

Dispelling Mathematical Doubts.

Assessing Mathematical Correctness of Algorithms in Bhāskara's Commentary on the Mathematical Chapter of the *Āryabhaṭīya*.

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Abstract

In his commentary on the mathematical chapter of the *Āryabhaṭīya*, a Sanskrit astronomical treatise of the fifth century, Bhāskara (629 CE) provides explanations, demonstrations and verifications to check the computational accuracy and logical consistency of some procedures. These unsystematic attempts at justification are often fragmentary and perhaps mere echoes of an activity carried out orally. Nonetheless, one can identify recurring modes of reasoning. To ground mathematically a procedure, a 're-interpretation' via the Rule of Three and the 'Pythagorean Theorem' could be provided. Another way of assessing the validity of a procedure was to establish an independent alternative procedure arriving at the same result.

Introduction

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Contrary to the perception prevalent at the beginning of the twentieth century, a concern for the mathematical correctness of algorithms existed

¹I would like to thank Karine Chemla and Micah Rosss for their attentive and helpful scrutinizing of this article.

in the mathematical tradition in Sanskrit. Reflections on the systematic *upapattis* of Kṛṣṇa's (fl. ca. 1600–1625) commentary on the *Bījagaṇita*, the explorations of the Mādhava school (fourteenth-sixteenth century) or other traditions of mathematical validity have already been published.² Still, the variations among this tradition of justification and explanation need to be studied.

In the following sections, the *Āryabhaṭīyabhāṣya* of Bhāskara (BAB) is analyzed with regard to its reasoning and vocabulary. The second chapter of *Āryabhaṭīya* (Ab)—an astronomical *siddhānta* composed in verse at the end of the fifth century—treats mathematics (*gaṇita*). Respecting the requirements of the genre, these aphoristic *āryas* usually provide the gist of a procedure, such as an essential relationship or the main steps of an algorithm. The BAB is not only the earliest known commentary on this treatise but also the oldest known text of mathematics in Sanskrit prose that has been handed down to us. The BAB thus gives us a glimpse into the reasonings used in the scholarly mathematical tradition in Sanskrit at the beginning of the seventh century³. Very little is known about who practiced scholarly mathematics in classical India, and why scholarly texts were elaborated. The BAB provides information on the intellectual context in which both the Ab and the BAB were composed. First, the commentator's defense of Āryabhaṭa's treatise (and the commentator's own interpretations of the verses) will provide a backdrop for reflections on the mathematical correctness of procedures. Next, the arguments behind the algorithms of mathematical justification will be

²Ikeyama 2003, Jain 2001, Patte 2004, Srinivas 1990. Some of Kṛṣṇa's demonstrations are noted, among others, in the footnotes of Colebrooke 1817.

³Keller 2006, Introduction.

clarified. Afterwards, Bhāskara’s vocabulary including explanations, proofs and verifications will be more precisely characterized.

1 Defending the Treatise

Bhāskara’s commentary, a prolix prose text, gives us a glimpse into the intellectual world of scholarly astronomers and mathematicians. The commentary records their intellectual debates. For the opening verse in which the author of the treatise mentions his name, Bhāskara’s commentary explains:

...as a heroic man on battle fields, whose arms have been copiously lacerated by the strength of vile swords, having entered publicly a battle with enemies, who proclaims the following, as he kills: ‘This Yajñadatta here ascended, a descendant of the Aditis, having undaunted courage in battle fields, ⟨now⟩ strikes. If someone has power, let him strike back!’ In the same way, this master also, who has reached the other side of the ocean of excessive knowledge about Mathematics, Time-reckoning and the Sphere, having entered an assembly of wise men, has declared: ‘Āryabhaṭa tells three: Mathematics, Time-reckoning, the Sphere’.⁴

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*...yaḥ tejasvī puruṣaḥ samareṣu nikṛṣṭāsītejovitānacchuritabāhuḥ
śatruśaṅghātam prakāśam praviśya praharan evam āha ‘ayam asau
uditaḥ aditikulaprasūtaḥ samareṣu anivāritavīryaḥ yajñadattaḥ praharati
/ yadi kasyacit śaktiḥ pratipraharatu’ iti / evam asau api ācāryaḥ
gaṇitakālakriyāgolātīśayajñānodadhipāragaḥ vitsabhām avagāhya ‘āryabhaṭaḥ*

Within this hostile atmosphere, Bhāskara's commentary attempts to convince the reader of the coherence and validity of Āryabhaṭa's treatise. To this end, the commentary dispels 'doubts' (*sandeha*) that arise in the explanations of Āryabhaṭa's verses. Thus, the analysis provides refutations (*parihāra*) to objections and establishes (*sādhya*, *siddha*) Bhāskara's readings of Āryabhaṭa's verse. This commentary presents mainly syntactical and grammatical discussions which debate the interpretation of a given word in the treatise. More often than not, the discussion of the meaning and use of a word defines and characterizes the mathematical objects in question. (Are squares all equal sided quadrilaterals? Do all triangles have equally halving heights? etc.).⁵ Bhāskara's commentary adopts technical words, and the specialized readings of the verses show that the Ab cannot be understood in a straightforward way. The verses need interpretation and the interpretation should be the correct one.

The search for the proper interpretation thus defines the commentator's task. The importance of interpretation becomes especially clear when Bhāskara criticizes Prabhākara's exegesis of the Ab.⁶ For instance, in his comment on the rule for the computation of sines, Bhāskara explains that the expression *samavṛtta* refers to a circle, not a disk as Prabhākara understood it.⁷ More crucially, through his understanding of the word *agra*

trīṇi gadati gaṇitam kālakriyām golam' iti uktavān /.

Shukla 1976:5. Unless otherwise specified, the text follows the critical edition published in Shukla 1976. I would like to thank T. Kusuba, T. Hayashi and M. Yano for the help they provided in translating this paragraph, during my stay in Kyoto in 1997.

⁵Keller 2006, Introduction.

