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2006.81



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# Attitude toward imprecise information \*

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## Abstract

This paper presents an axiomatic model of decision making which incorporates objective but imprecise information. We axiomatize a decision criterion of the multiple priors (or maxmin expected utility) type. The model achieves two primary objectives. First, it explains how subjective belief varies with information. Second, it identifies an explicit attitude toward imprecision that underlies usual hedging axioms. Information is assumed to take the form of a probability-possibility set, that is, a set  $P$  of probability measures on the state space. The decision maker is told that the true probability law lies in  $P$ . She is assumed to rank pairs of the form  $(P, f)$  where  $P$  is a probability-possibility set and  $f$  is an act mapping states into outcomes. The representation result delivers multiple-priors utility at each probability-possibility set. There is a mapping that gives for each probability-possibility set the subjective set of priors. This allows both subjective expected utility when the subjective set of priors is reduced to a singleton and the other extreme where the decision maker takes the worst case scenario in the entire probability-possibility set. We show that the relation “more averse to imprecision” is characterized by inclusion of the sets of priors, irrespective of the utility functions that capture risk attitude. We characterize, under extra axioms, a more precise functional form, in which the subjective set of priors is obtained by (i) solving for the “mean value” of the probability-possibility set, and (ii) shrinking the probability-possibility set toward the mean value to a degree determined by preference.

**Keywords:** Imprecise information, imprecision aversion, multiple priors, Steiner point.

**JEL Number:** D81.

## Résumé

Nous présentons un modèle dans lequel le décideur a accès à une information objective imprécise. Nous axiomatisons un critère de décision de la famille dite à “croyances multiples”. Nous caractérisons une notion d’aversion à l’imprécision et donnons également une caractérisation de la notion “être plus adversaire de l’imprécision”.

**Mots Clé:** Information imprécise, croyances multiples.

**JEL Number:** D81.

# 1 Introduction

In many problems of choice under uncertainty, some information is available to the decision maker. Yet, this information is often far from being sufficiently precise to allow the decision maker to come up with an estimate of a probability distribution over the relevant states of nature. The archetypical example of such a situation is the so-called Ellsberg paradox (Ellsberg (1961)), in which subjects are given some imprecise information concerning the composition of an urn and are then asked to choose among various bets on the color of a ball drawn from that urn.

In this paper, we model a decision maker who reacts to imprecision of the available data in a given choice problem. We do so assuming that data can be represented by sets of probability distributions. Our main concern is twofold. First, can we identify an explicit attitude toward such imprecision and, second, can we establish a relationship between objective (but imprecise) information and subjective beliefs?

We present an axiomatic model of decision making which incorporates objective but imprecise information as a variable. The model permits the analyst to relate the choices made under different information and to apprehend which type of information is valued by the decision maker. Further, it explains how subjective belief is related to objective information.

Thus, we define preferences as a binary relationship on the cross product of acts (mappings from states of the world to outcomes) and available information (sets of probability distributions over the state space). Denoting  $P$  the set of probability distributions over the state space that represents the information available to the decision maker (hereafter *probability-possibility sets*), preferences bear on couples  $(P, f)$  where  $f$  is an act in the usual sense. This means that, at least conceptually, we allow decision makers to compare the same acts in different informational settings.

The motivation for this formalization can be best understood on Ellsberg's two urns example. In urn 1 there is a known proportion of black and white balls (50-50) while in urn 2, the composition is unknown. The decision maker has the choice to bet on black in urn 1 or on black in urn 2. Thus, the action (bet on black) itself is the same in the two cases and the information has changed from a given probability distribution (1/2,1/2) (urn 1) to the simplex (urn 2). This interpretation can be extended to non experimental situations. Consider an investor contemplating building an oil platform in either of two locations: the North sea or the Mexican Gulf. In both cases, the platform is susceptible to climatic incidents such as storms and the like. One could well argued that probabilistic information is available in both cases for the immediate future but that over the longer run, there is a large amount of imprecision as to how climatic change will affect the frequency and the severity of tropical storms, while there are good reasons to believe that tempest in

the North Sea will not be affected by climatic change. Then, one can argue that the main features of the investor’s problem here are comparable to Ellsberg’s two urns. The possible actions are the same (build or not the platform) at the two locations, but the available information is different.

We place ourselves in an Anscombe and Aumann (1963) setting, in which outcomes are probability distributions over a set of prizes. Our general representation theorem axiomatizes a class of functionals of the maxmin expected utility type à la Gilboa and Schmeidler (1989), where the set of priors is a subset of the available information. Hence, compared to Gilboa and Schmeidler (1989) we enrich the space on which preferences are defined: in their setting, the (un-modelled) prior information that the decision maker has is fixed. If we adapt Gilboa and Schmeidler (1989) axioms to our setting, we can axiomatize the following functional form. For two probability-possibility sets  $P$  and  $Q$  and two acts  $f$  and  $g$ ,  $(P, f) \succeq (Q, g)$  if, and only if,

$$\min_{p \in \varphi(P)} \int u \circ f dp \geq \min_{p \in \varphi(Q)} \int u \circ g dp.$$

The function  $\varphi$  takes the probability-possibility set and transforms it into some subjective set of “beliefs”.

This representation is obtained using Gilboa and Schmeidler’s axiom of uncertainty aversion which states that mixing two indifferent acts can be strictly preferred to any of these acts, for hedging reasons. In our setting, we can provide a more direct way of modeling the decision maker’s attitude toward imprecision, which also provides an easy way of experimentally testing the axiom. We show in particular that uncertainty aversion is implied by an axiom of aversion toward imprecision which compares the same act under two different probability-possibility sets. Aversion toward imprecision states, loosely speaking, that the decision maker always prefers to act in a setting in which he possesses more information, i.e., the decision maker is averse toward a “garbling” of the available information. At this stage, we simply remark that the notion we adopt of what it means for a probability-possibility set to be more imprecise than another one is rather weak and partial in the sense that it does not enable one to compare many sets (this will be discussed in Section 3.)

The next step in the paper is to put more discipline on the belief. This is done under an extra axiom that captures some invariance properties, which will be discussed in Section 4. The subjective set of priors is obtained by (i) solving for the “mean value” of the probability-possibility set, and (ii) shrinking the probability-possibility set toward the mean value according to a degree given by preference. The mean value is the *Steiner point* (see Schneider (1993)). For polytopes, the Steiner point is the weighted average of extreme points in which the weight for each is proportional to its outer angle. For cores of beliefs functions for instance, it coincides with the Shapley value. Denoting  $s(P)$  the Steiner point

of  $P$ , we obtain that  $\varphi(P) = (1 - \varepsilon)s(P) + \theta P$ . This functional form, already suggested in Ellsberg (1961), consists of taking the convex combination of the minimum expected utility with respect to all the probability-possibility set, with the expected utility with respect to a particular probability distribution in this set. The parameter  $\varepsilon$  is obtained as part of the representation result and can be interpreted as a (subjective) degree of imprecision aversion. When  $\varepsilon = 0$ , we obtain subjective expected utility. When  $\varepsilon = 1$  the functional form expresses the extreme case where the decision maker takes the worst case scenario in the entire probability-possibility set.

We then proceed to define a notion of *comparative imprecision aversion* with the feature that it can be completely separated from risk attitudes. Loosely speaking, we say that a decision maker  $b$  is more imprecision averse than a decision maker  $a$  if whenever  $a$  prefers to *bet* on an event when the information is given by a (precise) probability distribution rather than some imprecise information,  $b$  prefers the bet with the precise information as well. This notion captures in rather natural terms a preference for precise information, which does not require the two decision makers that are compared to have the same risk attitudes, the latter being captured, as we show, by the concavity of the utility function. Our result states that two decision makers can be compared according to that notion if and only if the transformed set of one of them is included in the other's. We also define a notion of *imprecision premium* that is consistent with this notion of comparative imprecision aversion

## Relationship with the literature

We conclude this introduction by mentioning some related literature, whose precise relationship with our model and results will be discussed further in the text. We also make clear what are the main conceptual differences between our approach and much of the recent literature.

Our model incorporates explicitly information as an object on which the decision maker has well defined preferences. To the best of our knowledge, Jaffray (1989) is the first to axiomatize a decision criterion that takes into account “objective information” in a setting that is more general than risk. In his model, preferences are defined over belief functions. The criterion he axiomatizes is a weighted sum of the minimum and of the maximum expected utility. This criterion prevents a decision maker from behaving as an expected utility maximizer, contrary to ours, which obtains as a limit case the expected utility criterion. Interest in this approach has been renewed recently, in which object of choices are sets of lotteries (Ahn (2005), Olszewski (2002), Stinchcombe (2003)). Olszewski (2002) characterizes, under a weakening of the independence axiom, a version of the  $\alpha$ -maxmin expected utility in which the decision maker puts weights both on the best-case and the worst-case scenarios. Stinchcombe (2003) characterizes a general class of expected utility

for sets of lotteries. Ahn (2005) characterizes a conditional subjective expected utility in which the decision maker has a priori probability over lotteries and updates it according to each objective set. Our model however does not reduce to one of choice over sets of lotteries.

More closely related to our analysis is Wang (2003). In his approach the available information is explicitly incorporated in the decision model. That information takes the form of a set of probability distributions together with an anchor, i.e., a probability distribution that has particular salience. As in our analysis, he assumes that decision makers have preferences over couples (information,act). However, his axiom of ambiguity aversion is much stronger than ours and forces the decision maker to be a maximizer of the minimum expected utility taken over the entire information set. There is no scope in his model for less extreme attitude towards ambiguity. Following Wang's approach, Gajdos, Tallon, and Vergnaud (2004) proposed a weaker version of aversion towards imprecision still assuming that information was coming as a set of priors together with an anchor. The notion of aversion toward imprecision that developed in section 3 is based on the one analyzed in Gajdos, Tallon, and Vergnaud (2004) and is different from the one defined in Gilboa and Schmeidler (1989) and Schmeidler (1989) and the subsequent literature.

The notion of comparative imprecision aversion could itself be compared to the one found in Epstein (1999) and Ghirardato and Marinacci (2002). The latter define comparative ambiguity aversion using constant acts. They therefore need to control for risk attitudes in a separate manner and in the end, can compare (with respect to their ambiguity attitudes) only decision makers that have the same utility functions.<sup>1</sup> Epstein (1999) uses in place of our bets in the definition of comparative uncertainty aversion, acts that are measurable with respect to an exogenously defined set of unambiguous events. As a consequence, in order to be compared, preferences of two decision makers have to coincide on the set of unambiguous events. If the latter is rich enough, utility functions then coincide. Our notion of comparative imprecision aversion, based on the comparison of bets under precise and imprecise information does not require utility functions to be the same when comparing two decision makers. Said differently, risk attitudes are simply irrelevant to the imprecision aversion comparison.

The functional form axiomatized in Section 4 appears in some previous work (Gajdos, Tallon, and Vergnaud (2004) and Tapking (2004)), based on a rather different set of axioms and in a more limited setting. Kopylov (2006) also axiomatizes this functional form, for a fixed information-possibility set. In a setting similar to ours, Giraud (2006) axiomatizes a model in which the decision maker has non additive second order beliefs.

Finally, we compare our approach with Klibanoff, Marinacci, and Mukerji (2005). They

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<sup>1</sup>They actually mention that if one wants to compare two decision makers with different utility functions, one has first to completely elicit them.



provide a fully subjective model of ambiguity aversion, in which attitude towards ambiguity is captured by a smooth function over the expected utilities associated with a set of priors. The latter, as in Gilboa and Schmeidler (1989) is subjective. Hence, although their model allows for a flexible and explicit modeling of ambiguity attitudes, there is no link between the subjective set of priors and the available information. Interestingly, part of Klibanoff, Marinacci, and Mukerji (2005)’s motivation is similar to ours, that is disentangling ambiguity attitude from the information the decision maker has. Formally, however, this separation holds in their model only if one makes the extra assumption that subjective beliefs coincide with the objective information available. In particular, comparative statics are more transparent in our model, as information can be exogenously changed. At a more conceptual level, Klibanoff, Marinacci, and Mukerji (2005)’s approach assumes that all uncertainty is eventually reduced to subjective probabilities, although on two different levels: essentially, the decision maker has in mind a second order probability distribution, but does not perform reduction of lotteries. The criterion they obtained is smooth and appeals only to probabilistic tools, which should make it easy to use in economic applications. Besides the different specific modeling choices, our conceptual departure from their approach is that we do not assume that, even subjectively, imprecise information can be reduced to probabilities (even of a second or higher order). In that sense we are more in line with Ellsberg (2001)’s view, that when a decision maker lacks a determinate probability distribution over states, “there will correspond [to any available option], in general, a set of expected utility values, among which he cannot discriminate in terms of definite probabilities”.

## 2 Extended multiple-priors model

We start with a benchmark model that extends the multiple-priors model by Gilboa and Schmeidler (1989) into the variable information setting.

Let  $\Omega = \mathbb{N}$  be the countable set of all the *potential* states of the world. Let  $\mathcal{S}$  be the family of nonempty and finite subsets of  $\Omega$ . For each  $S \in \mathcal{S}$ , denote the set of probability measures over  $S$  by  $\Delta(S)$ . Let  $\mathcal{P}(S)$  be the family of compact and convex subset of  $\Delta(S)$ , where the compactness is defined with regard to the Euclidian space  $R^S$ . Let  $\mathcal{P}$  be the family of *probability-possibility sets*, that is defined by  $\mathcal{P} = \bigcup_{S \in \mathcal{S}} \mathcal{P}(S)$ . For each  $P \in \mathcal{P}$ , its support is denoted by  $supp(P)$ .

When told  $P$ , the DM is assumed to know only that the true probability lies in  $P$ . When a probability-possibility set is given as a singleton typically denoted by  $\{p\}$ , the DM knows the true probability precisely and we say there is *precise information*.

