrowed against it, can be computed using the set Π of probabilities that transform $\frac{Z_1(t)}{R^t}$ into a supermartingale and $\frac{Z_2(t)}{R^t}$ into a submartingale. Indeed, we have already recalled that the interval defined by these bounds on the price of a derivative security is equal to the set of expectations of the normalized payoff of the security with respect to all these probabilities. These probabilities are of the form α for state "up" tomorrow (and $1-\alpha$ for state "down" tomorrow), where $\alpha_2 \leq \alpha \leq \alpha_1$, at each date in every state. This suggests the following algorithm to compute the upper bound on the derivative security price, i.e. the minimum cost of hedging it: start from the vector of payoffs of the derivative security at the final date T divided by R^T and for each event at date $T-1$ compute the

[27] I.e. We shall assume that the return on a long position in the stock is negatively correlated with the return on a short position in the stock. This is not required by the mathematics and the other (awkward) case could be treated equally well.

[28] In fact, it will be clear that any distribution of payoffs in the "tree" defined by the movements of the stock price can be priced using our algorithm.

maximum conditional expected value of these normalized payoffs, where probabilities are taken in Π. For instance, if in a given event at time $T-1$ the normalized payoffs at time T are x_u if "up" occurs and $x_d > x_u$ if "down" occurs, the maximum conditional expected value is $\alpha_2 x_u + (1-\alpha_2)x_d$ since $\alpha_2 \leq \alpha_1$. Once this vector of expected "payoffs" at time $T-1$ in every event is computed, do the same thing at time $T-2$ and so on down to date 0. The number obtained at time 0 is hence the upper bound on the ask price of the derivative security that pays $h(Z_1(T))$ at date T. Note that this algorithm also gives the risk-neutral probabilities for which this upper bound is the expectation of the normalized payoff of the derivative security. To compute the lower bound on its bid price, perform the same algorithm but consider minimum conditional expected values at each stage instead.

We shall now turn to the approximation of the Black and Sholes model with short sales costs by a sequence of binomial models of this sort, in the spirit of Cox et al. (1979). If T is the horizon of the economy (in some unit of time: days, weeks, years...) we shall denote by $\frac{T}{n}$ the amount of time between stock price movements, and hence n represents the number of periods. As $n \to +\infty$ we have that $\frac{T}{n} \to 0$ and we must adjust R, u_i and d_i in a way that approximates the Black and Sholes model with short sales costs in the limit. Since the riskless return is R^T over the horizon of the economy, the riskless return $\tilde{R}$ over one period of length $\frac{T}{n}$ must satisfy $\tilde{R}^n = R^T$ and hence $\tilde{R} = R^{\frac{T}{n}} = exp(r\frac{T}{n})$. We shall also choose $u_1 = exp(\sigma\sqrt{\frac{T}{n}})$, $d_1 = exp(\sigma\sqrt{\frac{T}{n}})$, $u_2 = exp(c\frac{T}{n} + \sigma\sqrt{\frac{T}{n}})$, $d_2 = exp(c\frac{T}{n} - \sigma\sqrt{\frac{T}{n}})$, and $q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}$, where q is the probability of the stock price moving "up", $\sigma > 0$ is the volatility of the stock, μ its drift, and $c \geq 0$ the cost (in terms of expected return) of going short in the stock. This leads to the following expectations and variances: $E(ln(\frac{Z_1(T)}{Z_1(0)})) = \mu T$, $E(ln(\frac{Z_2(T)}{Z_2(0)})) = (\mu + c)T$, $Var(ln(\frac{Z_1(T)}{Z_1(0)})) = Var(ln(\frac{Z_2(T)}{Z_2(0)})) = \sigma^2 T - \frac{\mu^2 T^2}{n}$. Therefore, the limits of the means and the variances of the (compounded) returns in a long and a short position in the stock coincide with the means and variances of the returns in the Black Sholes model. It can also be shown using a Central Limit Theorem as in Cox et al. (1979) that the whole distributions coincide as well in the limit.

We shall compute the arbitrage bounds on the prices of two types of financial instruments: straddles and butterfly spreads (see Cox and Rubinstein [1985] for the use of these instruments). A straddle is a combination of a put option and a call option written on the same stock, with the same expiration date T and the same strike price K. Therefore its payoff at date T is of the form:

$$h_s(x) = Max(0, K - x) + Max(0, x - K).$$

On the other hand, a butterfy spread is a combination of a call option of strike price K_1, a call option of strike price K_3 larger than K_1, and a short position in two call options of strike price K_2 with $K_1 < K_2 < K_3$, all written on the same stock and of same expiration date T. Therefore its payoff at date T is of the form:

$$h_b(x) = Max(0, x - K_1) - 2Max(0, x - K_2) + Max(0, x - K_3).$$

In our numerical computations we have adopted the parameters $\sigma = 30\%$ for the volatility of the stock, $r = 5\%$ for the riskless lending rate, $Z(0) = 100$ for the initial value of the stock, and $T = 6$ months for the time to expiration of the puts and calls.

We computed the numerical values of the arbitrage bounds on an at-the-money straddle (i.e. with strike price $K = 100$) when $c = 0\%$ and s ranges between 0% and 5%, when $s = 0\%$ and c ranges between 0% and 5%, and when $s = c$ range between 0% and 5%. We find that the spread between the arbitrage bounds ranges between 0% and 3.4% in the first case, between 0% and 3.5% in the second case, and between 0% and 6.6% in the third case.

We also computed the numerical values of the arbitrage bounds on a butterfly with strike prices $K_1 = 90$, $K_2 = 100$, and $K_3 = 110$ when $c = 0\%$ and s ranges between 0% and 5%, when $s = 0\%$ and c ranges between 0% and 5%, and when $s = c$ range between 0% and 5%. We find that the spread between the arbitrage bounds ranges between 0% and 11.2% in the first case, between 0% and 10.7% in the second case, and between 0% and 20.6% in the third case.

