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Many pieces of evidence converge towards the conclusion that generality was the main theoretical value prized by the practitioners of mathematics in ancient China and that it was valued more than abstraction (Chemla 2003). More precisely, these scholars regularly aimed at the greatest generality possible, but did not always achieve or express it through abstract terms. This does not mean, however, that abstraction played no role for them and that they did not find it necessary in some cases to carry out operations of abstraction. In fact, the earliest Chinese mathematical sources handed down through the written tradition, *The Gnomon of the Zhōu* (*Zhōubì* 周髀, probably 1st century C. E., Cullen 1996), *The Nine Chapters on Mathematical Procedures* (*Jiǔzhāng suànshù* 九章算術, below: *The Nine Chapters*; probably 1st century C. E.),[1] as well as their commentaries, bear witness to several uses of abstraction. On the other hand, the relationship between generality and abstraction that they evince differs from what can be found in, say, Euclid's *Elements*.[2] In correlation with this, both the generality

[1] My introduction to chapter 6 in (Chemla&Guo 2004) summarizes my arguments in favor of this dating. In chapter B, my co-author, Prof. Guō Shūchūn, gives arguments in favor of an earlier dating, placing the end of the compilation of *The Nine Chapters* towards the end of the first century B.C.E. In what follows, I shall rely on our joint critical edition and translation of the Classic and its commentaries published in this book. It is my pleasure to wholeheartedly thank the editors of the volume as well as Irfan Habib and Ben Marsden for their generous help in polishing the English of the paper.

[2] In this respect, let us recall that, for the geometer Michel Chasles who, in his *Aperçu historique sur l'origine et le développement des méthodes en géométrie* (1837), developed a history of geometry that takes generality as the main theoretical value, the mathematical objects with which Euclid dealt were 'concrete'.

achieved and the abstractions carried out in the two contexts are not all of the same type.

The contrast provides a most interesting basis for further inquiry into these two ways of theoretical endeavor. Clearly, however, any comprehensive inquiry of this kind exceeds what can humanly be addressed within the context of a paper. As a contribution to the launching of this discussion, therefore, I shall limit myself here to showing how the new evidence yielded in China by excavated manuscripts allows us not only to observe the use of abstraction, but also to document *processes* of abstraction in the mathematics of ancient China. In 1984, a book devoted to mathematics was found in tomb 247, sealed in ca. 186 B.C.E., at Zhāngjiāshān 張家山 in Húběi province (Peng Hao 2001). Through the differences it shows compared with the texts handed down through the written tradition, this *Book of Mathematical Procedures* (*Suànshùshū* 算數書) highlights the modalities according to which abstraction was carried out in Hàn China. I shall describe part of what appear to me the most striking pieces of evidence in this respect. In addition to allowing us to examine how, in ancient China, abstraction was used in the context of mathematics, analysis of these paragraphs in the new source material provides useful insight regarding the relationship between the excavated material and what was handed down through the written tradition.

The statement of the problem

The *Book of Mathematical Procedures* (*Suànshùshū* 算數書), which was excavated from a tomb sealed in ca. 186 B.C.E., predates the source material that has been handed down through the written tradition, such as *The Nine Chapters on Mathematical Procedures*, by at least two centuries. A clear reflection of the mathematics needed by an administration in charge of taxation and other managerial matters (Peng Hao 2001:4-12), the book deals with mathematical topics that are for the most part to be found in *The Nine Chapters*. In fact, the two writings share much in the way of subject matter (grain, customs, to mention just a few), mathematical concepts (that of fractions, for instance), and practices (computing on a surface with rods, mixing abstract and measured numbers in computations, using paradigms). These elements suggest that the books belonged to the same tradition, even though they may have been written down for different purposes and hence for different uses, perhaps even in different milieux. More precisely, the distribution of similarities and dissimilarities between them is such that it seems difficult to consider the *Book of Mathematical Procedures* to have been one of the documents on the basis of which *The Nine Chapters* was later to be simply compiled. In fact, we shall see that the new evidence available raises questions regarding the authorial acts that produced the text of *The Nine Chapters*. Whether *The Nine Chapters* stands in an ascendant-descendant relationship to the *Book of Mathematical Procedures,* or the two writings simply present family resemblances, this question must be kept in mind when discussing the new possibilities that the discovery of the *Book of Mathematical Procedures* has opened up for grasping processes of transformation in the mathematics of ancient China. Conversely, these discussions may help resolve these unclear matters.

One difference between the two books is striking, in relation precisely to the topic of this paper. Despite the fact that they are both mainly composed of problems and algorithms,[3] *The Nine Chapters* makes much more use of abstraction than the *Book of Mathematical Procedures*.[4] I do not mean, by this statement, to contradict my introductory remarks about the relatively minor role played by abstraction in the mathematics of Ancient China. Abstraction is not to be met with everywhere in *The Nine Chapters*. On the contrary, the authors regularly expressed general statements in the form of paradigms. However, abstraction is on the whole much more present, and in a specific way. This fact hence calls for a description, to account for why and how abstraction was carried out.

In what follows, I shall provide examples to back this statement and deal with these issues on the basis of three intimately related examples: the division between quantities of the kind 'integer increased by a fraction';[5] the rule of three; determining the standard price. I could have chosen other, simpler examples, but they would not have displayed the wealth of transformations that allows us to go beyond merely taking note of a change.

A specific process of abstraction

Interestingly enough, the name of the operation for dividing between integers (possibly) increased by fractions is the same in both books: 'Directly

[3] The relationship between problems and algorithms is different in the two books and would require a separate treatment.

[4] This remark regarding abstraction struck many present-day commentators on the *Book of Mathematical Procedures* (Horng 2001, Guo Shuchun 2004). Let me stress that the remark holds true in general, which does not mean that this is the case on every count. See the entry on *lèi* 類 'category', in my glossary (Chemla&Guo 2004:948-949) - below to be referred to as *Glossary*. (Horng 2001) also has reservations, even though they are of a different nature, I come back to his main point below. In this respect, the overall comparison I would make differs from that of (Guo Shuchun 2004). My aim here is to go beyond noticing the use of abstraction.

[5] The fact that this is the main concept of quantity occurring in the *Book of Mathematical Procedures* is yet another hint of its close relationship to the tradition to which *The Nine Chapters* also belongs.

sharing' (*jìngfēn* 徑分).[6] Moreover, the problems in relation to which the procedure for executing the operation is described also employ the same type of situation: dividing cash between persons. The section bearing this title in the *Book of Mathematical Procedures* can be interpreted as follows:[7]

> 徑分　徑分以一人命其實，故曰： 五人分三有(又)半、少半，各受卅（三十）分之廿（二十）三。其術曰：下有少半，以一為六，以半為三[8]， 以少半為二，// 并之為廿（二十）三，即值(置)一[9]數，因而六之以命其實。[10]
> […]
> Directly sharing: When directly sharing, one takes each unit of person to be given the name of that (the unit) of the corresponding dividend.[11] This is why

[6] In fact, this is the name as given by the *Book of Mathematical Procedures*. *The Nine Chapters* has the phonetically equivalent *jìngfēn* 經分, which, in his gloss, the Táng commentator Lǐ Chúnfēng interprets as 'Directly sharing' (*jìngfēn* 徑分).

[7] Bamboo slips 26-27 (Peng Hao 2001:46). In what follows, unless otherwise mentioned, I rely on the critical edition of the *Book of Mathematical Procedures* provided in Peng Hao 2001. Guo Shirong (2001), Guo Shuchun (2001), and Cullen (2004) also provide critical editions, based on this book or related publications by the research group working on Zhāngjiāshān source material. Cullen (2004) and Dauben (forthcoming) give an English translation of the text. Note that a similar problem is to be found in the context of adding up fractions (bamboo slips 23-24). This may be explained by the fact that most of the problems solved by such division have data consisting of an integer increased by two fractions and hence relate to the addition of fractions. The same remark holds true when, in the context of a different kind of situation (given the area of a rectangle and its width, compute its length), the same mathematical problem is taken up again under the title 'Small width' (slips 164-181). Note that, in all these cases, the fractions are all unit fractions. Is this a hint of an older algorithm for division? This question would be worth pursuing. On 'Small width' and comparable algorithms, see the introduction to chapter 4 in Chemla&Guo 2004.