The space of probability-possibility sets  $\mathcal{P}$  is a mixture space under the operation defined

by

$$\lambda P + (1 - \lambda)Q = \{\lambda p + (1 - \lambda)q : p \in P, q \in Q\}.$$

The set of pure outcomes is denoted by  $X$ . Let  $\Delta^*(X)$  be the set of simple lotteries (probability measures with finite supports) over  $X$ . Let  $\mathcal{F} = \{f : \Omega \rightarrow \Delta^*(X)\}$  be the set of *lottery acts*. Without loss, any lottery is viewed as a constant act which delivers that lottery regardless of states.

The domain of objects of choice is  $\mathcal{P} \times \mathcal{F}$ . The DM has a preference relation over  $\mathcal{P} \times \mathcal{F}$ , which is denoted by  $\succsim$ . The DM compares pairs of probability-possibility sets and acts. When

$$(P, f) \succsim (Q, g),$$

the DM prefers choosing  $f$  under  $P$  to choosing  $g$  under  $Q$ . When  $Q = P$ , the preference relation represents the ranking of acts given the information embodied by  $P$ . When  $g = f$ , the preference relation represents the ranking of probability-possibility sets given the action embodied by  $f$ .

We introduce the axioms. The first two axioms are quite standard.

**Axiom 1** (Order) The preference relation  $\succsim$  is complete and transitive.

**Axiom 2** (Act Continuity) For every  $P \in \mathcal{P}$  and  $f, g, h \in \mathcal{F}$ , if  $(P, f) \succ (P, g) \succ (P, h)$ , then there exist  $\alpha$  and  $\beta$  in  $(0, 1)$  such that

$$(P, \alpha f + (1 - \alpha)h) \succ (P, g) \succ (P, \beta f + (1 - \beta)h).$$

The third axiom states that the preference over lotteries is independent of information sets and is nondegenerate. When a lottery is given regardless of states of the world, information about their likelihood is irrelevant. Also, we exclude the case in which the DM is indifferent between all lotteries.

**Axiom 3** (Outcome Preference) (i) For every  $P, Q \in \mathcal{P}$  and  $l \in \Delta^*(X)$ ,  $(P, l) \sim (Q, l)$ , and (ii) there exist  $P \in \mathcal{P}$  and  $l, m \in \Delta^*(X)$  such that  $(P, l) \succ (P, m)$ .

For probability  $\{p\} \in \mathcal{P}$  and act  $f \in \mathcal{F}$ , define the induced distribution over outcomes by

$$l(p, f) = \sum_{\omega \in \text{supp}(\{p\})} p(\omega) f(\omega).$$

The next axiom states that the evaluation of an act under *precise* information depends only on its induced distribution. Notice that we do not assume the counterpart of this for general information sets. In general, two probability-possibility set/act pairs may be differently evaluated even if they induce the same sets of distributions over outcomes.

**Axiom 4** (Reduction under Precise Information) For every  $\{p\} \in \mathcal{P}$  and  $f \in \mathcal{F}$ ,

$$(\{p\}, f) \sim (\{p\}, l(p, f)).$$

The following two axioms are parallel to those in Gilboa and Schmeidler (1989).

**Axiom 5** (c-Independence) For every  $f, g \in \mathcal{F}, l \in \Delta^*(X), \lambda \in (0, 1)$  and  $P \in \mathcal{P}$ ,

$$(P, f) \succeq (P, g) \implies (P, \lambda f + (1 - \lambda)l) \succeq (P, \lambda g + (1 - \lambda)l)$$

**Axiom 6** (Uncertainty aversion): For every  $f, g \in \mathcal{F}, \lambda \in (0, 1)$  and  $P \in \mathcal{P}$ ,

$$(P, f) \sim (P, g) \implies (P, \lambda f + (1 - \lambda)g) \succeq (P, f)$$

The next axiom states that if one act is preferable to another under every element of the information set, the ranking is unchanged under the whole set.

**Axiom 7** (Dominance) For every  $f, g \in \mathcal{F}$  and  $P \in \mathcal{P}$ ,

$$(\{p\}, f) \succeq (\{p\}, g) \text{ for every } p \in P \implies (P, f) \succeq (P, g).$$

We show in the Appendix that this axiom, together with (Reduction under Precise Information), implies Gilboa and Schmeidler's monotonicity axiom.

The last axiom for the benchmark is a von-Neumann Morgenstern type independence condition for information sets.

**Axiom 8** (Information Independence) For every  $P, P', Q \in \mathcal{P}, f \in \mathcal{F}$ , and  $\lambda \in (0, 1)$ ,

$$(P, f) \succ (P', f) \implies (\lambda P + (1 - \lambda)Q, f) \succ (\lambda P' + (1 - \lambda)Q, f).$$

To interpret, consider an object ' $\lambda \circ P + (1 - \lambda) \circ Q$ '. Given this, the DM knows that ' $P$  is true with probability  $\lambda$  and  $Q$  is true with probability  $1 - \lambda$ .' Suppose the DM is to evaluate the two objects  $\lambda \circ P + (1 - \lambda) \circ Q$  and  $\lambda P' \circ + (1 - \lambda) \circ Q$  in choosing an act. Then, the difference between them is only in  $P$  and  $P'$  which are true 'with probability  $\lambda$ '. It is intuitive that the ranking of these should depend only on the ranking of  $P$  and  $P'$ , and the common information set  $Q$  being true 'with probability  $1 - \lambda$ ' should not matter.

We explain why the axiom permits the above interpretation, by arguing how the DM identifies the object  $\lambda \circ P + (1 - \lambda) \circ Q$  with  $\lambda P + (1 - \lambda)Q$ . The argument consists of two steps.

First, the DM is indifferent between  $\lambda \circ P + (1 - \lambda) \circ Q$  and ' $\{\lambda \circ p + (1 - \lambda) \circ q : p \in P, q \in Q\}$ .' When the latter is given, the DM knows it is possible that  $p$  is true with probability  $\lambda$  and  $q$  is true with probability  $1 - \lambda$ , for each  $p \in P$  and  $q \in Q$ .

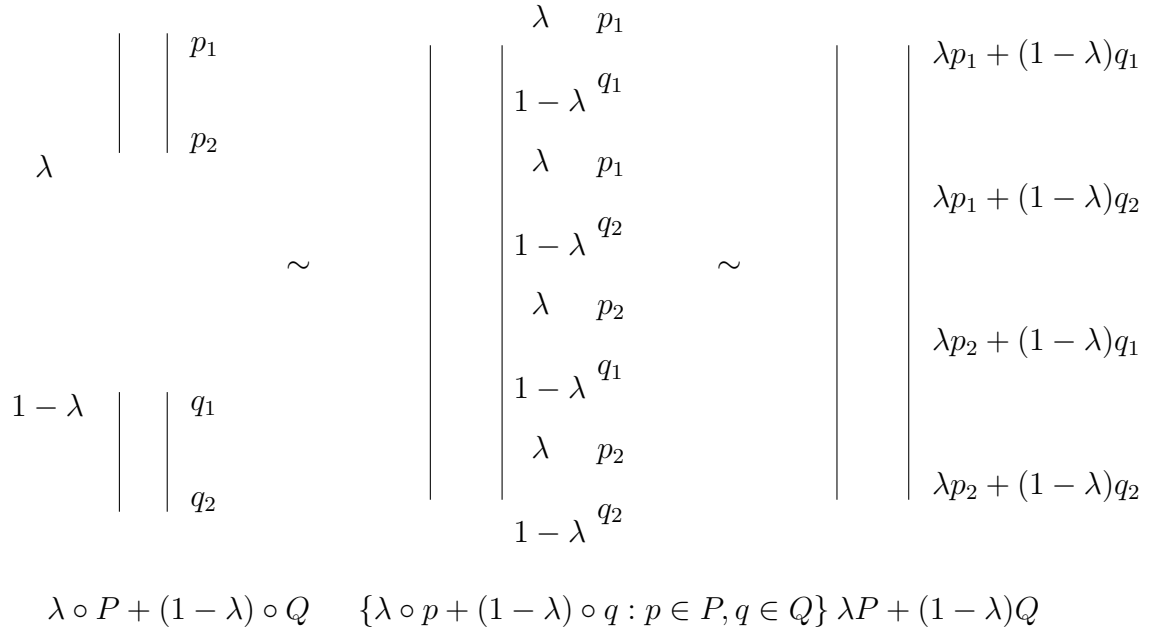


Figure 1: Information Independence

There are two kinds of uncertainty here. One is about outcome of randomization, which is risk, and the other is about ultimate realization of true probability law. We assume that the DM is indifferent in the order of these two uncertainties (see the left half of Figure 1).

Second, compare  $\{\lambda \circ p + (1 - \lambda) \circ q : p \in P, q \in Q\}$  and  $\lambda P + (1 - \lambda)Q$ . The former is the set of compound probabilities, and the latter is that of their reduced ones. We assume that the DM is indifferent in the reduction of compound probabilities. That is, she is indifferent in the timing of resolution of risk, which is assumed in the standard theory (see the right half of Figure 1).

Thus, we deduce that the DM views three objects  $\lambda \circ P + (1 - \lambda) \circ Q$ ,  $\{\lambda \circ p + (1 - \lambda) \circ q : p \in P, q \in Q\}$  and  $\lambda P + (1 - \lambda)Q$  to be the same, and that the axiom allows the interpretation discussed above.

**Theorem 1** The preference relation  $\succsim$  satisfies Axioms 1 to 8 if and only if there exists a function  $U : \mathcal{P} \times \mathcal{F} \rightarrow R$  which represents  $\succsim$  and a mixture-linear and continuous function  $u : \Delta(X) \rightarrow R$  and a mapping  $\varphi : \mathcal{P} \rightarrow \mathcal{P}$  such that

$$U(P, f) = \min_{p \in \varphi(P)} \sum_{\omega \in \Omega} u(f(\omega)) p(\omega)$$

where  $\varphi$  satisfies

1. (Selection):  $\varphi(P) \subset P$  for every  $P \in \mathcal{P}$ ,

2. (Mixture-linearity):  $\varphi(\lambda P + (1 - \lambda)Q) = \lambda\varphi(P) + (1 - \lambda)\varphi(Q)$  for every  $P, Q \in \mathcal{P}$  and  $\lambda \in [0, 1]$ ,

Moreover,  $u$  is unique up to positive linear transformations and  $\varphi$  is unique.

We purposely kept as close as possible to the original axioms of Gilboa and Schmeidler (1989). In particular, we kept their two key axioms (c-Independence) and (Uncertainty Aversion). We will argue in the next section that the latter can be replaced by a more explicit representation of the agent's attitude toward imprecision of the available information. (Dominance) and (Independence) are more or less orthogonal to the rest of the axioms, in the sense that they are used only to put some discipline on the  $\varphi$  function. It might be worth mentioning that (Independence) does not imply (c-Independence).

### 3 Imprecision aversion

In this Section, we take advantage of the setting developed so far to give a new foundation for the uncertainty aversion axiom, showing that it is implied by an axiom of aversion toward imprecision. The latter compares an act under two different probability-possibility settings and states that the decision maker always prefers the more precise information. We therefore have to define a notion of imprecision on sets of probability distributions. The most natural definition would be that  $P$  is more precise than  $Q$  whenever  $P \subset Q$ . This is actually the definition proposed by Wang (2003). However, this definition turns out to be much too strong. Indeed, the idea behind the notion of aversion toward imprecision is that an imprecision averse decision maker should *always* prefer a more precise information, whatever the act under consideration. Consider an act  $f$  for which the worst outcome is obtained, say, in state 1. Then, Wang's notion of precision would force the decision maker to prefer  $(\{(1, 0, \dots)\}, f)$ , that is, putting probability one on the worst outcome to  $(\Delta(\{1, 2\}), f)$ , that is, being totally uncertain about the state; a feature of the axiom which is very unlikely and unappealing.

On the other hand, it is clear that a set being more precise than another has something to do with set inclusion. The following definition restricts the inclusion condition to some sets of probability distributions that are comparable in some sense, exactly as the comparison of two distributions in terms of risk focusses on distributions that have the same mean.

**Definition 1** Let  $P, Q \in \mathcal{P}$ . Say that  $P$  is *conditionally more precise* than  $Q$  if

- $P \subset Q$  and,
- there exists a partition  $(E_1, \dots, E_n)$  of  $S$  such that
  - (i)  $\forall p \in P, \forall q \in Q, p(E_i) = q(E_i)$  for all  $i = 1, \dots, n$ ,

(ii)  $co\{p(\cdot|E_i); p \in P\} = co\{q(\cdot|E_i); q \in Q\}$  for all  $i = 1, \dots, n$ .

Note that this notion is very weak in the sense that it is very incomplete. For instance, an  $n$ -dimensional simplex cannot be compared through this definition with any of its subsets. Indeed, two sets  $\mathcal{P}$  and  $\mathcal{Q}$ , ordered by set inclusion, can be compared only if there exists a partition of the state space on which they agree *and* have precise probabilities (item (i) of the definition), and furthermore, conditionally on each cell of this partition, they give the same information (item (ii) of the definition). This means that the extra information contained in  $\mathcal{P}$  is about some correlation between what happens in one cell  $E_i$  with what happens in another cell  $E_j$ . Said differently, the extra information is orthogonal to the “initial” probabilistic information, reflected in the fact that the cells of the partition have precise probabilities attached to them.

Take for instance

$$\mathcal{Q} = \left\{ \left( p, \frac{1}{2} - p, q, \frac{1}{2} - q \right) \mid p \in \left[ 0, \frac{1}{2} \right], q \in \left[ 0, \frac{1}{2} \right] \right\}$$

and consider

$$\mathcal{P}_\alpha = \left\{ \left( p, \frac{1}{2} - p, q, \frac{1}{2} - q \right) \mid p \in \left[ 0, \frac{1}{2} \right], q \in \left[ 0, \frac{1}{2} \right], |q - p| \leq \alpha \right\}$$

where  $\alpha \in [0, \frac{1}{4}]$ .

One obviously has  $\mathcal{P}_\alpha \subset \mathcal{P}_{\alpha'}$  for all  $\alpha' \geq \alpha$ , and  $\mathcal{P}_{1/4} = \mathcal{Q}$ .