⁶Keller 2006, BAB.2.11; BAB.2.12.

⁷Shukla 1976: 77; Keller 2006, volume 1: 57.

(remainder) as a synonym of *saṅkhyā* (number), Bhāskara provides a new interpretation of the rule given in BAB.2.32-33⁸: the verse giving the rule for a ‘pulverizer with remainder’ (*sāgrakuṭṭakāra*) can now be read as giving a rule for the ‘pulverizer without remainder’ (*niragrakuṭṭakāra*)⁹. This peculiar reading of the word *agra* is an extreme example of the technical and inventive devices commentators use for their interpretations.

Outside the syntactical discussion of a verse, Bhāskara sometimes considers the mathematical content of the procedure directly. Defending Āryabhaṭa’s approximation of π against those of competing schools, he undertakes a refutation (*parihāra*) of the jaina value of $\sqrt{10}$ (*daśakaraṇī*), claiming that the value rests only on tradition and not on proof.

In this case also, it is just a tradition (*āgama*) and not a proof (*upapatti*) (...) But this also should be established (*sādhya*).¹⁰

The above statement should not induce a romantic vision of an enlightened Bhāskara using reason to overthrow prejudices transmitted through (religious) traditions. Although here he criticizes the reasoning which cites ‘tradition’ to justify a rule, in other cases Bhāskara accepts this very argument as evidence of the correctness of a mathematical statement¹¹. The

⁸Shukla 1976: 77; Keller 2006:132-133.

⁹Both rules are mathematically equivalent but do not follow the same pattern. Furthermore, the second reading also involves omitting the last quarter of verse 33. See Keller 2006: Volume II, Appendix on BAB.2.32-33.

¹⁰

atrāpi evāgamaḥ naivopapatih ... cetad api sādyaṃ eva.

Shukla 1976: 72.

¹¹Keller 2006, Introduction.

question nonetheless is raised: Bhāskara argues that the procedures of the *Āryabhaṭīya* are correct, but how does Bhāskara “establish” a rule? Moreover, what does Bhāskara consider a “proof” ? The answer to these questions presents difficulties. Indeed, the rational behind the fragmentary arguments that BAB sets forth is at times hard to grasp. The point of this article is to show that two specific commentarial techniques, the ‘re-interpretation’ of procedures and establishing an alternative independent procedure, were used to ground the Ab’s rules. To establish this point, a characterization of these commentarial techniques will be necessary. This characterization will be followed by a description of the different ways Bhāskara explicitly tries to establish the mathematical validity of Ab’s rules.

2 ‘Re-interpretating’ of Procedures

Bhāskara, in an attempt to elucidate Āryabhaṭa’s rules, gives interpretations of Āryabhaṭa’s verses. He thus makes clear what are the different steps required to carry out a procedure, or the word used to define a mathematical object. In certain cases, having put forth such an interpretation, Bhāskara re-invests his understanding of the rule with an additional meaning. This is what I call a ‘re-interpretation’. A ‘re-interpretation’ does not invalidate a previous interpretation. It is somehow like the poetic process of *śleṣa* which reads several meanings in a same compound, creating thus a poetic aura. A ‘re-interpretation’ adds a layer of meaning, gives depth, to the interpretation of a rule. A ‘re-interpretation’ provides a new mathematical context for the different steps of a procedure which is not modified. Another name for this

commentarial technique could be ‘re-reading’ a procedure.

The next section describes how an ‘explanation’, a ‘proof’ or a ‘verification’ consisted of providing either an alternative independent procedure or a ‘re-interpretation’ of a given procedure via the Rule of Three or the Pythagorean Theorem. In both cases, these arguments would provide a mathematical justification for what alone could appear as an arbitrary succession of operations. Before examining ‘re-interpretations’ of procedures in Bhāskara’s commentary, the expression of the Rule of Three and the Pythagorean Theorem in BAB must be explained.

2.1 Rule of Three

The Rule of Three (*trairāśika*¹²) appears in verse Ab.2.26.

Now, when one has multiplied that fruit quantity of the Rule of
Three by the desire quantity|

The quotient of that divided by the measure should be this fruit
of desire||¹³

In other words, if M (the measure) produces a fruit F_M , and D is a desire for which the fruit, F_D , is sought, the verse may be expressed in modern algebraic notation as:

$$F_D = \frac{F_M \times D}{M} \quad (1)$$

¹²For a general overview on the Rule of Three in India see Sarma 2002.
¹³

trairāśikaphalarāśiṃ tam atheccchārāśinā hataṃ kṛtvā|
labdhaṃ pramāṇabhajitaṃ tasmād icchāphalam idaṃ syāt||

(Shukla 1976: 112-223).

Obviously, this expression can also be understood as a statement that the ratios are equal:

$$\frac{F_D}{D} = \frac{F_M}{M} \quad (2)$$

The procedure given in the verse provides an order for the different operations to be carried out. First, the desire is multiplied by the fruit. Next, the result is divided by the measure. This order of operations causes the procedure to appear as an arbitrary set of operations¹⁴. Bhāskara provides a standard expression to define the kind of problem which the Rule of Three solves. When the commentator thinks that a situation involves proportional quantities and thus the Rule of Three is (or can be) applied, he brings this fact to light by using a verbal formulation (*vāco yukti*) of the Rule of Three. This verbal formulation is a syntactically rigid question which reads as follows:

If the measure produces the fruit, then with the desire what
is produced? The fruit of desire is produced.

This question, when it appears, shows that Bhāskara thinks that the Rule of Three can be applied. I believe that for Bhāskara the Rule of Three invokes proportionality.