In each case we also computed the arbitrage bounds that would be obtained by adding up the arbitrage bounds given by using the Black and Scholes formula for the puts and calls separately. It appears that the spread between the actual arbitrage bounds is substantially smaller than the spread between these suboptimal bounds. For the straddle, the spread between our bounds is roughly equal to half the spread between the suboptimal bounds. For the butterfly, the effect is even more dramatic and the spread between our bounds is roughly equal to one twentieth of the spread between the suboptimal bounds. The magnitudes of these results are quite robust to changes in parameter values although (as one would expect) they are sensitive to the strike prices of the puts and calls in the portfolios. We may conclude that the arbitrage bounds in the presence of (reasonable) shortselling and borrowing costs are quite sharp, and substantially sharper than the bounds derived from the Black and Scholes formula.

4.4 Incomplete markets

In both Theorems 3.1 and 3.2 we have not assumed that the markets are complete and all our results encompasses this particular case. Nevertheless it seems to be interesting to derive a result concerning incomplete markets without other imperfections. We then have,

Theorem 0.4.6 (El Karoui-Quenez) *In an incomplete market without other imperfections the arbitrage interval for some contingent claim x is given considering all the normalized expectation values of x relatively to probability measures P^* for which all the price processes of the traded securities are martingales.*

This result is a direct consequence of Theorem 3.1 or Theorem 3.2. Note that, in the general case, we have two sources of indeterminacy for P^* : we have to choose Z^* between the bid and the ask processes and for each Z^* we have many probability measures for which Z^* is a martingale. In the case considered by Theorem 4.4.1, there is only one source of indeterminacy : Z^* is known (equal to Z and to Z') and we only have to find P^* for which this Z^* is a martingale.

0.5 Trading strategies with market frictions

We saw in the previous sections that in the presence of market frictions duplication can be suboptimal and conversely some nonoptimal strategies in the frictionless framework become optimal if we introduce frictions in the model. In this section we characterize efficient consumption bundles in dynamic economies with uncertainty, taking market frictions into account. We define an efficient consumption bundle as one that is an optimal choice of at least a consumer with increasing, state-independent and risk-averse Von Neumann-Morgenstern preferences. We incorporate market frictions into the analysis, including dynamic market incompleteness, bid-ask spreads, short sales constraints, different borrowing and lending rates and taxes.

For the perfect market case Dybvig (1988 a) develops a new model, the payoff distribution pricing model (PDPM), and shows that the size of the inefficiency of a contingent claim can be measured by the difference between the investment it requires and the minimum investment necessary to obtain at least the same utility level for all possible agents (utility price). This utility price is equal to the minimum investment necessary to obtain the same distribution of payoffs (its "distributional price"). In the presence of market frictions, however, simple examples show that some distributions of payoffs are inefficient as a whole : there might not exist any efficient consumption bundle with a given distribution of payoffs. Hence, in general the PDPM would ignore a piece of the potential inefficiency of a consumption bundle. On the positive side, we show that the inefficiency of a consumption bundle can be measured by the difference between the investment it requires and the minimum investment necessary to obtain a claim with the same distribution of payoffs *or* a convex combination of such claims (the "utility price"). We also show that the utility price of a consumption bundle is in fact the largest of its distributional prices in the underlying frictionless economies defined by the underlying linear pricing rules.

We consider a multiperiod economy with uncertainty, where consumers can trade at each intermediate date a finite number of securities that give the right to a contingent claim to consumption at the final date. We shall assume that consumption (of a single good, the numeraire) takes place at the initial and the final date only. For expositional purposes we shall also assume that there are a finite number of trading dates and of possible states of the world [29]. The states of the world are numbered by $i = 1$ to n and a contingent claim that gives the right to c_i units of consumption in state i (for $i = 1, \ldots, n$) at the final date is represented by the $n-$ dimensional vector $c = (c_1, \ldots, c_n)$. We do *not* assume that markets are dynamically complete (i.e. that any contingent claim to consumption can be achieved through dynamic securities trading), neither do we assume that short sales are unrestricted, and we allow for bid-ask spreads, different borrowing and lending rates, and possibly other types of market frictions. [30]

As we have already seen, in the presence of market frictions (including market incompleteness, bid-ask spreads, different borrowing and lending rates, short sale constraints and taxes), there exists a convex set K of state price vectors[31] such that the minimum cost today of achieving any contingent claim to consumption $c = (c_1, \ldots, c_n)$ tomorrow is equal to $\pi(c) = \max\{p \cdot c = \sum_{i=1}^{n} p_i c_i : p \in K\}$. Once we normalize prices and consumption bundles by the price of the security that serves as a numeraire[32]

[29] We shall use tools that make the extension of our results to continuous time economies possible (although technical).

[30] We shall rely on theoretical results of previous sections.

[31] One of which is strictly positive for every state of the world.

[32] This could be a riskless bond.

the state prices add up to one across states of the world (i.e. $\sum_{i=1}^{n} p_i = 1$) and K is the set of so called "risk-neutral probabilities". The normalized price of any contingent claim is then the largest expected value of its payoff with respect to the risk-neutral probabilities. Also, as we shall see, in such an economy the vectors of intertemporal marginal rates of substitution of maximizing agents can be identified with some element of the set of candidate state price vectors K.

