Euler's Introductio in analysin infinitorum and the program of algebraic analysis: quantities, functions and numerical partitions

Marco Panza

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Among the many differences severing Newton's and Leibniz's mathematical approaches to the calculus, the most general one concerns the very conception of mathematics as a discipline. On the one hand, we have the Newtonian ideal of mathematics conceived of as a unitary and systematic body of knowledge, whose basis is constituted by the solution of a number of classical problems adequately rephrased within a new conceptual framework including the theory of fluxions[1]. On the other hand, we have the Leibnizian view of differential calculus as a new theory, detached from classical mathematics and distinct from it, both for its methods and for the problems it deals with. Leibniz's conception does not seem to receive confirmation in the development of Continental mathematics, at least from the middle of the century onwards. Rather, it is the Newtonian insight that seems gradually to become successful. On the Continent, unity of mathematics becomes, at the middle of the 18th century, a crucial requirement to be meet through the elaboration of a general theory of power series, and a systematic application to these series of the method of indeterminate coefficients. In the second half of the century, it is thus the unification program that Newton had foreseen in the *De methodis*[2] that gets gradually realized thanks to the enquiries and results of Continental mathematicians, though it is generally rephrased in a Leibnizian language. This program was not only grounded on an increasing autonomy of analysis, which transformed itself from a method for the solution of problems into an all-embracing theory. More radically, it took the form of a project of algebraization: to reduce the whole of mathematics to a theory of infinite polynomials. Quite soon, this revealed itself to be a hopeless enterprise. Nevertheless, many results obtained in the course of such an enterprise provided the successive mathematical theories with some of their bases and remains until today unaltered.

The *Introductio in Analysin infinitorum*[3], published by Euler in 1748 and composed of two books, has a central role in this course of events: on the one hand, it is the first systematic account of the new algebraic analysis[4]; on the other hand, it is an extraordinary concentrate of results that

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[1] Cf. [22].

[2] Cf. [20], vol. III, part **1**, pp. 3-372.

[3] Cf. [5]. The two French translations that appeared in the course of the 18th century ([9]) and [12]) witness to the influence of this treatise. An English translation of this treatise, made by J. D. Blanton, appeared some years ago: cf. [11]. I shall use this translation as a base for my quotations drawn from the *Introductio*. Still, I shall amend it in some places to stay closer to Euler's original text. In some cases, namely when Euler's claims do not admit a quite simple and unquestionable interpretation, I shall add the original text.

[4] The term 'algebraic analysis' is not used by Euler in the *Introductio*. It become quite common at the end of the

grounded mathematical researches in the following decades and lay at the core of many different mathematical theories still today. Differently from Lagrange[5], Euler does not aim to reduce the whole of mathematics to algebra of power series. Rather, he seems to consider the calculus (both differential and integral) as a non algebraic part of analysis. Still, he aims to extend analysis as far as possible without relying on the principles, the algorithm and the results of the calculus, so as to assign to the calculus itself nothing but its right scope and to enlarge algebraic analysis as much as possible. In the first book of the *Introductio*, Euler provides all the necessary algebraic tools for the calculus to be appropriately established on analytical bases, that is as a part of the analysis. In the second, he applies some of the results thus obtained to the study of algebraic curves, showing how their theory can be extensively developed without relying on the calculus.

This program is quite clearly exposed in the preface of the first book, where Euler motivates it by claiming that the very "idea of infinite" emerges from the solution of algebraic problems. Here is what he writes[6]:

> Although infinitesimal analysis does not require an exhaustive knowledge of common algebra, even of all the algebraic techniques so far discovered, still there are topics whose consideration prepares a student for an understanding of such a higher science. However in the ordinary treatise on the elements of algebra, these topics are either completely omitted or are treated carelessly. For this reason, I am certain that the material I have gathered in this book is quite sufficient to remedy that defect. I have striven to develop more adequately and clearly than is the usual case those things which are absolutely required for infinitesimal analysis. Moreover, I have also unraveled quite a few knotty problems so that the reader gradually and almost imperceptibly become acquainted with the idea of the infinite.

The six books composing the *Introductio* and of the *Institutiones*[7] should thus be considered as a unitary enterprise, conferring a new autonomous structure and a vast scope to analysis, both algebraic and infinitesimal, so as to let one believe that the aim of an unification of mathematics had largely been achieved. Indeed, even though in the first book of the *Introductio* Euler deals only with a few classical problems of algebraic analysis—namely those that have direct applications for infinitesimal analysis—, his approach is so general that the reduction of the whole of algebraic analysis to the principles that he sets forward could almost be viewed as an exercise left for the reader to go through.

However, in order to get a grip on the real significance of the unification that Euler proposed, we must look beyond the simple unitary structure of the enterprise. In the first book of the *Introductio* Euler never tackles a problem in isolation from the others; on the contrary, different issues intersect and apply to each other in a system whose detailed character, cohesion and extension are amazing. Thus, for example, the character of limited function of cosine allows employing some trigonometric relations in order to properly express the trinomial factors of an integer function, and therefore to

18th century and the beginning of the 19th, after having being used by Lacroix in his *Traité du calcul différentiel et du calcul intégral* [17], for denoting that part of analysis that can be developed independently of the *calculus*—that is just Euler's introduction to infinitesimal analysis—and having occurred in the same title of Lagrange's *Théorie des fonctions analytiques* [18], to denote the mathematical domain to which the calculus should be reduced, according to the program exposed in this treatise. The same term also appeared later in the titles of Garnier's and Cauchy's treatises ([14] and [1]) devoted to the matter that Lacroix had denoted with such a term.

[5] Cf. [18] and the previous footnote (4).

[6] Cf. *Introductio*, Preface: [5], vol. I, pp. VII-VIII; and [11], vol. I, p. V.

[7] Cf. [6] and [8].

study the imaginary roots of algebraic equations up to the point of giving the roots of any real number. Alternatively, just to make another example (on which I shall come back later), the law of the combination of coefficients of certain infinite products is proved to be able to represent, in general terms, the addition of positive integer numbers, thus allowing for the neat application of several results about recurring series to the theory of numbers. One could point to many other examples, but the following quotation from the preface can, maybe more than anything else, suggest the width of the connections that Euler brings forth in the eighteen chapters of the first book of his treatise[8].

> Thus, in the first book, since all of infinitesimal analysis is concerned with variable quantities and functions of such variables, I have given a full treatment to functions. I have also treated the transformation of functions and their resolution and development in infinite series. Many kinds of functions whose characteristic qualities are discovered by higher analysis are classified. First I have distinguished between algebraic and transcendental functions [...]. Both [...][rational and irrational (algebraic) functions] can be developed in infinite series, and this method is usually applied with the greatest usefulness also to transcendental functions. It is clear that the theory of infinite series has greatly extended higher analysis. Several chapters have been included in which I have examined the properties and summation of many infinite series; some of these are arranged in such a way that it can be seen that they could hardly be investigated without the aid of infinitesimal analysis. Series of this type are those whose summations are expressed either through logarithms or circular arcs [...]. After that I shall have progressed from powers of quantities to exponential quantities, which are simply powers whose exponents are variables. From the inverse of these I have arrived at the most natural and fruitful concept of logarithms. [...] In a like manner I have turned my attention to circular arcs. This type of quantity, although quite different from logarithms, nevertheless, there is such a close mutual relationship that when the latter is viewed as a complex quantity, it is converted into the former. [...] I have expressed the sinus and cosinus of a very small and almost evanescent arc starting from the sinus and cosinus of any arc, and this has led me to infinite series [...]. Just as logarithms have their own particular algorithm, which has most useful applications in all of analysis, I have derived algorithms for the circular quantities, so that one could calculate with these quantities as easily as with the circular and the algebraic ones. [...] But this investigation brings the greatest help to the resolution of fractional functions into real factors. Since this is so important for integral calculus, I have given this diligent attention. I have investigated those infinite series which arise from the development of this type of functions, and are known as recurrent series. For these I have given both summations and general terms and also other important properties. Since the resolution into factors, so in turn, I have pondered to what extent the product of several factors , and even infinite products, can be expressed in a series. This business opened the way to knowledge of a myriad of series. Since a series can be expressed as an infinite product, I have found rather convenient numerical expressions with the aid of which the logarithms of sines, cosines, and tangents can easily be computed. Furthermore, from this same source we can derive the solution of many problems which are concerned with the partition of numbers.

Even though Euler's treatise includes much more than this, the foregoing quotation should point

[8]Cf. *Introductio*, Preface: [5], vol. I, pp. VIII-X; and [11], vol. I, pp. VI-VIII.

to the structural character of Euler's mathematics. As I have said, Euler's program of unification adopts algebra as its ideal framework. But it does not aim to construct a symbolic language rich and powerful enough to deductively express the whole body of mathematical knowledge. On the contrary, Euler's analysis is only locally deductive, and the connections it displays seem to be irreducible to the rules of a symbolic game. Algebra is not for him a deductive tool apt to provide new proofs of already known results, and indeed he does not put any effort into proving every result he employs, nor in justifying all of his conclusions in formal terms (just to cite the most astonishing example, we find in the *Introductio* several references to, and applications of, the generalized binomial theorem, without any effort on the part of Euler to offer any proof for it). Algebra, or better polynomial algebra, is rather for him a theoretical context apt to frame the whole edifice of mathematics. His treatise is definitively not an elementary text-book. Rather, it is an advanced text whose subject matter is unity of mathematics. Its real aim is to present a new architecture for mathematics and provide its main grounds.

1 Functions as the objects of analysis

The main novelty of the *Introductio* is apparent from the very title of its first chapter: "On Functions in General". Functions are for Euler the objects of the analysis. A similar point of view had never been expressed so clearly before 1748, and, of course, it had never given rise to a systematic treatment capable of turning such an idea into a mathematical building.

What marks the difference between Euler's approach and that proper to mathematicians belonging to the Leibnizian tradition—of whom Euler is the heir—is not simply the expository choice of preferring a certain arrangement of the subject over another. Rather, what changes is the nature of the objects that mathematics is about. Analysis is no more described as a form of reasoning: it is neither conceived as the first stage of the method of analysis and synthesis, nor it is seen as set of methods for the study of external objects having their own specific nature, like numbers or magnitudes. Moreover, it is not understood as a common language that is appropriate to handle all of these objects. It is rather presented as a theory determining its own objects from the inside. These objects are functions and the explicit aim of the first chapter of the first book of the *Introductio* is to introduce and account for them.

However explicit (and explicitly aiming to define functions as appropriate mathematical objects), this account displays an ambiguity that points to a persisting hiatus between Euler's informal notion of function and the definition he provides for the corresponding objects. Euler seems to conceive a function as an abstract quantity, or a quantity as such: a quantity that is not defined after its specific nature, like numbers or particular magnitudes are, but rather after its operational relations with other similar quantities. Moreover, he seems to believe that, so conceived, a function is the same as the solution of an equation. The way a function is explicitly defined, and some crucial properties are ascribed to it, are however far from avoiding ambiguities and departures from this informal notion.

Here is how J. Dhombres has made this point[9]:

> [...] dans son *Introductio in analysin infinitorum*, Euler plaçait le concept the fonction comme un des piliers de l'analyse et cette discipline, au fond, prenait por but la détermination des functions au moyen d'équations, comme les équations différentielles,

[9] Cf. [3], pp. 179-181.

les equations aux dérivées partielles ou encore les équations que nous qualifions de fonctionnelles aujourd'hui. [...] au milieu du XVIII$^{\text{ème}}$ siècle nous constatons la présence d'un concept moteur—celui de fonctions—et la validité d'une méthode—la méthode fonctionnelle. Simultaneament, nous percevons des réflexes anciens restreignant subrepticement l'idée générale de fonctions en la réduisant à un comportement polynomial [...]. La généralité nécessaire du concept de function s'oppose à la restriction non moins nécessaire des fonctions concernées.

[...] in the *Introductio in analysin infinitorum*, Euler placed the concept of function as one of the pillars of analysis and this discipline, after all, took for its object the determination of functions by means of equations, such as differential equations, partial differential equations, or again, the equation we call 'functional' today [...] in the middle of the 18th century, we notice the presence of a driving concept—that of function—and the validity of a method—the functional method. Simultaneously, we perceive old reflexes, surreptitiously restraining the general idea of functions by reducing them to polynomial behavior [...]. The necessary generality of the concept of functions opposes the restrictions that are also needed for the functions in question.]