[8] Peng Hao 2001:4, fn. 3, indicates that the '一 (one)' that the manuscript contains here is a mistake for '三 (three)'. This suggestion is adopted by Guo Shirong (2001:277), Guo Shuchun (2001:205), and Cullen (2004:120, fn. 11). I follow them.

[9] Guo Shuchun (2001:205) and Cullen (2004:120, fn. 13) suggest that '一 (one)' needs to be emended into '人 (person)'. Like Péng Hào and Guō Shìróng, I take the text transmitted as correct. There are reasons to believe that this procedure is parallel to the one described below under the title 'Determining the standard price on the basis of the *shí lǜ*' (石率). In this other context, a similar use of the unit is attested to. See the discussion of this point below.

[10] Peng Hao 2001:49, fn 4, considers that the text presents here an omission: "以人數為法，實如法而一" ("One takes the quantity of persons as divisor, the dividend is divided by the divisor"). If one compares to slip 24, one may rather suggest that, after "并之為廿（二十）三", "為實" was omitted. The whole sentence would hence read: "One adds these up to make 23, which makes the dividend." However, here no division is necessary. The way in which one emends the text depends on one's conception of the generality of the procedure. Guo Shuchun (2001:205) considers that the section regarding division in bamboo slips 23-24 was ill-placed by the copyist and should be inserted here.

[11] The key technical word here is *mìng* 命. It recurs below and on bamboo slip 140 with a meaning that in my view must be distinguished from, even if it relates to, the ordinary technical meaning of 'naming' the units of the remaining dividend with the divisor taken as

> one says: if 5 persons share 3 and one half and one third, each receives 23/30. Corresponding procedure: In the lowest (row), there is one third,[12] one hence takes 1 as[13] 6, one takes one half as 3, one takes one third as 2, one adds these up to make 23. Hence[14] one places the quantity of units and on this basis one multiplies it by[15] 6 so that they be given the name of that of the corresponding dividend.[16]

The procedure hence first expresses all the units entering in the quantity of cash (1 cash, one half of cash, one third of cash) with respect to the same unit —an *abstract* one—, which allows them all to be transformed into integers and hence added up. It then carries out the same transformation of the unit with respect to which the quantity of persons is expressed. As a consequence, the division to be carried out is that of an integer by an integer.

'denominator' (see *Glossary*). Here my temporary conclusion is that it refers to the fact that the units of one quantity are to be modified as a consequence of the modification of the units of a related quantity. Cullen (2004:43) interprets the term here and below as 'counting off'.

[12] On the surface on which computations are carried out, evoked later on by the verb *zhì* 置 'to place', this 'lowest (row)' refers to the lowest sub-row in the position in which the dividend is placed. Since the description of the procedure makes sense with this interpretation, conversely, we have here a hint that the principles for using the surface for computing (i.e., the ways of placing quantities or terms of operations) that can be reconstructed on the basis of later texts were already in use at the time of the *Book of Mathematical Procedures*. One is thus incited to put forward the hypothesis that, like later, when a given quantity composed of an integer and various fractions was placed in a given position of the surface for computing, fractions were put, each in a row, below the integers, the lower the row, the greater the denominator. As a consequence, in the paradigm considered, ½ is in a row below, 3, and 1/3 is placed in the row further below. Moreover, by the same token, one is led to assume that, at a higher level, the dividend was placed in the middle position whereas the divisor was put in the lower position. See the introduction to Chapter 4 in (Chemla&Guo 2004).

[13] Such expressions that reflect the change of unit are quite common in the *Book of Mathematical Procedures*. Note the key fact that the unit is expressed here as abstract. This change of unit has the property that all components in $3 + 1/2 + 1/3$ are transformed into integers simultaneously.

[14] Note the stress on the fact that a computation logically derives from another one. The unit of the dividend having been changed, the transformation of the unit of the divisor is given as following from this. (Horng 2001) notes that such logical connectors are much more numerous in the *Book of Mathematical Procedures* than in *The Nine Chapters*. He concludes from this that the former book is much more theoretical than the latter. Even though I believe that he is right in stressing this feature of the *Book of Mathematical Procedures*, I think it should be placed in a context of similar remarks regarding the two books, before one draws such a conclusion. This paper develops part of the background against which Horng's thesis could be reformulated.

[15] Literally, one 'sextuples' it, an expression which, in contrast to that of using *chéng* 乘 ('multiply'), refers to a multiplication that derives from a change of unit.

[16] Here, one could also interpret *míng qí shí* 名其實 as 'to name the dividend', since the divisor is to become the denominator corresponding to the dividend taken as numerator. I prefer an interpretation that conforms to that for the beginning of the text. This is how 'the quantity of units' is to be interpreted in relation to persons, without having to consider the text as corrupted.

However, in the paradigm given, the result is obtained by the mere statement of a fraction.[17] Even though the procedure bearing the same name in *The Nine Chapters* can be interpreted —and is actually interpreted by the 3rd century commentator Liú Huī— as involving exactly the same operations,[18] it is described in a markedly different way. It is found in chapter 1 and reads:

> 經分 術曰：以人數爲法，錢數爲實，實如法而一。有分者通之； 重有分者同而通之。
>
> Directly sharing Procedure: One takes the quantity of persons as divisor, the quantity of cash as dividend, and one divides the dividend by the divisor. If there is one type of part, one *makes* them *communicate*. If there are several types of parts, one *equalizes* them and hence *makes* them *communicate*. (Chemla&Guo 2004:166-169)

We need not enter into the details of the interpretation here.[19] Suffice it to note that in contrast to the previous text analyzed, the procedure does not

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[17] The procedure described in bamboo slips 23-24 can be interpreted as follows: "Since, below there are thirds, one takes 1 as 6. Hence, on this basis one sextuples persons, which is taken as divisor, one sextuples *also* cash, which is taken as dividend". The description of how to shape a dividend and a divisor is correlated to the fact that the paradigm in relation to which the procedure is described is solved by division. Moreover, note that the related transformation of the quantities of persons and cash is here marked by an 'also'. It is carried out in both cases by a multiplication by 6 that is given as translating the change of unit. The change of abstract unit brings about a change in the units of cash and persons, respectively. Lastly, this use of 'also' to mark parallel, correlated, computations in a procedure is quite frequent in the *Book of Mathematical Procedures* precisely in cases similar to this one. It represents hints of a metadiscourse on procedures.

[18] Let us derive three observations here from the fact that the translation of the algorithm of *The Nine Chapters* in operational terms, offered by Liú Huī, matches the procedure as described in the *Book of Mathematical Procedures*. First, as regards the procedure referred to in *The Nine Chapters*, the commentator's exegesis is here in agreement with Hàn documents. This, together with many other hints, seems to indicate that commentators relied on other source material to compile their exegesis of the Canon. Secondly, this fact confirms that the interpretation Liú Huī gave for the procedure, which I followed, is documented to have been a procedure actually used in Hàn times. One should add that, in the *Book of Mathematical Procedures*, two other algorithms are given for carrying out the same operation. One was mentioned above (Peng Hao 2001:164-181) and relates to the procedure having the same title in *The Nine Chapters*: 'Small width.' The other procedure (Peng Hao 2001:113-114, bamboo slips 159-163) is not found in *The Nine Chapters,* but was added to it by Liú Huī in the context of his commentary. Consequently, there were several possible algorithms for carrying out the same operation in ancient China. Yet, those entitled 'Directly sharing' in the two books clearly correspond to each other, and this is also the identification carried out by the commentators. Thirdly, this confirms that the operation that led to the description as found in *The Nine Chapters* was obtained through a process of abstraction on the basis of the procedure with the same name which is to be found in the *Book of Mathematical Procedures*. The exegesis somehow bridges the gap between the two states of description of the procedure, developing further the theoretical dimensions introduced in *The Nine Chapters*.