In the perfect market case, Dybvig (1988 a and b) develops the payoff distribution pricing model (PDPM). Assuming a finite number of equiprobable states of the world, he finds that a consumption bundle c is efficient (i.e., choosen by some maximizing agent) if and only if it gives the right to at least as much consumption in states of the world where consumption is strictly cheaper to obtain, i.e if $p_i^* > p_j^*$ implies $c_i \leq c_j$, where p^* is the vector of state prices that represents the linear pricing rule. This is equivalent to the fact that the consumption bundle c minimizes the cost of achieving the distribution of its payoffs, i.e.

$$p^* \cdot c = \min\{p^* \cdot c' : c' \text{ is distributed as } c\}.$$

Dybvig (1988a) then defines the distributional price of an arbitrary contingent claim c (efficient or not) as the minimum cost of achieving the distribution of its payoffs. A claim is then efficient if and only if its market price is equal to its distributional price and the inefficiency of a claim c can be measured by the difference $p^* \cdot c - P(c, p^*)$. This is the difference between the cost of achieving a consumption bundle c and the minimum cost of achieving the same distribution of payoffs (that gives the same utility as c to every agent).

In the presence of market frictions, however, simple examples show that the "distributional approach" does not apply in a straightforward manner. As opposed to the frictionless case, in the presence of market frictions some distributions are in fact inefficient and are never chosen by any maximizing agent. Therefore, the distributional price of a claim does not reveal all its potential inefficiency. In other words, there might be contingent claims that give at least as much utility as c to every agent, are not distributed as c, and are strictly cheaper than the cheapest claim distributed as c.

We shall illustrate this fact by analyzing a two-period economy with two equiprobable states of the world (1 and 2), and where the opportunity set is represented by the set of state price vectors $K_{a,b} = \{(p, 1-p) : p \in [a, b]\}$ with $0 < a < b < 1$. This is the case, for instance, of an economy with a zero riskless rate and where agents can buy and sell a stock with payoffs (S_1, S_2), $S_1 > S_2$ at an ask price $S^a = aS_1 + (1-a)S_2$, and a bid price $S^b = bS_1 + (1-b)S_2$. In this case, the minimum cost to obtain a consumption bundle (c_1, c_2), is $ac_1 + (1-a)c_2$ if $c_1 \leq c_2$ and it is $bc_1 + (1-b)c_2$ otherwise. Suppose that $c_1 < c_2$, then the distributional price of (c_1, c_2) (i.e. the minimum cost to get a consumption claim distributed as (c_1, c_2)) is $\min\{ac_1 + (1-a)c_2, bc_2 + (1-b)c_1\}$. It is then easy to check that if $a < \frac{1}{2}$ then $\min\{ac_1 + (1-a)c_2, bc_2 + (1-b)c_1\} > \frac{c_1+c_2}{2}$ and since any maximizing agent (with preferences satisfying (i)-(iii) above) weakly prefers the consumption bundle $(\frac{c_1+c_2}{2}, \frac{c_1+c_2}{2})$ to (c_1, c_2) (and to (c_2, c_1)) this shows that the distribution of payoffs of (c_1, c_2) as a whole is inefficient. Moreover, note that this example is not a degenerate one. Both consumption bundles (c_1, c_2) and (c_2, c_1) are in the opportunity set and neither of them is dominated[33] by a consumption bun-

[33] In the sense of a weakly larger payoff in every state of the world and strictly larger in some state.

dle that costs the same amount to achieve.[34] Note that, as we shall see, this does not mean that the presence of market frictions makes efficiency a tighter criterion as a general rule.

In the next subsection we shall characterize efficient portfolios in the presence of market frictions and give a preference-free evaluation of the inefficiency cost. We shall also relate this measure of inefficiency to the measure given by the distributional approach in the underlying frictionless economies. We shall also see how these results apply to the measurement of portfolio performance.

The reader can find all the proofs of this section and more examples in Jouini and Kallal (1996a).

0.5.1 Efficient trading strategies

Recall that an economy with market frictions in securities trading can be represented by a convex set K of probability measures (state price vectors or linear pricing rules), where at least one element of K assigns a strictly positive weight to every state of the world.[35] In such an economy, achieving at least a consumption bundle c (through securities trading) requires a minimum investment of $\pi(c) = \max\{E(c) : E \in K\}$. We shall say that a linear pricing rule E of K "prices" the consumption bundle c if it satisfies $\pi(c) = E(c)$.

Note that we have implicitly assumed a zero interest rate. This assumption turns out to be innocuous: it only amounts to normalize the state prices by the discount factor (i.e. the sum of the state prices across all the states of the world). In order to analyze an economy with a nonzero interest rate, we only need to multiply all the payoffs by the discount factor in our analysis.

For convenience, we shall assume that there is a finite number n of equiprobable states of the world. The class of weakly concave and strictly increasing Von Neumann-Morgenstern preferences will be denoted by $\mathcal{U}$.

We shall say that a consumption vector is efficient if there exists a risk-averse and strictly increasing Von Neumann - Morgenstern utility function and an initial wealth for which it is an optimal choice. More formally

Definition 0.5.1 *A contingent claim $c^* \in R^n$ is efficient if there exists $u \in \mathcal{U}$ such that c^* solves* $\max\{u(c) : \pi(c) \leq \pi(c^*)\}$.

This is the same definition as in the frictionless case except that the budget constraint is expressed in terms of a nonlinear pricing operator π, where the nonlinearity comes from the presence of market frictions. However, this pricing operator is of a particular form : it is the supremum of a family of linear positive pricing rules. Hence, the budget constraint can be viewed as a collection of linear budget constraints with linear pricing rules ranging in K. Also, since agents are assumed to have strictly increasing preferences, an efficient claim c^* makes the budget constraint binding and the initial wealth for which it is an optimal choice is necessarily $\pi(c^*)$.

[34]In the presence of market frictions this does not violate the absence of arbitrage.

[35]It is easy to see that in such an economy, the existence of such a strictly positive vector of state prices is both necessary and sufficient for the absence of arbitrage opportunities. It also means that K, which we assume to be closed, is the closure of the set of its strictly positive elements.

Similarly, we shall say that c^* is *strictly* efficient if it is an optimal choice for an agent with a *strictly* concave and strictly increasing Von Neumann - Morgenstern utility function. We shall denote by $\mathcal{U}_{sc}$ the class of such preferences.