Furthermore,  $\{\{1, 2\}, \{3, 4\}\}$  is a partition of the state space such that  $\forall p \in \mathcal{P}, \forall q \in \mathcal{Q}, p(E_i) = q(E_i)$  since  $p(\{1, 2\}) = q(\{1, 2\}) = \frac{1}{2}$ ,  $p(\{3, 4\}) = q(\{3, 4\}) = \frac{1}{2}$ ,  $\forall p \in \mathcal{P}_\alpha, \forall q \in \mathcal{Q}$ . It is also easily checked that the set of probabilities conditional on  $\{1, 2\}$  is the same when computed starting from  $\mathcal{P}_\alpha$  and from  $\mathcal{Q}$ . The same is true for conditionals with respect to  $\{3, 4\}$ . Thus, the two requisite of the definitions are met and we can assert that  $\mathcal{P}_\alpha$  is conditionally more precise than  $\mathcal{Q}$ . The nature of the extra information that is present in  $\mathcal{P}_\alpha$  is maybe clearest for  $\alpha = 0$ . In that case, one has  $q = p$  and the extra information that is present in  $\mathcal{P}_0$  is a strong correlation between the different cells of the partition. More generally, we can look at upper and lower probabilities for events according to  $\mathcal{P}_\alpha$  and  $\mathcal{Q}$ . We know they agree on the partition  $\{\{1, 2\}, \{3, 4\}\}$ . One can also check that the upper and lower probabilities on the events  $\{1, 3\}$  and  $\{2, 4\}$  are the same for the two sets (0 and 1 respectively). However, the lower and upper probability of events  $\{2, 3\}$  and  $\{1, 4\}$  do differ for the two sets. One has, with obvious notation,  $\underline{p}_\alpha(\{2, 3\}) = 1/2 - \alpha$  and  $\bar{p}_\alpha(\{2, 3\}) = 1/2 + \alpha$  while  $\underline{q}(\{2, 3\}) = 0$  and  $\bar{q}(\{2, 3\}) = 1$ , and similarly for event  $\{1, 4\}$ . The fact that  $\underline{p}_\alpha > \underline{q}$  and  $\bar{p}_\alpha < \bar{q}$ , is another way to see that  $\mathcal{P}_\alpha$  is more precise than  $\mathcal{Q}$ .

We can now state our axiom.

**Axiom 9** (Aversion toward Imprecision) Let  $P, Q \in \mathcal{P}$  be such that  $P$  is conditionally more precise than  $Q$ , then for all  $f \in \mathcal{F}$ ,  $(P, f) \succeq (Q, f)$ .

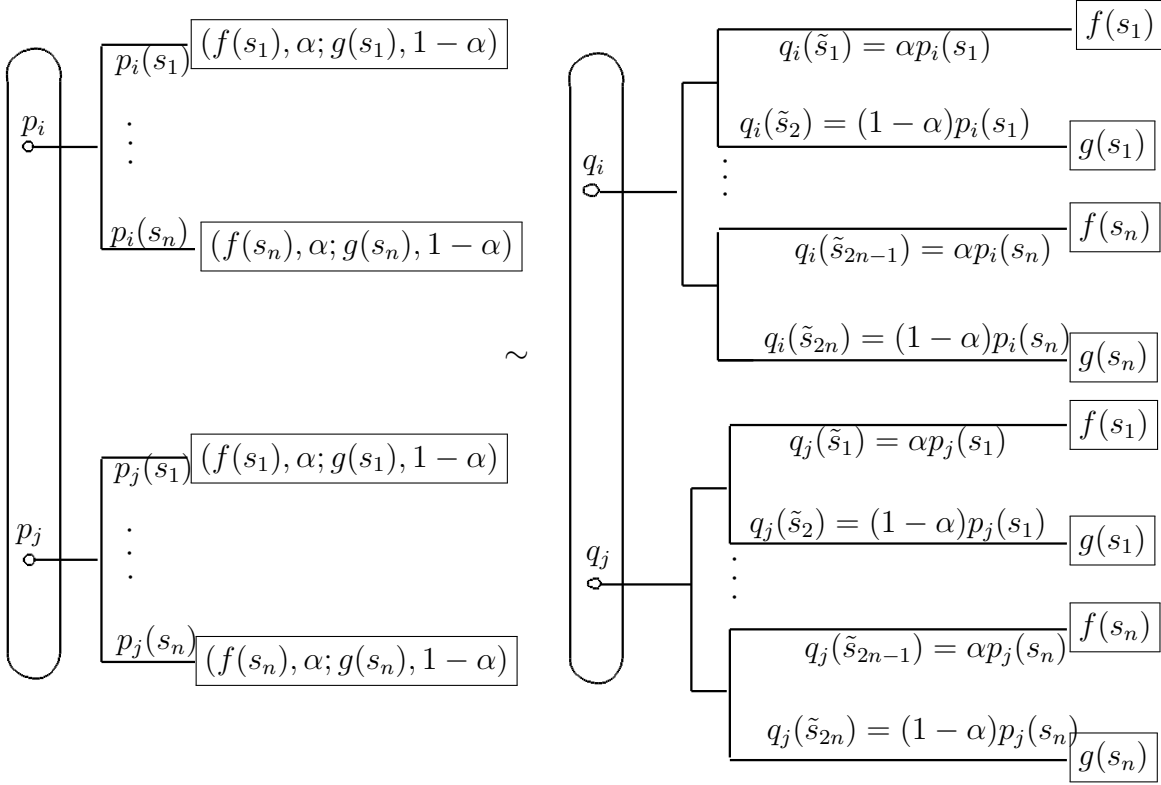


Figure 2: Decomposition Indifference

**Remark 1** Assume Theorem 1 and (Aversion toward Imprecision) hold. Then, for any  $P, Q \in \mathcal{P}$  such that  $P$  is conditionally more precise than  $Q$ ,  $\varphi(P) \subseteq \varphi(Q)$ .

To fully exploit this axiom we need an extra axiom, which embeds some mild invariance property.

**Axiom 10** (Decomposition Indifference) Let  $f, g, h \in \mathcal{F}$  and  $P, Q \in \mathcal{P}$ . If

- $h(\omega) = f(\frac{\omega+1}{2})$  if  $\omega$  is odd, and  $h(\omega) = g(\frac{\omega}{2})$  if  $\omega$  is even and,
- $Q = \{q \mid \exists p \in P \text{ s.th. } q(\omega) = \alpha p(\frac{\omega+1}{2}) \text{ if } \omega \text{ is odd and } q(\omega) = (1-\alpha)p(\frac{\omega}{2}) \text{ if } \omega \text{ is even}\}$

for some  $\alpha \in [0, 1]$ , then  $(P, \alpha f + (1-\alpha)g) \sim (Q, h)$

The act  $(P, \alpha f + (1-\alpha)g)$  can be interpreted by saying that a state  $s$  is determined, according to some unknown probability  $p(s)$  that belongs to  $P$ . Then, once the state is realized, a roulette lottery or a coin flip takes place with odds  $\alpha : (1-\alpha)$ . As illustrated on figure 2, the act  $(Q, h)$  can then be seen as “collapsing” this roulette lottery in the probability distribution that determines the state. Said differently, the state now incorporates whether the coin toss ended up heads or tails. Each state is now split in two: state  $s$  is split into (state  $s$ , heads) and (state  $s$ , tails). Thus, the axiom says, this operation is neutral for the decision maker as it does not modify the timing of the process: uncertainty first and

then risk. In spirit, this axiom is very similar to the usual reduction of compound lottery axiom.

**Theorem 2** Under (Decomposition Indifference) and (Independence), (Aversion toward Imprecision) implies (Uncertainty Aversion).

## 4 Contraction representation: Axiomatic foundation

In this section we provide an axiomatic characterization of the contraction representation.

The contraction representation is characterized by a stronger invariance axiom and an additional continuity axiom. The invariance condition roughly says that the DM's attitude toward information should not change under some transformations of the state space (and probability simplex) that do not change attitude toward any imprecise information. This is interpreted as a requirement for a sophisticated attitude toward imprecise information.

Recall that for each  $S \in \mathcal{S}$ ,  $\Delta(S)$  is a compact subset of the Euclidian space  $\mathbb{R}^{|S|}$ , and  $\mathcal{P}(S)$  is a compact metric space with regard to the Hausdorff metric.

For each  $S \in \mathcal{S}$ , let  $\psi : \Delta(S) \rightarrow \Delta(S)$  be such a transformation, and let  $\tilde{\psi} : \mathcal{F} \rightarrow \mathcal{F}$  be the transformation of acts associated with  $\psi$ . Transformation  $\psi$  is assumed to satisfy

$$(\{p\}, f) \succeq (\{q\}, f) \Rightarrow (\{\psi(p)\}, \tilde{\psi}(f)) \succeq (\{\psi(q)\}, \tilde{\psi}(f)).$$

for every  $p, q \in \Delta(S)$  and  $f \in \mathcal{F}$ . Then the axiom takes the form that for every  $S \in \mathcal{S}$ , every  $P, Q \in \mathcal{P}(S)$  and  $f \in \mathcal{F}$ ,

$$(P, f) \succeq (Q, f) \Rightarrow (\psi(P), \tilde{\psi}(f)) \succeq (\psi(Q), \tilde{\psi}(f)).$$

We take such a class of transformations as a subclass of bistochastic matrices, that are stochastic generalization of permutations. An  $|S| \times |S|$ -matrix  $\Pi$  is *S-bistochastic* if it is nonnegative and  $\sum_{\omega \in S} \Pi_{\omega\omega'} = 1$  for each  $\omega' \in S$ , and  $\sum_{\omega' \in S} \Pi_{\omega\omega'} = 1$  for each  $\omega \in S$ . For an *S-bistochastic* matrix  $\Pi$  and  $f \in \mathcal{F}$ , define the transformed act  $\Pi f \in \mathcal{F}$  by  $(\Pi f)(\omega) = \sum_{\omega' \in S} \Pi_{\omega\omega'} f(\omega')$  for each  $\omega \in S$ , and  $(\Pi f)(\omega) = f(\omega)$  for each  $\omega \notin S$ .

Any bistochastic matrix may be expressed as a convex combination of permutation matrices (see Birkoff (1946)). In that sense, it is a stochastic generalization of permutation.

We restrict attention to a subclass of bistochastic matrices that do not change attitude toward *any precise information*.

**Definition 2** A bistochastic transformation  $\Pi$  is *S-unitary* if for every  $p, q \in \Delta(S)$  and  $f \in \mathcal{F}$ ,

$$(\{p\}, f) \succeq (\{q\}, f) \implies (\{\Pi p\}, \Pi f) \succeq (\{\Pi q\}, \Pi f).$$

Denote the set of all *S-unitary* transformations by  $\mathcal{T}(S)$ .



The following lemma shows that the class of unitary transformation is non-empty and characterized as a conjunction of bistochastic transformation and similarity transformation.

**Lemma 1** Assume axioms 1 to 4 and 8. Then, any bistochastic transformation  $\Pi$  is  $S$ -unitary if and only if there exists  $\lambda \geq 0$  such that  $\|\Pi p - \Pi q\|_S = \lambda \|p - q\|_S$  for any  $p, q \in \Delta(S)$ , where  $\|\cdot\|_S$  denotes the Euclidian norm in  $\mathbb{R}^{|S|}$ .

We state the axiom.

**Axiom 11** (Invariance to Unitary Transformations): For every  $S \in \mathcal{S}$ , any  $\Pi(S) \in \mathcal{T}(S)$ ,  $f \in \mathcal{F}$  and  $P, Q \in \mathcal{P}$ ,

$$(P, f) \succsim (Q, f) \implies (\Pi P, \Pi f) \succsim (\Pi Q, \Pi f).$$

To interpret, suppose that the DM prefers  $P$  to  $Q$  given  $f$ . Then under axioms 1 to 4 and 8,  $\Pi f$  induces the same ranking of probabilities in the new simplex as  $f$  does in the original one. That is, for any  $p \in P$  and  $q \in Q$ ,

$$(\{p\}, f) \succsim (\{q\}, f) \implies (\{\Pi p\}, \Pi f) \succsim (\{\Pi q\}, \Pi f),$$

and

$$(\{q\}, f) \succsim (\{p\}, f) \implies (\{\Pi q\}, \Pi f) \succsim (\{\Pi p\}, \Pi f).$$

Thus in the new probability simplex when given  $\Pi f$ ,  $\Pi P$  and  $\Pi Q$  play the same roles as  $P$  and  $Q$  do in the original one when given  $f$ . Therefore, it is intuitive that the ranking of information sets is unchanged, which leads to the ranking  $(\Pi P, \Pi f) \succsim (\Pi Q, \Pi f)$ .

We also consider a continuity axiom with regard to probability-possibility sets.

**Axiom 12** (Information Continuity): For every  $S \in \mathcal{S}$ ,  $f \in \mathcal{F}$  and  $P \in \mathcal{P}(S)$ , the sets  $\{Q \in \mathcal{P}(S) : (Q, f) \succsim (P, f)\}$  and  $\{Q \in \mathcal{P}(S) : (P, f) \succsim (Q, f)\}$  are closed with regard to the Hausdorff metric.

Now we provide the contraction representation in which the subjective set of priors is obtained by (i) solving for the ‘center’ of the probability-possibility set, and (ii) shrinking the set toward the center to a degree given by preference. The ‘center’ is the *Steiner point*. Imagine that a vector  $v$  is drawn from the unit sphere around the origin according to the uniform distribution. Then the Steiner point of set  $P$ , denoted by  $s(P)$ , is the expected maximizer of  $pv$  over  $p \in P$ .

More formally, fix  $S \in \mathcal{S}$  and let  $e = (\frac{1}{|S|}, \dots, \frac{1}{|S|})$  and  $V = \{v \in R^S : (v - e)e = 0, \|v - e\| = 1\}$  be the  $|S| - 2$  dimensional unit sphere around  $e$ . For  $P \in \mathcal{P}(S)$ , its Steiner point is defined by

$$s(P) = \int_V \arg \max_{p \in P} \langle p, v - e \rangle \mu(dv)$$

where  $\mu$  is the uniform distribution over  $V$ .<sup>2</sup>

<sup>2</sup>Multiplicity of maximizers inside the integral does not matter since uniform distribution is non-atomic.