This claim points to the heart of the issue, I believe. But, at least as concerns the *Introductio*, it stands in need of clarification. First of all, differential and partial differential equations never occur in this treatise. The only equations to which Euler devotes his attention are algebraic ones. These are, however, conceived of—in a broad sense, at least—as functional equations, *i.e.* as implicit representations of functions. Secondly, it seems to me that behind Euler's proof procedures there is the belief that functions not only can be determined through equations, but are solutions of equations (*i.e.*. they are the explicit counterpart of an implicit representation, possibly unexpressed). Thirdly, the restriction of the range of functions is not a surreptitious limitation of Euler's arguments and his functional method; it is rather an essential presupposition of Euler's theory. Hence, it is not only necessary for these arguments and this method to work; it is also necessary for his theory to be coherently and appropriately established.

The third point deserves a further discussion. Transcendental functions, or at least some of them, seem to be conceived by Euler as parts of algebraic analysis. They are so conceived, insofar as they are viewed as compact expressions of appropriate infinite polynomials, and thus as the expressions of standard analytical operations, namely of those operations that are also expressed by these polynomials. Still, these functions are also the solutions of non-algebraic equations, equations that depend on the introduction of non-algebraic operators on functions themselves, like differentiation. The *Introductio* seems thus to fall prey to a difficulty which is independent of the application (and legitimacy) of the functional method, and has rather to do with the very architectural project that this treatise expounds, that is, with the program of an algebraic analysis, preliminary to the calculus but, at the same time, extended enough for all mathematics that can be done without relying on the algorithm of the calculus and its principles and results fall inside its boundaries.

Indeed, three possibilities seems to be open, and each of them is open to unfortunate consequences. Functions can firstly be identified with nothing but solutions of algebraic equations, but then algebraic analysis should not consider, at least at its very beginning, exponentials, logarithms and sines among its objects, thus falling back to a limitation similar to that derived from the Cartesian exclusion of mechanical curves from the domain of geometry. Secondly, one can surreptitiously introduce new objects into the domain of algebraic analysis, independently of their original tie with the key notion of algebraic equation, thus introducing, at the same time (apart from a certain lack of organization and unity) an element of arbitrariness, and contradicting the idea that mathematics extends itself gradually, according to a genetic order, starting from the elementary operation of addition (originally defined on numbers and magnitudes, then appropriately generalized so to apply to abstract quantities). Thirdly, one can abandon the project of a maximal extension of algebraic analysis and admit the possibility of employing differential equations—and thus the algorithm of the calculus—in order to introduce transcendental functions.

Clearly, Newton advised to take the third path; in the very possibility of introducing a new analytic operation like quadrature, he had spotted the way of breaking with the Cartesian discrimination, thus endowing analysis with the scope that Descartes had denied to his geometry. Euler's route seems rather to be the second one. He generally takes functions to be equations' solutions, but he characterizes the elementary transcendental functions—apparently assumed as already given objects, delivered to the new analysis by the very history of mathematics—wholly independently of reference to any equation: by means of an arbitrarily conditioned and largely intuitive extension of the algebraic operation of power, as far as exponential and logarithm are concerned; and by making recourse to the circle's properties, in the case of trigonometric functions. But in this way, functions come to lose, as a matter of fact, their nature of objects internal to analysis, and turn into representations of entities originally thought of in geometrical terms. The very concept of a function, then, become ambiguous, as long as it answers to different needs and different conceptions, whose unification is only achieved *a posteriori*, by admitting that, given the elementary functions (x^n, a^x, $\log_a x$, $\sin x$, $\arcsin x$), any other function can be got as a finite arrangement of them under the algebraic operations and the operation of composition defined on them. The body of algebraic analysis is presented as unitary, but it displays a persistent ambiguity, as concerns the original nature of its basic objects, that Euler does not seem capable of clearing away.

1.1 Quantities and functions

This being said, in general, let us consider Euler's definition of functions and the immediate consequences he draws from it.

The *Introductio* opens with an explicit definition of constant quantities[10]:

> A constant quantity is a determined quantity which always keeps the same value.

It is not clear whether, for Euler, the value of a certain constant quantity depends on the fact that it is such a determined quantity, or *vice versa*, this quantity is determined because it takes a fixed value. In case of specific quantities, like segments, it seems natural to think that the first possibility holds: a segment is determined before a value is assigned to it and thus its determinateness is perfectly independent of the fact that it has a certain value, and depends, instead, on its particular nature. But, here a quantity is not a specific quantity, but rather a quantity *tout court*. What is not clear is thus how its determinateness is conceived: is it independent of the fact that such a quantity has a certain value, or does it depend on that? This question is of course connected to another one: what a value of a quantity is, provided that such a quantity is not a specific one? The value of a specific quantity can be conceived either as a numerical measure for this quantity, or as a common feature of all quantities of the same sort that are equal to it (that is, in modern terms, as an equivalence class under such a relation of equality). In both cases, to give a clear sense to the notion of value of a certain quantity, it is necessary to rely on a relation of equality defined on quantities of the same

[10]Cf. *Introductio*, § 1: [5], vol. I, p. 3 and [11], vol. I, p. 2.

sort. And, in the first case, it is also necessary to rely on an operation of addition, also defined on these same quantities. The question is thus the following: how the notion of value can be conceived with respect to a quantity that is not a specific one?

A simple possibility for answering these questions is of course that of thinking a quantity that is not a specific one as a quantity whose specific nature has not been established yet, that is, as a quantity that is either a number, or a segment, or any other sort of magnitude, even if it has not been said what it is, yet. If so, the notions of determinateness and value can be, so to say, associated to such a quantity according to a general schema that takes, in any specific case, a specific content. But is this the way as Euler understands a quantity in the *Introductio*? I think not. I rather think that when he speaks of quantities *tout court*, in this treatise, he understands it as an abstract quantity, a quantity whose determination does not depend on the identification of its specific nature. But, if it is so, what is a value of a similar quantity, and how can such a quantity be determined and take a certain value? Euler gives no explicit answer to these questions. So, how can we admit that he is understanding a quantity *tout court* as an abstract quantity? I suggest the following answer: in his first definition he is not defying a constant quantity in terms of the notions of determinateness and value; he is rather defining at the same time, constant quantities, determined quantities, and values: a constant quantity is a determined quantity, though it is not a quantity of a specific nature, and, insofar as it is determined, it has a fixed value. In such a way, Euler links to each other three distinct notions, and, so to say, opens a space of possibilities[11].

Such a space of possibilities can of course be interpreted, and it works, in this case, as a general schema to be applied to any sort of specific quantities. Thus, according to Euler, numbers (or better, as Euler says, any sort of numbers) are constant quantities, since they "keep the same constant value, once they have been obtained[12]." But such a space of possibilities can also be studied as such. The result of this study is common analysis, since, Euler says[13], "in common analysis only determined quantities are considered", and the only pertinent distinction is that between known and unknown (constant) quantities. Common analysis seems thus to be identified by Euler with the theory of algebraic equations, or algebra understood in a restricted way. Higher analysis begins, instead, when the space of possibilities of common analysis is enlarged with the introduction of variable quantities, since, Euler continues[14], in higher analysis, the distinction between known and unknown quantities "is not so much used", and quantities are rather distinguished according whether they are constant or variables. Higher analysis, in Euler's sense, is thus not to be confused with infinitesimal analysis or calculus. It is rather algebraic analysis insofar as it is understood as an extension of the mere theory of algebraic equations, that is, insofar as it takes functions as its own objects.

But what is a function? To answer, Euler has first to say what a variable quantity is. This is the

[11] *Mutatis mutandis*, this understanding is similar to the understanding that D. Reed has suggested for Euclid's plane goemetry (as it is exposed in the first four books of *Elements*) cf. [23], part. I. Though I do not think it applies to this matter, I argue that it does in the case of Euler's analysis. To open a space of possibilities, as Euler does, is of course not the same as providing an implicit axiomatic definition of a structure, in Hilbert's sense. The obvious difference is that, in the first case, the definitions cannot be detached from a system of more or less explicit and clear meanings inherited by the previous history of mathematics. In the case of the first definition of Euler, the term 'quantity', though used as a mere component of the terms 'constant quantity' and 'determined quantity' is, for example, far from being perfectly neutral. Hence, one could say that, whereas in the case of Euler, the history of mathematics is an actual part of mathematics itself, in the case of Hilbert, it is part, at most, of the heuristic of mathematics.

[12] Cf. *Introductio*, § 1: [5], vol. I, p. 3 and [11], vol. I, p. 2. Euler is of course referring to determined fixed numbers, like 3, $\frac{7}{4}$, $\sqrt{5}$ or e.

[13] Cf. *ibid.*

[14] Cf. *ibid.*

job of his second definition[15]:

> A variable quantity is an undetermined or universal quantity which encompasses within itself, in general, any determined value.

This second definition is not simply correlative to the first one: it does not result from such a definition by the mere introduction of appropriate negations. This does not depend only on the introduction of the notion of universality, but also and overall on the introduction of the notion of determined value and of the implicit admission that determined values (or better all determined values) are encompassed within a certain quantity. To explain this quite complex definition, Euler says something more about determined values: "all determined values—he claims[16]—can be expressed by numbers". This is the same as admitting that they are numerical measures or are at least expressed by them. As it seems obvious to suppose that a determined value is the value of a constant quantity, it follows that any constant quantity can be associated to one and only one number as to its value or to an expression of its value. A variable quantity, instead, Euler explains[17], "involves all numbers of any sort", that is, if I understand well, it can be associated to any number of whatever sort as one of its possible values, or as an expression of one of its possible values. But, if the second definition is taken literally, this association is a quite strong one, since a variable quantity "encompasses within itself any determined value." Thus, if values are numbers, a variable quantity "encompasses within itself" all numbers of any sort, or, if values are only expressed by numbers, a variable quantity "encompasses within itself" the values expressed by all numbers of any sort, that is, all values. Still—insofar as numbers are constant quantities (providing that the space of possibilities relative to such quantities is interpreted on numbers), but constant quantities do not identify with numbers (since this same space of possibilities can be interpreted in another way and can also be studied as such)—, this cannot be the same as admitting that variable quantities vary as numbers, that is, that the determination of a variable quantity consists of its identification with a certain number, or that a possible value of a variable quantity is necessarily a number. In other terms, though any number can be a possible value of any variable quantity, it has to be admitted that a possible determination of a variable quantity does not consist in its identification with a number or with a value actually expressed by a certain number. A variable quantity has to be able to be determined through its identification with a constant quantity as such.

This is just what Euler seems to mean by claiming that[18]:

> Just as from the ideas of individuals the idea of species and genus are formed, so a variable quantity is the genus in which are contained all determined quantities.

As constant quantities are determined quantities, this is the same as claiming that a variable quantity is the genus in which are contained all constant quantities. A variable quantity is thus a sort of a formal characterization of quantity as such. Its concept responds to a need for generality, *i.e.* a need of studying the essential properties of any object of a certain genus, the properties that this object has insofar as it belongs to such a genus. But, according to Euler, this study has to have its own objects.

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[15]Cf. *Introductio*, § 2: [5], vol. I, p. 4, and [11], vol. I, p. 2-3: "Quantitas variabilis est quantitas indeterminata seu univsersalis, quæ omnes omnino valores determinatos in se complectitur."

[16]Cf. *ibid.*.

[17]Cf. *ibid*: "quantitas variabilis omnes numeros cujusvis generis involvit".

[18]Cf. *ibid*: "Quemadmodum scilicet ex ideis individuorum formatur idæ specierum & generum; ita quantitas variabili est geneus, sub quo omens quantitates determinatæ continentur."

In order to identify these objects, it is necessary to sever these essential properties from any other property that characterizes any object falling under the same genus. If the genus is that of quantities, one has thus to identify some objects that are not specific (and, *a fortiori*, particular) quantities and pertain thus to a higher ontological order than that to which specific quantities pertain.

The notion of generality that seems to be at stake here has a clear Aristotelian flavor.

In an extremely lucid passage, J. Chevalier has thus characterized the difference between the Platonic and the Aristotelian conceptions of science[19]:

> En écartant le devenir sensible de la science, Platon s'est interdit à jamais d'établir entre le devenir et l'être, objet de la science, une liason nécessaire; Aristote en l'y faisant rentrer dépasse le dualisme du principe et du fait, du *διότι* et de l'*ὅτι*: le particulier, que la démonstration a pour but de ramener au principe générale, contient ce principe en puissance et lui est lié par un rapport d'inhérence logique, de telle sorte qu'ils sont perçus spontanément l'un dans l'autre [...].
>
> [In separating the sensible becoming from science, Plato proscribed the establishing of a necessary link between the becoming and the being as the object of science; Aristotle, bringing this back in, went beyond the dualism of principle and fact, of *διοτι* and of *ὅτι*: the particular, that this proof aims to reduce to a general principle, contains this principle potentially, and is linked to it by a relation of logical inherence, so that each is spontaneously perceived within the other.]