The following theorem characterizes the efficiency of a given contingent claim in terms of a particular state price vector in K : the linear pricing rules that price it.

Theorem 0.5.1 *A contingent claim $c^* \in R^n$ is (strictly) efficient if and only if there exists a strictly positive probability measure $E^* \in K$ such that*
(i) $E^*(c^*) = \pi(c^*)$,
(ii) c^* is in (strict) reverse order of E^*.

We say that $c^* = (c_1^*, \ldots, c_n^*)$ is in reverse order of $E^* = (e_1^*, \ldots, e_n^*)$ if : $c_i^* > c_j^*$ implies $e_i^* \leq e_j^*$. This means that the payoff is not lower in a "cheaper" state of the world. Similarly, we say that $c^* = (c_1^*, \ldots, c_n^*)$ is in strict reverse order of $E^* = (e_1^*, \ldots, e_n^*)$ if : $c_i^* > c_j^*$ implies $e_i^* < e_j^*$. This means that the payoff is not lower in a "cheaper or as expensive" state of the world. The Theorem then says that a claim is (strictly) efficient if and only if it is in (strict) reverse order of a strictly positive state price vector that "prices" it.

Roughly speaking, this result follows from the first-order conditions: marginal utilities of consumption in each state of the world are proportional to the state price vector representing one of the binding linear budget constraints (that is binding at the cost of the optimal consumption bundle). From the assumption that agents are risk-averse, marginal utilities are decreasing, which implies that payoffs must be higher in cheaper (relative to the binding linear pricing rule) states of the world. The difficulty is that we are dealing with a maximization problem under a continuum of constraints. This Theorem generalizes the price characterizations obtained by Peleg and Yaari (1975) and by Dybvig and Ross (1982) in the incomplete markets case.

The previous result provides us with a diagnostic test: given a contingent claim we are now able to determine whether it is an optimal choice for some maximizing agent. What we need now is an evaluation of the inefficiency cost, i.e. a measure of how far a claim is from being efficient.

A simple measure of the (potential) inefficiency of a consumption bundle c^* is given by $\pi(c^*) - V(c^*)$ where

$$V(c^*) = \sup_{u \in \mathcal{U}} \{\min\{\pi(c) : u(c) \geq u(c^*)\}\}.$$

Indeed, $V(c^*)$ represents the larger amount that is required by rational consumers (with preferences in $\mathcal{U}$) in order to get the same utility level as with c^*. Of course, if c^* is efficient, then $V(c^*) = \pi(c^*)$ and our measure of inefficiency is equal to zero. On the other hand, if c^* is inefficient, the difference $\pi(c^*) - V(c^*)$, which is equal to $\inf_{u \in \mathcal{U}} \{\pi(c^*) - \min\{\pi(c) : u(c) \geq u(c^*)\}\}$, represents the smallest discrepancy, across rational consumers, between the actual cost of c^* and the price at which it would be an optimal choice. Hence our measure of inefficiency $\pi(c^*) - V(c^*)$ does not depend on the choice of a specific utility function.[36]

[36]It depends though on the class of preferences that we use. As we shall see, however, it is quite robust to changes in the choice of this class.

We shall call $V(c^*) = \sup_{u \in \mathcal{U}}\{\min\{\pi(c) : u(c) \geq u(c^*)\}\}$ the "utility price of c^*". It turns out that in dynamically complete perfect markets, the utility price of a consumption bundle coincides with its distributional price (see Dybvig [1988a]), i.e. the minimum cost of achieving the same distribution of payoffs. Our example in the introduction clearly shows that this is not the case in the presence of market frictions. In other words, even though an efficient claim obviously minimizes the cost of achieving the distribution of its payoffs (since agents have state-independent preferences), minimizing the cost of a achieving a given distribution of payoffs does not imply efficiency. In order to be efficient a claim c^* also needs to minimize the cost of achieving the distribution of its payoffs in another economy : the frictionless economy represented by a positive linear pricing rule E^* (in K) that prices c^* in the original economy with market frictions.

In the frictionless case, the pricing rule is linear and hence there always exists a minimum cost consumption bundle in the set $\{c : \forall u \in \mathcal{U}, u(c) \geq u(c^*)\}$ that has the same distribution of payoffs as c^* (i.e. that is a permutation of c^*). However, in the presence of market frictions the pricing rule is not linear. Hence, it might be strictly cheaper to obtain a convex combination of consumption bundles that are distributed as c^* than to obtain any claim distributed as c^*. We shall denote by $\Sigma(c^*)$ the set of convex combinations of consumption bundles that are distributed as c^*. We then have,

Theorem 0.5.2 *for all $c^* \in R^n$, the utility price of c^* is equal to*

$$V(c^*) = \min\{\pi(c) : u(c) \geq u(c^*), \forall u \in \mathcal{U}\} = \min\{\pi(c) : c \in \Sigma(c^*)\}.$$

This says that the utility price of c^* is in fact the cost of the cheapest consumption bundle that is distributed as c^* *or* that is a convex combination of consumption bundles distributed as c^*. Equivalently, according to our Lemma above, the utility price of c^* is then the cost of the cheapest consumption bundle that makes every rational agent (with preferences in $\mathcal{U}$ or in $\mathcal{U}_{sc}$) at least as well off as with c^*. In proving this Theorem we also prove that the utility price $\sup_{u \in \mathcal{U}_{sc}} \min\{\pi(c) : u(c) \geq u(c^*)\}$ defined relatively to the smaller class of preferences $\mathcal{U}_{sc}$ coincides with the utility price $V(c^*)$.