[19] The reader is referred to the notes to the French translation (Chemla&Guo 2004:766-767).

mention the actual value of the greatest denominator. Moreover, it clearly articulates the several cases that can be encountered.[20] What is more important for us here, the description prescribes the operations to be carried out using verbs that I placed in italics and that are highly abstract: 'make communicate', 'equalize'. Since, as we argued above, the computations described are the same, this means that this description derives from an operation of abstraction on the basis of the former. How was this abstraction carried out?

First, let us focus on the abstract terms chosen and the kind of entity they identify. The choice of terms implies that in the process of division, the authors of the procedure recognized general operations that are at play everywhere in the cosmos, and not only in mathematics. By using abstract terms to refer to the computations carried out, they highlighted the general transformations that were effected when they were applied. A first result of the abstraction carried out is hence to reveal how the transformations involved in mathematics can be apprehended through concepts that can be relevant for any other transformation. In fact, the commentators will show that in several distinct procedures one can read the aim of 'making' mathematical realities of various categories 'communicate' - in other words, they show that the operations carried out are efficient by the very fact that they make realities communicate. Links are thereby established between mathematics and the rest of the cosmos, but also, within mathematics, between different procedures. Such is the kind of generality achieved by the use of the abstract term of 'making communicate', which does not occur in the *Book of Mathematical Procedures*.

Secondly, to go back to the division, let us examine how the abstract terms relate to the computations to be performed. The operations on the dividend and the divisor that were described separately in the procedures of the *Book of Mathematical Procedures* are now prescribed together, as distinct facets of 'making communicate'. This is a second way in which the abstraction carried out introduces a form of generality, specific to operations. The commentator Liú Huī analyzes that this term captures several ways in which the procedure 'makes' entities 'communicate'. Once the fractions are all reduced to the same denominator - this is what 'equalizing' accomplishes -, the integers are 'made to communicate' with the fractions following them, by the fact that their units are disaggregated in as many parts as the common denominator has (in the example of the *Book of Mathematical Procedures*, examined above, this corresponds to transforming 1, 1/2 and 1/3 into 6, 3 and 2, respectively). They are thus all transformed into integers

[20] These are, successively: dividing an integer by an integer; dividing between quantities in which there appears one type of fraction; dividing between quantities in which there appear more than one type of fraction. The different paradigms to be found in the *Book of Mathematical Procedures* illustrate the last case.

with respect to the same unit and can be added to each other. Moreover, the quantities of cash and men, Liú Huī notes, are 'brought into relation' (*xiāngyǔ* 相與) by the situation, and hence in the operation of division. This entails - the commentator continues - that they 'are made to communicate', any modification in the units of one thus having to be reflected in those of the other for the relation to be maintained. This property is adduced to account for the validity of the simultaneous transformation of their values (what corresponds to the operation, carried out above, of taking 1 as 6 for both the dividend and the divisor).[21] Thus this second transformation is also approached as, and encapsulated into, 'making communicate'. It is interesting to see how the term of 'making communicate' simultaneously prescribes a set of operations, which the commentator dissociates and explains as a matter of bringing all the relevant entities into communication.

In this context, to designate with full generality such quantities that 'are in relation with each other' - a state of affairs entailing the property of 'communicating' - , the commentator introduces an abstract term that, in *The Nine Chapters*, will constitute one of the main themes of chapter 2: *lǜ* 率.[22] This is the essential point that makes this procedure for division important for our purposes: it appears to be carried out in a specific way that can be captured by a key theoretical term and that makes it similar, precisely in this respect, to other procedures. On this basis, more generality is to be achieved.

A second manifestation of phenomena later linked to lǜ

In the context of the commentary regarding division, *lǜ* appeared to designate the property of quantities that caused their 'being in communication'. In the related procedure of the *Book of Mathematical Procedures*, the manifestation of what was to be later designated as *lǜ* took the shape of correlated transformations of different entities in a computation. In Chapter 2 of *The Nine Chapters*, the term *lǜ* is used to prescribe the rule of three and a procedure that, in *The Nine Chapters*, is presented as deriving from it: 'Directly determining the standard price'. Both algorithms are to be found in the *Book of Mathematical Procedures*, but again described - and this time entitled - in a completely different way. Let us therefore examine these differences to see the part played by the concept of *lǜ* in shaping them

[21] As a consequence, their values are transformed in relation to each other. They can become, in correlation with each other, abstract integers and even integers that are relatively prime with each other.

[22] I prefer not to translate the term and use the transcription *lǜ*, because I cannot find any term that, for all occurrences, could do justice to the concept it designates. Since it represents a major concept of the mathematics of ancient China, an inaccurate translation would cause an important prejudice.

and reshaping, at a higher level, the relationship between different procedures. They will reveal, in the *Book of Mathematical Procedures*, a second kind of manifestation of what later appeared to be grasped as *lǜ*.

Chapter 2 of the Classic is organized in such a way that it is shown to 'derive' globally - in a fashion to be explained - from the rule of three. Its title, 'Foxtail Millet and Husked Grain' (*sùmí* 粟米), refers to the main paradigm in relation to which the central procedure of the chapter is described: equivalences between grain, mainly in the context of taxation carried out by the imperial administration. Even though, in the *Book of Mathematical Procedures*, rules of three are brought into play in various situations, there are reasons to believe that, there too, the procedure has a specific relation to taxation in grains.

Both books yield values that official regulations enacted to govern equivalences between different types of grain. In *The Nine Chapters*, they are organized in a table, the beginning of which reads as follows:

> 粟米之法　　粟率五十　糲米三十　　粺米二十七　糳米二十四 […]
> The norms for foxtail millet and husked grains:
> The *lǜ* of the foxtail millet is 50, that of coarsely husked grain 30
> that of fairly husked grain 27, that of finely husked grain 24 […][23]

Note that *The Nine Chapters* introduces the concept of *lǜ* in relation to these values - we shall come back to this point below. By contrast, here is the way in which the regulation is mentioned in the *Book of Mathematical Procedures* (Peng Hao 2001:80, bamboo slip 88):

> 程禾　　　　程曰：禾黍一石為粟十六斗泰（大）半斗；舂之為糲=米=一=石=，（糲米一石)為糳=米=九=斗=（糳米〔九〕斗）為毀（毇）米八斗。
> Applying the standards to millets　　The standards say: 1 *shí* of broomcorn millet (?) makes 16 *dǒu* 2/3 *dǒu* of foxtail millet.[24] Husking this makes 1 *shí* of husked grain; 1 *shí* of husked grain makes 9 *dǒu* of finely husked grain; 9 *dǒu* of finely husked grain makes 8 *dǒu* of highly finely husked millet (?).

Peng Hao (2001:80, fn.1) shows how close this passage is to the text of the Qín *Regulations for Granaries* (*cānglǜ* 倉律).[25] I suggest limiting our comparison of the two quotations to their structure and the values they

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[23] Chemla&Guo 2004:222-223; it is my pleasure to thank Georges Métailié for his help in identifying the grains in *The Nine Chapters*.

[24] Note that, even though 10 *dǒu* is 1 *shí*, the unit *shí* is not used to state the value.