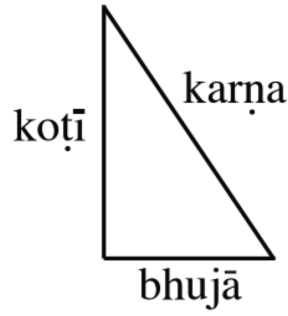
2.2 The Pythagorean Theorem

Bhāskara, like other medieval Sanskrit mathematicians, does not use the concept of angles. In his trigonometry, Bhāskara uses lengths of arcs. As

¹⁴If the division was made first (resulting in the “fruit” of one measure) and then the multiplication, the computation would have had a step-by-step meaning, but this is not the order adopted by Ab.

for right triangles, Bhāskara distinguishes them from ordinary triangles by giving to each side a specific name. Whereas scalene, isosceles and equilateral triangles have sides (*aśra*, for all sides), flanks (*pārśva*, a synonym) and sometimes earths (*bhū*, for the base), right angle triangles have a “base” (*bhujā*), an “up-right side” (*koṭī*) and a “hypotenuse” (*karṇa*), as shown in Figure 2.2. In the first half of Ab.2.17, Bhāskara states the Pythagorean

Figure 1: Names of the Sides of a Right Triangle



Theorem:

That which precisely is the square of the base and the square of the upright side is the square of the hypotenuse.¹⁵

Therefore, in order to indicate that a situation involves a right-triangle, Bhāskara gives the names of the sides of a right-triangle to the segments

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yaś caiva bhujāvargaḥ koṭāvargaś ca karṇavargaḥ saḥ

Shukla 1976:96.

concerned by his reasoning. Two examples of Bhāskara's 're-interpretation' will demonstrate how he employed this theorem.

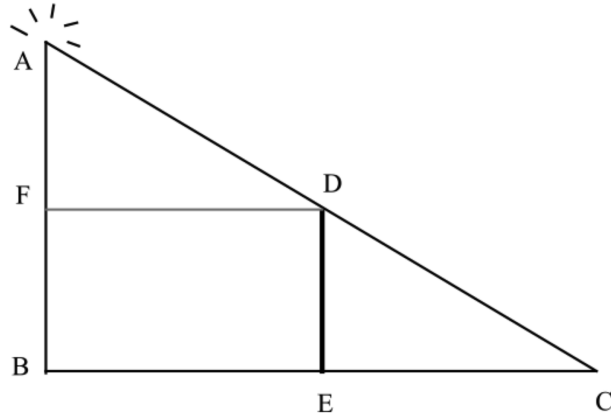
2.3 'Re-interpretation' with gnomons

The section devoted to gnomons (*śaṅku*) contains two illuminating cases.

2.3.1 A Gnomon and a Source of Light

The standard situation is as follows: A gnomon (*śaṅku*, DE) casts a shadow (EC), produced by a source of light (A), as illustrated in Figure 2.

Figure 2: A Schematized Gnomon and Light



First, consider the procedure given in Ab.2.15:

The distance between the gnomon and the base, with ⟨the height of⟩ the gnomon for multiplier, divided by the difference of the heights of the⟩ gnomon and the base.|

Its computation should be known indeed as the shadow of the
gnomon $\langle$ measured $\rangle$ from its foot. $\|$ ¹⁶

This procedure involves a multiplication and a division. In modern algebraic notation:

$$EC = \frac{BE \times DE}{AF}$$

The procedure given in the verse appears to be an arbitrary set of operations. Bhāskara begins with a general gloss. Then, as in all his verse commentaries, Bhāskara's commentary provides a list of solved examples. These examples have a standard structure: first comes a versified problem, then a 'setting down' (*nyāsaḥ*) section, and finally a resolution (*karaṇa*). Thus, in his 're-interpretation' of the above procedure after a solved example, Bhāskara writes:

This computation is the Rule of Three. How? If from the top of the base which is greater than the gnomon $[AF]$, the size of the space between the gnomon and the base, which is a shadow, $[FD = BE]$ is obtained, then, what is $\langle$ obtained $\rangle$ with the gnomon $[DE]$? The shadow $[EC]$ is obtained¹⁷.

¹⁶

śaṅkuguṇaṃ śaṅkubhujāvivaraṃ śaṅkubhujayor viśeṣaḥṛtam|
yal labdaṃ sā chāyā jñeyā śaṅkoḥ svamūlāt hi||

Shukla 1976:90.

¹⁷

etatkarma trairāśikam/ katham ? śaṅkuto 'dhikāyā uparibhujāyā yadi
śaṅkubhujāntarālapramāṇaṃ chāyā labhyate tadā śaṅkunā keti chāyā labhy-
ate

The standard formulation of the Rule of Three, applied to the similar triangles AFD and DEC, can be recognized here. The standard expression of the Rule of Three provides the proportional elements on which the computation is based. Here the rule indicates that the ratio of AF to FD is equal to the ratio of DE to EC. The ‘re-interpretation’ of the rule thus gives the arbitrary set of operations a mathematical significance. Rather than just a list of operations, the rule in Ab.2.15 becomes a Rule of Three.

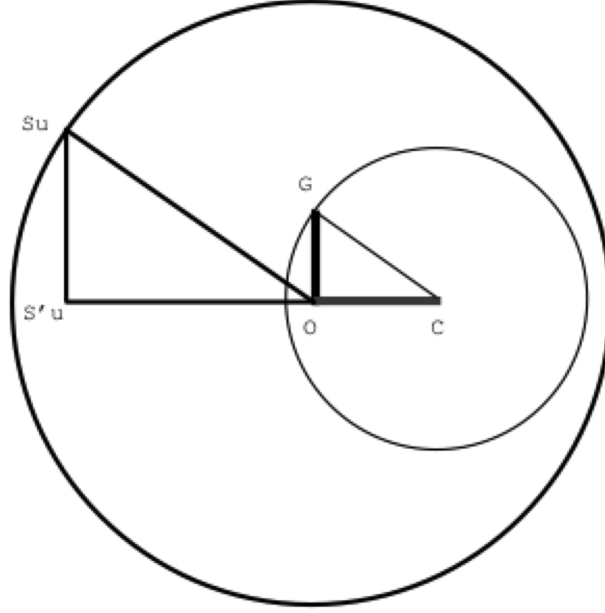
2.3.2 A Gnomon in Relation to the Celestial Sphere

In the previous commented verse (BAB.2.14), Bhāskara sets out two procedures. Both rest on the proportionality of the right triangle formed by the gnomon and its midday shadow with the right triangle composed by the Rsine of the altitude and the zenithal distance. In the present example, one procedure uses only the Rule of Three, while the other uses the Rule of Three with the Pythagoras Theorem. Both procedures compute the same results.