**Example 1** Steiner point of a segment is its midpoint.

**Example 2** Steiner point of a polytope is the weighted average of its vertices, in which the weight for each vertex is proportional to its outer angle.

**Example 3** When a probability-possibility set is given as the core of a lower probability (convex capacity), its Steiner point coincides with the Shapley value of the lower probability. This is not surprising since in the domain of convex capacities Shapley value is the unique single-valued selection of the core that satisfies mixture independence and permutation invariance.

We state the contraction representation result.

**Theorem 3** The preference relation  $\succsim$  satisfies Axioms 1 to 8, and 11 if and only if we have the representation as in Theorem 1 with the additional property that for every  $S \in \mathcal{S}$ , and  $P \in \mathcal{P}(S)$

$$\varphi(P) = (1 - \varepsilon)\{s(P)\} + \varepsilon P$$

with  $\varepsilon \in [0, 1]$  that is unique.

Notice that the rate  $\varepsilon$  is constant for every probability-possibility set with finite support.

## 5 Comparative imprecision aversion

One of the advantage of our setting is that it allows a clean separation between imprecision neutrality and the absence of imprecision. The latter is a feature of the information the decision maker possesses, while imprecision neutrality is characterized by the fact that the decision maker subjective set of priors is reduced to a singleton, i.e., even though faced with imprecise information, the decision maker behaves as a subjective expected utility maximizer. In this Section, we characterize a notion of comparative imprecision aversion.

### 5.1 Definition and characterization

In line with the notion of comparative risk aversion, one can define comparative imprecision aversion by saying that a decision maker  $b$  is more averse toward imprecision if whenever he prefers an act under a singleton probability-possibility set over the same act under a general probability-possibility set, so does decision maker  $a$ . Furthermore, one would like to separate out this attitude toward imprecision from the traditional attitude toward risk. In order to do that, one has to define carefully the set of acts for which the definition applies.

A fairly weak notion is the following:

**Definition 3** Let  $\succeq_a$  and  $\succeq_b$  be two preference relations defined on  $\mathcal{P} \times \mathcal{F}$ . Suppose there exist two prizes,  $\bar{x}$  and  $\underline{x}$  in  $X$  such that both  $a$  and  $b$  strictly prefer  $\bar{x}$  to  $\underline{x}$ . We say that  $\succeq_b$  is *more averse to bet imprecision than*  $\succeq_a$  if for all  $E \subset S$ ,  $P \in \mathcal{P}$ , and  $\{p\} \in \mathcal{P}$ ,

$$(\{p\}, \bar{x}_E \underline{x}) \succeq_a [\succ_a](P, \bar{x}_E \underline{x}) \Rightarrow (\{p\}, \bar{x}_E \underline{x}) \succeq_b [\succ_b](P, \bar{x}_E \underline{x})$$

In this definition, the comparison bears only on bets.

**Theorem 4** Assume  $\Omega = \mathbb{N}$ . Let  $\succeq_a$  and  $\succeq_b$  be two preference relations defined on  $\mathcal{P} \times \mathcal{F}$ , satisfying Axioms 1 to 8 as well as Axiom 10. Then, the following assertions are equivalent:

- (i)  $\succeq_b$  is more averse to bet imprecision than  $\succeq_a$ ,
- (ii) for all  $P \in \mathcal{P}$ ,  $\varphi^a(P) \subset \varphi^b(P)$ .

**Remark 2** In Theorem 4, one uses (Decomposition Indifference) to be able to “duplicate” information sets, which is why we need countably infinitely many states.

**Remark 3** Under the representation Theorem 3, a decision maker  $b$  who is more averse to bet imprecision than a decision maker  $a$  will indeed have  $\varepsilon^b > \varepsilon^a$ .

This notion of aversion to imprecision ranks preferences that do not necessarily have the same attitudes toward risk. This is of particular interest in applications if one wants to study the effects of risk aversion and imprecision aversion separately. For instance, one might want to compare portfolio choices of two agents, one being less risk averse but more imprecision averse than the other. This type of comparison cannot be done if imprecision attitudes can be compared only among preferences that have the same risk attitude, represented by the utility function. To the best of our knowledge, there is no available result in the literature that achieves this separation of the characterization of comparative ambiguity or imprecision attitudes from risk attitudes.

## 5.2 Imprecision premium

We define here a notion of imprecision premium which captures how much an agent is “willing to lose” when betting on an event in order to be in a setting that has no imprecision. More precisely, consider a preference relation  $\succeq$  and let  $\bar{x}$  and  $\underline{x}$  be two prizes in  $X$  such that  $\bar{x} \succ \underline{x}$ . For any event  $E \subset S$ , let  $q^E$  be a probability distribution such that  $(P, \bar{x}_E \underline{x}) \sim (\{q^E\}, \bar{x}_E \underline{x})$ . Under Axioms 1 to 6, such a probability distribution exists and is independent of  $\bar{x}$  and  $\underline{x}$ , since  $(P, \bar{x}_E \underline{x}) \sim (\{q^E\}, \bar{x}_E \underline{x})$  if, and only if,  $q^E(E) = \min_{p \in \varphi(P)} p(E)$ .

**Definition 4** For any  $P \in \mathcal{P}$  and for any event  $E \subset S$ , let

- the *absolute imprecision premium*,  $\pi^A(E, P)$  be defined by  $s(P)(E) - q^E(E)$ ,

- the *relative imprecision premium*,  $\pi^R(E, P)$  be defined by  $\frac{\pi^A(E, P)}{s(P)(E) - \text{Min}_{p \in \mathcal{P}} p(E)}$  whenever  $s(P)(E) \neq \text{Min}_{p \in \mathcal{P}} p(E)$ .

The absolute premium is thus the mass of probability on the good event that the agent is willing to forego in order to act in precise situation rather than with the imprecise probability-possibility set  $P$ . The precise reference in that definition is rather naturally taken to be the Steiner point of the set  $P$ . An analogy with the risk premium can be drawn as follows:  $s(P)(E)$  plays the role of the expectation of the risky prospect while  $q^E(E)$  plays the role of the certainty equivalent.

**Theorem 5** Let  $\succeq_a$  and  $\succeq_b$  be two preference relations defined on  $\mathcal{P} \times \mathcal{F}$ , satisfying axioms 1 to 8 and 10. Then, the following assertions are equivalent:

- (i)  $\succeq_b$  is more averse to bet imprecision than  $\succeq_a$ ,
- (ii) for all  $P \in \mathcal{P}$ ,  $\varphi^a(P) \subset \varphi^b(P)$ .
- (iii) for all  $P \in \mathcal{P}$ , for all event  $E \subset S$ ,  $\pi_b^A(E, P) \geq \pi_a^A(E, P)$ .

Hence, as in the theory of risk aversion, one can capture by a single number the comparison of imprecision attitude. Furthermore, one can also define a notion of constant relative imprecision premium, again in a similar fashion as for the relative risk premium.

**Definition 5** A decision maker is said to have constant relative imprecision premium  $\theta$  if for any  $P \in \mathcal{P}$  and for any event  $E \subset S$  such that  $s(P)(E) \neq \text{Min}_{p \in \mathcal{P}} p(E)$ ,  $\pi^R(E, P) = \theta$ .

Note that the functional form axiomatized in Section 4 satisfies constant relative imprecision premium. Actually, the converse is true under Axiom 10.

**Theorem 6** Consider a decision maker satisfying Axioms 1 to 8 and 10. The following assertions are equivalent:

- (i) the decision maker has constant relative imprecision premium  $\varepsilon$ ,
- (ii) for all  $P \in \mathcal{P}$ ,  $\varphi(P) = \varepsilon P + (1 - \varepsilon) \{s(P)\}$ .

This Theorem therefore gives another foundation for the functional form of Theorem 3, based this time not on invariance properties, but rather on constant relative attitude toward imprecision.

## 6 Example

We develop in this section a simple application of our analysis to portfolio choice that is similar in spirit to Klibanoff, Marinacci, and Mukerji (2005)'s. There are three assets,  $a$ ,

$b$ , and  $c$ . The following table gives the payoff matrix

s	1	2	3	4
a	k	k	k	k
b	$\bar{b}$	$\bar{b}$	1	1
c	$\bar{c}$	1	1	$\bar{c}$

We put the following restrictions on the parameters:  $\bar{c} > \bar{b} > k > 1$ . The information available is given by the set

$$\mathcal{P}_\alpha = \left\{ \left( p, \frac{1}{2} - p, q, \frac{1}{2} - q \right) \mid p \in \left[ 0, \frac{1}{2} \right], q \in \left[ 0, \frac{1}{2} \right], |q - p| \leq \alpha \right\}$$

where  $\alpha \in [0, \frac{1}{2}]$ . Hence, the probability of  $\{1, 2\}$  is precise, equal to  $1/2$  and similarly for  $\{3, 4\}$ .  $\alpha$  is a measure of how “imprecise the set is”: a higher  $\alpha$  corresponds to a higher degree of imprecision. Taken with this information, the assets have a natural interpretation: asset  $a$  is the safe asset,  $b$  is the “risky” asset as its payoffs are measurable with respect to the partition  $\{\{1, 2\}, \{3, 4\}\}$ , and asset  $c$  is the “imprecise” asset.

We consider a decision maker with CARA utility function  $u(w) = -e^{-\gamma w}$ , where  $\gamma$  is the coefficient of absolute risk aversion. The transformed set is given by:

$$\mathcal{F}(\mathcal{P}_\alpha) = \left\{ \left( p, \frac{1}{2} - p, q, \frac{1}{2} - q \right) \mid p \in \left[ \frac{1}{4} - \theta, \frac{1}{4} + \theta \right], q \in \left[ \frac{1}{4} - \theta, \frac{1}{4} + \theta \right], |q - p| \leq \alpha \right\}$$

$\theta$  is the parameter of imprecision aversion, in the sense that it gives the rate of contraction for the simplex  $\Delta(\{1, 2\})$ . For simplicity, we assume that the constraint on the distance between  $p$  and  $q$  is the same in the transformed set as in the information set (it is easy to generalize to a constraint of the type  $|q - p| \leq \beta(\alpha)$  with  $\beta(\cdot)$  increasing in  $\alpha$ .) To make things interesting, we assume that  $\theta \geq \alpha/2$ , so that the constraint  $|q - p| \leq \alpha$  is effective in the computation of the optimal portfolio (although see Remark 4 below.)

The decision maker has one unit of wealth that he has to allocate among the three assets. We allow for short sales. We consider successively three cases depending on which assets are actually available, the first case being the benchmark situation of choice between the safe and the risky asset.

Case 1: choice between safe and risky asset.

This case is the usual one and one gets that  $b^* = \frac{1}{\gamma(1-b)} \log \left( \frac{k-1}{b-k} \right)$ , which is naturally independent from the parameters  $\theta$  and  $\alpha$ . Under the parameter restrictions, it is easy to see that increasing risk aversion decreases holding of the risky asset.

Case 2: choice between safe and imprecise asset.

The problem to be solved here is to find the optimal amount of the imprecise asset, i.e., the solution to:  $\max_c \min_{\pi \in \mathcal{F}(\mathcal{P}_\alpha)} - [(\pi(1) + \pi(4))e^{-\gamma((1-c)k+c\bar{c})} + (\pi(2) + \pi(3))e^{-\gamma((1-c)k+c)}]$ , or rewritten in terms of  $p$  and  $q$ :

$$\max_c \min_{\mathcal{F}(\mathcal{P}_\alpha)} - [(p + 1/2 - q)e^{-\gamma((1-c)k+c\bar{c})} + (1/2 - p + q)e^{-\gamma((1-c)k+c)}]$$

As long as  $c > 0$ ,  $-e^{-\gamma((1-c)k+c\bar{c})} > -e^{-\gamma((1-c)k+c)}$  and hence the decision maker will “use” the probability in  $\mathcal{F}(\mathcal{P}_\alpha)$  that put the highest weight on the event  $\{2, 3\}$  and lowest weight on  $\{1, 4\}$ . Hence, one wants to minimize  $p - q$ . Let therefore  $q = 1/4 + \theta$  and  $p = 1/4 + \theta - \alpha$ .<sup>3</sup> Solving for the optimal solution yields

$$c^* = \frac{1}{\gamma(\bar{c} - 1)} \log \left( \frac{(\bar{c} - k)(1/2 - \alpha)}{(k - 1)(1/2 + \alpha)} \right)$$

One can check that  $c^*$  is positive as conjectured if  $(k - 1)/(\bar{c} - k) < (1/2 - \alpha)/(1/2 + \alpha)$ . Here, the comparative statics with respect to  $\gamma$  works as in the single risky asset case. What is more interesting, although intuitive, is that the imprecise asset holding is decreasing in  $\alpha$ : an increase in imprecision of the information provided reduces the amount of asset the decision maker wants to hold. Note also that imprecise asset holding does not depend, in this example, on the imprecision aversion parameter  $\theta$  (as long as  $\theta \geq \alpha/2$ ).

### Case 3: choice among all three assets.

This is the more general case and is a bit more tedious to study. Let's write  $u_s$  the utility of the portfolio in state  $s$ . As long as  $b > 0$  and  $c > 0$ , one has that  $u_1 > u_2$  and  $u_4 > u_3$  and furthermore,  $u_4 - u_3 > u_1 - u_2$ . Hence, the minimizing probability that belongs to  $\mathcal{F}(\mathcal{P}_\alpha)$  is given by  $p = 1/4 + \theta - \alpha$  and  $q = 1/4 + \alpha$ .

Let  $K = \frac{(\bar{c}-k)(\bar{b}-1)}{(\bar{c}-b)(k-1)}$ . Under our assumption,  $K > 1$ . Then, the optimal solution can be written:

$$\begin{aligned} b^* &= \frac{1}{\gamma(\bar{b} - 1)} \log \left[ (K - 1) \frac{1/4 - \theta + \alpha}{1/4 + \theta} \right] \\ c^* &= \frac{1}{\gamma(\bar{c} - 1)} \log \left[ \frac{\bar{c} - \bar{b}}{\bar{b} - 1} \left( (K - 1) \frac{1/4 - \theta}{1/4 + \theta} + \frac{1/4 + \theta - \alpha}{1/4 - \theta + \alpha} \right) \right] \end{aligned}$$

Under some further (uninteresting) restrictions on the parameters, one can check that  $b^* > 0$  and  $c^* > 0$  as conjectured when picking the minimizing probability distribution.