It also turns out that, even though the utility price does not coincide with the distributional price in the presence of market frictions, there is a link between the utility price and the set of distributional prices in the underlying frictionless economies. Indeed, we find that the utility price of a claim is the largest of its distributional prices in the underlying frictionless economies defined by the underlying linear pricing rules that belong to K. The following Theorem states this result

Theorem 0.5.3 *for all $c^* \in R^n$, the utility price of c^* is equal to*

$$V(c^*) = \max\{P(c^*, E) : E \in K\},$$

where $P(c^*, E) = \min\{E(c) : c \text{ is distributed as } c^*\} = \min\{E(c) : c \in \Sigma(c^*)\}$.

Note the analogy with the price at which a consumption bundle is available in this economy (the amount an agent needs to invest to get it) which is the largest of its prices in the underlying frictionless economies with pricing rules belonging to K.

Morever it is shown in Dybvig (1988a) that the distributional price $P(c^*, E)$ can be expressed using the cumulative distribution functions of the payoff c^* (denoted F_{c^*}) and of the state price E (denoted F_E). Recall that $F_{c^*}(x)$ is equal to the probability that the random variable c^* is less than or equal to x (and similarly for F_E). Also, let the inverse of a cumulative distribution function F be defined by $F^{-1}(y) = \min\{x : F(x) \geq y\}$ for all $y \in (0, 1)$ (the values at 0 and 1 will be irrelevant to us). We then have that $P(c^*, E) = \int_0^1 F_{c^*}^{-1}(y)F_E^{-1}(1-y)dy$, which implies that the utility price is equal to

$$V(c^*) = \max\{\int_0^1 F_{c^*}^{-1}(y)F_E^{-1}(1-y)dy : E \in K\}.$$

0.5.2 Portfolio performance

As in Dybvig (1988a), in measuring performance we follow the tradition of comparing some investment strategy (and the distribution of payoffs it leads to) to the alternative of trading in a market. However, we do not assume that this market is frictionless. This means that we allow it to be (dynamically) incomplete, to have restricted short sales, different borrowing and lending rate and positive bid-ask spreads. Ignoring these frictions would make the benchmark market available to investors more attractive than it actually is, and would lead to an underestimation of the performance of the investment strategy being analyzed. Of course, this effect is mitigated by the fact that the investment strategy itself is subject to transaction costs (and other frictions) and therefore leads to lower payoffs than it would in a perfect market.

An investment strategy is evaluated on the basis of the distribution F_c of its payoff c, where the actual payoff c might depend on information that is not available to the agents (but only to the portfolio manager), allowing for information-trading and private investments outside the benchmark market. The benchmark market itself is described by the set K of linear pricing rules that summarize the investment opportunities available to investors. As far as utility pricing is concerned the relevant characteristic of the benchmark market is the set of cumulative distributon functions of the underlying linear pricing rules $\{F_E : E \in K\}$. The following Corollary is similar to Theorem 4 in Dybvig (1988 a) for the frictionless case, and is a consequence of our Theorem 5.1.3.

Corollary 0.5.4 *Suppose that an investment strategy leads from an initial wealth w_0 to a distribution of payoffs F_c. Let $V(c) = \max\{\int_0^1 F_c^{-1}(y)F_E^{-1}(1-y)dy : E \in K\}$. Then,*

(i) If $w_0 < V(c)$, we have superior performance, i.e. there exists a rational agent[37] *who prefers receiving the distribution of payoffs F_c to trading in the benchmark market.*

(ii) If $w_0 = V(c)$, we have ordinary performance, i.e. every rational agent weakly prefers trading in the benchmark market to receiving the distribution of payoffs F_c.

(iii) If $w_0 > V(c)$, we have inferior performance, i.e. every rational agent strictly prefers trading in the benchmark market to receiving the distribution of payoffs F_c.

Hence, by comparing the initial investment to the utility price of the distribution of payoffs obtained by the investment strategy, one is able to evaluate the performance of

[37] I.e. with a utility function in $\mathcal{U}$.

the portfolio. If the utility price is lower than the initial investment, then we conclude that the portfolio is not well-diversified and is underperforming. If the utility price is equal to the initial investment, then the portfolio is well-diversified and it is performing as it should. If the utility price is larger than the initial investment, the manager has superior ability and/or information and the portfolio is overperforming.

As argued by Dybvig (1988 a) this provides an alternative to the Security Market Line (SML) in measuring portfolio performance. As opposed to the SML analysis, this alternative gives a correct evaluation even when superior performance is due to private information. Indeed, the SML is based on mean-variance analysis,[38] and even if securities returns are assumed to be jointly normally distributed, they will typically not be normal once conditioned on information (see Dybvig and Ross [1985 a and b]).

0.5.3 Efficient hedging strategies

We now assume that an agent has some contingent liability x (suppose, for instance, that he has written an option contract) and some initial wealth w_0 to hedge it. A rational agent with a utility function $u \in \mathcal{U}$ will then solve the following maximization problem

$$\max\{u(c-x) : \pi(c) \leq w_0\}.$$

In the case of dynamically complete markets without frictions, this problem can be separated into two different stages: first duplicate the claim x, at a cost $\pi(x)$, then solve for the optimal investment with an initial wealth $w_0 - \pi(x)$. Formally, our agent will solve the maximization problem

$$\max\{u(c) : \pi(c) \leq w_0 - \pi(x)\}.$$

Hence, in this case there is no real difference between the efficiency of a hedging strategy and of an investment strategy since optimal hedging consists in duplication of the liability followed by an optimal investment strategy of the remaining funds.

In the presence of market frictions, however, this is no longer true: the hedging problem cannot be separated from the investment problem. This means that the efficiency of hedging strategies is an issue. We propose the following definition of efficient hedging strategies :

Definition 0.5.2 *A strategy leading to a payoff $c^* \in R^n$ is an efficient hedging strategy of a contingent claim x if there exists $u \in \mathcal{U}$ and $w_0 \in R$ such that c^* solves* $\max\{u(c-x) : \pi(c) \leq w_0\}$.