However, the "sensible becoming" of Aristotelian physics does not seem reducible to a sum of accidental motions for all of which what is true of the first and necessary motion remains true. Rather, it is such a first and necessary motion, the motion triggered by the first motor, of which no particular motion can be taken to be a particular instance. The generality (and necessity) of the science of nature—that, according to Aristotle, is tantamount to a science of motion—does not reside in the fact that its claims are true for each single motion, but lies rather in the capacity of describing that motion to which any particular motion pertains even if it cannot be one of its instances (just because of its particular nature).

Euler's analysis seems to result from a structurally similar conception. For him, variable quantities have a universal form displaying necessary properties of any quantity. These properties depend on an absolute indeterminacy that is lacking in any particular quantity that embodies this form. Still, whereas Aristotle's first motion is one, variable quantities are much. They have thus to be, at the same time, absolutely undetermined and distinct to each other. Hence, they cannot be distinguished from each other because of their particular nature, but only because they are considered in different times and designated with different names; their distinction does not depend on their identification with particular quantities, but only on their separation in our internal intuition. This is exactly what Kant emphasized some years later, speaking of mathematics in general. In Euler's analysis, this typical character of mathematics is all the more evident in that its main objects—that is, variable quantities—are not only distinguished from each other because they are separated in our internal intuition, but have also no other property than the essential properties that a quantity has to have.

Not only does this preclude any specification, but it also leads to thinking of a variable quantity as something that is capable, in any circumstance, of any possible determination that is compatible to its being a quantity. For Euler, any variable quantity varies over the totality of its possible

[19] Cf. [2], pp. 102-103.

determinations. This is the essential point that he makes in the third article of the first book of the *Introductio*.

This article opens with the following claim[20]:

> A variable quantity is determined when some definite value is assigned to it

Hence, if it is admitted that a variable quantity can be determined through its identification with a constant quantity as such, then it has also to be admitted that a constant quantity can be a value of a variable one. And, if so, and constant quantities do not identify with numbers, values are not numbers, but can only be expressed by numbers. Still, as any value can be expressed by a number, the fact that any variable quantity can take any value is a consequence of the fact that any number can be assigned to it. This is just what Euler claims[21]:

> Hence a variable quantity can be determined in innumerable ways, since all numbers can certainly be substituted for it. Nor is the meaning of a variable quantity exhausted until all determined values have been substituted for it. Thus a variable quantity encompasses within itself absolutely all numbers , both positive and negative, integers and fractions, irrationals and transcendentals. Even zero[22] and imaginary numbers are not excluded from the meaning of a variable quantity.

It is apparent that such a widening in scope of the notion of a variable quantity does not go together with the edification of complex analysis. It is clear that, for Euler, imaginary numbers were not the elements of our complex field, and might even be doubted that Euler really saw the possibility of geometrically representing these numbers and the implications of an extension of functions to their domain. His aim is only to ensure the closure of the domain of quantities over the solution of any algebraic equation, and thus the possibility of thinking of a quantity as a solution of such an equation. However, if the equation is taken as functional, its coefficients can be conceived of as whatever functions of a variable whose unknown is, in its turn, a function; and, in order to remain within the domain of algebraic equations, one must take those coefficients to be real quantities, thus implicitly limiting the range of the independent variable. Here we have an example of a tension between the generality of Euler's notions and the constraints they have to satisfy in order to become fully operational. This tension is just an aspect of an additional difficulty intrinsic to Euler's project. On the one hand, the, so to say, philosophical need for generality is incompatible with any restriction of the possibility of determination of the variables, which would moreover appear as an external act, alien to analytic ideals; on the other hand, the attempt to respect the framework of classical mathematics, by unifying it on new grounds, requires that some restrictions be in fact admitted.

[20] Cf. *Introductio*, § 3: [5], vol. I, p. 4, and [11], vol. I, p. 3.

[21] Cf. *ibid.:* "Quantitas ergo variabili innumerabilibus modis determinari potest, cum omnes omnino numeros ejus loco substituere liceat. Neque significatus quantitatis variabilis exhauritur, nisi omnes valore determinati ejus loco fuerint substituti. Quantitas ergo variabilis in se complectitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales & transcendentes. Quinetiam cyphra & numeri imaginarii a significatu quantitatis variabilis non excluduntur."

[22] Euler's term is here 'cyphra' [cf the previous footnote (21)]. The etymology of this late-Latin term is quite complex, and the Arab derivation itself is uncertain [cf., for example, the *Glossarium mediæ et infimæ latinitatis*, F. Didot, Paris 1840-46]. It is nonetheless clear that Euler, by a peculiar metonym, is here referring to zero: cf., in this regard, also [6], ch. III.

It is within this framework, marked by a need for generality that goes hand in hand with intrinsic ambiguities, that Euler's notion of a function take place. This notion cannot, clearly, be reduced to the sharp formulation with which Euler introduces it[23]:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of this variable quantity and numbers or constant quantities[24].

Though quite general (and, indeed, for this very reason), a similar definition appears, within the economy of the *Introductio*, all but exhaustive. More than on what it literally says, Euler seems, indeed, constantly to rely on its intended interpretation to justify his claims, among which the first undoubtedly appears as the most astonishing to a modern reader,[25]:

> Hence a function itself of a variable quantity will be a variable quantity.

If we understand a variable quantity according to Euler's previous definitions and claims, that is, as an absolutely undetermined quantity whose variation can in no way be limited, this means that any function $f(z)$, when interpreted on numbers—that is, when its argument is taken to vary on numbers—, is, in modern terms, a surjective function $\mathbb{C} \to \mathbb{C}$, *i. e.* for any function $y = f(z)$, so interpreted, and any complex value y^*, there is a complex value z^* such that $y^* = f(x^*)$.

Here is how Euler justifies his claim[26]:

> Since it is permitted to substitute all determined values for the variable quantity, the function takes innumerable determined values; nor is any determined value excluded from those which the function can take, since the variable quantity involves also imaginary values. Thus , although the function $\sqrt{9-zz}$, with real numbers substituted for z, never attains a value greater than 3, nevertheless by giving z imaginary values like $5\sqrt{-1}$, there is no determined value which cannot be obtained from the expression $\sqrt{9-zz}$.

Now, in order to find the value that must be assigned to z for the function $f(z) = \sqrt{9-z^2}$ to take the complex value α, it is necessary to solve the equation $z^2 - \beta = 0$, where β is a complex coefficient equal to $9 - \alpha^2$. But, this kind of equation is never accounted in the *Introductio*, and the very value chosen as an example, $z = 5\sqrt{-1}$, clearly corresponds to a real value both of β and of $\sqrt{9-zz}$. Moreover, it appears evident that Euler implicitly refers to real values of $\sqrt{9-zz}$ when he claims that this function "never attains a value greater than 3." His example, therefore, gives us just the case of an algebraic function, whose surjectivity is trivial, with respect to the tools that Euler accounts in the *Introductio*, only when it is understood as a function $\mathbb{C} \to \mathbb{R}$. Even though the introduction of the values $+\infty$ and $-\infty$ in $\mathbb{C}$ can avoid easy counterexamples, such as those given by the functions $y = 1/z$ and $y = e^z$ for $y = 0$, Euler's claim appears thus to be absolutely unjustified.

Furthermore, Euler's mathematical practice, and, generally, 17th-century mathematicians' practice, suggests a different interpretation of the term 'variable quantity', an interpretation that, even without rejecting the foregoing remarks or the possibility of understanding every function as a variable quantity, does not necessarily lead to such a claim. The term seems to be used, in the 17th and

[23] Cf. *Introductio*, § 4: [5], vol. I, p. 4, and [11], vol. I, p. 3.

[24] Notice that, according to this definition, an analytical expression is composed of quantities, rather than of symbols for quantities. I do not think this is simply a slip of the pen. Still here, I have not the space to account for it in a comprehensive way. For a discussion of this matter, in relation with Lagrange, cf. [13], sect 1.

[25] Cf. *Introductio*, § 5: [5], vol. I, p. 4, and [11], vol. I, p. 3.

[26] Cf. *Introductio*, § 5: [5], vol. I, pp. 4-5, and [11], vol. I, pp. 3-4.

18th centuries, with two different meanings, in order to refer either to independent variables—which constitute the primary objects of analysis and are, in this sense, nothing but transpositions of the absolutely undifferentiated notion of quantity—or to dependent variables—which, although they still are abstract entities, cannot be conceived of, in general, as absolutely undetermined. In this last case, the law of dependence, expressed by a function in Euler's sense—that is, by "an analytic expression composed in any way whatsoever of this variable quantity and numbers or constant quantities"—can, indeed, involve a partial determination of the dependent variable even when the independent variable is absolutely undetermined. Envisaging this possibility does not stand in contradiction with the original need for generality that simply leads to denying the possibility of any limitation that is not implicitly expressed through the identification of a variable quantity with a certain function, in Euler's sense.

It follows that Euler's claim—undeniably strengthened by its justification—results from an *a priori* denial of such a possibility, or, in other words, from the supposition that the well-known limitations relative to the real field are lacking when "imaginary" values are taken into account.

But, apart from the intuitive reasons that might have made Euler lean towards this claim, it is important to stress that such a claim plays in the *Introductio* nothing but a marginal role. In the great majority of cases, when Euler assigns numerical values to his functions, he only takes real numbers into account, and this imposes handling functions that are not defined for certain intervals within the range of its arguments. These functions seem to self-determine the domain to which their values belong, and are thus, so to say, trivially surjective (and continuous) with respect to it. Euler's characterization of variable quantities as absolutely undetermined quantities has thus no counterpart in an actual extension to the domain of "imaginary" numbers of the range of the independent variable and the value of every function that is interpreted on numbers. On the contrary, such an extension seems to be only locally active and is always governed, more than by a mature theory of the complex field, by an extraordinary insight that allows drawing, each time, substantially correct (though often formally inaccurate) conclusions.

The restriction of the range of variable quantities—both denied and required by the very same program of the *Introductio*—does not go as far as allowing, either for Euler or for other 18th-century mathematicians, an external limitation (different from the implicit one to real values) of the range of the independent variable of a function, as an integral part the characterization of this same function. Therefore, insofar as a function is an analytic expression, it can never take a form such as the following:

$$y = \left\{ \begin{array}{ll} f(z) & \text{if } z \leq a \\ g(z) & \text{if } z > a \end{array} \right\}$$

where '$f(z)$' and '$g(z)$' denote two distinct expressions where z occurs. The explicit specifications $z \leq a$ and $z > a$ seem indeed to correspond to illegitimate limitations of generality, inconsistently with the analytic ideal of Euler's program. For the very same reason, no analytic expression expressing a relation whose output is, for operational reasons, independent of its input, can be considered as a function[27]:

> There do occur, however, some functions which retain the same value no matter in what way the variable quantity is changed, for example z^0; 1^z; $\frac{aa-az}{a-z}$, which assume the appearance of function, but really are constant quantities.

[27]Cf. *Introductio*, § 5: [5], vol. I, p. 5, and [11], vol. I, p. 4.

Though any function is thus, for Euler, an analytic expression where a variable quantity occurs, not every analytic expression where a variable quantity occurs is for him a function. For it to be so, it has to be the expression of a variable quantity. A function is thus an expression only insofar as it expresses a variable quantity. *Vice versa*, it is a variable quantity, only insofar as this is expressed by an analytic expression where another variable quantity occurs.

1.2 The classification of functions

Insofar as they are analytic expressions, functions can be classified with respect to their form, or, as Euler says, with respect the "the modality of composition according to which they are formed by variable and constant quantities", that "depends[, in its turn,] on the operations by which the quantities can be composed and mixed together[28]." Among such operations, Euler mentions addition and subtraction, multiplication and division, raising to a power, extraction of a root, and solution of (algebraic) equations—that are algebraic operations—, plus exponential, logarithm, and "innumerable others which integral calculus supplies in abundance", that are transcendental operations, instead[29]. Whereas a function which is "composed only by algebraic operations" is algebraic, a function that involves transcendental operations is transcendental[30].

Euler's list of operations is both surprising and symptomatic of the internal difficulties of Euler's program. Firstly, it is so because it openly includes the solution of (algebraic) equations and covertly evokes the solution of differential equations, without making any distinction between them and the other operations it includes. Secondly, it is so because it lacks trigonometric operations, whereas it includes exponential and logarithm.