Consider Figure 3. Here, GO represents a gnomon and OC indicates its midday shadow. The circle of radius OSu (Su symbolising the sun) represents the celestial meridian. The radius OSu is thus equal to the radius of the celestial sphere. S’u designates the projection of the Sun onto the horizon. The segment SuS’u illustrates the Rsine of altitude. Bhāskara I notes that the triangle SuS’uO is similar to GOC. Therefore the segment S’uO (that is, the Rsine of the zenithal distance) is proportional to the shadow of the gnomon at noon and the Rsine of the altitude is proportional to the length of the gnomon. This propotionality is further illustrated in Figure 4.

Shukla 1976:92.

Figure 3: Proportional Astronomical Triangles



In modern algebraic notation,

$$\frac{SuS'u}{GO} = \frac{S'uO}{OC} = \frac{SuO}{GC}$$

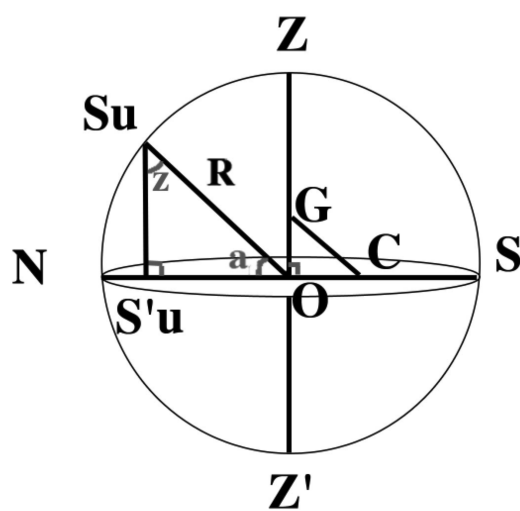
The mathematical key to this situation is the relationship between the celestial sphere and the plane occupied by the gnomon, which Bhāskara and Āryabhaṭa call ‘one’s own circle’ (*svavṛtta*). This relationship is highlighted here by a set of puns. Thus, the gnomon and the Rsine of the altitude have the same name (*śaṅku*), as do the shadow of the gnomon and the Rsine of zenith distance (*chāya*). *GC* is the ‘half-diameter of one’s own circle’.

Bhāskara states this relationship by considering the Rule of Three¹⁸:

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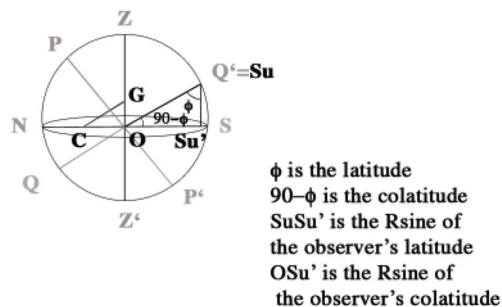
trairāśikaprasiddhyartham- yady asya svavṛttaviṣkambhārdhasya ete śaṅkuc chāye tadā gola-viṣkambhārdhasya ke iti śaṅkuc chāye labhyete

Figure 4: Altitude and Zenith



z is the zenith distance
a is the altitude

Figure 5: Latitude and colatitude on an equinoctial day



In order to establish the Rule of Three – ‘If for the half-diameter of one’s own circle both the gnomon and the shadow ⟨are obtained⟩, then for the half-diameter of the ⟨celestial⟩ sphere, what are the two ⟨quantities obtained⟩?’ In that way the Rsine of altitude and the Rsine of the zenith distance are obtained.

He also adds¹⁹:

Precisely these two [i.e. the Rsine of the sun's altitude and the Rsine of the sun's zenith distance] on an equinoctial day are said to be the Rsine of colatitude (*avalambaka*) and the Rsine of the latitude (*aksajyā*).

Indeed, as illustrated in Figure 5, on the equinoxes the sun is on the celestial equator. At noon, the sun occupies the intersection of the celestial

Shukla 1976: 89.

19

tāv eva visuvati avalambakāksajye ity ucyete/

Shukla 1976: 89.

equator and the celestial meridian. At that moment, the zenithal distance z equals the latitude of the gnomon (ϕ) and the altitude (a) becomes the co-latitude ($90 - \phi$). Once again, the similarity of $SuS'uO$ and OGC is underlined by a certain number of puns. Here, the Rsine of latitude ($SuSu'$) is called ‘perpendicular’ (*avalambaka*).

Now, Bhāskara considers an example for an equinox in which $OG = 13$, $OC = 5$, and the radius of the celestial sphere (SuO) is the customary 1348. Bhāskara writes²⁰:

When computing the Rsine of latitude (*akṣajyā*) the Rule of Three is set down: 13, 5, 3438. What is obtained is the Rsine of latitude, 1322²¹. That is the base (*bhujā*) the half-diameter is the hypotenuse (*karṇa*); the root of the difference of the squares of the base and the hypotenuse is the Rsine of co-latitude (*avalambaka*), 3174²²

In this case, Bhāskara uses the fact that the triangles are both right and similar. Bhāskara then uses this similarity to compute $SuS'u$. Bhāskara employs the ‘Pythagorean Theorem’ to compute $OS'u$. In order to identify the right triangle, Bhāskara renames the Rsine of latitude (*akṣajyā*, $SuS'u$)

²⁰

akṣajyā “nayane trairāśīkasthāpanā- 13/ 5/ 3438/ labdham akṣajyā 1322/ eṣā bhujā, vyāsārdha karṇaḥ, bhujākarṇavargaviśeṣamūlam avalambakaḥ 3174.