[25] The names of the grains in the sequence starting from the foxtail millet are the same. By contrast, they differ slightly from what is found in the quotation given above from *The Nine Chapters*. However, the ratios between the various types of grain are exactly the same in all sources. We shall not dwell on these issues here.

contain. The standards mentioned in the *Book of Mathematical Procedures* form a sequence of grains that are obtained one from the other, by means of distinct operations. By contrast, in 'The Norms for Foxtail Millet and Husked Grains', the values associated with the grains form a mere list, each being equivalent to any of the others. In addition, the values given in the former are measured numbers, expressed with respect to different units, whereas, in the latter, the values are all integers, expressed abstractly.[26] These two features can be correlated to the use, in *The Nine Chapters*, of the term *lǜ* to designate the values. In an echo of this term, Liú Huī comments on them by stressing that "being all put into relation, they communicate globally (諸率相與大通)". In the context of division, evoked above, the Classic made use of the term 'making communicate', which led Liú Huī to introduce the concept of *lǜ*. Conversely, here, *The Nine Chapters* introduces the concept of *lǜ*, and in the commentary on this Liú Huī uses the notions of 'put into relation' and 'communicate'. The three notions hence appear to be mutually referential, but with respect to the grains, the 'relation' as well as the 'communication' are now meant for a whole set of entities, and not only, as previously, a dividend and a divisor. In other words, the use, in *The Nine Chapters* and the commentaries, of the abstract concept of *lǜ* and the related operation of 'making communicate' goes along with the establishment of links between distinct mathematical contexts, in which the same formal phenomena are shown to appear. By contrast, not only is the term for the operation absent from the *Book of Mathematical Procedures*, but also, in this context, so is its result in the form of a list of abstract integers globally expressing all the equivalences.[27]

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[26] An exception is to be mentioned: 103 + 1/2. Interestingly enough, several hints indicate, as C. Harbsmeier suggested to me as we were working together on the TLS, that ½ may have been conceived in ancient China as 'close to' integers. Note that, in addition, the values are, as a whole, relatively prime - their greatest common divisor (in the Chinese terminology of the time, *děng shù* 等數 'equal number', see *Glossary*) being 1.

[27] Although the *Book of Mathematical Procedures* contains no such set of numbers, it nonetheless provides evidence that may be interpreted as documenting the emergence of an interest in such values and the way in which they were established. As will be shown below, these values, sometimes adequately simplified, are used in rules of three that are converting grains one into the other. Bamboo slips 98-104 (Peng Hao 2001: 84-85) bear witness to how different equivalences between actual amounts of distinct grains were transformed into abstract numbers to be used in the corresponding rules of three - I shall comment on the mode of describing such rules of three in another paper. In addition, they betray a clear interest in *linking* these transformations to each other, in such a way that the same grain is associated to the same integer in different rules of three. Otherwise, one could not explain the values occurring in some of them. It is precisely in this detail that one may read the emergence of an interest that, consistently pursued, would lead to the formation of a table similar to the one found in *The Nine Chapters*. However, in the *Book of Mathematical Procedures,* this concern does not result in having a unique number associated to a given grain, as in the list provided by *The Nine Chapters*. For instance, foxtail millet is, according to the cases, associated with

Despite these differences, the rules of three allowing conversion of one kind of grain into another present the same specific features in both books. As an example, let us mention one of those gathered on bamboo slip 111 in the *Book of Mathematical Procedures* (Peng Hao 2001:89), which reads:

粟求米　　粟求米三之五而一。
(Having) foxtail millet, to look for[28] (coarsely) husked grain　　If, (having) foxtail millet, one looks for (coarsely) husked grain, triple this (i.e., the quantity one has) and divide by 5.

Exactly the same procedure is provided by *The Nine Chapters*, with the difference that, in the Classic, it follows problem 2.1, which reads as follows:

今有粟一斗，欲為糲米。問得幾何。
荅曰：為糲米六升。
Suppose that, having 1 *dǒu* of foxtail millet, one wants to make coarsely husked grain. One asks how much it yields.
Answer: It makes 6 *shēng* of coarsely husked grain. (Chemla&Guo 2004:224-225)

This context sheds light on the characteristic feature that these rules of three share: they mix in the same computation a quantity expressed with respect to units of measure (the quantity of the grain one has; in the case of problem 2.1: 1 *dǒu*) with abstract —co prime— numbers (that is, numbers sharing no common divisor except 1, like 3 and 5 in the previous quotation) to yield the quantity of the grain into which one wishes to convert the former.[29] A dissymmetry is thereby introduced between the various values entering into a rule of three. However, as far as I can see, in the *Book of Mathematical Procedures*, the procedure that relies on the data given in 'Applying the standards to millets' to yield the abstract numbers entering in rules of three is

either 10 or 50, whereas wheat can be linked to either 3, 9, or 15. In addition, to be complete, we should mention that bamboo slips 109-110 (Peng Hao 2001:88) express equivalences between grains taken two at a time with abstract and relatively prime integers. For example, one reads there "If foxtail millet makes (coarsely) husked grain […] 5 of foxtail millet makes 3 of (coarsely) husked grain […] (粟為米[…]粟五為米三[…])." Yet, again, this does not lead to the establishment of a list of globally equivalent values that can serve as the general source of all rules of three.

[28] The term 'looking for' refers to the search carried out in the context of a problem. See *Glossary*. The expression of the multiplication and the division at the end of the sentence, 'triple this', '5 becomes 1', strongly evokes changes of unit.

[29] To my knowledge, no other ancient tradition ever described the rule of three in this way. It involves a concept of abstract number that is also quite specific to ancient China: such a number being defined in relation to other quantities, they can all be simultaneously transformed into integers that, in addition, are relatively prime to each other.

not described explicitly. As a temporary conclusion, let us stress that although in this book the equivalences between grains are given with measured quantities, the rules of three employ for them abstract numbers that are integers relatively prime with each other, as in *The Nine Chapters*. It is precisely with this reality that the Classic will later associate the qualifier of *lǜ,* which is where we meet with our second manifestation, in the *Book of Mathematical Procedures*, of what came to be designated as *lǜ*.

Indeed, *The Nine Chapters* provides another procedure for solving all similar problems, right after the table of values that opens Chapter 2, but outside the framework of any problem. Its formulation is accordingly completely abstract. It captures the dissymmetry just described by designating two of the quantities entering into the procedure (3 and 5 in the example) with the term *lǜ*, whereas the remaining one is referred to as 'quantity' (*shù* 數). Its formulation, for which no equivalent is to be found in the *Book of Mathematical Procedures*, can be translated as follows:

> 今有
> 術曰：以所有數乘所求率為實。以所有率為法。實如法而一。
> Suppose.[30]
> Procedure: One multiplies, by the quantity of what one has, the *lǜ* of what one looks for, what makes the dividend; one takes the *lǜ* of what one has as divisor. One divides the dividend by the divisor.

Let us sketch briefly some specificities of this procedure. On the one hand, it introduces the abstract designations of 'what one has' and 'what one looks for' to oppose what is linked to the thing owned and what is linked to the thing sought for. The rule of three will rely on two equivalent quantities, one for each of them. The specific procedures translated above show that they are used in the form of abstract numbers. It is to these values that this new procedure grants the status of *lǜ*.

By making use, in an essential way, of the opposition between 'quantities' (*shù* 數) and *lǜ* (率), this abstract procedure hence captures the specificity of the exposition of a rule of three in both the *Book of Mathematical Procedures* and the specific procedures of *The Nine Chapters*. With it, as in the case above, we have a clear example of the description of a procedure produced by abstracting a pattern from a set of specific algorithms. The procedure derived by abstraction is of the same nature as that of 'directly sharing' as described in *The Nine Chapters*. In contrast to the example discussed in the section "A specific process of abstraction" above, however,

[30] The name of the operation carried out by the procedure is quite surprising. It may either stress the character of the operation to be essential or suggest an interpretation for it that involves making a supposition on the unknown.

in this case *The Nine Chapters* bears witness to both the specific procedures and the abstracted one. We can observe that the specific algorithms are exactly the same as those contained in the *Book of Mathematical Procedures*. Moreover, we can see how it is now for a manifestation different from the one examined in the previous section that *The Nine Chapters* opts for the abstract name of *lǜ*.

First of all, this remark supports the idea that a mathematical development has taken place that leads from one to the other —a development in which an operation of abstraction plays a part. What is important here is that in the later writings this development does not *only* concern the rule of three: it also introduces a concept - that of *lǜ* - which *The Nine Chapters*, and even more the commentaries, will show to have a remarkable range of extension. In particular, via its introduction a link is established between the rule of three and 'directly sharing', for the commentary on which, as we saw above, Liú Huī made use of the concept of *lǜ*. In fact, it will allow bridging further different branches of mathematics. This raises a first question about the way in which the search for the actual range of extension of the concept was conducted and which changes it brought about.