One can thus perform comparative statics exercises. As  $\alpha$  increases, that is as the information available is less precise, the decision maker will hold *more* of the risky asset and *less* of the imprecise asset. Thus, there is some form of substitution among assets

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<sup>3</sup>Actually, it is easy to see that this is not the only possible choice of a minimizing probability.  $q = 1/4 - \theta + \alpha$  and  $p = 1/4 - \theta$  would also minimize  $p - q$ . The optimal solution however does not depend on which one of these probability distributions is used, as the objective function depends only on  $p - q$ .

as imprecision increases. This suggests that the observed under diversification of decision makers' portfolio might be a consequence of how imprecision affects different assets. More specifically, consider parameter values such that  $b^* > c^*$  (in our toy example this is the case for a large range of parameter values.) Note that if one were to ignore the effect of uncertainty on asset holding by wrongly setting  $\alpha = 0$ , the predicted holding of the risky asset would be lower than  $b^*$  while the predicted holding of the imprecise asset would be higher than  $c^*$ , i.e., the predicted holdings would appear to be more diversified. Thus if one fails to identify which assets are affected by imprecision, one could overestimate the predicted weight of these assets in the optimal portfolio.

Finally, it is also easy to show that the holdings of the risky as well as the imprecise assets are decreasing in the risk aversion parameter  $\gamma$ , as well as with the imprecision aversion parameter  $\theta$ . This might help explaining phenomenon like the equity premium puzzle, as imprecision aversion essentially reinforces the effect of risk aversion. Interestingly, these two very tentative hints as to how to account for the under-diversification puzzle *and* the equity premium puzzle in our model are linked to two different parameters (imprecision and imprecision aversion) and could therefore be incorporated in the *same* model.

**Remark 4** The comparative static exercises performed were done under the assumption that  $\theta \geq \alpha/2$ . If this were not the case, then one can show that the minimizing probability used to evaluate the portfolio returns does not depend on  $\alpha$  (when looking at the choice among all three assets.) Hence, over the full range of parameters there is a discontinuity in how imprecision affects holding of the risky and imprecise assets.

**Remark 5** Note that all the action in this example does not take place because of the non-differentiability introduced by the min operator, as for instance in Epstein and Wang (1994) or Mukerji and Tallon (2001). Rather, the comparative statics were done at points where, locally, the decision maker behaves like an expected utility maximizers. More precisely, in usual maxmin expected utility models, decision makers look like expected utility maximizers away from the 45 degree line and there is no sense in which one can change the set of priors as there is no explicit link with the available information. In our model, there is some leverage in that respect even away from the kinks, as we have a way to link changes in the set of priors to changes in available information and to changes in imprecision attitudes. Thus, although non smooth, our model remains tractable in applications.

## Appendix

### Proof for the Extended Multiple-priors Representation

Necessity of the axioms is routine. We show sufficiency.

Fix an information set  $P^*$ . By Outcome Preference, we can define the preference over lottery outcomes  $\succsim^*$  by

$$l \succsim^* m \quad \text{if} \quad (P^*, l) \succ (P^*, m).$$

By Act Continuity and c-Independence,  $\succsim^*$  satisfies the condition for the standard expected utility à la von-Neumann and Morgenstern. Denote the vNM utility by  $u$ , which is mixture-linear and unique up to positive linear transformations.

We extend this to the variable information setting where information is precise.

**Lemma 2** There exists a mixture linear and continuous function  $u : \Delta^*(X) \rightarrow R$  such that  $(\{p\}, f) \succ (\{q\}, g)$  if and only if

$$\sum_{\omega \in \text{supp}(\{p\})} u(f(\omega)) p(\omega) \geq \sum_{\omega \in \text{supp}(\{q\})} u(g(\omega)) q(\omega).$$

Moreover,  $u$  is unique up to positive linear transformations.

**Proof.** By Reduction under Precise Information and mixture-linearity of  $u$ , we obtain

$$\begin{aligned} (\{p\}, f) \succ (\{q\}, g) &\Leftrightarrow (\{p\}, l(p, f)) \succ (\{q\}, l(q, g)) \\ &\Leftrightarrow (P^*, l(p, f)) \succ (P^*, l(q, g)) \\ &\Leftrightarrow l(p, f) \succ^* l(q, g) \\ &\Leftrightarrow u(l(p, f)) \geq u(l(q, g)) \\ &\Leftrightarrow \sum_{\omega \in \text{supp}(\{p\})} u(f(\omega)) p(\omega) \geq \sum_{\omega \in \text{supp}(\{q\})} u(g(\omega)) q(\omega). \end{aligned}$$

■

We extend the above representation to the entire domain. For this purpose we first prove a monotonicity condition.

**Lemma 3 (Monotonicity)** For any  $P \in \mathcal{P}$  and  $f, g \in \mathcal{F}$ ,

$$(P, f(\omega)) \succ (P, g(\omega)) \quad \text{for every } \omega \in \text{supp}(P)$$

implies  $(P, f) \succ (P, g)$ .



**Proof.** Take  $P \in \mathcal{P}$  and let  $(P, f(\omega)) \succsim (P, g(\omega))$  for every  $\omega \in \text{supp}(P)$ . Under the previous result, this implies  $(\{p\}, f) \succsim (\{p\}, g)$  for every  $p \in P$ . By Dominance, we get  $(P, f) \succsim (P, g)$ . ■

**Lemma 4** Given  $u$ , there exists a unique continuous function  $U : \mathcal{P} \times \mathcal{F} \rightarrow R$  such that:

1.  $(P, f) \succsim (Q, g)$  if and only if  $U(P, f) \geq U(Q, g)$ ,
2.  $U(\{p\}, f) = \sum_{\omega \in \text{supp}(P)} u(f(\omega)) p(\omega)$  for every  $\{p\} \in \mathcal{P}$  and  $f \in \mathcal{F}$

**Proof.** Define  $U : \mathcal{P} \times \mathcal{F} \rightarrow R$  by

$$U(P, f) \equiv u(l)$$

such that  $(P, f) \sim (P, l) \sim (P^*, l)$ .

Monotonicity and Act Continuity guarantee the existence of such  $l$ , and c-Independence guarantee the uniqueness of such  $l$  up to payoff. ■

**Lemma 5** Given  $u$  and  $U$ , there exists a unique function  $I : \mathcal{P} \times R_u \rightarrow R$ , where  $R_u$  is the range of  $u$ , such that

1.  $I(P, u \circ f|_{\text{supp}(P)}) = U(P, f)$  for any  $P \in \mathcal{P}$  and  $f \in \mathcal{F}$ .
2.  $I(P, c\mathbf{1}) = c$  for every  $c \in \{u(l) : l \in \Delta^*(X)\}$ .

**Proof.** By Monotonicity,  $I : \mathcal{P} \times R_u \rightarrow R$  is well-defined by

$$I(P, u \circ f|_{\text{supp}(P)}) \equiv U(P, f).$$

Monotonicity again delivers  $I(P, c\mathbf{1}) = c$ . ■

The following three lemmata can be shown by applying the argument in Gilboa and Schmeidler (1989)

**Lemma 6** For any  $P \in \mathcal{P}$ ,  $x \in R^{|\text{supp}(P)|}$ ,  $c \in R$  and  $\lambda \geq 0$ ,

$$I(P, \lambda x) = \lambda I(P, x).$$

**Lemma 7** For any  $P \in \mathcal{P}$ ,  $x \in R^{|\text{supp}(P)|}$ ,  $c \in R$  and  $\lambda \in [0, 1]$ ,

$$I(P, \lambda x + (1 - \lambda)c\mathbf{1}) = \lambda I(P, x) + (1 - \lambda)c.$$

**Lemma 8** For any  $P \in \mathcal{P}$ , the function  $I(P, \cdot)$  is quasi-concave.

Thus, we can apply the Gilboa and Schmeidler (1989) argument so that we obtain

**Lemma 9** For any  $P \in \mathcal{P}$ , there exists a unique closed (hence compact) convex set  $\varphi(P)$  such that

$$I(P, x) = \min_{p \in \varphi(P)} \sum_{\omega \in \text{supp}(P)} x(\omega) p(\omega)$$

for every  $x \in R^{|\text{supp}(P)|}$ .

**Proof.** Fix  $P \in \mathcal{P}$ . Then,  $I(P, \cdot)$  satisfies the condition of GS. Thus, there is a closed convex set  $\varphi(P) \in \mathcal{P}$  such that

$$I(P, x) = \Phi \left( \min_{p \in \varphi(P)} \sum_{\omega \in \text{supp}(P)} x(\omega) p(\omega); P \right)$$

where  $\Phi(\cdot; P)$  is a monotone transformation depending on  $P$ .

Recall that  $I(P, c\mathbf{1}) = c$  for any  $c$ . Therefore,  $\Phi(c; P) = c$  for any  $c$ , which implies  $\Phi(\cdot, P)$  is an identity map, and it is true for any  $P \in \mathcal{P}$ . ■

### *Selection Property*

For later use, we prove an alternative dominance condition.

**Lemma 10** (Set Dominance): For every  $f \in \mathcal{F}$ ,  $P \in \mathcal{P}$  and  $\{p\} \in \mathcal{P}$ ,

$$(\{p\}, f) \succeq (\{p'\}, f) \text{ for every } p' \in P \Rightarrow (\{p\}, f) \succeq (P, f),$$

and

$$(\{p'\}, f) \succeq (\{p\}, f) \text{ for every } p' \in P \Rightarrow (P, f) \succeq (\{p\}, f).$$

**Proof.** Without loss, we just prove the first statement. Suppose  $(\{p\}, f) \succeq (\{p'\}, f)$  for every  $p' \in P$ . Since  $(\{p\}, f) \sim (\{p\}, l(p, f)) \sim (\{p'\}, l(p, f))$  follows from Reduction under Precise Information and Outcome Preference, we have  $(\{p', l(p, f)\}) \succeq (\{p'\}, f)$  for every  $p' \in P$ . By Dominance,  $(P, l(p, f)) \succeq (P, f)$ . By Reduction under Precise Information and Outcome Preference again,  $(P, l(p, f)) \sim (\{p\}, l(p, f)) \sim (\{p\}, f)$ . Therefore,  $(\{p\}, f) \succeq (P, f)$ . ■

**Lemma 11** For every  $P \in \mathcal{P}$ ,  $\varphi(P) \subset P$ .

**Proof.** Suppose  $\varphi(P) \not\subset P$ . Then, there exists  $f \in \mathcal{F}$  such that

$$\begin{aligned} U(P, f) &= \min_{p \in \varphi(P)} \sum_{\omega} u(f(\omega)) p(\omega) \\ &< \min_{p \in P} \sum_{\omega} u(f(\omega)) p(\omega) \\ &= U(\{p\}, f), \end{aligned}$$

where  $(\{p\}, f) \succeq (\{p\}, f)$  for every  $p \in P$ . However, this contradicts (Set Dominance). ■

### *Mixture Linearity*

**Lemma 12** Under Information Independence: for every  $P, Q \in \mathcal{P}$ ,  $f \in \mathcal{F}$  and  $\lambda \in [0, 1]$ ,

$$U(\lambda P + (1 - \lambda)Q, f) = \lambda U(P, f) + (1 - \lambda)U(Q, f).$$

**Proof.** Let  $\{p\}, \{q\} \in \mathcal{P}$  be such that  $(P, f) \sim (\{p\}, f)$  and  $(Q, f) \sim (\{q\}, f)$ , respectively. Then, repeated application of Information Independence delivers

$$(\lambda P + (1 - \lambda)Q, f) \sim (\lambda\{p\} + (1 - \lambda)Q, f) \sim (\lambda\{p\} + (1 - \lambda)\{q\}, f)$$

Since  $(\lambda\{p\} + (1 - \lambda)\{q\}, f) = \lambda U(\{p\}, f) + (1 - \lambda)U(\{q\}, f)$  is true for precise information, we obtain the claim. ■

**Lemma 13** For every  $P, Q \in \mathcal{P}$  and  $\lambda \in [0, 1]$ ,  $\varphi(\lambda P + (1 - \lambda)Q) = \lambda\varphi(P) + (1 - \lambda)\varphi(Q)$ .

**Proof.** By construction,

$$U(\lambda P + (1 - \lambda)Q, f) = \min_{p \in \varphi(\lambda P + (1 - \lambda)Q)} \sum_{\omega} u(f(\omega)) p(\omega).$$

for any  $f \in \mathcal{F}$ .

By mixture-linearity of  $U$  over  $\mathcal{P}$ , the above is equal to

$$\begin{aligned} \lambda U(P, f) + (1 - \lambda)U(Q, f) &= \lambda \min_{p \in \varphi(P)} \sum_{\omega} u(f(\omega)) p(\omega) + (1 - \lambda) \min_{p \in \varphi(Q)} \sum_{\omega} u(f(\omega)) p(\omega) \\ &= \min_{p \in \lambda\varphi(P) + (1 - \lambda)\varphi(Q)} \sum_{\omega} u(f(\omega)) p(\omega) \end{aligned}$$

for any  $f \in \mathcal{F}$ . By uniqueness of  $\varphi(\cdot)$ , we obtain the result. ■

## Proof of Theorem 2

Let  $f, g \in \mathcal{F}$  and  $P \in \mathcal{P}$  and assume  $(P, f) \sim (P, g)$ . Define  $h$  by  $h(s) = f(\frac{s+1}{2})$  if  $s$  is odd, and  $h(s) = g(\frac{s}{2})$  if  $s$  is even. For any  $\alpha \in [0, 1]$ , define  $Q(\alpha) = \{q | \exists p \in P \text{ s. th. } q(s) = \alpha p(\frac{s+1}{2}) \text{ if } s \text{ is odd and } q(s) = (1 - \alpha)p(\frac{s}{2}) \text{ if } s \text{ is even}\}$ .