Shukla 1976:90.

²¹This is an approximate value. For more on this value, see Keller 2006, BAB.2.14.

²²This value is also an approximation.

as the base of a right triangle (*bhujā*) and he identifies the radius of the celestial sphere as the hypotenuse. Thus, the Rsine is identified with the up-right side of a right-triangle. This identification implicitly explains how the computation is carried out. However, Bhāskara immediately adds²³:

With the Rule of Three also 13, 12, 3438; what has been obtained
is the Rsine of the colatitude, 3174²⁴.

In this way, Bhāskara again computes $OS'u$ by using the similarity of $OSuSu'$ and OGC . Bhāskara thus computes the same value twice, using two different methods. The most likely explanation is that he verifies the results obtained with one algorithm by using another independent process.

The mathematical key to both these computations is the prior relationship between the gnomon and the celestial sphere. A syntactical connection establishes the relationship between these two spaces. The invocation of the Rule of Three begins with a standard question. The naming of two of its segments identifies a right triangle. This identification indicates not only one of the mathematical properties underpinning the procedure but also maps the specific astronomical problem onto a more general and abstract mathematical situation. (That is, Rsines of altitudes and zenithal distances become the legs of a simple right-triangle.) Since this mathematical interpretation is linked to a set of operations (first multiplication and division, then squaring the lengths with subsequent additions or subtractions of the results), the

²³

trairāśikenāpi 13/ 12/ 3438/ labdham avalambakaḥ 3174/

Shukla 1976, 90.

²⁴This value is an approximation again.

unexplained steps of the procedure are given a mathematical grounding that may serve as a justification of the algorithm itself.

This analysis thus brings to light two kinds of reasoning: the confirmation of a result by using two independent procedures and the mathematical grounding of a set of operations via their ‘re-interpretation’ according to the Rule of Three and/or the ‘Pythagoras Theorem’. These kinds of mathematical reasoning are also found in the parts of BAB which explicitly have a persuasive aim, attempting to convince the reader that the algorithms of the Ab are correct.

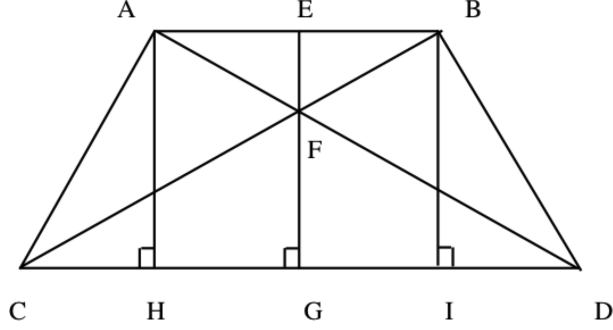
3 Explanations, Verifications and Proofs

Bhāskara uses specific names when referring to a number of arguments: ‘explanations’, ‘proofs’ (*upapatti*) and ‘verifications’ (*pratyāyakaraṇa*). These arguments do not appear systematically in each verse commentary and – as will be seen below – are always fragmentary. The following description of explanations, proofs and verifications will attempt to highlight how they are structured and the different interpretations they can be subject to.

3.1 Explanations

Bhāskara’s commentary on verse 8 of the mathematical chapter of the Ab presents an example of explanation. Verse 8 describes two computations concerning a trapezoid. (See Figure 3.1.) The first calculation evaluates the length of two segments (*svapātalekha*, EF and FG) of the height of a trapezoid. In this case, the height is bisected at the point of intersection

Figure 6: Inner segments and fields in a trapezoid



for the diagonals. The procedure is made of a multiplication followed by a division²⁵:

Ab.2.8. The two sides, multiplied by the height ⟨and⟩ divided by their sum are the ‘two lines on their own fallings’.
When the height is multiplied by half the sum of both widths, one will know the area.||

In other words, with the labels used in Figure 3.1, we have:

$$EF = \frac{AB \times EG}{AB + CD};$$

$$FG = \frac{CD \times EG}{AB + CD}.$$

Likewise, the area $\mathcal{A}$ is:

$$\mathcal{A} = EG \times \frac{(AB + CD)}{2}.$$

25

āyāmaguṇe pārśve tadyogahṛte svapātalekhe te|
vistarayogārdhaguṇe jñeyam kṣetraphalam āyāme||

(Shukla 1976: 63).

On the first part of the verse, Bhāskara comments²⁶:

The size of the ‘lines on their own fallings’ should be explained (*pratipādayitavya*) with a computation of the Rule of Three on a figure drawn by ⟨a person⟩ properly instructed. Then, by means of just the Rule of Three with both sides, a computation of ⟨the lines whose top is⟩ the intersection of the diagonals and a perpendicular ⟨is performed⟩.

This explanation consists of ‘re-interpreting’ the procedure—which is a multiplication followed by a division—according to the Rule of Three. The explanation contains two steps. The first step considers the proportionality in a diagram, then ‘re-interprets’ the set of operations of the algorithm as the application of the Rule of Three. As previously, the seemingly arbitrary set of operations is endowed with a mathematical meaning.

The second computation in verse 8 determines the area of the trapezoid. As shown in Figure 3.1, the area of the trapezoid can be broken into the summation of the areas of several triangles. Alternately, the trapezoid can be decomposed into a rectangle and two triangles.

Although no figure is explicitly drawn, Bhāskara seems to have such a diagram in mind. Indeed, he seems to refer to such a drawing when he writes²⁷:

26

samyagādiṣṭena ālikhite kṣetre svapātalekhāpramaṇaṃ
trairāśikagaṇitena pratipādayitavyam/ tathā trairāśikenaivobhaya pārśve
karṇāvalambakasampātānayanam/

Shukla 1976: 63.