These observations lead me to a second remark. We concluded from the previous analysis that, in the earlier stage attested to, the *Book of Mathematical Procedures* did not show any hint of 'having made communicate' a large set of related quantities - we find neither the name, nor the reality. However, several features of the mathematical practices outlined above, to which the book testifies, show that the author(s) must have made use of the transformation of two related measured quantities into two abstract co prime numbers. This is why the abstract formulation found in *The Nine Chapters* captures the way in which the procedure is carried out in the *Book of Mathematical Procedures* so perfectly. This practice once again manifests a concept, even though it is never designated —neither as *lǜ* nor otherwise—. Moreover, this manifestation does not seem to have the extension that *The Nine Chapters*, let alone its commentaries, later gave to the concept of *lǜ*.[31] Perhaps, in fact, the shaping of the practice turned out to be one of the ingredients that led to the thematization and abstraction of the concept of *lǜ* and the related operation of 'making communicate'. It may have been a basis for noticing an analogy with how 'directly sharing' was carried out. If that were the case, the introduction of the abstract terms may have been concomitant with the observation of the generality of the phenomena designated. In any event, if we read the actual text of the *Book of*

[31] Several hints suggest that, in the *Book of Mathematical Procedures,* the abstract integers into which measured quantities were transformed were conceived to be of the type of a numerator and a denominator, respectively. I shall explore this hypothesis systematically elsewhere.

Mathematical Procedures retrospectively, on the basis of *The Nine Chapters*, we find that what came later to be grasped as *lǜ* corresponded earlier to distinct manifestations. The question of how an actual link between them was established remains open.

Abstracting further

Examining our last piece of evidence will provide a basis for further inquiry into these questions. Moreover, it may reveal what was at stake in producing the abstract statement of the procedure to carry out the operation 'Suppose', translated above. In correlation with the fact that *The Nine Chapters* formulates such an abstract algorithm for the rule of three, other procedures in it will be described in such a way as to show that they are in fact simple kinds of it —some instancing, if you will.[32] It is actually in this way that Chapter 2, entitled 'Foxtail millet and husked grains', can be said to derive from this procedure. It will therefore be useful to observe how a procedure found in the *Book of Mathematical Procedures* was rewritten in relation to its introduction into the framework of Chapter 2.

In fact, the procedure in which we shall be interested constitutes one of the main topics of the *Book of Mathematical Procedures*, in which it recurs several times. A whole paper would be required to analyze the set of its occurrences and the distribution of nuances they present. Let us limit ourselves here to examining its formulation in a section, the title of which contains a character that seems to be one of the three occurrences of *lǜ* 率 in the book —all in the same context. There, as well as in the corresponding sections of *The Nine Chapters*, it appears to take on a verbal meaning, which can be translated as 'determining the standard', or, more precisely, as 'determining the standard price'. The problem solved under this title consists in determining the cash equivalent for a given unit of a commodity, when one knows the cash exchanged for a given amount. The reason why such a problem is not dealt with through simple division is precisely what brings about the specificity of the procedure. Moreover, this is also precisely the reason why, despite its close link to division, it will be made part of Chapter 2 in *The Nine Chapters*.

Although determining the price of a given unit of measure is clearly the task that the procedure provided carries out, it is given in bamboo slips 74-

[32] In addition, for procedures the statement of which betrays no relation to the rule of three, the commentators regularly reveal how translating their terms into the abstract terms of this procedure brings to light their meaning, and thereby the reasons for their correctness.

75 (Peng Hao 2001:73) without the framework of any specific problem and without reference to any specific commodity.[33] Here is how the text reads:

> 石衛（率） 石衛(率)之術日：以所買=（賣）為法，以得錢乘一石數以為實。其下有半者倍之，少半者三之，有斗、升、斤、兩、朱（銖）者亦皆//破其上，令下從之為法。錢所乘亦破如此。
>
> Determining the standard price on the basis of the *shí* Procedure for determining the standard price on the basis of the *shí*: One takes what is exchanged[34] as divisor. One multiplies, by the cash obtained, the quantity of 1 *shí*,[35] which is taken as dividend.[36] Those for which, in their lower (rows), there

[33] A related problem, dealing with the price of units of weight for gold, can be found on bamboo slips 46-47 (Peng Hao 2001:60). Was it in fact stated, in the original version of the *Book of Mathematical Procedures*, in relation to the procedure that is examined immediately below? Several hints indicate that this may have been the case. First, as can be seen below, the statement of this procedure contains a list of units that includes not only units of capacity such as *shí*, in echo of the title of the section, but also units of weight (*jīn, liǎng* and *zhū*). Again in correlation with the title, the units of capacity are the topic of the following problem (slips 76-77). Yet, there appears no reason why units of weight should be mentioned here. It would be accounted for if bamboo slips 46-47 were originally part of the group 74-77. Incidentally, this indicates that, despite its title, the procedure is felt to be general. Secondly, if the problem of slips 46-47 was stated in relation to the general procedure given in slips 74-75, one would have, represented in the *Book of Mathematical Procedures*, the first case in relation to which *The Nine Chapters* formulates a more abstract procedure clearly similar to the one discussed here (Chemla&Guo 2004:246-251). This question would require a deeper analysis and will be left unanswered here. The problem dealt with on bamboo slips 76-77 (Peng Hao 2001:74) aims, in relation to pricing salt, to 'determine the standard price on the basis of the *shí* (unit of capacity)'. This is the second occurrence of *lǜ*, and it is clearly linked to the first one mentioned. In relation to the fact that the task suggested is formulated with the expression forming the title of the section to which we turn immediately below, the procedure is an instancing of the general one described there. How were the particular problem and procedure placed with respect to the general and abstracted algorithm corresponding to them? This is a question to be addressed systematically in relation to the *Book of Mathematical Procedures* (Guo Shuchun 2004).

[34] Cullen (2004:128) suggests this character may be read as *jià* 價 'price, cost'.

[35] That is, the quantity corresponding to the unit on the basis of which the unitary price is determined. In the generic case, it is 1 *shí*. One may wonder why one multiplies by 1: a multiplication by 1 does not change anything; moreover a simple division would seem to solve the problem. From the two perspectives the multiplication appears dispensable. As will be argued below, this operation characterizes the algorithm and distinguishes the whole procedure from a division. In fact, strangely enough for the description of an algorithm, the final statement of the procedure will suggest a transformation of precisely this 1 that it is prescribed to multiply here. In other words, the quantity multiplied here is *retrospectively* modified. Such a description of the procedure allows the author to highlight its structure as well as the reasons for which the modification is to be carried out: it will appear that the retrospective modification is parallel to the one the divisor must undergo. As a result, the units in both the dividend and the divisor are expressed with respect to the lowest unit appearing in the quantity acquired. In fact, at this point in the computations, the transformation required because of the possible presence of different units in the divisor has not yet been addressed. This feature of the algorithm relates to its structure, described in the

is a half,[37] one doubles them;[38] those for which there is a third, one triples them. Those for which there are *dǒu* and *shēng, jīn, liǎng* and *zhū*,[39] one *also* breaks up

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following footnote. The multiplication by 1 is hence prescribed at a moment of the computation when the lowest unit and the correlative multiplication it entails —the multiplier by which 1 *shí* should be multiplied— have not yet been determined.

[36] Note that the division is not prescribed explicitly. The structure of this procedure is similar to the structure of the procedure for 'directly sharing' in *The Nine Chapters* (Chemla&Guo 2004:767, fn. 68). First, a basic procedure is described, for the simplest cases. Such cases cover that of the problem dealt with in slips 46-47. Then, other possible cases are introduced, such as that of the problem covered in slips 76-77, for which a correlative modification of the divisor and part of the dividend is prescribed before the basic procedure can be used. These cases are presented in a series of conditionals: operations are to be applied in succession, in relation to the sequence of conditionals that are fulfilled for the case dealt with.