By (Decomposition Indifference)  $(Q(1), h) \sim (P, f)$  and  $(Q(0), g) \sim (P, g)$ . By assumption,  $(P, f) \sim (P, g)$  and thus  $(Q(0), g) \sim (P, f)$ . By (Independence) for any  $\lambda \in [0, 1]$ ,  $(\lambda Q(1) + (1 - \lambda)Q(0), h) \sim (P, f)$ .

Now,  $Q(\lambda)$  is conditionally more precise than  $\lambda Q(1) + (1 - \lambda)Q(0)$ . Indeed, (i)  $Q(\lambda) \subset \lambda Q(1) + (1 - \lambda)Q(0)$  and (ii) take as a partition of the state space  $\{E_1, E_2, \dots\}$  where  $E_n = \{2n - 1, 2n\}$ , then the condition holds. Hence,  $(Q(\lambda), h) \succ (\lambda Q(1) + (1 - \lambda)Q(0), h) \sim (P, f)$  and therefore, since (Decomposition Indifference) implies that  $(Q(\lambda), h) \sim (P, \lambda f + (1 - \lambda)g)$ , we have:  $(P, \lambda f + (1 - \lambda)g) \succ (P, f)$

## Proof of the Contraction Representation result

For a while we fix  $S \in \mathcal{S}$ . We first prove continuity of  $\varphi$  in  $P(S)$ .

**Lemma 14** The mapping  $\varphi : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$  is continuous with respect to the Hausdorff metric.

**Proof.** Let  $\{P^n\}$  be a sequence in  $\mathcal{P}(S)$  converging to  $P \in \mathcal{P}(S)$ . Because  $\mathcal{P}(S)$  is compact, it is without loss of generality to assume that  $\{\varphi(P^n)\}$  is convergent. Suppose  $\varphi^* \equiv \lim_{n \rightarrow \infty} \varphi(P^n) \neq \varphi(P)$ . Then there exists  $f \in \mathcal{F}$  such that

$$\begin{aligned} U(P, f) &= \min_{p \in \varphi(P)} \sum_{\omega \in S} u(f(\omega)) p(\omega) \\ &> \min_{p \in \varphi^*} \sum_{\omega \in S} u(f(\omega)) p(\omega) \\ &= \lim_{n \rightarrow \infty} \min_{p \in \varphi(P^n)} \sum_{\omega \in S} u(f(\omega)) p(\omega) \\ &= \lim_{n \rightarrow \infty} U(P^n, f), \end{aligned}$$

which is a contradiction to Information Continuity. ■

**Lemma 15** For any bistochastic matrix  $\Pi$ , the following two statements are equivalent:

- (i) There exists  $\lambda \in [0, 1]$  such that  $\|\Pi p - \Pi q\| = \lambda \|p - q\|$  for any  $p, q \in \Delta(S)$ , where  $\|\cdot\|$  denotes the Euclidian norm;
- (ii) There exists  $\lambda \in [0, 1]$  such that  $\Pi^t \Pi = \lambda I + \frac{1-\lambda}{|S|} E$ , where  $I$  is the identity matrix and  $E$  is a matrix in which all the entries are 1.

**Proof.** Let  $e = (\frac{1}{|S|}, \dots, \frac{1}{|S|})$ .

(i)  $\Rightarrow$  (ii): Take any  $p, q \in \Delta(S)$ . Then,

$$\begin{aligned} \langle \Pi p, \Pi q \rangle &= 2 \left\langle \frac{\Pi p + \Pi q}{2} - e, \frac{\Pi p + \Pi q}{2} - e \right\rangle - \frac{1}{2} \langle \Pi p - e, \Pi p - e \rangle - \frac{1}{2} \langle \Pi q - e, \Pi q - e \rangle \\ &\quad + \langle \Pi p, e \rangle + \langle \Pi q, e \rangle - \langle e, e \rangle \end{aligned}$$

Since  $\Pi e = e$  and  $\langle \Pi p, e \rangle = \langle p, e \rangle = \frac{1}{|S|}$  for any  $p$ , by assumption the right-hand-side of the above equality becomes

$$\begin{aligned} &\lambda \left[ 2 \left\langle \frac{p+q}{2} - e, \frac{p+q}{2} - e \right\rangle - \frac{1}{2} \langle p - e, p - e \rangle - \frac{1}{2} \langle q - e, q - e \rangle + \langle p, e \rangle + \langle q, e \rangle - \langle e, e \rangle \right] \\ &+ (1 - \lambda) [\langle p, e \rangle + \langle q, e \rangle - \langle e, e \rangle] \\ &= \lambda \langle p, q \rangle + (1 - \lambda) / |S| \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \langle \Pi p, \Pi q \rangle &= p^t \Pi^t \Pi q \\ &= \lambda \langle p, q \rangle + (1 - \lambda) / |S| \end{aligned}$$

for any  $p, q \in \Delta(S)$ . Pick  $p = \delta_\omega$ ,  $q = \delta_{\omega'}$  where  $\delta_\omega$  and  $\delta_{\omega'}$  are probabilities degenerated on  $\omega$ ,  $\omega'$ , respectively. Then, for all the column vectors  $\Pi_1, \dots, \Pi_{|S|}$ , we obtain

$$\begin{aligned}\Pi_\omega^t \Pi_{\omega'} &= \lambda \quad \text{when } \omega \neq \omega' \\ \Pi_\omega^t \Pi_\omega &= \lambda + (1 - \lambda)/|S|.\end{aligned}$$

(i)  $\Leftarrow$  (ii): By the converse argument of the above.  $\blacksquare$

We now prove Lemma 1.

**Proof.** ( $\Leftarrow$ ): Take any  $\omega \in S$  and let  $e_\omega$  be the vector which assigns 1 on the  $\omega$ -th coordinate and 0 on the others. Without loss of generality, we assume that  $e_\omega$  is obtained as the payoff vector of some act  $f_\omega$ , that is,  $e_\omega = u \circ f_\omega$ .

Now take  $p, q \in \Delta(S)$  such that  $(p - q)^t e_\omega = 0$ . This means  $p_\omega = q_\omega$ . Without loss of generality, let  $p_\omega = q_\omega = 0$ . By assumption,  $(p - q)^t \Pi^t \Pi e_\omega = 0$ . Let  $\Gamma = \Pi^t \Pi$ . Then,  $\sum_{\omega' \neq \omega} (p_{\omega'} - q_{\omega'}) \Gamma_{\omega' \omega} = 0$ . For any  $\omega', \omega'' \neq \omega$ , take  $p_{\omega'} = 1$  and  $q_{\omega''} = 1$ . Then, we have  $\Gamma_{\omega' \omega} = \Gamma_{\omega'' \omega}$ . This is true for any  $\omega$  and  $\omega', \omega'' \neq \omega$ . Since  $\Gamma = \Pi^t \Pi$  is a symmetric matrix, we obtain that all the off-diagonal entries of  $\Gamma$  are the same. Therefore, all the diagonal entries of  $\Gamma$  are the same.

Finally, we show that the diagonal entries cannot be smaller than the off-diagonal entries. Let  $\Gamma_d$  be the diagonal entry and  $\Gamma_{nd}$  be the off-diagonal entry. Let  $p_\omega^\varepsilon$  be the vector which assigns  $1 - (|S| - 1)\varepsilon$  on the  $\omega$ -th coordinate and  $\varepsilon$  on the others, where  $0 < \varepsilon < \frac{1}{|S| - 1}$ . Then,  $(\delta_\omega - p_\omega^\varepsilon)^t e_\omega = (|S| - 1)\varepsilon > 0$ . Then,  $(\delta_\omega - p_\omega^\varepsilon)^t \Gamma e_\omega = (|S| - 1)\varepsilon(\Gamma_d - \Gamma_{nd}) \geq 0$ , which implies  $\Gamma_d \geq \Gamma_{nd}$ .

( $\Rightarrow$ ): Obvious.  $\blacksquare$

**Lemma 16** For any  $P \in \mathcal{P}(S)$  and  $\Pi \in \mathcal{T}(S)$ ,  $\varphi(\Pi P) = \Pi \varphi(P)$ .

**Proof.** Suppose  $\varphi(\Pi P) \not\subseteq \Pi \varphi(P)$ . Then, there is  $y \in R^S$  such that

$$\min_{p \in \Pi \varphi(P)} \sum_{\omega \in S} y(\omega) p(\omega) > \min_{p' \in \varphi(\Pi P)} \sum_{\omega \in S} y(\omega) p'(\omega)$$

By taking  $y = \Pi x$ , both sides are written as

$$\min_{p \in \varphi(P)} \sum_{\omega \in S} (\Pi x)(\omega) (\Pi p)(\omega) > \min_{p' \in \varphi(\Pi P)} \sum_{\omega \in S} (\Pi x)(\omega) p'(\omega) \quad (*)$$

By homogeneity with respect to  $x$ , without loss of generality, we can set  $x = u \circ f$  and for some  $f \in \mathcal{F}$ . Take  $p^* \in \arg \min_{p \in \varphi(P)} \sum_{\omega \in S} x(\omega) p(\omega)$ . Since

$$\sum_{\omega \in S} (\Pi x)(\omega) (\Pi p)(\omega) = \lambda \sum_{\omega \in S} x(\omega) p(\omega) + \frac{1 - \lambda}{|S|} \sum_{\omega \in S} x(\omega),$$

we have  $p^* \in \arg \min_{p \in \varphi(P)} \sum_{\omega} (\Pi x)(\omega) (\Pi p)(\omega)$ . Thus, the left hand side of (\*) is equal to  $U(\{\Pi p^*\}, \Pi f)$ . On the other hand, the right hand side of (\*) is  $U(\Pi P, \Pi f)$ . Thus,  $U(\{\Pi p^*\}, \Pi f) > U(\Pi P, \Pi f)$

By definition of  $p^*$ , we have  $U(\{p^*\}, f) = U(P, f)$ . This contradicts to the Invariance to Unitary Transformations.

We similarly obtain a contradiction for the case  $\varphi(\Pi P) \not\subseteq \Pi \varphi(P)$ . ■

Let  $\mathbf{1} = (1, \dots, 1)$  and  $e = \frac{1}{|S|} \mathbf{1}$ . For later use, we show the lemma below.

**Lemma 17** Let  $F : \Delta(S) \rightarrow \Delta(S)$  be a mixture-linear mapping satisfying  $F(e) = e$ . Then there is a unique bistochastic matrix  $\Pi$  such that  $F(p) = \Pi p$  for every  $p \in \Delta(S)$ .

**Proof.** Given such  $F$ , define  $\Pi$  by  $\Pi_{ij} = F_i(\delta_j)$  where  $\delta_j$  is a probability which assigns unit mass on state  $j \in S$ . By mixture linearity,  $\Pi$  represents  $F$ .

Suppose there are two matrices  $\Pi$  and  $\Pi'$  which represent  $F$ . If  $\Pi_{ij} \neq \Pi'_{ij}$  for some  $i, j \in \Omega$ , this leads to  $F_i(\delta_j) = \Pi_{ij} \neq \Pi'_{ij} = F_i(\delta_j)$ , a contradiction. Thus  $\Pi$  is unique.

If  $\Pi_{ij} < 0$  for some  $i, j \in S$ , this leads to  $F_i(\delta_j) < 0$ , which is a contradiction.

For any  $j \in S$ ,  $\Pi \delta_j = (\Pi_{ij})_{i \in S} \in \Delta(S)$ . Therefore,  $\sum_{i \in S} \Pi_{ij} = 1$  for each  $j \in S$ .

Since  $\Pi e = e$ , for each  $i \in N$ ,  $\frac{1}{|S|} \sum_{j \in S} \Pi_{ij} = \frac{1}{|S|}$ . Therefore,  $\sum_{j \in S} \Pi_{ij} = 1$  for each  $i \in S$ . ■

Now define  $\varphi^* : \mathcal{P}(S) - \{e\} \rightarrow \mathcal{P}(S) - \{e\}$  by

$$\varphi^*(K) = \varphi(K + \{e\}) - \{e\}$$

**Lemma 18** For any  $K \in \mathcal{P}(S) - \{e\}$  and  $\lambda \geq 0$  with  $\lambda K \in \mathcal{P}(S) - \{e\}$ ,  $\varphi^*(\lambda K) = \lambda \varphi^*(K)$ .

**Proof.** The case of  $\lambda = 0$  or  $1$  is obvious. Let  $\lambda \in (0, 1)$ . Then,

$$\begin{aligned} \varphi^*(\lambda K) &= \varphi(\lambda K + \{e\}) - \{e\} \\ &= \varphi(\lambda(K + \{e\}) + (1 - \lambda)\{e\}) - \{e\} \\ &= \lambda \varphi(K + \{e\}) + (1 - \lambda) \varphi(\{e\}) - \{e\} \\ &= \lambda \varphi(K + \{e\}) + (1 - \lambda) \{e\} - \{e\} \\ &= \lambda(\varphi(K + \{e\}) - \{e\}) \\ &= \lambda \varphi^*(K). \end{aligned}$$

The case of  $\lambda > 1$  is immediate from the above. ■

Let  $H_e$  be the  $|S| - 1$  dimensional linear subspace of  $R^S$  which is orthogonal to  $e$ . Let  $\mathcal{K}_e$  be the family of compact convex subsets of  $H_e$ , endowed with the Hausdorff metric. By the above lemma, we extend  $\varphi^*$  to  $\mathcal{K}_e$ . This preserves continuity in the Hausdorff metric.

**Lemma 19** For any  $K, K' \in \mathcal{K}_e$ ,  $\varphi^*(K + K') = \varphi^*(K) + \varphi^*(K')$ . In particular,  $\varphi^*(K + \{z\}) = \varphi^*(K) + \{z\}$ .