Concerning the first point, Euler's list suggests that the definition of functions as analytical expressions should be taken *cum grano salis*. If his list is compared with this definition, two possibilities appear. Firstly, one can admit that both an algebraic equation $F(x, y) = 0$ and a differential equation $F(x, y, dy, \ldots d^n y) = 0$ are "analytical expression[s] composed in any way whatsoever of [...][the] variable quantity [x] and numbers or constant quantities". Secondly, one can admit that a function is either an analytic expression that has been actually set down (insofar as it expresses a variable quantity), or a similar analytic expression that is actually unknown but that it is supposed to comply with certain appropriate conditions (insofar as these conditions are such that this expression cannot but express a variable quantity). The first possibility seems to be highly implausible. And, as a matter of fact, Euler explicitly opts for the second, at least in the case of algebraic functions. Here is what he writes, indeed[31]:

> Indeed, frequently algebraic functions cannot be exhibited explicitly. For example, consider the function Z of z defined by the equation $Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1$. Even if this equation cannot be solved, still it remains true that Z is equal to some expression composed of the variable z and constants, and for this reason Z shall be a function of z.

In the *Introductio*, Euler does not justify this conclusion: he clearly takes here for granted what he had only conjectured some years later[32], namely that the roots of every algebraic equation can be algebraically expressed. It follows not only that an algebraic function can be identified for Euler with

[28]Cf. *Introductio*, § 6: [5], vol. I, p. 5, and [11], vol. I, p. 4.
[29]Cf. *ibid.*
[30]Cf. *ibid.*
[31]Cf. *Introductio*, § 7: [5], vol. I, pp. 5-6, and [11], vol. I, p. 5.
[32]Cf. [4]. On this matter, cf. also [7].

an analytic expression that is actually unknown, but also that addition and subtraction, multiplication and division, raising to a power and extraction of a root exhausts for him the domain of algebraic operations, even if this domain includes solutions of algebraic equations.

The second part of Euler's list of analytic operations suggests that also a transcendental function can for him be identified with an analytic expression that is actually unknown. Still, this second part differs from the first one on a crucial aspect, since the domain of transcendental operations is not exhausted by exponential and logarithm. Moreover, these last operations do not differ from the other transcendental operations from their elementary character. Any transcendental operation, is, for Euler, a (unary) operation that, if applied to a variable quantity x, gives a variable quantity y which is the solution of an appropriate differential equation. And, insofar as a differential equation includes nothing but algebraic operations applied to two variable quantities and to some differentials of one of them, there is no need to admit transcendental operations in order to introduce differential equations. Hence, any transcendental operation can be defined—and, thus, introduced in analysis—without relying on any other transcendental operation, as a (unary) operation that, if applied to a variable quantity x, gives the solution of an appropriate differential equation. The lack of trigonometric operations in Euler's list has thus a quite simple explication: Euler limits himself to mention two examples of transcendental operations that do not differ from other ones for some relevant structural reasons, and simply avoids continuing the list of examples by mentioning also trigonometric operations. But, if it is so, why are exponential, logarithm and trigonometric operations, for Euler, part of algebraic analysis, whereas the other transcendental operations are not? Euler provides no convincing answer to such a question. He merely treats a^x, $\log_a x$, $\sin x$, and $\cos x$ as elementary functions[33], and seems to admit that any function could be composed algebraically by these functions, together with x^n.

Algebraic functions, being distinct from transcendental functions, can on their turn be subdivided into rational and irrational, the former being subdivisible, then, into integer and fractional. These distinctions parallel those we apply today (integer functions being, of course, those that are today called 'polynomial') and there is no need to insist on them.

It is rather important to emphasize that, for Euler, functions can be either *uniform* or *multiform*[34]:

> A uniform function is one which takes a single determined value, no matter what value is assigned to the variable z. On the other hand, a multiform function is one which, for some determined value substituted for the variable z, exhibits several determined values.

Although Euler proposes radical functions and the arcsine as examples of multiform functions, his privileged reference is clearly given by algebraic functions implicitly represented by means of (algebraic) equations of degree greater than one. This example is so privileged that, when introducing it, he offers a new definition, that implicitly limits the distinction between uniform and multiform values to the domain of algebraic functions[35]:

> Thus Z is a multiform function of z which, for each value of z, exhibits so many values as many unities the number n contains, if Z is defined by this equation
>
> $$Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \&c. = 0. \tag{1}$$

[33] As a matter of fact, Euler does not define $\cos x$ in terms of $\sin x$, or *vice versa* (though he remarks of course that $\sin^2 x + \cos^2 x = 1$) and does not include any inverse trigonometric functions among the elementary ones.

[34] Cf. *Introductio*, § 10: [5], vol. I, p. 7, and [11], vol. I, p. 7.

[35] Cf. *Introductio*, § 14: [5], vol. I, p. 9, and [11], vol. I, p. 9. Here the coefficients P, Q, R, &c. are, of course, uniform functions of z, as Euler specifies some lines later.

It is clear that, in the first definition, what is called 'function' is the analytic expression, which, for undetermined z, maintains its unique form, even though this form expresses (or exhibits, as Euler says) different values for every appropriate substitution of z, as is the case with the function $f(z) = \sqrt{z}$. In the second definition, on the contrary, what is called 'function' is the quantity Z that, for any determination of z, takes (or can take, at least) more than one form: the unaltered form is rather that of the equation, of which the function is a solution.

Apart from this, what is maybe more interesting, in order to grasp the intended interpretation that Euler assigns to his definition, is the following remark[36]:

> If Z is a multiform function of z that always exhibits a single real value, then Z imitates a uniform function of z, and frequently can take the place of a uniform function. Functions of this kind are $\sqrt[3]{P}$, $\sqrt[5]{P}$, $\sqrt[7]{P}$, &c. which indeed give only one real value, the others all being imaginary, provided that P is a uniform value of Z. For this reason, an expression like $P^{\frac{m}{n}}$, whenever n is a odd number, can be counted as a uniform function, whether m is a number even or odd.

Here P is clearly not only a uniform function, but also a function having real values. But what is even more evident, and indeed wholly explicit, in Euler's claim, is the restriction of the distinction between uniformity and multiformity to the domain of real values. Therefore, the equation (1) defines a multiform equation only when the coefficients are such that they assign to Z a plurality of real values[37]. The distinction under consideration shows thus its intuitive nature, appearing as an analytic transposition of the distinction between simple and ramified curves[38], a transposition that squares badly with the need for generality that had been advocated a few pages earlier, and gives thus rise to another example of the structural ambiguity in the *Introductio*.

The same ambiguity is even more manifest when Euler claims—by relying on the identification of a function with a solution of an (algebraic) equation, rather than on the surjectivity of all functions, and without any specification as regards the domain of invertibility—that, if y is a function of z, then z is a function of y[39]:

> If y is any function of z, then likewise, z will be a function of y. Since, if y is a function of z, whether uniform or multiform, there is given an equation by which y is defined through z and constant quantities. From the same equation, z can be defined through y and constants.

Clearly, Euler's argument is cogent only with respect to algebraic equations[40], that only define, according to him, algebraic functions. Even though the conclusion so obtained can thus be extended to functions of any sort (provided that adequate specifications as regards the domain of invertibility are made), the argument neatly shows how Euler is not strictly reasoning relying on his own definitions, but rather on his intended interpretation of them.

[36] Cf. *Introductio*, § 15: [5], vol. I, pp. 9-10, and [11], vol. I, p. 10.

[37] As a matter of fact, Euler seems to refer to polyndrome functions in the domain of imaginary values that possess several real values for some real value of the independent variable. Thus it appears that, for example, a function $Z = Z(z)$, defined by the equation $Z^3 + zZ + z = 0$, must be globally considered as a triform function, even though it has two complex roots and a single real one for z real and less than $-4/27$ or greater than zero.

[38] Cf. cf. the previous footnote (37).

[39] Cf. *Introductio*, § 16: [5], vol. I, p. 10, and [11], vol. I, p. 10.

[40] Euler gives the example of the equation $y^3 = ayz - bz^2$.

Once this additional consequence is drawn, Euler closes the first chapter of his treatise by introducing the notions of odd and even functions and specifying the main properties they reciprocally have. The implicit reference seems again to be to functions that are restricted to the real values and conceived of as analytic expressions of determinate classes of curves. And, indeed, Euler relies on such a distinction only in the second book of the *Introductio*, where he deals with algebraic curves.

1.3 Functions of several variables

The foregoing definitions and claims can all be generalized to the case of functions of several variables[41]. If the variables x, y, z, &c. are such that "the determination of one does not in any way limit the meaning of the others"[42], then "an expression composed in any way from these quantities" is a function of them[43]. It follows that if one of the variables of such a function is determined, this function is still a "variable quantity", *i.e.* a function of the remaining variables. The possible determinations of a function of more than one variable constitute thus a hierarchy of infinities[44]:

> Since any variable quantity can be determined in an infinity of ways, a function of two variables, that for each determination of one of them is susceptible of an infinity of determinations, admits an infinity of infinities of determinations. Further, in a function of three variables, the number of determinations will be greater by an infinity; and it grows in this manner for a greater number of variables.

The functions of several variables can be classified, like those of one variable, into algebraic and transcendental functions, even though such a classification must be relativized. A function $f(x, y, z, \&c)$ can thus be algebraic with respect to x if this variable stands only for the argument of algebraic operations, and be transcendental with respect to y if some of the operations applied to y are transcendental. Also in the present case, however, uniformity is an absolute feature of a function: a function of several variables is said to be uniform if, whatever the way in which all of its variables have been determined[45], it receives only one value, and it is said to be multiform in the opposite case. As with the case of functions of one variable, Euler transposes this definition to the case of implicit functions, claiming that V is a function of n-forms of x,y,z, &c. if:

$$V^n - PV^{n-1} + QV^{n-2} - ... \pm S = 0,$$

where $P, Q, ..., S$ are uniform functions of the same variables $x, y, z, \&c$.

As a two-variable equation $F(z, y) = 0$ supplies a function of a variable $y = y(z)$, an equation of n variables $F(x_1, x_2, .., x_n) = 0$ supplies a function of $n-1$ variables. *Vice versa*, given a(n explicit) function of n variables it is always possible to get a(n implicit) function of $n-1$ variables from it, simply by letting it be equal to zero.

[41] Cf. *Introductio*, ch. V: [5], vol. I, pp. 60-69, and [11], vol. I, pp. 64-74.

[42] Cf. *Introductio*, § 77: [5], vol. I, p. 60, and [11], vol. I, p. 64.

[43] Cf. *Introductio*, § 78: [5], vol. I, p. 60, and [11], vol. I, p. 64.

[44] Cf. *Introductio*, § 78: [5], vol. I, p. 61, and [11], vol. I, p. 65. The insight of a hierarchy of infinities, fairly common in the 17th and 18th centuries, should not be seen as an anticipation of the Cantorian distinction between infinities of successive orders and, in particular, between the countable infinite and the uncountable infinities. This essential distinction between countable and uncountable infinities is, indeed, wholly alien to 17th and 18th century mathematicians.

[45] Euler does not mention a relativization of the uniformity to appropriate intervals in the domain of the function; uniformity remains, in his view, a global property of functions: cf. the previous footnote (37).

Finally, peculiar to the (algebraic) functions of several variables is the distinction between homogeneous and heterogeneous functions[46]:

> A homogenous function is one for which the same number of dimensions of variables occurs everywhere.

An integer function will thus be homogeneous —and of dimension n —if the sum of the exponents of the different component variables of each of its terms is everywhere the same —and is equal to n—, whereas a fractional function will be homogeneous when its numerator and its denominator are, on their turn, (integer) homogeneous functions, and the dimension of such a function is given by the difference of the dimension of the numerator and the dimension of the denominator. Finally, if P is an homogeneous function (either integer or fractional) of dimension n, then $\sqrt[m]{P}$ is an irrational homogeneous function of dimension n/m.

2 ***Partitiones numerorum***

The definition of functions, the attribution to them of their more general properties, and their classification provide the basis on which Euler erects the whole edifice of analysis. As said, the first book of the *Introduction* is devoted to the first part of this enterprise, the edification of algebraic analysis, namely of its pure part, the second book being rather devoted to its application to the geometry of plane curves. It is of course impossible to account here for such an enterprise, by entering into its more relevant details. Rather than limiting myself to a quite broad outline, I prefer to offer an example of it, so as to show, so to say, *in concreto*, its structural character.

The first book of the *Introductio*—and, consequently, the pure part of Euler's algebraic analysis—can be divided in six parts. The first of them includes chapters I and V. It is just what I have accounted for in the previous section. As said, it provides the basis of the edifice. The second part provides its breast walls; it includes chapters II-IV and VI-VIII and concerns the mutual transformations of algebraic functions and the development of them and of the elementary transcendent functions (a^x, $\log_a x$, $\sin x$, and $\cos x$) in power series. The other parts complete the edifice in different ways: chapters IX-XII are devoted to the study of "trinomial factors" of a polynomial (they are, in modern language, the second-degree real factors of it); chapters XIII, XIV and XVII concern recurrent series; chapters XV and XVI deal with the resolution of infinite products of linear factors into power series and the applications of the relative rules to combinatorics; finally, chapter XVIII is about continued fractions. I consider here the third of these last four parts, by insisting in a special way on Euler's (elements of) combinatorics.