This means that we say that a hedging strategy is efficient if it leads to a net payoff that is efficient. As for investment strategies we shall say that a hedging strategy is *strictly* efficient if it is optimal for a rational agent with a strictly concave utility function (i.e. with preferences in $\mathcal{U}_{sc}$). We refer to Jouini and Kallal (1996a) for a characterization of efficient hedging strategies and, in the case of general hedging strategies, for a measurement of the inefficiency.

[38] Mean-variance analysis can be justified either by assuming normally distributed returns or by assuming quadratic utility. However, the latter assumption implies undesirable properties such as nonmonotonic preferences and increasing absolute risk aversion.

0.5.4 Numerical results

Let us consider an economy where there is a riskless bond and a stock that follows a stationary multiplicative binomial model, with an actual probability of 0.5 of going "up" or "down" at each node. This stock can be sold short, paying a constant cost in terms of expected return. In this case we have as it is shown in subsection 4.3 $K = [\alpha_1, \alpha_2]^{\text{number of nodes}}$ where each component of an element of a measure in K is the conditional probability of going "up" at the corresponding node. Then let us define the scalar $\beta = \max\{([\alpha_1, \alpha_2] \cup [1-\alpha_2, 1-\alpha_1]) \cap [0, 0.5]\}$ and the associated probability measure E_β on our tree defined by a constant conditional probability β of going "up" at each node. We then have

Theorem 0.5.5 *For all $c^* \in R^n$, its utility price is equal to $V(c^*) = E_\beta(\tilde{c})$ where $\tilde{c}$ is distributed as c^* and is in reverse order of E_β.*

In Jouini and Kallal (1996a) we present a simple algorithm for computing the utility price of a payoff and evaluating the inefficiency cost of a trading strategy. Of course, if holding the stock is an efficient strategy, going short in the stock is *not* an efficient strategy. Also, shortselling costs would have an impact on the inefficiency of a trading strategy only if this strategy and/or the strategies that dominate it involve some shortselling. We could examine, for instance, a stop-loss strategy in a setup where investors expect a negative return for the stock and go short in it, liquidating their position as soon as the price of the stock reaches a certain level (say 110% of the initial price). This strategy is inefficient (for reasonable parameter values of the stock and bond price processes), but the strategies that dominate it involve more shortselling than the strategy itself, and hence we expect its inefficiency to be smaller than in the absence of shortselling costs. For a high enough cost of shortselling the strategy can even be shown to be efficient.

0.6 Imperfections and stationarity

In this section, we consider a model in which agents face investments opportunities (or investments) described by their cash flows as in Gale (1965), Cantor and Lippman (1983,1995), Adler and Gale (1993) and Dermody and Rockafellar (1991,1995). These cash flows can be at each time positive as well as negative. It is easy to show that such a model is a generalization of the classical one with financial assets. We impose that the opportunities can not be sold short and that the opportunities available today are those that are available tomorrow and the days after in an infinite horizon model...(stationarity). In such a model we prove in Carassus and Jouini (1996b) that the set of arbitrage prices is smaller than the set obtained without stationarity. This result does not contradicts previous results because we have now an infinite horizon and each security can start at each date (for example, there is new options at each date). Then when we consider some price for a given security, this price can induce arbitrages between this security starting today, tomorrow,... If we impose that there is no transaction costs on the call option, the previous sections imply that the price of the option can be equal to any price in the non arbitrage interval. Dubourg (1994) and Soner and al. (1995) proved that this interval is too large and equal to $[0, S]$. If we impose stationarity in the model there is only one price compatible with the no arbitrage condition : the Black and Scholes one or more generally the frictionless one.

In order to give an idea of the result we will consider in the next the simplest case defined by a deterministic framework.

The model we consider entails the absence of risk, stationarity, and short sales constraints. In the general theory of arbitrage formalized by Harrison and Kreps (1979), Harrison and Pliska (1981), and Kreps (1981), securities markets are assumed to be frictionless, and the main result is that the absence of arbitrage opportunities (or no arbitrage) is equivalent to the existence of an equivalent martingale measure. The existence of state prices follows. In our framework, we will prove that the state prices must have a particular form : e^{-rt}. In fact, our main result is basically that no arbitrage implies the existence of a yield curve, and that the only yield curve process consistent with no arbitrage, in a deterministic and stationary setup, is flat.

In our model we allow short sales constraints, but only in order to give an intuition of our result, let us consider a simple frictionless setup. The absence of arbitrage opportunities implies the existence at any time t of a positive discount function D_t. $D_t(s)$ is the market value at time t of one dollar paid at time $t+s$ (this is in discrete settings just an implication of the separating hyperplane theorem). No arbitrage means no arbitrage even for contracts that may not be present, including forward contracts and zero coupon bonds. Following Cox, Ingersoll, and Ross (1981), the consequence of the no arbitrage condition in a deterministic setting is that the spot bond price is equal to the forward bond price. So the forward price at time t of a bound delivered at time $t+s$ and paying one dollar at time $t+T$, for $s<T$, that is $D_t(T)/D_t(s)$, is equal to the price at time $t+s$ of one dollar paid at time $t+T$, $D_{t+s}(T-s)$. Roughly speaking, stationarity in the model would imply stationarity for D, i.e. $D_t = D_{t+s}$, for all t and s. Hence we get $D(T)/D(s) = D(T-s)$, and the unique solution to this equation is $D(t) = e^{-rt}$, for some constant r. In fact, the stationarity for D is not straightforward and we prove that there exists many discount functions but a unique of this form.

Moreover it is well known that the classical notion of no arbitrage is not always equivalent to the existence of an equivalent martingale measure. Since Kreps (1981), Back and Pliska (1990), and more recently, Delbaen (1992) and Schachermayer (1994), we know that it is necessary to eliminate possibilities of getting arbitrarily close to something positive at an arbitrarily small cost, and therefore we will use this free lunch concept to derive our result.