27

Here, with a previous rule [Ab.2.6.ab] the area of isosceles and uneven trilaterals should be shown/explained (*darśayitavya*). Or, with a rule which will be stated [Ab.2.9.] the computation of the area of the inner rectangular field ⟨should be performed⟩;

Even though it has not survived, such a figure shows how areas can be added to give the area of the trapezoid. This time, a collection of already known procedures, those computing the area of triangles and rectangles, is mobilised. We do not know if they are used to ‘re-interpret’ the procedure or to establish an alternative independent procedure. The procedures of Ab.2.9 will be analyzed below.

Both of Bhāskara’s explanations in BAB.2.8 consist of

1. an explanation of a diagram, and
2. either a ‘re-interpretation’ of the procedure or exposing an independent alternative procedure. This ‘re-interpretation’ either confirms or verifies the reasoning by looking at a diagram.

Three words refer to an explanation: *vyākhyāna*, *pradarśaṇa*, and *pratipādita*. The word *vyākhyāna* indicates that the commentary gives an explanation, but it is also used for an argument connected with a diagram²⁸:

Or else, all the procedures ⟨used⟩ in the production of chords are in the realm of a diagram, and a diagram is intelligible ⟨only⟩

pūrvasūtreṇātra dvisamaviṣamatryaśrakṣetrāphalaṃ darśayitavyam/
vakṣyāṇasūtreṇāntarāyatacaturaśrakṣetrāphalānāyanam (. . .) vā/

Shukla 1976: 63.

²⁸*athavā jyotpattau yatkarāṇaṃ tatsarvaṃ chedyakaviṣayaṃ, chedyakaṃ ca vyākhyānagamyamiti na pratipāditam/* (Shukla 1976: 79).

with an explanation (*vyakhyāna*). Therefore it has not²⁹ been put forth (*pratipādita*) (by Āryabhaṭa in the *Āryabhaṭīya*).

Note that this passage emphasizes that explanations belong to the genre of commentary and, at least according to Bhāskara, should not be exposed in a treatise.

The word *pradarśana* is derived from the verbal root *dr̥ṣ-*, ‘to see’. It has a similar range of meaning as the English verb ‘to show’. It is often hard to distinguish if the word refers to the visual part of an explanation or to the entirety of the explanation. For instance, in BAB.2.11, Bhāskara uses a diagram and writes³⁰:

In the field drawn in this way all is to be shown/explained
(*pradarśayitavya*).

Finally, the word *pratipādita* is more technical and straightforward. It commonly appears in lists of solved examples found in most of the commented verses in the mathematical part of BAB.

In the illustrations of explanations presented above, the commentator ‘re-interprets’ geometrical procedures according to the Rule of Three or the ‘Pythagorean Theorem’. Only geometrical procedures receive such arguments. Each time, the commentary omits a diagram to which the text seemingly refers. Among the geometrical processes, explanations are ‘seen’, as

²⁹*na* has been added by the editor, K. S. Shukla and is not found in the manuscripts. Another possible interpretation of the sentence reads: ‘Therefore it has been put forth (by Bhāskara in his commentary)’

³⁰*evam ālikhite kṣetre sarvaṃ pradarśayitavyam* (Shukla 1976: 79).

will be seen in the only example from the BAB in which the word ‘proof’ occurs.

3.2 The Only Two Occurrences of the Word “Proof”

The Sanskrit word *upapatti* refers directly to a logical argument. This word is used twice in Bhāskara’s commentary as noted by Takao Hayashi³¹. The gender of this word is feminine and it is derived from the verbal root *upa-PAD-*, meaning ‘to reach’. Thus, an *upapatti*, is literally ‘what is reached’ and has consequently been translated as ‘proof’. In both instances, some ambiguity surrounds this word, and the meaning of the word is not certain. One occurrence has been quoted above, wherein proofs (*upapatti*) are described as opposed to tradition. The other instance refers to the reasoning whereby the height of a regular tetrahedron is determined from its sides. In this case, Bhāskara understands Āryabhaṭa’s rule in the second half of verse 6 of the mathematical chapter as the computation of the volume of a regular tetrahedron. Such a situation is described in Figure 7.

Given a regular tetrahedron $ABCD$, AH is the line through A perpendicular to the plane defined by the triangle BDC . AH is called the ‘upward-side’ (*ūrdhvabhujā*). AC is called *karṇa* (lit. ear) because it is the hypotenuse of AHC . Bhāskara explains how to compute the upright side by using the Pythagorean Theorem and the Rule of Three. The determination of CH , from which the upright side AH may be computed, rests upon the proportional properties of similar triangles, illustrated in Figure 8. The triangles

³¹Hayashi 1985: 75-76.

Figure 7: An Equilateral Pyramid with a Triangular Base

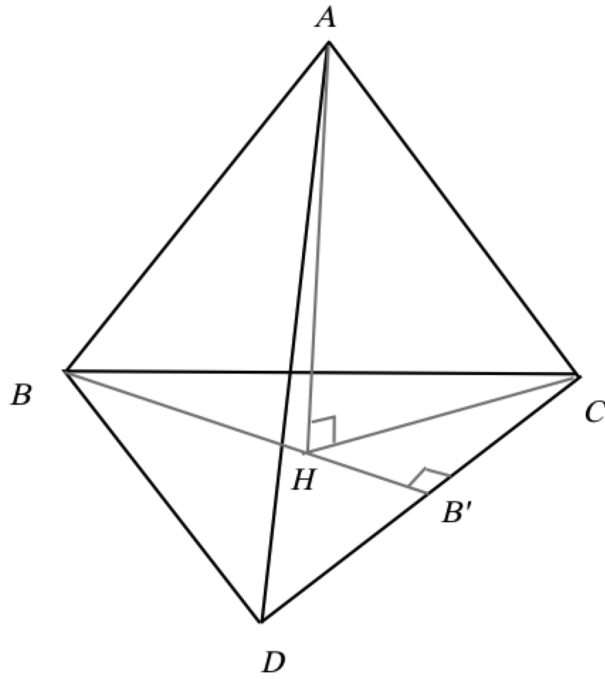
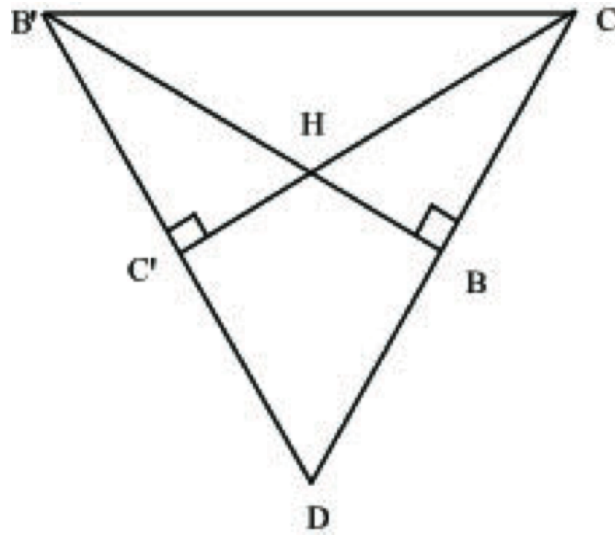