2.1 Combinations with and without repetitions

The step from a product of linear factors to a polynomial or a power series obeys, in general, very simple rules that it is easy to establish through the method of indeterminate coefficients. Supposing that[47]

[46] Cf. *Introductio*, § 83: [5], vol. I, p. 63, and [11], vol. I, p. 67.

[47] Cf. *Introductio*, § 165: [5], vol. I, p. 128, and [11], vol. I, p. 137. Of course Euler does not employ indices, relying rather on the lexicographic order of letters that denote coefficients. I introduce them for short, together with the symbols '$\sum\limits^{-}_{-=-}$' and '$\prod\limits^{-}_{-=-}$'. According to an habitual convention, Euler also employs the symbol '&*c*.' to indicate a

$$1 + \sum_{\nu=1}^{\&c.} A_\nu z^\nu = \prod_{\mu=1}^{\&c.} (1 + a_\mu z)\,,$$

where A_ν ($\nu = 1$, 2, &c.) are indeterminate coefficients, one gets, according to this method:

$$\begin{aligned} A_1 &= \sum_{i=1}^{\&c.} a_i, \\ A_2 &= \sum_{i=1}^{\&c.} a_i \sum_{j=i+1}^{\&c.} a_j, \\ A_3 &= \sum_{i=1}^{\&c.} a_i \sum_{j=i+1}^{\&c.} a_j \sum_{h=j+1}^{\&c.} a_h \\ \&c. &\quad \&c. \end{aligned} \tag{2}$$

If we let $z = 1$ we easily draw that the product $\prod_{\mu=1}^{\&c.} (1 + a_\mu)$ is equal to a sum $1+\mathcal{P}_1+\mathcal{P}_2+$&c., whose generic term $\mathcal{P}_m$ is constituted by the sum of all products of m factors that can be formed by choosing these factors among the terms a_μ ($\mu = 1$, 2, &c.) and taking each of them only once. If addition and multiplication are replaced by (or conceived as) simple enumerations, $\mathcal{P}_m$ is reduced, then, to a combination without repetitions: namely, it is reduced to the combination without repetitions of m objects chosen among the objects a_μ ($\mu = 1$, 2, &c.). Employing the modern combinatorial notation, and supposing that these objects are k in number, we would have, then: $\mathcal{P}_m = C\,(k, m)$. Here is Euler's way of putting it[48]:

> Now, if we let $z = 1$, then the product
>
> $$(1 + a_1)\,(1 + a_2)\,(1 + a_3)\,(1 + a_4)\,\&c.$$
>
> is equal to the unity together with the series of all numbers that results from a_1, a_2, a_3, a_4, a_5, &c. either taken singly, or multiplied among them two or more, different to each other, at a time.

On the other hand, if one lets[49]:

$$\frac{1}{\prod_{\mu=1}^{\&c.} (1 - b_\mu z)} = 1 + \sum_{\nu=1}^{\infty.} B_\nu z^\nu$$

reiteration of a sequence of objects or operations that can either continue up to any finite bound, or be actually infinite. I shall employ this symbol according to the same convention and the unusual symbols '$\sum\limits_{-=-}^{\&c}$', '$\prod\limits_{-=-}^{\&c.}$' and '$\{--\}_{-=-}^{\&c.}$' to indicate, respectively, sums, products, and sequences that can, also, either continue up to any finite bound, or be actually infinite. I shall employ, instead, the symbols '...', to indicate a reiteration of an infinite sequence of objects or operations and the symbols '$\sum\limits_{-=-}^{\infty}$', '$\prod\limits_{-=-}^{\infty}$' to indicate respectively a series and an infinite product.

[48] Cf. *Introductio*, § 265: [5], vol. I, p. 221, and [11], vol. I, p. 228. Euler writes 'α', 'β', 'γ', 'δ', 'ε' where I write 'a_i' ($i = 1, \ldots, 5$).

[49] Cf. *Introductio*, § 270: [5], vol. I, pp. 223–224, and [11], vol. I, p. 230.

one gets, thanks to a simple application of the method of indeterminate coefficients:

$$
\begin{array}{lcl}
B_1 & = & \sum\limits_{i=1}^{\&c.} b_i, \\
B_2 & = & B_1 \sum\limits_{i=1}^{\&c.} b_i - \sum\limits_{i=1}^{\&c.} b_i \sum\limits_{j=i+1}^{\&c.} b_j = \sum\limits_{i=1}^{\&c.} b_i \sum\limits_{j=1}^{i} b_j, \\
B_3 & = & B_2 \sum\limits_{i=1}^{\&c.} b_i - B_1 \sum\limits_{i=1}^{\&c.} b_i \sum\limits_{j=i+1}^{\&c.} b_j + \sum\limits_{i=1}^{\&c.} b_i \sum\limits_{j=i+1}^{\&c.} b_j \sum\limits_{h=j+1}^{\&c.} b_h = \sum\limits_{i=1}^{\&c.} b_i \sum\limits_{j=1}^{i} b_j \sum\limits_{h=1}^{j} b_h, \\
\ldots & & \ldots,
\end{array}
\tag{3}
$$

from which, letting $z = 1$, one draws that the product $\prod\limits_{\mu=1}^{\&c.} (1 - b_\mu)^{-1}$ is equal to a sum $1 + \mathcal{Q}_1 + \mathcal{Q}_2 + \&c.$, whose generic term $\mathcal{Q}_m$ is constituted by the sum of all products of m factors that can be formed by choosing such factors among the terms b_μ ($\mu = 1$, 2, &c.), taking each of them many times, possibly. Again, if addition and multiplication are replaced by (or conceived as) simple enumerations, $\mathcal{Q}_m$ is reduced, then, to a combination with repetitions: namely, it is reduced to the combination with repetitions of m objects chosen among the objects b_μ ($\mu = 1$, 2, &c.). Employing the modern combinatorial notation, and supposing that these objects are k in number, we would have, then: $\mathcal{Q}_m = K(k, m)$. Here is, again, Euler's way of putting it[50]:

> When we let $z = 1$, the expression
>
> $$\frac{1}{(1 - b_1)(1 - b_2)(1 - b_3)(1 - b_4)(1 - b_5)\,\&c.}$$
>
> is equal to the unity together with the series of all numbers that results from b_1, b_2, b_3, b_4, b_5, b_6, &c. either taken singly, or multiplied among them two or more, at a time, without excluding those which are equal.

These results are, as such, quite trivial, and were perfectly known before the publication of the *Introductio*. They receive however, in such a treatise, a new interpretation. Rather than reading the equalities (2) and (3) as rules allowing one to determine the unknown coefficients A_ν and B_ν ($\nu = 1, 2, \&c.$) with the help of appropriate combinatorial tools, he understands them as basic results starting from which it is possible to offer an analytic account of combinatorics: the general theory of (algebraic) functions proves to be rich enough to express and study in abstract terms the combinations of any multiplicity of objects, or, better—admitting Euler's explicit interpretation—of numbers. Indeed, after deriving, in Chapter XV, from the equalities (2) and (3) many particular results, he devotes Chapter XVI to the edification of an analytic theory of numerical partitions.

Among the results that Euler gets in Chapter XV, I choose only a few examples.

If one considers the infinitary version of (3) and replaces the coefficients b_μ ($\mu = 1$, 2, &c.) with

[50] Cf. *Introductio*, § 271: [5], vol. I, p. 221, and [11], vol. I, p. 224. Euler writes again 'α','β', 'γ', 'δ', 'ε', 'ξ' where I write 'b_i' ($i = 1, \ldots, 6$).

the successive prime numbers 2, 3, 5, &*c*., one straightforwardly derives the following two equalities[51]:

$$\begin{aligned} i) \quad & \prod_{\mu=1}^{\infty}\left(1-\frac{1}{p_\mu^n}\right)^{-1} &=& \sum_{\nu=1}^{\infty}\frac{1}{\nu}, \\ ii) \quad & \prod_{\mu=1}^{\infty}\left(1-\frac{1}{p_\mu^k}\right)^{-1} &=& \sum_{\nu=1}^{\infty}\frac{1}{\nu^k}, \end{aligned} \tag{4}$$

where, for any positive integer μ, p_μ is the μ-th prime number and k is, in its turn, any positive integer. From the second of these equalities and the infinitary extension of (2), it is then easy to draw[52]:

$$\frac{\sum_{\nu=1}^{\infty}\frac{1}{\nu^n}}{\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2n}}} = \prod_{\mu=1}^{\infty}\frac{1-\frac{1}{p_\mu^{2n}}}{1-\frac{1}{p_\mu^n}} = \prod_{\mu=1}^{\infty}\left(1+\frac{1}{p_\mu^k}\right) = \sum_{\nu=1}^{\&c.}\frac{1}{q_\nu^k}$$

where q_ν $(\nu = 1, 2, ...)$ are all positive integer numbers but those which have two or more equal prime factors. From the equality $(4.i)$ and the power series expansion of $\log\frac{1}{1-z}$, that is,

$$\log\frac{1}{1-z} = -\log(1-z) = \sum_{\nu=1}^{\infty}\frac{z^\nu}{\nu},$$

Euler draws, then[53], by supposing that $z = 1$,

$$\log(\infty) = \infty = \sum_{\nu=1}^{\infty}\frac{1}{\nu} = \prod_{\mu=1}^{\infty}\left(1-\frac{1}{p_\mu}\right)^{-1},$$

and thus:

$$\prod_{\mu=1}^{\infty}\left(1-\frac{1}{p_\mu}\right) = 0.$$

These few results are revealing of Euler's approach. Being aware that the infinitary versions of the equalities (2) and (3) provide nothing but formal associations, he only chooses, in his examples, substitutions that allow him to determine the limit of converging series or infinite products or to equalize diverging series or infinite products to ∞ (which is a sort of formal expression of their divergence).

2.2 Basic principles of an analytic theory of generalized partitions

Leaving aside the particular applications of the equalities (2) and (3), let's move on to Euler's treatment of numerical partitions, in Chapter XVI, and begin with the basic principles on which this treatment is based.

[51] Cf. *Introductio*, § 272–274: [5], vol. I, pp. 224–226, and [11], vol. I, pp. 221–232.
[52] Cf. *Introductio*, § 277: [5], vol. I, pp. 227-228, and [11], vol. I, pp. 233–234.
[53] Cf. *Introductio*, § 277, ex. I: [5], vol. I, pp. 228-230, and [11], vol. I, pp. 234-236.

By supposing that[54]

$$\prod_{\mu=1}^{\&c.}\left[1+x^{\alpha_\mu}z\right]=1+\sum_{\nu=1}^{\&c.}N_\nu z^\nu, \tag{5}$$

where α_μ $(\mu=1,2,\&c.)$ are any sorts of exponents, and N_ν $(\nu=1,2,\&c.)$ are indeterminate coefficients, from the equalities (2) it follows that:

$$\begin{array}{rcl}
N_1 & = & \displaystyle\sum_{i=1}^{\&c.}x^{\alpha_i}, \\
N_2 & = & \displaystyle\sum_{i=1}^{\&c.}x^{\alpha_i}\sum_{j=i+1}^{\&c.}x^{\alpha_j}=\sum_{i=1}^{\&c.}\left[x^{\alpha_i+\alpha_{i+1}}+x^{\alpha_i+\alpha_{i+2}}+x^{\alpha_i+\alpha_{1+3}}+\&c.\right], \\
N_3 & = & \displaystyle\sum_{i=1}^{\&c.}x^{\alpha_i}\sum_{j=i+1}^{\&c.}x^{\alpha_j}\sum_{h=j+1}^{\&c.}x^{\alpha_h}=\sum_{i=1}^{\&c.}\left[\begin{array}{l}x^{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}+x^{\alpha_i+\alpha_{i+1}+\alpha_{i+3}}+x^{\alpha_i+\alpha_{i+1}+\alpha_{i+4}}+\&c.+\\ x^{\alpha_i+\alpha_{i+2}+\alpha_{i+3}}+x^{\alpha_i+\alpha_{i+2}+\alpha_{i+4}}+x^{\alpha_i+\alpha_{i+2}+\alpha_{i+5}}+\&c.+\\ x^{\alpha_i+\alpha_{i+3}+\alpha_{i+4}}+x^{\alpha_i+\alpha_{i+3}+\alpha_{i+5}}+x^{\alpha_i+\alpha_{i+3}+\alpha_{i+6}}+\&c.+\\ +\&c.\end{array}\right], \\
\&c. & & \&c.
\end{array}$$

from which it is easy to understand that the generic coefficient N_m is constituted by the sums of the powers of x whose exponents are all the sums of m addenda which can be formed by choosing such addenda among the terms α_μ $(\mu=1,2,\&c.)$ and taking each of them just once. But, some of these sums can be equal to each others. It follows that such a generic coefficient complies with the equality

$$N_m=\sum_{i=1}^{\&c.}\Re_{n_i,m}x^{n_i} \tag{6}$$

where $\Re_{n_i,m}$ are natural numbers indicating how many times the appropriate exponent n_i can be formed as the sum of m different addenda chosen among the terms α_μ $(\mu=1,2,\&c.)$. If we abstract from the index of n_i and take this exponent as a number, we conclude, with Euler, that[55]:

> [...] if one wants to know how many different ways the number n can be the sum of m different terms of this sequence [that is, $\{\alpha_\mu\}_{\mu=1}^{\&c.}$], one has to search for the term $x^n z^m$ in the expression expanded, and its coefficient will indicate the number that is searched for.