We assume that every investment is available at each period of the investment horizon, one can subscribe to the investment at each date (stationarity). We will also assume the number invested in each time period to be nonnegative. This requirement here is that no investment can be sold. Notice that this short sale constraint is not a restriction and our model includes the case without constraints (see Corollary (3.1)). Because the investor wants to become rich in a finite time, we also will constrain the strategies to end in a finite time.

In the discrete case, an investment project m will be characterized by $(m_0, ..., m_T)$ where the real number m_t represents the cash received from the project in the t^{th} period. If m_t is nonpositive, the investor must pay for the project, and if m_t is nonnegative, the investor is paid by the project. In this formalization, not matter if assets have a price or not. If m_0 is negative it could represent the price to pay in order to assure the cash flow $m_1, ..., m_T$. Here, we choose to include the price in the cash flow sequence, this is to say investments have price zero.

In this subsection, an investment will be represented as a Radon measure (for example see Bourbaki (1965) or Rudin (1966)). Roughly, for an investment represented by a Radon measure μ, $\int_{t_1}^{t_2} d\mu$ represents the investment payment between times t_1 and t_2. This choice allows us to describe investments with discrete as well as continuous cash flows. The previous discrete payment, $m = (m_0, ..., m_T)$, will be represented by the discrete measure $\mu = \sum_{t=0}^{T} m_t \delta_t$, where δ_t is the Dirac measure in t. But it also allows us to treat investment having continuous payoff, that is investment represented by a function m. In this case, $m(t)dt$ represents the investment payment in the short period dt, and the Radon measure μ associated to this investment will be given by the following measure defined by a density, $d\mu(t) = m(t)dt$.

We allow our model to contain an infinite number of investments. Notice that a continuous rate is modeled by an infinite number of investments, because one should consider all the possible repayment dates. The set of investment income streams is modeled by a family of Radon measure $(\mu_i)_{i \in I}$ with I finite or infinite. We suppose that all investment i have a finite horizon T_i, that is the support of measure μ_i lies in $[0, T_i]$. Otherwise, assuming the existence of an investment with an infinite horizon, it will always be possible to suspend repayment of the debt to infinity. This is not an arbitrage opportunity, because the investor wants to become rich in a finite time, which implies the time horizon to be finite. In this model, the investor is only allowed to choose a finite number of investments. There is an infinite number of possibilities but only a finite number of choices. Let us consider the example of a single investment m. At each time t, we must choose the number of subscriptions to investment m. Let l_t be the chosen number. At time 0, we buy l_0 investments which assures a payoff of $l_0 m_0$. At time 1, the total payoff is $l_0 m_1 + l_1 m_0$, and at time t, it will be $l_0 m_t + l_1 m_{t-1} + ... + l_{t-1} m_1 + l_t m_0$, which can be described by the convolution product $l * m(t)$. In the general case, after selecting a finite subset J of the set I of investments, the investor chooses the number of subscriptions from each element of J. For the same reasons as before, these numbers will be modeled by a family $(l_j)_{j \in J}$ of Radon measures. Roughly, $\int_{t_1}^{t_2} dl_i$ represents the number of investments i bought between times t_1 and t_2. We also require that the support of all measures l_i is in a fixed compact set. Moreover, the no sell assumption requires all the l_i to be positive. The previous payoff calculus is easily generalized and the choice of a finite subset J of I, and a strategy $(l_j)_{j \in J}$ leads to the payoff $\sum_{j \in J} l_j * \mu_j$.

The following example, from Adler and Gale (1993), intended to show whether it is possible to make an arbitrarily large profit in a finite time. Consider an investment which pays \$1 today. The investor must pay \$2 tomorrow and finally receives \$1.01 the day after. We denote this investment by $m = (1, -2, 1.01)$. As previously, the investor has no money to begin with, so the only way to pay the second day's installment on a unit of investment is by initiating a second investment at level two. It is straightforward to show that in order to get a zero payoff, the investor must subscribe at time t to $l_t = -(l_{t-2} m_2 + l_{t-1} m_1)$ investments. A simple calculus leads to a positive payoff after 32 periods. So, with this investment, it is possible to become arbitrarily rich after 32 periods (assuming one can buy an arbitrarily large number of investment m).

As we saw before a strategy will be defined as follows :

Definition 0.6.1 *A strategy is defined by the choice of :*
- a finite number of investments indexed on a finite subset J included in I,
- an investment horizon n,
- a buying strategy for the set of investments J modeled by a family of positive Radon measures l_j which support is included in $[0, n - T_j]$, for all j in J.

We now want to define the absence of arbitrage opportunities. In fact, we will consider a general notion of no arbitrage, which has been developed by Kreps (1981) : no free lunches. We recall that an arbitrage opportunity is the possibility to get something positive in the future for nothing or less today. The no free lunches concept allows us to eliminate the possibility of getting arbitrarily close to something positive at an arbitrarily small cost. In fact, Back and Pliska (1990) provide an example of a securities market where there is no arbitrage and where the classical theorem of asset pricing does not hold, essentially where there is no linear pricing rule. More recently, Schachermayer (1994) has introduced a more precise version of the no free lunches concept : no free lunches with bounded risk, which makes more sense from an economic point of view. This concept is also equivalent to the existence of a martingale measure for discrete time process. In this work, we will say that the set of investments $(\mu_i)_{i \in I}$ admits no free lunches if it is possible to get arbitrarily close to a nonnegative payoff in a certain way. To define the type of convergence that we use, we have to recall some properties of the Radon measure (see for example Bourbaki (1987)). We denote by E_n the space of continuous functions with support in $[0, n]$, and we attribute to E_n the topology $\mathcal{T}_n$ of the uniform convergence on $[0, n]$ (E_n is a classical Banach space). Recall that the strict inductive limit topology $\mathcal{T}$ is defined such as, for all n, the topology induced by $\mathcal{T}$ on E_n is the same as $\mathcal{T}_n$. More precisely, a sequence (φ_j) in E is said to be converging to φ in the sens of the topology $\mathcal{T}$ if there exists n, such that all the considered functions have their support in $[0, n]$ and such that the considered sequence converges to φ in the sens of the topology $\mathcal{T}_n$, i.e. in the sens of the uniform convergence on $[0, n]$. The completeness of E is shown in Bourbaki (1987), and we recall that with this topology on E, the space E^* of continuous linear forms on E is the Radon space measure. Notice that, using one of the Riesz representation theorem, a positive Radon measure is uniquely associated to a Borel-Radon measure, and we will use the same notation for both of them. We will now consider the weak-* topology on the space E^* of the Radon measures, which is called the vague topology. This means that the sequence (π_n) of Radon measures converges vaguely to π if for all continuous function φ with compact support $\pi_n(\varphi)$ converges to $\pi(\varphi)$. In fact as in Schachermayer, we will consider only the limit of weak-* sequences and not all the weak-* closure as in the classical definition of free lunches. Our definition of a free lunch will be :