Figure 8:



$BB'C$ and $B'CH$ are similar:

$$BB' : CB = CB' : CH.$$

From this relationship it is known that:

$$CH = \frac{CB \times CB'}{BB'}.$$

Bhāskara expressed this relationship as the Rule of Three. The text does not give a precise argument, but it alludes to the properties as being clear from a diagram. It is in this context that the word *upapatti* appears³²:

In order to show the proof (*upapatti*) of ⟨that⟩ Rule of Three, a field is set-down.

The argument implied by this word depends on the diagram. As in the case of the explanations, the proof must have been presented orally. This situation differs from the acts of ‘re-interpretation’ seen above. In the present case, an argument is created, and there is no pre-existing algorithm to ‘re-interpret’. However, the foundations of this new argument are set out in a diagram. Furthermore, the procedure used is the Rule of Three, as in the ‘explanations’ seen above. Another type of argument concerns the correctness of algorithms: verification.

3.3 Verification

Verifications are distinguished from explanations and proofs by their name, *pratyayakarāṇa*. Indeed, *pratyaya* has an etymological root in a verb meaning ‘to come back’, which has connotations of conviction. *Pratyayakarāṇa*

³²*trairāśīkopapattipradarśanārthaṃ kṣetranyāsaḥ*-(Shukla 1976: 59).

thus means ‘enabling to come back’ or ‘producing conviction’. Historians of Indian mathematics usually understand this word as a type of verification and translate it accordingly.³³

A verification resembles an explanation in that a verification ‘re-interprets’ a given procedure according to another rule and establishes a mathematical grounding. The arguments that the commentator labels “verifications” sometimes present difficulties, and currently our understanding of them is not at all certain. Below are set out several hypotheses about how these verifications can be understood.

3.3.1 Verification of an Arithmetical Computation

Bhāskara states a verification by the Rule of Five for the rule given in Ab.2.25.

Āryabhaṭa states the rule in Ab.2.25 as follows³⁴:

The interest on the capital, together with the interest ⟨on the
interest⟩, with the time and capital for multiplier, increased
by the square of half the capital|

The square root of that, decreased by half the capital and divided
by the time, is the interest on one’s own capital||

This passage can be formalized as follows: Let m (*mūla*) be capital; let p_1 (*phala*) be the interest on m during a unit of time, $k_1 = 1$ (*kāla*), usually

³³Hayashi 1995:73-74.

³⁴

mūlaphalaṃ saphalaṃ kalamūlaguṇam ardhāmūlakṛtiyuktam|
tanmūlaṃ mūlārdhonaṃ kālahṛtaṃ svamūlaphalam||

(Shukla 1976: 114).

a month. Let p_2 be the interest on p_1 at the same rate for a period of time k_2 . If $p_1 + p_2$, m , and k_2 are known, the rule can be expressed in a modern mathematical notation as:

$$p_1 = \frac{\sqrt{mk_2(p_1 + p_2) + (\frac{m}{2})^2} - \frac{m}{2}}{k_2}.$$

This rule is derived from a constant ratio:

$$\frac{m}{p_1} = \frac{p_1}{p_2} k_2.$$

The Rule of Five, described in BAB.2.26-27.ab, rests on the same ratio as the rule given in Ab.2.25. In the former instance though, k_1 may be a number other than 1:

$$\frac{m}{p_1} k_1 = \frac{p_1}{p_2} k_2.$$

The Rule of Five indicates an expression equal in value to p_2 :

$$p_2 = \frac{p_1^2 k_2}{m k_1}$$

The Rule of Five may therefore be used in the opposite direction to find a value for p_1 .

In BAB.2.25 Bhāskara gives an example³⁵:

1. I do not know the ⟨monthly-⟩interest on a hundred. However,
the ⟨monthly-⟩interest on a hundred increased by the interest
⟨on the interest⟩ |

35

*jānāmi śatasya phalam na ca kintu śatasya yatphalam saphalam | māsaṅ
caturbhiḥ āptam ṣaḍ vada vṛddhim śatasya māsothām||*

Obtained in four months is six. State the interest of a hundred
produced within a month||

This example states a case in which:

$$\begin{aligned} m &= 100 \\ k_2 &= 4 \\ p_1 + p_2 &= 6 \end{aligned}$$

By the procedure given in Ab.2.25, the value of p_1 is 5.

Bhāskara then adds³⁶:

Verification (*pratyayakaraṇa*) with the Rule of Five : “If the monthly
interest (*vṛddhi*³⁷) on a hundred is five, then what is the interest of
the interest [of value (*dhana*)-five] on a hundred, in four months?”

$$\begin{array}{cc} 1 & 4 \end{array}$$

Setting down: 100 5 The result is one. This increased by the

$$\begin{array}{cc} 5 & 0 \end{array}$$

⟨monthly⟩ interest on the capital is six *rūpas*, 6.