Said in other terms, the coefficient $\Re_{n,m}$ of $x^n z^m$, in the polynomial which is equal to the product $\prod_{\mu=1}^{\&c.}\left[1+x^{\alpha_\mu}z\right]$, indicates how many times the number n can be formed as the sum of m different addenda chosen among the numbers α_μ $(\mu=1,2,\&c.)$.

If one sets $z=1$, in the polynomial $\sum_{\nu=1}^{\&c.}N_\nu z^\nu$, all the powers of x with the same exponent can be added to each other. Hence, the coefficients R_{n_i} $(i=1,2,...)$ of the sum which one gets in this way, that is,

$$1+\sum_{i=1}^{\&c.}R_{n_i}x^{n_i}=\prod_{\mu=1}^{\&c.}\left[1+x^{\alpha_\mu}\right], \tag{7}$$

[54]Cf. *Introductio*, § 297-301: [5], vol. I, pp. 253–256, and [11], vol. I, pp. 256–258.
[55]Cf. *Introductio*, § 299: [5], vol. I, p. 254, and [11], vol. I, p. 257.

indicate how many times the exponent n_i can be formed as the sum of any number of different addenda chosen among the terms α_μ ($\mu = 1, 2, \&c.$). If we abstract from the index of n_i and take this exponent as a number, we conclude that the coefficient R_n of x^n, in the polynomial which is equal to the product $\prod_{\mu=1}^{\&c.} (1 + x^{\alpha_\mu})$, indicates how many times the number n can be formed as the sum of any number of different addenda chosen among the numbers α_μ ($\mu = 1, 2, \&c.$).

Applying similar considerations to the consequences that can be drawn from the equalities (3), we will have similar results in connection with sums that are formed by a finite number of not necessarily different addenda[56]. By supposing that

$$\prod_{\mu=1}^{\&c.} [1 - x^{\alpha_\mu} z]^{-1} = 1 + \sum_{\nu=1}^{\infty} Q_\nu z^\nu, \tag{8}$$

where α_μ ($\mu = 1, 2, \&c.$) are, as before, any sorts of exponents, and Q_ν ($\nu = 1, 2, \&c.$) are indeterminate coefficients, from these equalities it follows that:

$$\begin{array}{lcl}
Q_1 & = & \sum_{i=1}^{\&c.} x^{\alpha_i}, \\
Q_2 & = & \sum_{i=1}^{\&c.} x^{\alpha_i} \sum_{j=1}^{i} x^{\alpha_j} = \sum_{i=1}^{\&c.} \left[x^{\alpha_i+\alpha_1} + x^{\alpha_i+\alpha_2} + \ldots + x^{2\alpha_i} \right], \\
Q_3 & = & \sum_{i=1}^{\&c.} x^{\alpha_i} \sum_{j=1}^{i} x^{\alpha_j} \sum_{h=1}^{j} x^{\alpha_h} = \sum_{i=1}^{\&c.} \left[\begin{array}{l} x^{\alpha_i+2\alpha_1} + \\ x^{\alpha_i+\alpha_1+\alpha_2} + x^{\alpha_i+2\alpha_2} + \\ x^{\alpha_i+\alpha_1+\alpha_3} + x^{\alpha_i+\alpha_2+\alpha_3} + x^{\alpha_i+2\alpha_3} + \\ + \ldots + \\ x^{2\alpha_i+\alpha_1} + x^{2\alpha_i+\alpha_2} + \ldots + x^{3\alpha_i} + \&c. \end{array} \right], \\
\ldots & & \ldots
\end{array}$$

The generic coefficient Q_m is thus constituted by the sums of the powers of x whose exponents are all the sums of m not necessarily different addenda which can be formed by choosing such addenda among the terms α_μ ($\mu = 1, 2, \&c.$). As some of these sums can be equal to each others, such a generic coefficient complies with the equality

$$Q_m = \sum_{i=1}^{\&c.} \mathfrak{S}_{n_i,m} x^{n_i} \tag{9}$$

where $\mathfrak{S}_{n_i,m}$ are natural numbers indicating how many times the appropriate exponent n_i can be formed as the sum of m not necessarily different addenda chosen among the terms α_μ ($\mu = 1, 2, \&c.$). Also in this case, we can, of course abstract from the index of n_i and take this exponent as a number. Hence, the coefficient $\mathfrak{S}_{n,m}$ of $x^n z^m$, in the polynomial which is equal to the product $\prod_{\mu=1}^{\&c.} [1 - x^{\alpha_\mu} z]^{-1}$, indicates how many times the number n can be formed as the sum of m not necessarily different addenda chosen among the numbers α_μ ($\mu = 1, 2, \&c.$). It is then enough to set $z = 1$ in the

[56] Cf. *Introductio*, § 302-305: [5], vol. I, pp. 256-258, and [11], vol. I, pp. 259-261.

polynomial $\sum_{\nu=1}^{\&c.} Q_\nu z^\nu$, to see that the coefficients S_{n_i} $(i = 1, 2, ...)$ such that

$$1 + \sum_{i=1}^{\infty} S_{n_i} x^{n_i} = \prod_{\mu=1}^{\&c.} [1 - x^{\alpha_\mu}]^{-1}, \tag{10}$$

indicate how many times the exponent n_i can be formed as the sum of any number of not necessarily different addenda chosen among the terms α_μ $(\mu = 1, 2, \&c.)$. Once again, if we abstract from the index of n_i and take this exponent as a number, we conclude that the coefficient S_n of x^n, in the polynomial which is equal to the product $\prod_{\mu=1}^{\&c.} [1 - x^{\alpha_\mu}]^{-1}$, indicates how many times the number n can be formed as the sum of any number of not necessarily different addenda chosen among the numbers α_μ $(\mu = 1, 2, \&c.)$.

2.3 Partitions of a positive integer number over the sequence of positive integer numbers

Clearly, the foregoing results are independent of the choice of the terms α_μ $(\mu = 1, 2, \&c.)$, and can thus be employed in the study of the "compositions"[57] of the kind considered for any choice of these terms. However, Euler only considers the case where these terms are positive integer numbers, and pays, in fact, a particular attention only to the case where they are the totality of these numbers. If we suppose to be in this last case—that is, we suppose that the terms α_μ $(\mu = 1, 2, \&c.)$ coincide with the totality of positive integer numbers, or $\alpha_\mu = \mu$, for any positive integer μ—then the coefficients $\mathfrak{R}_{n,m}$, R_n, $\mathfrak{S}_{n,m}$ and S_n are reduced to the numbers of different kinds of partitions[58] of any positive integer n: $\mathfrak{R}_{n,m}$ is the number of all possible partitions of n in m different parts; R_n is the number of all possible partitions of n in different parts; $\mathfrak{S}_{n,m}$ is the number of partitions of n in m equal or different parts; S_n is the number of all possible partitions of n in equal or different parts. If we suppose, instead, that the terms α_μ $(\mu = 1, 2, \&c.)$ coincide with the first λ positive integer numbers (where λ is any integer positive number), or $\alpha_\mu = \mu$, for any integer positive smaller that $\lambda + 1$, then: $\mathfrak{R}_{n,m}$ is the number of all possible partitions of n in m different parts smaller than $\lambda + 1$; R_n is the number of all possible partitions of n in different parts smaller than $\lambda + 1$; $\mathfrak{S}_{n,m}$ is the number of partitions of n in v equal or different parts smaller than $\lambda + 1$; S_n is the number of all possible partitions of n in equal or different parts smaller than $\lambda + 1$. In order to stress the fact that these different sorts of numbers result from quite particular application of Euler's previous results, I will denote them with appropriate symbols. Namely, I shall use:

'$[\mathfrak{R}_{n,m}]_{[1,2,...]}$' to denote the number of all possible partitions of n in m different parts;

'$[\mathfrak{R}_{n,m}]_{[1,2,...,\lambda]}$' to denote the number of all possible partitions of n in m different parts smaller than $\lambda + 1$;

'$[R_n]_{[1,2,...]}$' to denote the number of all possible partitions of n in different parts;

[57] This is Euler's term. Cf., for example, *Introductio*, § 306: [5], vol. I, p. 258, and [11], vol. I, p. 261: "After having given this general view, we now, with some diligence, investigate how to find this multitude of compositions."

[58] I am here employing the usual combinatorial language, and referring obviously to partitions of a positive integer into addenda that are constituted, in their turn, by positive integers.

'$[R_n]_{[1,2,\ldots,\lambda]}$' to denote the number of all possible partitions of n in different parts smaller than $\lambda+1$;

'$[\mathfrak{S}_{n,m}]_{[1,2,\ldots]}$' to denote the number of partitions of n in m equal or different parts;

'$[\mathfrak{S}_{n,m}]_{[1,2,\ldots,\lambda]}$' to denote the number of partitions of n in m equal or different parts smaller than $\lambda+1$;

'$[S_n]_{[1,2,\ldots]}$' to denote the number of all possible partitions of n in equal or different parts;

'$[S_n]_{[1,2,\ldots,\lambda]}$' to denote the number of all possible partitions of n in equal or different parts smaller than $\lambda+1$.

The problem that Euler tackles with is that of determining such numbers by relying on the formation laws of the series that are associated to them, and, namely, without making any recourse to any specifically combinatorial procedure.

Consider the equality (5), first[59]. If we let $z = xy$, suppose that the product is infinite, and $\alpha_\mu = \mu$ for any integer positive μ, we get

$$\prod_{\mu=1}^{\infty}\left[1+x^{\mu+1}y\right] = \frac{\prod\limits_{\mu=1}^{\infty}\left[1+x^{\mu}y\right]}{1+xy} = 1+\sum_{\nu=1}^{\infty} N_\nu\,(xy)^\nu$$

and thus:

$$\prod_{\mu=1}^{\infty}\left[1+x^{\mu}y\right] = [1+xy]\left[1+\sum_{\nu=1}^{\infty} N_\nu\,(xy)^\nu\right].$$

Comparing this identity with the same equality (5), for $\alpha_\mu = \mu$ for any integer positive μ, and $z = y$, we have

$$1+\sum_{\nu=1}^{\infty} N_\nu y^\nu = [1+xy]\left[1+\sum_{\nu=1}^{\infty} N_\nu\,(xy)^\nu\right],$$

from which, according to the method of indeterminate coefficients (applied to the powers of y), it is easy to get:

$$N_m = \frac{x^m}{1-x^m}N_{m-1} \qquad \text{with } N_0 = 1, \tag{11}$$

for any integer positive m.

The coefficient N_m is thus a fractional function of x. Thus, from the equality (6), it follows that, to find the number $[\mathfrak{R}_{n,m}]_{[1,2,\ldots.]}$, it is enough to apply to such a function the procedure for expanding a fractional function in a power series[60]. The coefficient of order n in the power series thus obtained will just be $[\mathfrak{R}_{n,m}]_{[1,2,\ldots.]}$.

Moreover, if one applies to the equality (11) a quite simple recursive procedure, one gets:

$$N_m = \frac{x^{\frac{m(m+1)}{2}}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)} = x^{\frac{m(m+1)}{2}}\prod_{\mu=1}^{m}\left[1-x^\mu\right]^{-1} = x^{\frac{m(m+1)}{2}}\left[1+\sum_{n=1}^{\infty}\left([S_n]_{[1,2,\ldots,m]}\right)x^n\right]. \tag{12}$$

[59] Cf. *Introductio*, § 307-312: [5], vol. I, pp. 258-262, and [11], vol. I, pp. 261-265.

[60] This procedure is presented by Euler in the first part of the chapter IV of the first volume of the *Introductio*: cf. § 59-70: [5], vol. I, pp. 46-55, and [11], vol. I, pp. 50-59.