[37] As above, the quantity by which one divides, as 'divisor', is presumably set up in the lowest position on the surface on which computations are carried out. As for the dividend, it is set up above, in the 'middle' position. However, such is not the position to which *xià* 下, translated here as 'the lower (rows)' refers. Rather, in the lowest zone, distinct sub-rows are further introduced for placing the various components of the quantity constituting the divisor. Like for 'Directly sharing', the finer the unit corresponding to them is, the lower the sub-row in the series of sub-rows in which the divisor is placed. However, there are reasons to interpret the indication 'below' here in a way slightly different from what has been suggested above in relation to 'Directly sharing'. The first reason is that, in what follows, the 'lower (rows)' of the quantity to be placed on the surface for computing are opposed to the 'upper rows' (I take *qí xià* 其下 'their lower (rows)' to be parallel to the following expression *qí shàng* 其上 'their upper (rows)', see below). As will become clearer, we are here dealing with a quantity, the integral part of which is expressed with respect to distinct units of measure and is increased by several fractions of the finest unit among them. It seems that the component of the quantity corresponding to the finest unit of measure is placed in the middle sub-row of the lowest position in which the whole quantity is placed. With respect to it, the components linked to coarser units are successively placed in sub-rows above (the reason for this plural is discussed below), in ascending order in the 'upper positions' (they form 'the upper (rows)'). In opposition to this, the successive fractions of this finest unit of measure are placed in sub-rows below the middle sub-row, in the 'lower positions'. The latter are, in my view, what the expression 'the lower (rows)' designates here. The second reason for interpreting in this way is that the multiplication derived from the presence of fractions differs from what was the case in 'directly sharing'. If 'there are thirds', the multiplication to be carried out here is not by 6, but by 3. In correlation with this, the operation described for such cases is not an overall change of unit, but a multiplication, which may be one in a sequence. These details indicate that the denominators are treated as independent from each other, and the multiplication to be carried out because of the presence of a denominator depends only on its value. In other words, the first set of transformations to be applied to the divisor is a sequence of multiplications depending on the sequence of denominators placed in the 'lower rows' of the divisor. When referring to 'the lower (rows)', the procedure hence does not refer to the greatest extant denominator of the fractional part, but to any denominator in the set of denominators. Note that, in contrast to 'Directly sharing', the quantities on which one operates keep their identity of measured quantities, and do not seem to be transformed into abstract numbers.

[38] As is made clear from the description of the remaining part of the procedure, 'them' refers here clearly, for any possible case, to the quantity placed in the position of the divisor, which has been reintroduced as a topic by *qí* 其 …*zhě* 者. In other terms, the anaphoras 'them' do not designate the value(s) obtained in the computations immediately preceding this one. Such a use of the anaphora differs from that of *The Nine chapters* (see Chapter A, Chemla&Guo

all their upper (rows),[40] one makes the (rows) below join them,[41] (yielding a result) which is taken as divisor.[42] What the cash was multiplying is *also broken up like this*.[43]

2004:25). Suppose the divisor is 3 + ½, the multiplication by 2 manages to get rid of the fraction and transforms it into an integer, yielding as a whole: 7. If, however, there are two denominators, 3 + ½ +1/3, a first multiplication, by 2, yields 7+ 2/3. A second multiplication, by 3, yields 23.

[39] The first two units are linked to capacities, the last three to weights. For the next operation described to be needed, it is required that the integral part of the divisor have at least two distinct units. The sequence of two units of a sort and three units of another sort may refer to any sequence with more than one unit. However, it is the same operation that has to be applied whether there are two or more units. And it is prescribed below in a general way. The generality of the expression fits with the fact that the procedure is given outside the framework of any problem.

[40] Let us comment successively on the terms 'break up', 'all their upper (rows)' and '*also*'. By contrast to the above, where, in cases involving fractions, the operations prescribed are multiplications and they are to be applied to all components of the divisor, here the prescription refers to the concrete operation of breaking up what must be understood as the various units of measure constituting the integral part of the quantity. More precisely, the units of measure to be broken up are those placed in the 'upper rows' —all of them—, and *not* the finest ones, placed in the middle row (or even those in the lower ones). This is the reason why I believe that the distinct components corresponding to different units were placed in different sub-rows. The result of breaking up the units is that their number is multiplied. The value by which they should be multiplied is the quantity of finest units contained in the unit of measure considered. This is precisely what bamboo slip 47 provides, with respect to the units of weight. This explains why it is linked to bamboo slip 46. Furthermore, this gives an additional argument in favor of the hypothesis that it was originally placed with slips 74-77. Note that the multiplication is prescribed through the *meaning* the operation has (to 'break up' units). On this basis, the 'also' is extremely interesting. Above, a multiplication was prescribed if the quantity forming the divisor had fractions. Here there is a prescription to 'break up' units. The 'also' expresses the fact that the two operations are similar. The 'multiplication' can certainly be interpreted as a 'breaking up' of all units, whereas the 'breaking up' of the units of measure leads to multiplying their number to transform them into units of the finest size. With this single word, the operations carried out because of the existence of fractions or because of that of different units are put on the same level and stated to be identical. Note that this 'also' is absolutely useless from the point of view of the prescription of the computations. Its only function is to introduce a structure in their description. The result of the breaking up of all upper units is such that all components of the original quantities are now expressed with respect to a unique unit. Another point is essential here: prescribing the multiplication through an invitation to 'break up quantities' is a way of referring to the operation by the material effect it has on units. Such a description gives a hint of the interpretation that accounts for the correctness of the procedure. I have already argued that such a constraint may have governed the modes of describing algorithms attested to in *The Nine Chapters*. The conclusion can apparently be extended to the *Book of Mathematical Procedures*. I shall gather pieces of evidence for this in another paper. (Horng 2001) noticed some ways in which the *Book of Mathematical Procedures* makes reasonings explicit. I believe that more can be done in that direction. However, I do not endorse the view that the *Book of Mathematical Procedures* would differ from *The Nine Chapters* in this respect.

[41] *Cóng* 'to join' is the prescription of a dissymmetric addition in which the content of some rows is emptied by the fact that it is made to add - 'to join' - the content of another row (see *Glossary*).

It should be stressed here that the correlative transformation of the dividend and the divisor that is underlined by the second 'also' recalls the parallel expressed by the same means in 'Directly sharing', even though there are differences.[44] The 'also' marks in both cases the same structure in the procedures —a fact that, in *The Nine Chapters*, will be captured by referring to both transformations with the unique operation of 'making communicate'. A slight difference between the two contexts should not be forgotten: the parallel transformations seem to yield abstract numbers in 'Directly sharing', whereas they seem to lead to measured numbers for the other case. By contrast, in the *Book of Mathematical Procedures*, there are several features opposing the way in which the rule of three and the procedure just examined

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[42] Here one gets a confirmation that the whole sequence of operations was applied to the quantity taken as divisor, which has hence been modified with respect to the first statement defining it, at the beginning of the procedure. It is quite interesting that the same term 'divisor' is used again here. Besides the fact that one has an assignment of variables, throughout the computation, it is understood that one is operating on the quantity that plays the part of the divisor, and, in the end, the value obtained retains the same name.

[43] Another 'also', underlined by the expression 'like this', marks a correlation between a transformation of a quantity entering into the composition of the dividend and the one just described. A dissymmetry is thereby introduced between the two factors of the dividend, one being transformed in parallel with the divisor. Again the 'also' is unnecessary for the prescription of the operation, but it introduces more structure and meaning into the description of the computation. Moreover, it is precisely the unit, 1 *shí*, that undergoes the transformation. This is highly interesting. The consequence is that the specific way in which the computation is carried out (multiplying the cash by 1) allows interpretations of the meaning of its successive steps in terms of units linked to the entity exchanged. In fact, this is exactly what happens in the procedure solving the concrete problem linked to the general procedure (slips 76-77): the last step is described as, "one triples *also* the quantity of *shēng* of 1 *shí*". The multiplication by 3 is linked to the denominator of the fraction contained in the original quantity bought. The transformation of 1 *shí* into *shēng* is the way chosen to prescribe what, in the general procedure, is described as a breaking up of units. As a consequence, the specificity of the procedure, which lies in the introduction of this multiplication by 1, finds its probable rationale here. It allows the procedure to be transparent as to the reasons why it is correct. This ends my interpretation of the procedure. It differs in several places from Cullen (2004:62-63) and Dauben (forthcoming). I gave my arguments above, but it would require too much space to discuss my choices in contrast to theirs.