Simply stated, the verification consists of knowing m , p_1 and k_2 , finding
 p_2 and confirming that its value increased by p_1 will give the same value for
 $p_1 + p_2$ as stated in the problem.

³⁶

*atyayakaraṇam pañcarāśīkena yadi śatasya māsikī vṛddhiḥ pañca tadā
caturbhiḥ māsaiḥ śatavṛddheḥ [pañcadhanasya] kā vṛddhiḥ iti/ nyāsaḥ -*

$$\begin{array}{cc} 1 & 4 \end{array}$$

100 5 labdham 1 / etatsahitā śatavṛddhiḥ ṣaḍ rūpāṇi 6/

$$\begin{array}{cc} 5 & 0 \end{array}$$

³⁷From now on, unless otherwise stated this is the word translated as ‘interest’.

The Rule of Five, as seen above, returns the value of p_2 . This procedure does not deliver the same result but gives a method of inverting the procedure to check independently that the result makes sense. In this case, an independent procedure is established. The use of the Rule of Five, which Bhāskara describes as a combination of two Rules of Three, also imbues the computation with a mathematical basis in proportionality.

3.3.2 Verification of the Area of Plane Figures

Bhāskara interprets the first half of Ab.2.9 as a way to verify procedures for areas given by Āryabhaṭa in the previous verses.

For all fields, when one has acquired the two sides, the area
is their product |³⁸

Bhāskara endows the verse with the goal of ‘verification’—a goal nowhere explicitly appearing in the verse itself. Two steps can be distinguished in the verifications of this verse commentary, each corresponding to a diagram. The first step constructs a diagram of the figure for which an area is verified. The length and width of a rectangle with the same area as the figure are identified. This ‘length’ and ‘width’ are usually values from Āryabhaṭa’s procedure for which verification is sought. For instance, to verify the area of a triangle, the length of the corresponding rectangle is identified as the height of the triangle, while the width of the rectangle is half the base of the triangle. Precisely, the area of a triangle is given elsewhere by Ab- in the first half of verse 6- as the product of half the base by the height of a triangle.

³⁸*sarveṣāṃ kṣetrāṇāṃ prasādhya pārśve phalaṃ tadabhyāsaḥ* | Shukla 1976: 66.

The second step of the argument presents a diagram of the rectangle and computes the multiplication.

How should this argument be understood? According to one means of understanding, this argument is a formal interpretation. The reasoning would consist of considering the rule one seeks to verify as the multiplication of two quantities. Each quantity is then interpreted geometrically as either the length or width of a rectangle with the same area as the initial figure. In this way, Bhāskara calculates the length and height of the rectangle, as required by verse 9.

Another way of understanding the argument begins with the fact that the verification for a given figure *produces* a rectangle of the same area as the given figure. The fact that all figures have a rectangle with the same area would then become an implicit assumption of Sanskrit plane geometry. T. Hayashi has interpreted this argument in such a manner.³⁹ The reasoning would produce a rectangle and verify that its area is equal to the area of the figure.

A third approach relies on the ‘setting down’ parts which contain diagrams. Such a verification consists of *constructing* a rectangle with the same area from a given figure. For instance, in the second step of the verification of the area of a triangle, Bhāskara specifies that when the parts of the area of such a triangle are rearranged (*vyasta*), they produce the rectangle which is drawn. The construction of a rectangle from the original figure is not described in Bhāskara’s commentary. However, such constructions could have been known, as shown by the methods exposed in BAB.2.13. Further-

³⁹Hayashi 1995: 73.

more, this process recalls the algorithms from the *sulbasūtras*, the earliest known texts of Sanskrit geometry. These algorithms produce a construction which, although not described in the text, corresponds with the discussion contained in the text. With just such a diagram, the argument in the text would arithmetically verify that the construction is correct.

These three interpretations can be combined if a verification is allowed to be simultaneously geometrical and arithmetical. Bhāskara relies on a geometrical strategy to produce a rectangle with the appropriate area, showing that he knows how to construct the corresponding rectangle from the initial figure. Because the construction is obvious, it would not be detailed, and only the lengths of the rectangle would be given. From an arithmetic perspective, this ‘re-interpretation’ provides a new understanding of the rule given by Āryabhaṭa. Through his arithmetical ‘verification,’ Bhāskara explains the geometrical verification. Bhāskara explains the link between the sides of the initial figure and the lengths and widths of the rectangle with the same area as the initial figure.

Regardless of which interpretation is accepted, the verification either ‘re-interprets’ a first algorithm (BAB.2.9) and produces a new understanding of the procedure, or it produces a new procedure that gives the same result (BAB.2.25). In either case, the so-called ‘verification’ confirms the numerical results and places the procedure in a secure mathematical context. Thus, after verification, the calculations do not appear to be a set of arbitrary steps.

Conclusion

This survey of the BAB has brought to light two kinds of reasonings checking the Ab rules and seeking to convince readers of their validity. One argument exhibits an independent alternative procedure. In one case the procedure exhibited arrives at the same result as the opposite direction procedure. The second reasoning, we have called it ‘re-interpretation’, uses the Rule of Three and the ‘Pythagoras Theorem’ to provide a new outlook onto the arbitrary steps of the procedure. How should the Rule of Three and the so-called ‘Pythagoras Theorem’ be described in this context? They are mathematical tools which enable astronomical situations or specific problems to be ‘re-interpreted’ as abstract and general cases, involving right triangles and proportionalities. The arbitrary steps of the procedure are thus given a mathematical explanation.

Nonetheless, the methods of reasoning are hard to understand and pin down. This difficulty may arise from their oral nature, of which Bhāskara’s written text preserves only a portion. For instance, the function of diagrams in these reasonings still remains mysterious. Further detailed explorations of how Sanskrit texts explain, prove and verify mathematical algorithms will advance understanding about how the mathematical correctness of algorithms was conceptualized by mathematicians in the Indian subcontinent.

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