Compared with the equality (6), the equality (12) expresses thus the following theorem, which Euler has proved without relying on any specifically combinatorial procedure:

Theorem 1 *If m is a integer positive number, the integer positive number n can be formed as the sum of any number of not necessarily different addenda chosen among the numbers $1, 2, ..., m$ as many times as the number $n + \frac{m(m+1)}{2}$ can be formed as the sum of m different addenda chosen among all integer positive numbers. In modern terms: the number of all possible partitions of n in equal or different parts smaller than $m + 1$ is equal to the number of all possible partitions of n in m different parts*[61]*:*

$$\left[\Re_{n+\frac{m(m+1)}{2},m}\right]_{[1,2,...]} = [S_n]_{[1,2,...,m]} . \tag{13}$$

Since the addenda into which any integer positive number n can be decomposed are necessarily smaller than $n + 1$, we have

$$[S_n]_{[1,2,...,\vartheta]} = [S_n]_{[1,2,...]} \qquad \text{provided that } \vartheta \geq n. \tag{14}$$

The following equality is thus an immediate corollary of this theorem[62]:

$$[S_n]_{[1,2,...]} = \left[\Re_{n+\frac{m(m+1)}{2},m}\right]_{[1,2,...]} .$$

Consider now[63] the equality (8), suppose that the product is infinite and $\alpha_\mu = \mu$ ($\mu = 1, 2, ...$), and let again $z = xy$. We get:

$$\prod_{\mu=1}^{\infty} \left[1 - x^{\mu+1}y\right]^{-1} = [1 - xy] \prod_{\mu=1}^{\infty} \left[1 - x^{\mu}y\right]^{-1} = 1 + \sum_{\nu=1}^{\infty} Q_\nu \, (xy)^\nu .$$

Comparing this equality with the same equality (8), for $\alpha_\mu = \mu$ for any positive integer μ, and $z = y$, we have:

$$[1 - xy] \left[1 + \sum_{\nu=1}^{\infty} Q_\nu y^\nu\right] = 1 + \sum_{\nu=1}^{\infty} Q_\nu \, (xy)^\nu ,$$

from which, according to the method of indeterminate coefficients (applied to the powers of y), it is easy to get:

$$Q_m = \frac{x}{1 - x^m} Q_{m-1} \qquad \text{with } Q_0 = 1,$$

for any positive integer m. It is then enough to compare this last equality to the equality (11), to get:

$$Q_m = \left[\frac{1}{x^{\frac{m(m-1)}{2}}}\right] N_m .$$

[61] Cf. *Introductio*, § 312: [5], vol. I, p. 262, and [11], vol. I, p. 265: "The number n can be produced through addition by the numbers 1, 2, 3, 4, ..., m as many times as the number $n + \frac{m(m+1)}{1 \cdot 2}$ can be split in different parts."

[62] Though combinatorially trivial, this corollary is not made explicit by Euler, who apparently wishes to remain within the bounds of a strictly analytical theory.

[63] Cf. *Introductio*, § 313–314: [5], vol. I, pp. 262–264, and [11], vol. I, pp. 265–267.

And from this last equality and the equalities (6), (9) and (13), it follows that:

$$[\Re_{n,m}]_{[1,2,...]} = \left[\mathfrak{S}_{n-\frac{m(m-1)}{2},m}\right]_{[1,2,...]} = \left[S_{n-\frac{m(m+1)}{2}}\right]_{[1,2,...,m]} \qquad \text{provided that } n > \frac{m\,(m+1)}{2},$$

that is,

$$[\mathfrak{S}_{n+m,m}]_{[1,2,...]} = [S_n]_{[1,2,...,m]}\,.$$

Also the following theorem has thus been proved by Euler without relying on any specifically combinatorial procedure:

Theorem 2 *If m is a integer positive number, the integer positive number n can be formed as the sum of any number of not necessarily different addenda chosen among the numbers $1, 2, ..., m$ as many times as the number $n+m$ can be formed as the sum of m not necessarily different addenda chosen among all integer positive numbers. In modern words: the number of possible partitions of n in equal or different parts smaller than $m+1$ is equal to the number of partitions of $n+m$ in m equal or different parts*[64].

According to the equality (14), also this theorem has of course an immediate corollary[65]:

$$[S_n]_{[1,2,...]} = [\mathfrak{S}_{2n,n}]_{[1,2,...]}\,.$$

From the two previous theorems, it is quite easy to draw four other corollaries, respectively expressed by the following equalities[66]

$$\begin{array}{rlcll}
i) & [\Re_{n,m}]_{[1,2,...]} & = & \left[S_{n-\frac{m(m+1)}{2}}\right]_{[1,2,...,m]} & \text{provided that } n > \frac{m(m+1)}{2};\\
ii) & [\mathfrak{S}_{n,m}]_{[1,2,...]} & = & [S_{n-m}]_{[1,2,...,m]}\,; & \\
iiv) & [\Re_{n,m}]_{[1,2,...]} & = & \left[\mathfrak{S}_{n-\frac{m(m-1)}{2},m}\right]_{[1,2,...]} & \text{provided that } n > \frac{m(m-1)}{2};\\
iv) & [\mathfrak{S}_{n,m}]_{[1,2,...]} & = & \left[\Re_{n+\frac{m(m-1)}{2},m}\right]_{[1,2,...]}\,. &
\end{array}$$

Through these theorems and corollaries it is thus possible to calculate the numbers $[\mathfrak{S}_{n,m}]_{[1,2,...]}$ and $[S_n]_{[1,2,...,m]}$ (as well as the number $[S_n]_{[1,2,....]}$) starting from the number $[\Re_{k,m}]_{[1,2,...]}$ relative to an appropriate positive integer k, that can, in its turn, be determined through the power series expansion of an appropriate fractional function. In many cases, however, it might be more convenient to reach the same result through a direct determination of the number $[S_n]_{[1,2,...,m]}$, according to a recurrent procedure based on the following equality[67]:

$$[S_n]_{[1,2,...,m]} = [S_n]_{[1,2,...,m-1]} + [S_{n-m}]_{[1,2,...,m]} \qquad \text{provided that } n > m > 1, \tag{15}$$

whose proof is analogous to that of the previous theorems. Indeed, if in the equality (10) we let $\alpha_\mu = \mu$ ($\mu = 1, 2, \ldots, m$), it is easy to get

[64] Cf. *Introductio*, § 314: [5], vol. I, p. 264, and [11], vol. I, p. 267: "The number n can be produced through addition by the numbers 1, 2, 3, 4, . . . , m as many times as the number $n+m$ can be split in m different parts."

[65] This corollary is not made explicit by Euler: cf. the previous footnote (62).

[66] Cf. *Introductio*, § 315: [5], vol. I, pp. 264-265, and [11], vol. I, pp. 267-268.

[67] Cf. *Introductio*, § 316-318: [5], vol. I, pp. 265-267, and [11], vol. I, pp. 268-270.

$$i) \quad 1+\sum_{n=1}^{\infty}[S_n]_{[1,2,\ldots,m]}\,x^n=\prod_{\mu=1}^{m}[1-x^{\mu}]^{-1}\,,$$

and thus

$$ii) \quad x^m+x^m\sum_{n=1}^{\infty}[S_n]_{[1,2,\ldots,m]}\,x^n=\prod_{\mu=1}^{m}x^m\,[1-x^{\mu}]^{-1}=\sum_{n=1}^{\infty}[S_{n-m}]_{[1,2,\ldots,m]}\,x^n,$$
$$\text{with } [S_\sigma]_{[1,2,\ldots,m]}=0 \text{ if } \sigma<0, \text{ and } [S_0]_{[1,2,\ldots,m]}=1,$$

which, when subtracted, gives the identity:

$$\prod_{\mu=1}^{m-1}[1-x^{\mu}]^{-1}=\sum_{n=1}^{\infty}\left([S_n]_{[1,2,\ldots,m]}-\sum_{n=1}^{\infty}[S_{n-m}]_{[1,2,\ldots,m]}\right)x^n,$$
$$\text{with } [S_\sigma]_{[1,2,\ldots,m]}=0 \text{ if } \sigma<0, \text{ and } [S_0]_{[1,2,\ldots,m]}=1.$$

It is then enough to apply, once again, the equality (10) for $\alpha_\mu=\mu$ ($\mu=1,2,\ldots,m-1$), to get

$$\prod_{\mu=1}^{m-1}[1-x^{\alpha_\mu}]^{-1}=1+\sum_{n=1}^{\infty}[S_n]_{[1,2,\ldots,m-1]}\,x^n,$$

and then

$$1+\sum_{n=1}^{\infty}[S_n]_{[1,2,\ldots,m-1]}\,x^n=\sum_{n=1}^{\infty}\left([S_n]_{[1,2,\ldots,m]}-\sum_{n=1}^{\infty}[S_{n-m}]_{[1,2,\ldots,m]}\right)x^n,$$
$$\text{with } [S_\sigma]_{[1,2,\ldots,m]}=0 \text{ if } \sigma<0, \text{ and } [S_0]_{[1,2,\ldots,m]}=1,$$

from which the equality (15) follows by equating the coefficients of any order n.

Even though Euler does not explicitly remark it, from the equality (14), it follows that

$$[S_n]_{[1,2,\ldots,\theta+n]}=[S_n]_{[1,2,\ldots,n]}\,, \tag{16}$$

for any positive integer θ[68]. Hence, the range of application of the equality (15) is not limited for it holds only if $n>m$.

This range of application is no more limited for such an equality applies only for $m>1$. Even without relying on strictly combinatorial arguments, it is possible, indeed, to use this same equality for providing a contextual determination of the number that would be denoted by the symbol

[68] After having constructed a table giving the numbers $[S_n]_{[1,2,\ldots,m]}$ for $n=1,2,\ldots,69$ and $m=1,2,\ldots,11$ [cf. *Introductio*, table annexed to the § 318: [5], vol. I, sheet included between p. 274 and p. 275, and [11], vol. I, pp. 280-282], Euler points out [cf. *ibid*, § 323: [5], vol. I, p. 269, and [11], vol. I, p. 273], in fact, that the different sequences $\left\{[S_n]_{[1,2,\ldots,m]}\right\}_{n=1}^{\&c.}$ for $m=1,2,\&c.$ have an initial common segment that tends to increase as long as m increases, which correspond to an *a posteriori* acknowledgment of the property expressed by the equality $[S_n]_{[1,2,\ldots,\theta+n]}=[S_n]_{[1,2,\ldots,n]}$. He also claims that from this it follows that "*in infinitum* these sequences will agree completely" [cf. *ibid*], which means that the sequence $\left\{[S_n]_{[1,2,\ldots]}\right\}_{n=1}^{\infty}=\left\{[S_1]_{[1]},[S_2]_{[1,2]},[S_3]_{[1,2,3]},\ldots\right\}$ can be perfectly determined, since it is the same as the sequence of coefficients of the power series expansion of the infinite product $\prod_{\mu=1}^{\infty}[1-x^{\mu}]^{-1}$.

‘$[S_n]_{[1,2,\ldots,m-1]}$’ if $m = 1$. According to this identity, we have, indeed, $[S_n]_{[1]} - [S_{n-1}]_{[1]} = [S_n]_{[1-1]}$, and thus, because of then equality (13), $[S_n]_{[1-1]} = [\Re_{n+1,1}]_{[1,2,\ldots]} - [\Re_{n,1}]_{[1,2,\ldots]}$, provided that $n > 1$. But, from the equalities (5) and (6) it is easy to draw, by calculating the coefficient of z in the power series expansion of $\prod_{\mu=1}^{\&c.} [1 + x^\mu z]$, that

$$[\Re_{n,1}]_{[1,2,\ldots]} = [\Re_{n+1,1}]_{[1,2,\ldots]} = 1, \tag{17}$$

for any positive integer n. It follows that $[S_n]_{[1-1]} = 1 - 1 = 0$, and, thus, $[S_n]_{[1]} = [S_{n-1}]_{[1]}$, for any positive integer n greater than 1. Hence, the equality (15) applies also for $m = 1$, and leads, in this case, together with the equalities (13) and (17), to

$$[S_n]_{[1]} = [S_{n-1}]_{[1]} = 1, \tag{18}$$

for any positive integer n greater than 1. For $n = 2$, one has, thus, $[S_1]_{[1]} = 1$.