**Proof.** Take sufficiently small  $\lambda > 0$ , then  $\lambda K, \lambda K' \in \mathcal{P}(S) - \{e\}$ . By homogeneity,

$$\varphi^*(K + K') = \frac{2}{\lambda} \varphi^* \left( \frac{\lambda K + \lambda K'}{2} \right)$$

Then, we have

$$\begin{aligned} \varphi^* \left( \frac{\lambda K + \lambda K'}{2} \right) &= \varphi \left( \frac{\lambda K + \{e\}}{2} + \frac{\lambda K' + \{e\}}{2} \right) - \{e\} \\ &= \frac{1}{2} \varphi(\lambda K + \{e\}) + \frac{1}{2} \varphi(\lambda K' + \{e\}) - \{e\} \\ &= \frac{\varphi(\lambda K + \{e\}) - \{e\}}{2} + \frac{\varphi(\lambda K' + \{e\}) - \{e\}}{2} \\ &= \frac{1}{2} \varphi^*(\lambda K) + \frac{1}{2} \varphi^*(\lambda K') \\ &= \frac{\lambda}{2} \varphi^*(K) + \frac{\lambda}{2} \varphi^*(K'), \end{aligned}$$

which gives the result. ■

For each  $\Pi \in \mathcal{T}(S)$ , define a mapping  $T_\Pi : \Delta(S) - \{e\} \rightarrow \Delta(S) - \{e\}$  by

$$T_\Pi(x) = \Pi(x + e) - e.$$

Since  $x + e \in \Delta(S)$  implies  $\Pi(x + e)$ , the mapping indeed satisfies  $T_\Pi(x) \in \Delta(S) - \{e\}$ . By nature of  $\Pi$ , we have (i)  $T_\Pi(\lambda x) = \lambda T_\Pi(x)$  for any  $\lambda$  with  $\lambda x \in \Delta(S) - \{e\}$ , (ii)  $T_\Pi(x + y) = T_\Pi(x) + T_\Pi(y)$ , and (iii) there exists  $\lambda_\Pi \in (0, 1)$  such that  $\|T_\Pi(x)\| = \lambda_\Pi \|x\|$  for every  $x \in \Delta(S) - \{e\}$ . By (i), we can extend  $T_\Pi$  to the whole linear subspace  $H_e$ .

We say that a linear transformation  $G$  is a *sub-similarity* if  $G(\Delta(S) - \{e\}) \subset \Delta(S) - \{e\}$  and there exists  $\lambda_G \in (0, 1]$  such that  $\|G(x)\| = \lambda_G \|x\|$  for any  $x \in H_e$ .

Conversely to the above, any sub-similarity  $G : H_e \rightarrow H_e$  has a corresponding unitary transformation. For  $G$ , define  $F_G : \Delta(S) \rightarrow \Delta(S)$  by

$$F_G(p) = G(p - e) + e.$$

Then, it is easy to see that  $F_G$  takes values in  $\Delta(S)$  and is mixture linear and  $F_G(e) = e$ . By the previous lemma, it has a representation by a doubly stochastic matrix  $\Pi_G$  and  $F_G(p) = \Pi(p)$ . Since  $F_G$  satisfies (iii),  $\Pi_G$  is an  $\mathcal{T}(S)$ .

Now we show that  $\varphi^*$  is equivariant in sub-similarities.

**Lemma 20** For any sub-similarity  $G : H_e \rightarrow H_e$ ,  $\varphi^*(GK) = G\varphi(K)$ .

**Proof.** By homogeneity of  $\varphi^*$ , without loss of generality we can take  $K \in \mathcal{P}(S) - \{e\}$ . By the above lemma  $G$  has a corresponding unitary transformation  $\Pi_G$  and  $G(x) = \Pi_G(x + e) - e$  for any  $x \in \Delta(S) - \{e\}$ .

Then,

$$\begin{aligned}
\varphi^*(G(K)) &= \varphi(G(K) + \{e\}) - \{e\} \\
&= \varphi(\Pi_G(K + \{e\}) - \{e\} + \{e\}) - \{e\} \\
&= \varphi(\Pi_G(K + \{e\})) - \{e\} \\
&= \Pi_G \varphi(K + \{e\}) - \{e\} \\
&= G(\varphi(K)).
\end{aligned}$$

■

A linear transformation  $I : H_e \rightarrow H_e$  is called *isometry* if  $\|I(x)\| = \|x\|$ . Let  $\mathcal{I}$  be the set of isometries. For any isometry  $I \in \mathcal{I}$ , its positive scalar multiplication  $\lambda I$  where  $\lambda > 0$  is chosen so that  $\lambda I(\Delta(S) - \{e\}) \subset \Delta(S) - \{e\}$  is a sub-similarity. Conversely, any isometry is obtained from a sub-similarity by reversing the above procedure.

By homogeneity of  $\varphi^*$ , we obtain

**Lemma 21** The mapping  $\varphi^*$  is equivariant in isometries. That is, for any isometry  $I \in \mathcal{I}$ ,  $\varphi^*(I(K)) = I(\varphi^*(K))$ .

The  $|S| - 1$  dimensional Euclidian space  $R^{|S|-1}$  is the image of the linear subspace  $H_e$  by some isometry. Let  $J : H_e \rightarrow R^{|S|-1}$  be such isometry. All the relevant operations are preserved under isometry. Let  $\mathcal{K}^{|S|-1}$  be the space of compact convex subsets of  $R^{|S|-1}$ . The space  $\mathcal{K}^{|S|-1}$  is also the image of  $\mathcal{K}_e$  by the isometry. Define  $\varphi^{**} : \mathcal{K}^{|S|-1} \rightarrow \mathcal{K}^{|S|-1}$  by

$$\varphi^{**}(K) = J(\varphi^*(J^{-1}(K))).$$

Then,  $\varphi^{**}$  is continuous, additive and equivariant in isometries in  $R^{|S|-1}$  and translations, and satisfies  $\varphi^{**}(K) \subset K$  for any  $K \in \mathcal{K}^{|S|-1}$ .

Let  $W = \{w \in R^{|S|-1} : \|w\| = 1\}$  be the  $|S| - 2$  dimensional unit sphere. For a compact convex set  $K \in \mathcal{K}^{|S|-1}$ , its *Steiner point* is defined by

$$s^{**}(K) = \int_W \arg \max_{p \in K} \langle p, w \rangle \nu(dw)$$

where  $\nu$  is the uniform distribution over  $W$ .<sup>4</sup>

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<sup>4</sup>Schneider (1974) has adopted a different definition of Steiner point, but it is equivalent to the current definition, which follows from Theorem 9.4.1 in Aubin and Frankowska (1990).



**Lemma 22** There exist  $\varepsilon \geq 0$  and  $\delta \geq 0$  such that

$$\varphi^{**}(K) = \varepsilon [K - s^{**}(K)] + \delta [-K + s^{**}(K)] + \{s^{**}(K)\}.$$

for every  $K \in \mathcal{K}^{|\mathcal{S}|-1}$ .

**Proof. Case 1**  $|\mathcal{S}| = 1, 2$ : Obvious.

**Case 2**  $|\mathcal{S}| = 3$ : Since image of a segment is its subsegment, we can apply Theorem 1.8 (b) in Schneider (1974) so that we obtain

$$\varphi^{**}(K) = \varepsilon T_1 [K - s^{**}(K)] + \delta T_2 [-K + s^{**}(K)] + \{s^{**}(K)\}$$

with  $\varepsilon \geq 0$ ,  $\delta \geq 0$  and  $T_1, T_2$  being some two dimensional rotation matrices.

Consider a segment with midpoint 0. Since its image is its subsegment, it must be the case that  $(T_1, T_2) = (1, 1)$  or  $(1, -1)$  or  $(-1, 1)$  or  $(-1, -1)$ . Thus, without loss of generality

$$\varphi^{**}(K) = \varepsilon [K - s^{**}(K)] + \delta [-K + s^{**}(K)] + \{s^{**}(K)\}$$

**Case 3**  $|\mathcal{S}| \geq 4$ : Since  $\varphi^{**}(K) \subset K$  for any  $K \in \mathcal{K}^{|\Omega|-1}$ , the image of any segment is its subsegment. Thus we can apply Theorem 1.8 (b) in Schneider (1974) so that we obtain

$$\varphi^{**}(K) = \varepsilon [K - s^{**}(K)] + \delta [-K + s^{**}(K)] + \{s^{**}(K)\}$$

with  $\varepsilon \geq 0$ ,  $\delta \geq 0$ . ■

Finally, we show  $\varepsilon \in [0, 1]$  and  $\delta = 0$ . Since  $\varphi^*(K) \subset K$  for any  $K$ ,  $\varepsilon$  cannot exceed 1. Now consider a family of triangles

$$K_\theta = \{(x_1, x_2, 0, \dots, 0) \in R^{|\mathcal{S}|-1} : x_2 \leq \frac{\cos \theta}{\sin \theta} x_1, x_2 \geq -\frac{\cos \theta}{\sin \theta} x_1, x_1 \leq \sin \theta\}$$

indexed by  $0 < \theta < \frac{\pi}{2}$ . Then we have  $s^{**}(K_\theta) = (\frac{\pi-\theta}{\pi} \sin \theta, 0, 0, \dots, 0)$ . Let  $\bar{x}_1(K_\theta) = \max_{x \in \varphi^{**}(K_\theta)} x_1$ . We get  $\bar{x}_1(K_\theta) = \frac{\pi-\theta}{\pi} \sin \theta + \varepsilon \frac{\theta}{\pi} \sin \theta + \delta \frac{\pi-\theta}{\pi} \sin \theta$ . Since  $\varphi^{**}(K_\theta) \subset K_\theta$ , this cannot exceed  $\sin \theta$ . Since  $\sin \theta$  is positive, we can divide both sides of  $\bar{x}_1(K_\theta) \leq \sin \theta$  by  $\sin \theta$  and by arranging we get

$$\delta \leq \frac{\frac{\theta}{\pi}}{1 - \frac{\theta}{\pi}} (1 - \varepsilon).$$

Since this is true for any  $\theta \in (0, \frac{\pi}{2})$ , we obtain  $\delta = 0$ .

Thus

$$\varphi^{**}(K) = \varepsilon [K - s^{**}(K)] + \{s^{**}(K)\}$$

with  $\varepsilon \in [0, 1]$ . Since Steiner point and every relevant operation are preserved by isometry, we obtain

$$\varphi(P) = \varepsilon [P - s(P)] + \{s(P)\}.$$

### Constancy of $\varepsilon$ with regard to $S$

Let  $\varepsilon_S$  be the rate corresponding to  $S$ . When  $S \subset S'$ , since  $P \in \mathcal{P}(S)$  implies  $P \in \mathcal{P}(S')$ , we must have  $\varepsilon_S = \varepsilon_{S'}$ . For every  $S, S'$ , since  $\varepsilon_S = \varepsilon_{S \cup S'}$ , and  $\varepsilon_{S'} = \varepsilon_{S \cup S'}$ , we obtain the desired claim.

## Proof for Comparative Imprecision Aversion

### Theorem 4

Let us first prove the following lemma

**Lemma 23** Let  $\succeq$  be a preference relation defined on  $\mathcal{P} \times \mathcal{F}$ , satisfying Axioms 1 to 8 as well as Axiom 10. For all  $P, Q \in \mathcal{P}$  and  $\alpha \in [0, 1]$ , such that

$$Q = \{q | \exists p \in P \text{ s.th. } q(\omega) = \alpha p\left(\frac{\omega+1}{2}\right) \text{ if } \omega \text{ is odd and } q(\omega) = (1-\alpha)p\left(\frac{\omega}{2}\right) \text{ if } \omega \text{ is even}\}$$

we have

$$\varphi(Q) = \{q | \exists p \in \varphi(P) \text{ s.th. } q(\omega) = \alpha p\left(\frac{\omega+1}{2}\right) \text{ if } \omega \text{ is odd and } q(\omega) = (1-\alpha)p\left(\frac{\omega}{2}\right) \text{ if } \omega \text{ is even}\}$$

#### Proof.

Let  $P, Q \in \mathcal{P}$  and  $\alpha \in [0, 1]$ , be such that

$$Q = \{q | \exists p \in P \text{ s.th. } q(\omega) = \alpha p\left(\frac{\omega+1}{2}\right) \text{ if } \omega \text{ is odd and } q(\omega) = (1-\alpha)p\left(\frac{\omega}{2}\right) \text{ if } \omega \text{ is even}\}$$

We first prove that

$$\varphi(Q) \subseteq Q^* = \{q | \exists p \in \varphi(P) \text{ s.th. } q(\omega) = \alpha p\left(\frac{\omega+1}{2}\right) \text{ if } \omega \text{ is odd and } q(\omega) = (1-\alpha)p\left(\frac{\omega}{2}\right) \text{ if } \omega \text{ is even}\}$$

Assume that there exists  $p^* \in \varphi(Q)$  such that  $p^* \notin Q^*$ . Since  $Q^*$  is a convex set, using a separation argument, we know that there exists a function  $\phi : \Omega \rightarrow \mathbb{R}$  such that  $\int \phi dp^* < \min_{p \in Q^*} \int \phi dp$ . Since  $\text{supp}(Q)$  is a finite set, there exist numbers  $a, b$  with  $a > 0$ , such that  $\forall \omega \in \text{supp}(Q)$ ,  $(a\phi(\omega) + b) \in u(\Delta(X))$ . Then, for all  $\omega \in \text{supp}(Q)$  there exists  $y(\omega) \in \Delta(X)$  such that  $u(y(\omega)) = a\phi(\omega) + b$ . Define  $h \in \mathcal{F}$  by  $h(\omega) = y(\omega)$  for all  $\omega \in \text{supp}(Q)$ ,  $h(\omega) = \delta_x$  for all  $\omega \in \Omega \setminus \text{supp}(Q)$ , where  $x \in X$ .

Then define  $f, g \in \mathcal{F}$  by  $f(\omega) = h(2\omega - 1)$  and  $g(\omega) = h(2\omega)$ . We have that  $\text{supp}(Q^*) \subseteq \text{supp}(Q)$  and therefore

$$\min_{p \in Q^*} \sum_{\omega \in \Omega} u(h(\omega)) p(\omega) = \min_{p \in \varphi(P)} \sum_{\omega \in \Omega} u(\alpha f(\omega) + (1-\alpha)g(\omega)) p(\omega)$$

while

$$\min_{p \in Q^*} \sum_{\omega \in \Omega} u(h(\omega)) p(\omega) > \sum_{\omega \in \Omega} u(h(\omega)) p^*(\omega) \geq \min_{p \in \varphi(Q)} \sum_{\omega \in \Omega} u(h(\omega)) p(\omega)$$

and thus

$$(P, \alpha f + (1 - \alpha)g) \succ (Q, h)$$

which contradicts Axiom 10.