[44] Remember that the correlative transformation there was likewise marked by an 'also' in one of the formulations of 'Directly sharing', see above, fn. 17. This link between the two procedures will be further emphasized by the way in which, in *The Nine Chapters*, both procedures are named, described and commented upon. The observation immediately leads to an interesting point. The interpretation given by the commentator Liú Huī of the various operations of 'directly sharing' is in terms of 'disaggregating'. This is therefore intimately related to the way in which, in the *Book of Mathematical Procedures*, within the context of 'Determining the standard price…', the operation is conceived of, and multiplications are stated to be similar to disaggregating. This network of similarities indicates that even for their interpretation of the effect of the operations and hence their proofs of correctness, the commentators certainly relied on earlier sources. This point will be discussed further in another paper.

are carried out. As we indicated above, the rule of three involves three numbers, two of which are abstract while one is a measured quantity. In the procedure for 'Determining the standard price for the *shí*', however, all the numbers involved are measured quantities. Yet, one quantity in the dividend and the divisor are transformed in correlation with each other. A dissymmetry is hence introduced between two of the three values that enter the procedure, which evokes that of the rule of three. It highlights a shape in the computation that seems to have paved the way for the reshaping of the relationship between the two procedures that is documented in *The Nine Chapters*. Let us observe how this reshaping is carried out.

The section of the *Book of Mathematical Procedures* just analyzed turns out to be the one with a title ('Determining the standard price on the basis of the *shí*') closest to the operation involved in several problems of *The Nine Chapters*. For instance, problem 2.34 deals with the task of how to 'determine the standard price on the basis of the *dǒu lǜ*' (斗率). The four problems 2.34—2.37 share the same type, none being immediately followed by a specific procedure solving it. In contrast, their sequence is concluded by an algorithm solving them all. The name of the operation it carries out is also close to the title of the corresponding procedure in the *Book of Mathematical Procedures*, since it is named 'Directly determining the standard price' (*jìng lǜ* 經率).[45] This name is quite interesting. In addition to reminding us of the more particular title of the section examined above, its syntactical structure and the word 'directly' evoke the name 'Directly sharing'. Liú Huī's commentary echoes this point since he states: "This procedure is like 'Directly sharing'." We shall come back to this point below. Through the parallelism of the two names, *lǜ* as verb, 'Determining the standard price', is put on a par with *fēn* 'sharing'. In the same way as *fēn* is also used as a noun, to designate the 'parts' that constitute fractions, in *The Nine Chapters*, *lǜ* is used as a noun, especially in Chapter 2 in which it is introduced. We saw some examples of this above, especially in the procedure for the operation named 'Suppose'. In *The Nine Chapters*, the structure of the description of the procedure for carrying out the operation of 'Directly determining the standard price' appears to derive from the latter, and, in particular, it also makes use of *lǜ* as a noun. Let us read the text:

[45] In fact, a first procedure having the same name – a phenomenon quite unique in *The Nine Chapters* - is stated after problem 2.33, for dealing with the standard price of discrete entities. The division that should conclude the solution is formulated in a quite special way. The whole section can be compared to the problem dealt with in slips 46-47. I cannot deal here with the question in all its details - see my introduction to Chapter 2 in Chemla&Guo 2004:205-206. Nor can I deal with philological problems, see Chemla&Guo 2004:244-251. I shall come back to these issues in the paper that will concentrate on this set of algorithms.

> 經率
> 術曰：以所求率乘錢數為實，以所買率為法，實如法得一。
> Directly determining the standard price
> Procedure: One multiplies, by the *lǜ* of what one looks for, the quantity of cash, which makes the dividend; one takes the *lǜ* of what is bought as divisor. Dividing the dividend by the divisor yields the result.

It is not necessary, I believe, to get into the technical details of the procedure described to make the comments needed for our reflection on abstraction. We shall rely on merely formal comparisons between formulations.

If we compare term by term this formulation with that of the procedure for 'Suppose', it is clear that the structure of the computations as well as the distribution of the qualifiers 'quantity' or '*lǜ*' for the terms show a similarity. However, a striking phenomenon presents itself here. If the procedure for 'Directly determining the standard price' were obtained as an application of the procedure for 'Suppose', the 'quantity of what one has' would *not* be the 'quantity of cash', as the similarity indicated would lead us to believe, but the unit for which the price needs to be found. Indeed, the values governing the conversion between things and cash are the quantity first bought and the cash spent in relation to this purchase. These should have been qualified as *lǜ*, with which one would 'look for' the 'quantity' of cash corresponding to buying the unit (the 'quantity of what one has'). It is extremely interesting that this is exactly how Lǐ Chúnfēng explains why the operation has the 'meaning' of 'Suppose'. But this is *not* the way in which the procedure for 'Directly determining the standard price' is described. How can this be possible?

The procedure examined has a dividend obtained by multiplying two data, and a divisor consisting of one datum. This is the structure that makes it possible to interpret it as a rule of three. However, one can read it as a rule of three in two ways, according to which factor of the dividend one interprets as the *lǜ*. In other words, there are two ways of introducing a dissymmetry in the interpretation of the algorithm. The commentator chooses one, which corresponds to the description of the terms of the rule of three as carried out in the formulation of the procedure for 'Suppose'. However, the description of the procedure for 'Directly determining the standard price' in *The Nine Chapters* opts for the other one. How can one account for this?

To answer this question, it is rewarding to compare the latter procedure to that described in the *Book of Mathematical Procedures*, under the title 'Determining the standard price on the basis of the *shí*'. One thereby comes to realize that the quantities designated as *lǜ* in *The Nine Chapters* are precisely those that are submitted to parallel transformations in the procedure of the *Book of Mathematical Procedures*. Let us recall that this parallelism was emphasized by the use of 'also' in the description of the procedure. We stressed above that these parallel transformations were the same as those affecting the dividend and the divisor in 'Directly sharing'. In one of the descriptions of the latter procedure, they are also explicitly marked by the use of 'also'. And we remember that, in reaction to the procedure of *The Nine Chapters*, Liú Huī qualified the dividend and the divisor as *lǜ*. From this set of observations, we may derive a number of conclusions.

First, one could quite easily account for the formulation of the procedure for 'Directly determining the standard price' in *The Nine Chapters* by putting forward the hypothesis that it was obtained *not* as an application of 'Suppose', but rather by abstraction from the procedure for 'Determining the standard price on the basis of the *shí*', as it can be found in the *Book of Mathematical Procedures*. We would hence have yet another example of the application of an operation of abstraction to account for the link between procedures contained in the two books.

Secondly, the question then arises of how the abstraction is carried out. It is interesting to compare it to what has been described above with respect to 'Directly sharing'. We saw how parallel computations applied to the dividend and the divisor were translated, in the description given by *The Nine Chapters*, by the grouping of their procedures in a unique operation: 'one makes them communicate'. In reaction to this, Liú Huī qualified the two terms as *lǜ*. In the case of the 'Procedure for determining the standard price on the basis of the *shí*', it is exactly the same feature of the procedure that is abstracted: the correlative transformations undergone by one factor of the dividend and the divisor. And, what is remarkable is that they are prescribed by the mere fact of referring to these two entities as *lǜ*. The process of abstraction is the same in that it bears on exactly the same part, the same feature of the procedure. Furthermore, the concept of *lǜ* and the operation of 'making communicate' associated with it are precisely those by which the abstraction shapes a link between the two procedures: the same structure is captured by the use of a concept qualifying the quantities submitted to the parallel computations. This is exactly the point at which a new kind of generality is achieved through abstraction. More precisely, at the same time as it refers to parallel transformations, the concept *lǜ* designates the property of the quantities taken together that accounts for the correctness of these transformations. All this is in complete agreement with the link shown in the names of the operations in *The Nine Chapters*: 'Directly sharing' and 'Directly determining the standard price'.[46] Note also that the operation of abstraction seems to go along with a modification in the 'Procedure for determining the standard price on the basis of the *shí*'. In fact, in the version

[46] On the relation between the two operations, see my fn. 77, in Chemla&Guo 2004:790-791.

of the *Book of Mathematical Procedures*, the quantities undergoing parallel transformations remained measured numbers. In the abstracted version, however, qualifying them as *lǜ* may indicate that the final stage which they reach consists of abstract integers. The similarity with the rule of three noted above would hence have been strengthened. This leads us to a second range of questions.