Definition 0.6.2 *There is a free lunch if and only if there exists an investment horizon n and a sequence of strategies $(l_j^p)_{j \in J}$ with the same investment horizon n such that the corresponding payoff sequence $(\sum_{j \in J} l_j^p * \mu_j)$ converges vaguely to a nonnegative and nonzero measure π.*

Note that the "limit payoff" π also has its support in $[0, n]$ and with this definition we do not include free lunches which occur in an infinite time. We will see, that we can use a weaker definition of free lunch in the case of investments having discrete or continuous cash flows, and also in the case of an investment set reduced to a single investment.

We want to show that the absence of free lunches is equivalent to the existence of a discount rate, such that the net present value of all projects is nonpositive. To prove this, we will assume that there exists at least one investment which is positive at the beginning, and another, at the end. Note that if we consider a discrete time model or even a continuous time model, this condition seems to be quite natural. If all the investments are negative at the beginning, it is straightforward to see that the payoff associated to a nonnegative strategy is necessarily negative at the beginning and then there is no free lunches. The same can be applied at the end and our condition seems therefore to be redundant. In fact, some particular situations are excluded by such a reasoning : the case of investments with oscillations in the neighborhood of the initial or final date such as we can not define a sign to the investment at these dates. Nevertheless, our condition is justified if we admit that such situations are pathological.

We say that a measure μ_k (resp. μ_l) is positive in zero (resp. T_l) if there exists a positive real ε_k (resp. ε_l) such that for all function φ with support contained in $[0, \varepsilon_k]$ (resp. $[T_l - \varepsilon_l, T_l]$), continuous and nonnegative on its support and positive in zero (resp. in T_l), the integral $\int \varphi d\mu_k$ (resp. $\int \varphi d\mu_l$) is positive.

Assumption 0.6.1 *There exist at least two investments k and l, and a positive real number ε, such that the measure μ_k is positive in zero , and the measure μ_l is positive in T_l.*

We will denote by ε the infimum of ε_k and ε_l. Under this assumption, our main result states as follows.

Theorem 0.6.1 *Under assumption 6.1, the absence of free lunches is equivalent to the existence of a discount rate r such that for all i in I, the net present value $\int e^{-rt} d\mu_i(t)$ is nonpositive.*

Another way to say the same thing is that there is a free lunch if and only if there exists a finite subset J of I, such that $\sup_{j \in J} \int e^{-rt} d\mu_j(t)$ is positive for all rate r. Furthermore, if we add for all investment μ_k the investment $-\mu_k$ in the model we obtain the situation where all investments can be sold, and the proof of the following result becomes straightforward.

Corollary 0.6.2 *If all investments can either be bought or sold, under assumption 6.1, the absence of free lunches is equivalent to the existence of a discount rate r, such that for all i in I, the net present value $\int e^{-rt} d\mu_i(t)$ is equal to zero.*

We recall that the lending rate r_0 (resp. the borrowing rate r_1) is the rate at which the investor is allowed to save (resp. to borrow). A lending rate is modeled by the following family of investments $\mu_t = -\delta_0 + e^{r_0 t}\delta_t$ (you lend one dollar at time zero and you will get back $e^{r_0 t}$ at all the possible repayment dates t), and similarly a borrowing rate can be represented as the family ${\mu'}_t = \delta_0 - e^{r_1 t}\delta_t$.

Corollary 0.6.3 *If there exists a lending rate r_0 and a borrowing rate r_1, under assumption 6.1, the absence of free lunches is equivalent to the existence of a discount rate r included in $[r_0, r_1]$ such that for all i in I, the net present value $\int e^{-rt} d\mu_i(t)$ is nonpositive.*

The proof of this result is given in Carassus and Jouini (1996).

In the case of a single investment or in the discrete case we can prove that it is sufficient to consider instead of the free lunch concept the classical arbitrage opportunity concept. Furthermore, in the discrete case we can prove that assumption 6.1 is meaningless. The main result of Adler and Gale (1993) appears then as a consequence of our results.

Before to end this subsection we give some situations where we can apply the previous results.

First, consider the case of a "plan d'épargne logement". In this case, and if we simplify, the product is divided in two stages. During the first stage, the investor saves at a fixed rate r. In the second stage, he can obtain a loan at a special rate r' near to r. More precisely, the bank receives 1^F today. After one period it returns $(1+r)^F$, and lends 1^F. Finally, at the last period the bank receives $(1+r')^F$. We denote this investment by $m = (1, -2-r, 1+r')$. Our main result is that there is an arbitrage opportunity if, for all positive real number x, $1-(2+r)x+(1+r')x^2$ is positive. A simple computation leads to the following condition $r'-r > \frac{r^2}{4}$. Considering a rate r of $5,25\%$, it is possible for the bank to construct an arbitrage opportunity if $r' > 5.32\%$.

Other examples are provided in Adler and Gale (1993).

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