By relying on this last equality, together with the equalities (15) and (16) and admitting that $[S_0]_{[1,2,\ldots,m]} = 1$, for any positive integer m[69], it is possible to calculate by recurrence all values of $[S_n]_{[1,2,\ldots,m]}$:

$$\begin{array}{ll}
\text{for } n = 1, & [S_1]_{[1]} = 1 = [S_1]_{[1,2,\ldots,\theta+1]} \\
\text{for } n = 2, & [S_2]_{[1]} = 1 \\
& [S_2]_{[1,2]} = [S_2]_{[1]} + [S_0]_{[1,2]} = 1 + 1 = 2 = [S_2]_{[1,2,\ldots,\theta+2]} \\
\text{for } n = 3, & [S_3]_{[1]} = 1 \\
& [S_3]_{[1,2]} = [S_3]_{[1]} + [S_1]_{[1,2]} = 1 + 1 = 2 \\
& [S_3]_{[1,2,3]} = [S_3]_{[1,2]} + [S_0]_{[1,2,3]} = 2 + 1 = 3 = [S_3]_{[1,2,\ldots,\theta+3]} \\
\text{for } n = 4, & [S_4]_{[1]} = 1 \\
& [S_4]_{[1,2]} = [S_4]_{[1]} + [S_2]_{[1,2]} = 1 + 3 = 3 \\
& [S_4]_{[1,2,3]} = [S_4]_{[1,2]} + [S_1]_{[1,2,3]} = 3 + 1 = 4 \\
& [S_4]_{[1,2,\ldots,4]} = [S_4]_{[1,2,3]} + [S_0]_{[1,2,\ldots,4]} = 4 + 1 = 5 = [S_4]_{[1,2,\ldots,\theta+4]} \\
\text{for } n = 5, & [S_5]_{[1]} = 1 \\
& [S_5]_{[1,2]} = [S_5]_{[1]} + [S_3]_{[1,2]} = 1 + 2 = 3 \\
& [S_5]_{[1,2,3]} = [S_5]_{[1,2]} + [S_2]_{[1,2,3]} = 3 + 2 = 5 \\
& [S_5]_{[1,2,\ldots,4]} = [S_5]_{[1,2,3]} + [S_1]_{[1,2\ldots,4]} = 5 + 1 = 6 \\
& [S_5]_{[1,2,\ldots,5]} = [S_5]_{[1,2,\ldots,4]} + [S_0]_{[1,2,\ldots,5]} = 6 + 1 = 7 = [S_5]_{[1,2,\ldots,\theta+5]} \\
\ldots & \ldots
\end{array} \tag{19}$$

that hold for any positive integer θ.

[69]From a purely combinatorial point of view, this equality cannot be but an appropriate convention, since 0 has, of course, no integer part. From Euler's analytic point of view, this same equality, as well as the equivalent one $[S_0]_{[1,2,\ldots]} = 1$, can instead be obtained as a consequence of the power series expansion of $\prod_{\mu=1}^{\&c.} [1 - x^\mu]^{-1}$, since the coefficient of x^0 in this expansion is just 1.

Once the basis of the recursive procedure has been established, the numbers $[S_n]_{[1,2,\ldots,m]}$ $(n, m = 1, 2, \ldots)$ can thus be calculated without relying on the determination of any particular power series expansion. Even though Euler does not stress this point[70], he has then shown that his analytic interpretation of the partitions of positive integer numbers makes it possible to obtain general results that lead to the determinations of all the numbers $[S_n]_{[1,2,\ldots,m]}$—and consequently of all the numbers $[\Re_{n,m}]_{[1,2,\ldots]}$ and $[\mathfrak{S}_{n,m}]_{[1,2,\ldots]}$—without going through the determination of the coefficients of power series expansion, the coefficient of z in the power series expansion of $\prod_{\mu=1}^{\&c.} [1 + x^\mu z]$ apart. The results provided by the equalities (19) can thus be used to determine the coefficients of the power series expansions of any product like $\prod_{\mu=1}^{\&c.} [1 \pm x^\mu z]^{\pm 1}$.

The foregoing results leave open the problem of determining the number $[R_n]_{[1,2,\ldots]}$ for any positive integer n. Concerning this number, Euler[71] remarks that

$$i) \quad \left[\prod_{\mu=1}^{\&c.} [1 + x^\mu]\right] \left[\prod_{\mu=1}^{\&c.} [1 - x^\mu]\right] = \prod_{\mu=1}^{\&c.} \left[1 - x^{2\mu}\right],$$

and thus

$$ii) \quad \prod_{\mu=1}^{\&c.} [1 + x^\mu] = \prod_{\mu=1}^{\&c.} \left[1 + x^{2\mu-1}\right]^{-1}.$$

It is then enough to let $\alpha_\mu = \mu$ $(\mu = 1, 2, \&c.)$ in the equality (7) and $\alpha_\mu = 2\mu - 1$ $(\mu = 1, 2, \&c.)$ in the equality (9), to draw, by equalizing the generic coefficient of order n of the power series expansions of $\prod_{\mu=1}^{\&c.} [1 + x^\mu]$ and $\prod_{\mu=1}^{\&c.} \left[1 + x^{2\mu-1}\right]^{-1}$:

$$[R_n]_{[1,2,\ldots]} = [S_n]_{[1,3,\ldots]}, \tag{20}$$

where the symbol '$[S_n]_{[1,3,\ldots]}$' denotes the number of all possible partitions of n in odd different parts. This equality expresses the following theorem:

Theorem 3 *The positive integer number n can be formed as the sum of any number of different addenda chosen among all the positive integer numbers as many times as it can be formed as the sum of any number of not necessarily different addenda chosen among all the odd numbers. In modern words: the number of all possible partitions of the number n in different parts is equal to the number of all possible partitions of the same number n in equal or different odd parts.*[72]

[70] The particular results provided by the equalities (19) are of course included in Euler's table giving the numbers $[S_n]_{[1,2,\ldots,m]}$ for $n = 1, 2, \ldots, 69$ and $m = 1, 2, \ldots, 11$ [cf. the previous footnote (68)], but Euler does not say explicitly that these numbers can be calculated by recursion using to the equality (15). He rather insists [cf. *Introductio*, § 319-322: [5], vol. I, pp. 267-269, and [11], vol. I, pp. 270-273.] on the relations that link the sequences $\left\{[S_n]_{[1,2,\ldots,m]}\right\}_{n=1}^{\&c.}$ ($m = 1, 2, \&c$) with the sequences of figurate numbers, so as to suggest that the terms of the sequences $\left\{[S_n]_{[1,2,\ldots,m]}\right\}_{n=1}^{\&c.}$ can be calculated starting from these last numbers.

[71] Cf. *Introductio*, § 326-327: [5], vol. I, pp. 271-272, and [11], vol. I, pp. 275-276.

[72] Cf. *Introductio*, § 326: [5], vol. I, p. 272, and [11], vol. I, pp. 275-276: "A given number can be formed through addition by all the integer numbers different to each other as many times as the same number can be formed through addition by only the odd numbers, whether the same or different."

The determination of the number $[S_n]_{[1,3,...]}$ is, however, less easy than that of the number $[R_n]_{[1,2,...]}$, which can, in any case, be determined through the power series expansion of $\prod_{\mu=1}^{\&c.}(1+x^\mu)$. From a purely calculatory point of view, the equality (20) should thus be read from the right to the left. The calculation of the coefficients of the power series expansion of $\prod_{\mu=1}^{\&c.}(1+x^\mu)$ can nevertheless be made easier by relying on the equalities (19). Since

$$\prod_{\mu=1}^{\&c.}(1+x^\mu) = \frac{\left[\prod_{\mu=1}^{\&c.}(1-x^\mu)\right]\left[\prod_{\mu=1}^{\&c.}(1+x^\mu)\right]}{\prod_{\mu=1}^{\&c.}(1-x^\mu)} = \left[\prod_{\mu=1}^{\&c.}\left(1-x^{2\mu}\right)\right]\left[\prod_{\mu=1}^{\&c.}(1+x^\mu)^{-1}\right], \qquad (21)$$

from the equalities (7), (10), (14), and (19), it follows that:

$$\begin{aligned} 1+\sum_{n=1}^{\&c.}[R_n]_{[1,2,...]}\,x^n &= \left[1-x^2-x^4+x^{10}+x^{14}-x^{24}-\&c.\right]\left[1+\sum_{n=1}^{\infty}[S_n]_{[1,2,...]}\,x^n\right] \\ &= \left[1-x^2-x^4+x^{10}+x^{14}-x^{24}-\&c.\right]\left[1+x+2x^2+3x^3+5x^4+7x^5+\&c.\right] \\ &= 1+x+2x^2+2x^3+2x^4+3x^5+\&c., \end{aligned}$$

for which Euler does not determine the generic term, however.

2.4 Concluding remarks on Euler's treatment of numerical partitions

The last four articles of the chapter XVI of the first book of the *Introductio*[73] are devoted to the study of the numerical partitions that arise when the terms α_μ ($\mu = 1, 2, \&c.$) are identified with the elements of some particular numerical sequences. There is no need to consider them, since the generality and richness of Euler's approach as well as the principles and the techniques he relies on should be already clear from the previous account. My use of a modern notation based on variable indices should not induce one to think that Euler's arguments consist of formal inferences depending on a fully developed symbolic calculus[74]. Still, Euler's treatment of numerical partitions is, for many reasons, quite modern. To conclude, let me emphasize some of these reasons.

The general idea of an analytic approach to combinatorics is firstly highly notable, since it goes together with a reduction of combinatorics to the theory of power series expansions of fractional (or, better, rational, in modern language) functions. This reduction is a quite particular one, however, since it leads to a recursive determination of the numbers $[S_n]_{[1,2,...,m]}$, $[\mathfrak{R}_{n,m}]_{[1,2,...]}$, and $[\mathfrak{S}_{n,m}]_{[1,2,...]}$ ($n, m = 1, 2, \ldots$) that is, in fact, independent not only of any specifically combinatorial procedure, but also of the actual determination of any particular power series.

[73] Cf. *Introductio*, § 328-331: [5], vol. I, pp. 272-275, and [11], vol. I, pp. 277-279.

[74] Here is, for example, how Euler state the result I have expressed through the equality (15) [cf. *Introductio*, § 318: [5], vol. I, p. 266, and [11], vol. I, p. 269]: "Let L be the number of ways in which a number n can be produced through addition by the numbers 1, 2, 3, . . . , $(m-1)$. Let M be the number of ways in which a number $n-m$ can be produced through addition by the numbers 1, 2, 3, . . . , m. And let N be the number of ways in which a number n can be produced through addition by the numbers 1, 2, 3, . . . , m. With these hypotheses [. . .] it will happen that $L = N - M$, that is, $N = L + M$."

It is secondly important to remark that all Euler's arguments depend on the same quite powerful technique. This consists in the comparison of the generic terms of distinct polynomials or power series whose coefficients are submitted to appropriate translations. Powers series and infinite products are thus understood as formal expressions to be appropriately handled term by term according to the rules of polynomial algebra. It is thus perfectly immaterial whether these series and products converge (on appropriate intervals). What is relevant is only that they satisfy appropriate conditions of unicity. Whereas Euler's analytic approach to combinatorics did not find any follower in the second half of the 18-th century (the so called "German combinatorial school" developed, rather, starting from Hindenburg's works[75], by relying on a diametrically opposed point of view), this technique was revived and generalized by Laplace in the context of his theory of generative functions[76].

Finally, the generality of Euler's approach is also quite noteworthy. It depends on the consideration of any sort of terms α_μ ($\mu = 1, 2, \&c.$) working as exponents occurring in two families of functions of two variables, $1 + x^{\alpha_\mu} z$ and $[1 - x^{\alpha_\mu} z]^{-1}$. Though Euler is mainly interested in the case where $\alpha_\mu = \mu$, so that these exponents are reduced to positive integer numbers and the partitions that are primarily taken into account are those of such numbers, this case is understood as a particular one, to be considered through an appropriate application of general principles susceptible to application to an infinity of other cases. For the functions $1 + x^{\alpha_\mu} z$ and $[1 - x^{\alpha_\mu} z]^{-1}$ to be algebraic, the exponents α_μ have of course to be rational numbers. Still, for Euler's argument to work it is, in fact, enough that they are elements of an appropriate algebraic structure $\langle N, + \rangle$ such that $x^{\alpha_i} x^{\alpha_j} = x^{\alpha_i + \alpha_j}$, for any i and j, and $+$ is an associative and commutative operation. As a matter of fact, Euler's approach applies, thus, to the study of the partitions of the elements of this structure relatively to such an operation, or, possibly, of an extension of it, if this structure is not closed under this same operation. Though Euler is quite far from any idea somehow similar to the ideas of algebraic structure and set, and it would be incongruous to see in his approach to combinatorics a direct antecedent of modern structural algebra or set theory, there is no doubt that he has grasped the possibility of studying a (finite or infinite) sequence $\{\alpha_\mu\}_{\mu=1}^{\&c.}$ as a mathematical object characterized by the mutual relations and composition laws defined on its terms, rather than by the specific nature of these same terms. This possibility is a quite natural outcome of Euler' general conception of analysis as a theory of abstract quantities.

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