So far, we have accounted for the reshaping of the relation between 'directly sharing' and 'Determining the standard price on the basis of the *shí*'. What about the transformation that the abstracting of the latter simultaneously yields in the relation between the procedure for 'Suppose' and that for 'Directly determining the standard price'? Dealing with this aspect of the situation will lead us to discuss yet another dimension of the operation of abstraction to which *The Nine Chapters* is testimony. Indeed, the procedure for 'Directly determining the standard price' is not only abstracted from another procedure to which the *Book of Mathematical Procedures* bears witness. It is abstracted in such a way that it appears to be a particular kind of rule of three, as is shown by the procedure and the names of the terms. In correlation with this, 'Directly determining the standard price' is placed in Chapter 2, which opens with the procedure for 'Suppose' and is entirely devoted to cognate algorithms. Note, however, that in the *Book of Mathematical Procedures* the two procedures did not appear to have anything in common.

Three necessary conditions made this transformation possible.

The first one was to abstract the procedure for 'Suppose' from more particular procedures in a way that also revealed its generality and, specifically highlighted how, in fact, different procedures were mere instances of it. Note that it is to this idea of abstraction and generality that the constitution of 'chapters' in *The Nine Chapters* corresponds.[47] Such a search, oriented towards the generality yielded by the shaping of fundamental operations, appears to guide the commentators (Chemla&Guo 2004: Chapter A). With the procedure for the operation 'Suppose', that is, an abstract algorithm, itself more abstract than other abstracted procedures, we have a procedure that seems to attest to a two-level abstraction. This is something revealed by its juxtaposition to 'Determining the standard price on the basis of the *shí*'.

The second condition is subtler. Remember that we stressed how, in fact, the procedure for 'Directly determining the standard price' was *not* a direct

[47] A key difference between the *Book of Mathematical Procedures* and *The Nine Chapters* is that the former has no chapters, whereas the latter is structured in chapters, as its title makes explicit. What a chapter is would require yet another paper. To make a long development short, this difference between the two books is correlated with the fact that *The Nine Chapters* bears witness to a specific idea of generality —procedures deriving from other procedures—, which its structure embodies and for which the commentaries account.

application of the rule of three. This is what the commentator Lǐ Chúnfēng shows while still asserting that it has the 'meaning' of the procedure for 'suppose'. How can the two claims be made simultaneously? For this, the procedure must be examined from two distinct perspectives. From the point of view of the reasoning establishing it and the interpretation of its steps, it does not correspond to the structure grasped by the formulation of the procedure for 'suppose'. But when the two procedures are compared *formally*, the latter appears to grasp the meaning of the former and emerges as more abstract. A formal view of procedures thus appears to be a condition for carrying out an abstraction that *in this way* extends the generality of the procedure for 'Suppose'.

The third condition relates to the concept of *lǜ* and the way in which it is introduced. For lack of more numerous sources, it appears that the shaping, in *The Nine Chapters*, of the set of procedures discussed, with the network of relations described, correlated with the introduction of the concept of *lǜ* and the related operation of 'making communicate'. As has been indicated, the relation between what the concept designates in the procedure and what is to be found in the corresponding procedure of the *Book of Mathematical Procedures* is not homogeneous. On the one hand, for 'directly sharing' and 'Directly determining the standard price', *lǜ* encapsulates two parallel sets of transformations undergone by two values (with the nuance that the final values are abstract in one case and measured quantities in the other). On the other hand, for the procedure for 'Suppose', *lǜ* designates what, in the particular procedures, are numbers that are abstract, generally relatively prime with each other. However, in the *Book of Mathematical Procedures*, nothing was said as to how they were obtained. In *The Nine Chapters*, the same name is given to the two types of reality: *lǜ* seems to have incorporated all these elements. It is the combination of these factors that define the concept: the parallel procedure to be applied to quantities designated as *lǜ* to transform them into abstract integers, and the final stage they can reach when further simplified and becoming relatively prime with each other. It was necessary that the concept of *lǜ* incorporate these distinct elements for it to be possible to establish the network we described between procedures that were previously unrelated. This is particularly true for shaping a relationship between the rule of three and 'Directly determining the standard price' as one being more general than the other. Moreover, several converging changes were necessary in the procedures to which the *Book of Mathematical Procedures* testified for this concept to allow the reshaping of the relationship between them to which *The Nine Chapters* attest.

How was the bridge between the two ideas built in the form of the concept of *lǜ*? What were the wider consequences of the emergence of such a concept? Could one of these be the extension of the concept of *lǜ* from designating two values to encompassing cases for which sets of *lǜ* could have an indeterminate number of terms? These are some of the questions that remain to be addressed. However, a conclusion can already be drawn: we can observe several ways in which carrying out abstractions accompanies producing a kind of generality that derives from shaping similarities between procedures that would otherwise seem to be distinct.

Conclusion

As has been stressed in the first part of this paper, there is no direct evidence of a historical link between the *Book of Mathematical Procedures* and *The Nine Chapters*. However, what this paper has described in terms of abstraction seems to me to provide interesting material for an inquiry into this question. For the three sets of procedures examined, the relationship between what is found in the two books is so intimate that it seems clear that the algorithms inserted in *The Nine Chapters* were obtained on the basis of the procedures gathered in the *Book of Mathematical Procedures*, whether they had been found in that particular book or in another one. For each of them, the description of the procedure included in *The Nine Chapters* was derived thanks to a process of abstracting and, in one case, a two-level process of abstracting. These processes were made possible thanks to the introduction of theoretical terms, one of the consequences of the introduction of these terms being to build bridges between procedures previously unrelated. It is hence a very specific form of generality that is achieved through this kind of abstraction, and one that must be distinguished from the generality brought about by the fact of addressing, say, 'the triangle', as Euclid did. One concept typical of the generality achieved in Hàn China is that of *lǜ*, a concept that encapsulates procedures applied to two numbers in parallel, and the multiple uses of which underline the generality. It is therefore clear how the *Book of Mathematical Procedures* helps us see not only the abstraction, but also the abstracting in *The Nine Chapters*. In correlation with this, it provides evidence allowing an interpretation in Hàn terms of what is to be read in these theoretical concepts. Lastly, the *Book of Mathematical Procedures* thereby allows us to grasp processes of change in Hàn mathematics. Was the abstracting carried out as part of the work that led to the composition of *The Nine Chapters*? Or was it earlier work that found its way into the Classic? We are not in a position today to answer such questions. However, the new evidence available clearly helps us to formulate more precise questions regarding the process of production of *The Nine Chapters*.

In contrast to this increase in abstraction, and hence generality, to which *The Nine Chapters* bears witness, in these cases the commentators elucidate the elementary procedures that the abstract terms encapsulate, thereby manifesting a knowledge of earlier, more elementary descriptions of the procedures. These elements of continuity between the commentaries and the Hàn source material thus raise questions regarding the exercise of exegesis. The commentators may be aiming to help bridge the gap between the ancient sources and the Canon, shedding light on the work involved in the making of *The Nine Chapters*. The way in which they may have compiled earlier documents to this end would require further research.

As far we can tell, however, the production of commentaries was not limited just to compiling earlier sources. They also introduced further abstract concepts and theoretical discussions in mathematics, on the basis of what could be read in the Canon. Interestingly enough, in this respect, several pieces in the commentary attributed to Liú Huī testify to the value the commentator attaches to one kind of abstraction, which can be grasped through his use of such formulations as 'abstract expressions *kōngyán* 空言'.[48] This kind of abstraction seems to address the fact of presenting or discussing a procedure independently of the context of any problem. It is thus to be distinguished from the one discussed here, and is connoted positively or negatively according to the context. This, however, raises a key question: how did the practitioners conceive of the form of abstraction that we have discussed and that they clearly carried out? I shall leave this unanswered by way of conclusion.

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[48] See *Glossary*:947.

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