The same kind of proof permits to show that  $Q^* \subseteq \varphi(Q)$ . ■

[(i)  $\Rightarrow$  (ii)] Let  $P \in \mathcal{P}$  and assume that  $\varphi^a(P) \not\subseteq \varphi^b(P)$ , i.e., there exists  $p^* \in \varphi^a(P)$  such that  $p^* \notin \varphi^b(P)$ . Using a separation argument, there exists a function  $\phi : \Omega \rightarrow \mathbb{R}$  such that  $\int \phi dp^* < \min_{p \in \varphi^b(P)} \int \phi dp$ . Note that we can choose by normalization  $u_a$  and  $u_b$  such that  $u_a(\bar{x}) = u_b(\bar{x}) > u_a(\underline{x}) = u_b(\underline{x})$ . Since  $\text{supp}(P)$  is a finite set, there exist numbers  $k > 0$  and  $\ell$ , such that for all  $\omega \in \text{supp}(P)$ ,  $k\phi(\omega) + \ell \in [u_a(\underline{x}), u_a(\bar{x})]$ . Denote  $\text{supp}(P) = \{\omega_1, \dots, \omega_n\}$ . Let  $\alpha_i = \frac{k\phi(\omega_i) + \ell - u_a(\underline{x})}{u_a(\bar{x}) - u_a(\underline{x})}$  and let suppose w.l.o.g that  $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ .

Let  $f^0 \in \mathcal{F}$  such that  $f^0(\omega_i) = \alpha_i \delta_{\bar{x}} + (1 - \alpha_i) \delta_{\underline{x}}$  for all  $i = 1, \dots, n$  and  $f^0(\omega) = \underline{x}$  for all  $s \in \Omega \setminus \text{supp}(P)$ .

Then define  $f^i \in \mathcal{F}$  for  $i = 1, \dots, n$  by:

- for all  $j = 1, \dots, i$ , for all  $\omega \in \left\{ 2^i \omega_j - \sum_{h=0, \dots, i-1} 2^h, \dots, 2^i \omega_j - \sum_{h=0, \dots, i-j-1} 2^h - 1 \right\}$ ,  $f^i(\omega) = \bar{x}$
- for all  $j = i + 1, \dots, n$ , for all  $\omega \in \left\{ 2^i \omega_j - \sum_{h=0, \dots, i-1} 2^h, \dots, 2^i \omega_j - 1 \right\}$ ,  $f^i(\omega) = \bar{x}$
- for all  $j = i + 1, \dots, n$ ,  $f^i(2^i \omega_j) = \frac{\alpha_j - \alpha_i}{1 - \alpha_i} \delta_{\bar{x}} + \frac{1 - (\alpha_j - \alpha_i)}{1 - \alpha_i} \delta_{\underline{x}}$
- for all other  $\omega$ ,  $f^i(\omega) = \underline{x}$

Let

- $Q^1 = \{q | \exists p \in P \text{ s.th. } q(\omega) = \alpha_1 p(\frac{\omega+1}{2}) \text{ if } \omega \text{ is odd and } q(\omega) = (1 - \alpha) p(\frac{\omega}{2}) \text{ if } \omega \text{ is even}\}$
- for  $j = 2, \dots, n$ ,  $Q^j = \{q | \exists p \in Q^{j-1} \text{ s.th. } q(\omega) = \alpha_1 p(\frac{\omega+1}{2}) \text{ if } \omega \text{ is odd and } q(\omega) = (1 - \alpha) p(\frac{\omega}{2}) \text{ if } \omega \text{ is even}\}$

Let

- $p^{*1}$  be such that  $p^{*1}(\omega) = \alpha_1 p(\frac{\omega+1}{2})$  if  $\omega$  is odd and  $p^{*1}(\omega) = (1 - \alpha) p(\frac{\omega}{2})$  if  $\omega$  is even
- for  $j = 2, \dots, n$ ,  $p^{*j}$  be such that  $p^{*j}(\omega) = \alpha_1 p^{*j-1}(\frac{\omega+1}{2})$  if  $\omega$  is odd and  $p^{*j}(\omega) = (1 - \alpha) p^{*j-1}(\frac{\omega}{2})$  if  $\omega$  is even.

By lemma 23, one can check that:

- for  $h = a, b$

$$\begin{aligned} \min_{p \in \varphi^h(P)} \sum_{\omega \in \Omega} u_h(f^0(\omega)) p(\omega) &= \min_{p \in \varphi^h(Q^1)} \sum_{\omega \in \Omega} u_h(f^1(\omega)) p(\omega) \\ \text{and } \sum_{\omega \in \Omega} u_h(f^0(\omega)) p^*(\omega) &= \sum_{\omega \in \Omega} u_h(f^1(\omega)) p^{*1}(\omega) \end{aligned}$$

- for  $h = a, b$ , for  $i = 2, \dots, n$

$$\begin{aligned} \min_{p \in \varphi^h(Q^{i-1})} \sum_{\omega \in \Omega} u_h(f^{i-1}(\omega)) p(\omega) &= \min_{p \in \varphi^h(Q^i)} \sum_{\omega \in \Omega} u_h(f^i(\omega)) p(\omega) \\ \text{and } \sum_{\omega \in \Omega} u_h(f^{i-1}(\omega)) p^{*i-1}(\omega) &= \sum_{\omega \in \Omega} u_h(f^i(\omega)) p^{*i}(\omega) \end{aligned}$$

- for  $i = 1, \dots, n$ ,  $p^{*i} \in \varphi^a(Q^i)$

Therefore,

$$\begin{aligned} \min_{p \in \varphi^b(P)} \sum_{\omega \in \Omega} u_b(f^0(\omega)) p(\omega) &= \min_{p \in \varphi^b(Q^n)} \sum_{\omega \in \Omega} u_b(f^n(\omega)) p(\omega) > \\ \sum_{\omega \in \Omega} u_b(f^0(\omega)) p^*(\omega) &= \sum_{\omega \in \Omega} u_b(f^n(\omega)) p^{*n}(\omega) \end{aligned}$$

while

$$\begin{aligned} \sum_{\omega \in \Omega} u_a(f^0(\omega)) p^*(\omega) &= \sum_{\omega \in \Omega} u_a(f^n(\omega)) p^{*n}(\omega) \geq \\ \min_{p \in \varphi^a(P)} \sum_{\omega \in \Omega} u_a(f^0(\omega)) p(\omega) &= \min_{p \in \varphi^a(Q^n)} \sum_{\omega \in \Omega} u_a(f^n(\omega)) p(\omega) \end{aligned}$$

However,  $f^n$  is of the form  $\bar{x}_E \underline{x}$ .

Therefore, we have

$$(\{p^{*n}\}, \bar{x}_E \underline{x}) \succeq_a (Q^n, \bar{x}_E \underline{x})$$

while

$$(Q^n, \bar{x}_E \underline{x}) \succ_b (\{p\}, \bar{x}_E \underline{x})$$

which contradicts the fact that  $\succeq_b$  is more averse to bet imprecision than  $\succeq_a$ .

[(ii)  $\Rightarrow$  (i)] Straightforward.

### Theorem 5

[(i)  $\Leftrightarrow$  (ii)] This equivalence was proved in Theorem 4.

[(ii)  $\Rightarrow$  (iii)] Since  $\pi_a^A(E, P) = s(P)(E) - \text{Min}_{p \in \varphi^a(P)} p(E)$  and  $\pi_b^A(E, P) = s(P)(E) - \text{Min}_{p \in \varphi^b(P)} p(E)$ ,  $\varphi^a(P) \subset \varphi^b(P)$  implies that  $\pi_b^A(E, P) \geq \pi_a^A(E, P)$ .

[(iii)  $\Rightarrow$  (i)] Consider prizes  $\bar{x}$  and  $\underline{x}$  in  $X$  such that both  $a$  and  $b$  strictly prefer  $\bar{x}$  to  $\underline{x}$ , and let  $P \in \mathcal{P}$ , and  $E \subset \Omega$ . For any  $p$ , for any agent  $i = a, b$ ,  $(\bar{x}_{E\underline{x}}, \{p\}) \succeq_i [\succ_i](\bar{x}_{E\underline{x}}, P)$  if, and only if,  $\pi_i^A(E, P) \geq [\succ]s(P)(E) - p(E)$ . Therefore since  $\pi_b^A(E, P) \geq \pi_a^A(E, P)$ , this implies that we have

$$(\bar{x}_{E\underline{x}}, \{p\}) \succeq_a [\succ_a](\bar{x}_{E\underline{x}}, P) \Rightarrow (\bar{x}_{E\underline{x}}, \{p\}) \succeq_b [\succ_b](\bar{x}_{E\underline{x}}, P)$$

which completes the proof that  $\succeq_b$  is more averse to bet imprecision than  $\succeq_a$ .

### Theorem 6

[(i)  $\Rightarrow$  (ii)] Let  $P \in \mathcal{P}$ , and  $p$  be a boundary point  $p$  of  $co(P)$ . Define:

$$\bar{\varepsilon} = Sup \{ \varepsilon' | \varepsilon' \in [0, 1] \text{ s.th. } (\varepsilon'p + (1 - \varepsilon')s(P)) \in \varphi(P) \}.$$

Then  $\bar{p} = \bar{\varepsilon}p + (1 - \bar{\varepsilon})s(P)$  is a boundary point of  $\varphi(P)$  since  $\varphi(P)$  is closed. Since it is convex as well, there exists a function  $\phi : S \rightarrow \mathbb{R}$  such that  $\int \phi d\bar{p} = \min_{p \in \varphi(P)} \int \phi dp$ .

Using the notation and definitions introduced in the proof of Theorem 4 in order to define  $f^n = \bar{x}_{E\underline{x}}$ ,  $p^n, \bar{p}^n$  and  $Q^n$ , we have that  $(f^n, \{\bar{p}^n\}) \sim (f^n, Q^n)$ . Note that  $\bar{p}^n = \bar{\varepsilon}p^n + (1 - \bar{\varepsilon})s(Q^n)$ . Thus

$$\pi^R(E, Q^n) = \frac{s(Q^n)(E) - \bar{p}^n(E)}{s(Q^n)(E) - Min_{q \in Q^n} q(E)} \leq \frac{s(Q^n)(E) - \bar{p}^n(E)}{s(Q^n)(E) - p^n(E)} = \bar{\varepsilon}.$$

If  $\varepsilon > \bar{\varepsilon}$  we get a contradiction with the fact that  $\pi^R(E, Q^n) = \varepsilon$ . Therefore, for any boundary point  $p$  of  $co(P)$ ,  $\bar{\varepsilon}(p) = Sup \{ \varepsilon' | \varepsilon' \in [0, 1] \text{ s.th. } (\varepsilon'p + (1 - \varepsilon')s(P)) \in \varphi(P) \}$  is such that  $\bar{\varepsilon}(p) \geq \varepsilon$ . Let  $p^*$  be a boundary point of  $co(P)$  such that  $\bar{\varepsilon}(p^*) \geq \bar{\varepsilon}(p)$  for all boundary point  $p$  of  $co(P)$ . Then, there exists a function  $\phi : S \rightarrow \mathbb{R}$  such that  $\int \phi dp^* = \min_{p \in P} \int \phi dp$ . Define  $\bar{p}^* = \bar{\varepsilon}(p^*)p^* + (1 - \bar{\varepsilon}(p^*))s(P)$  and consider now  $p' \in \varphi(P)$ . There exists a boundary point  $p$  of  $co(P)$  and  $\varepsilon' < \bar{\varepsilon}(p)$  such that  $p' = \varepsilon'p + (1 - \varepsilon')s(P)$ .

Let us use again the notation and definition introduced in the proof of Theorem 4. Since  $\int u \circ (\bar{x}_{E\underline{x}}) dp^{*n} \leq \int u \circ (\bar{x}_{E\underline{x}}) dp^n$  and  $\int u \circ (\bar{x}_{E\underline{x}}) dp^{*n} \leq \int u \circ (\bar{x}_{E\underline{x}}) ds(Q^n)$ , we have that  $\int u \circ (\bar{x}_{E\underline{x}}) d\bar{p}^{*n} \leq \int u \circ (\bar{x}_{E\underline{x}}) dp^{*n}$ . Thus  $\int u \circ (\bar{x}_{E\underline{x}}) d\bar{p}^{*n} = \min_{r \in \varphi(Q^n)} \int u \circ (\bar{x}_{E\underline{x}}) dr$  while  $\int u \circ (\bar{x}_{E\underline{x}}) d\bar{p}^{*n} = \min_{r \in \varphi(Q^n)} \int u \circ (\bar{x}_{E\underline{x}}) dr$ . Therefore

$$\pi^R(E, Q^n) = \frac{s(Q^n)(E) - \bar{q}^*(E)}{s(Q^n)(E) - Min_{q \in Q^n} q(E)} = \bar{\varepsilon}(p^*),$$

and thus  $\bar{\varepsilon}(p^*) = \varepsilon$ . Hence, for all boundary point  $p$  of  $co(P)$ ,  $\bar{\varepsilon}(p) = \varepsilon$  which proves that  $\varphi(P) = \varepsilon P + (1 - \varepsilon) \{s(P)\}$ .

[(ii)  $\Rightarrow$  (i)] Consider  $P \in \mathcal{P}$ , and  $E \subset \Omega$  such that  $s(P)(E) \neq Min_{p \in P} p(E)$ . We have

$$\min_{p \in \varphi(P)} p(E) = \varepsilon \min_{p \in \varphi(P)} p(E) + (1 - \varepsilon)s(P)(E),$$

and therefore

$$\pi^R(E, P) = \frac{s(P)(E) - \text{Min}_{p \in \varphi(P)} p(E)}{s(P)(E) - \text{Min}_{p \in \varphi(P)} p(E)} = \varepsilon.$$

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