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Developing into series and returning from series A note on the foundation of eighteenth-century analysis

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Summary. In this paper we investigate two problems concerning the theory of power series in eighteenth-century mathematics: the development of a given function into a power series and the converse problem, the return from a given power series to the function of which this power series is the development. The way of conceiving and solving these problems closely depended on the notion of function and in particular to the conception of a series as the result of a formal transformation of a function. After describing the procedures of the development considered acceptable by eighteenth-century mathematicians, we examine in detail the different strategies—both direct and converse, that is synthetic and analytical—they employed to solve these problems.

Sommario. In quest'articolo vengono analizzati due problemi relativi alla teoria delle serie di potenza nel secolo diciassettesimo: lo sviluppo di una funzione in serie di potenza e il problema inverso, il regresso dalla serie alla funzione di cui tale serie è lo sviluppo. Il modo in cui questi problemi erano concepiti e risolti dipendeva dalla nozione di funzione e, in particolare, alla concezione di una serie come il risultato di una trasformazione formale di una funzione. Dopo aver caratterizzato le procedure di sviluppo considerate accettabili, vengono esaminate le differenti strategie—dirette e inverse, ovvero sintetiche o analitiche—usate per risolvere tali problemi.

Résumé. Dans cet article nous étudions deux problèmes concernant la théorie des séries entières au XVIIIème siècle: le développement d'une fonction donnée en une série entière et le problème inverse, le retour d'une certaine série entière à la fonction dont cette dernière est le développement. La manière dont ces problèmes étaient conçus et résolus tenait à la notion de fonction, et en particulier à la conception d'une série comme le résultat d'une transformation formelle d'une fonction. Après avoir présenté et discuté les différentes procédures de développement employées par ces mathématiciens, nous examinons avec plus de détail les différentes stratégies de solutions de ces problèmes, en distinguant entre procédures directes et procédures inverses, c'est-à-dire synthétiques et procédures analytiques.

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1 Introduction

The eighteenth-century theory of series is the subject-matter of several studies, which approach the topic in different ways. Some of them insist on the main results; they show how and when such results were reached,but seem to dismiss the early procedures as naive or meaningless and to recast them directly in terms of modern formalisms[1]. Others highlight how certain results can be interpreted in terms of modern special theories (non-standard analysis or summability theory) and understand the results in light of this later context[2]. Finally, there are some writings which investigate the foundation and internal motivations of

[1]See, for example, Dutka (1984-1985).
[2]See, for example, MacMickizie and Tuckey (1997).

eighteenth-century theories[3]. Following this last approach, in the present paper we shall advance some historiographical theses which should serve as a possible key to the reading of eighteenth-century mathematical texts.

In the first part, we shall endeavour to establish the actual meaning of the equalities[4] of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \tag{1}$$

in eighteenth century. There exists, indeed, a radical difference between modern and eighteenth-century conception of series: even the fundamental terms, such as function, series, equality, have a significantly different meaning.

In the second part, we shall consider the problem of developing a function into a series and suggest that the problem of summing a series was conceived merely as the converse problem since it was viewed as the problem of the return from the series to the function. The relation between the problems of the sum and development was inverted with respect to today.

Finally, we shall investigate how these two problems were treated and try to classify different strategies for solving them.

In our inquiry, we shall attempt to identify those elements that seem to constitute evidence of a shared conception with respect to the foundation of analysis in eighteenth century and therefore focus our attention on common elements in the works of the main mathematicians who dealt with series[5]. We shall not discuss the differences between these mathematicians. Besides, we shall restrict ourselves to power series. Power series were not the only series considered in eighteenth-century analysis, however they were largely dominant[6].

[3]See, for example, Fraser (1989).

[4]In this paper, unless indicated otherwise, we shall use the term "equality" as a generic term to denote any written expression in which there are two members connected by the symbol "=", independently of the specific meaning this symbol possesses in different cases.

[5]In his (1989), Craig Fraser already tried " "to identify as clearly as possible those elements that are common" in eighteenth-century analysis; according to him "these elements constitute evidence of a shared conception significantly different from the modern one" [*ibid.*, 318]. With reference to eighteenth-century analysis, Fraser mainly refers to Euler's and Lagrange's conceptions but seems to suppose that these conceptions were largely shared by the entire mathematical community during the period that began roughly in the 1740s and lasted till the first years of the nineteenth century. We agree with such an opinion and would like to add some elements to Fraser's reconstruction, specially insisting on the earlier roots of such a "shared conception". Thus, we shall use the term "eighteenth-century" in a quite large sense, to refer to a period in the history of mathematics approximately starting from Newton's and Leibniz's research, and finishing with Lagrange's proposal to found the calculus on Taylor's expansions.

[6]Apart power series, from the end of the seventeenth century to about 1740s, mathematicians used only series of the type $\sum_{n=1}^{\infty} a_n x^{\alpha_n}$, where an could an be a negative integer or (in exceptional cases) a rational number. Only from the 1740s other function series and in particular trigonometric series began to be examined. The concepts and techniques originated from power series was applied to trigonometric series too; very interesting examples are in some of Euler's papers such as (1773). However, in certain cases, this application was rather problematic.

2 Convergence and power series

It is well known that in the first half of the eighteenth-century analysis gradually developed as a general theory of functions and was finally exposed as an organic theory by L. Euler in his *Introductio in analysin infinitorum* in 1748. The essential novelty of Euler's treatise consisted in the introduction of functions as autonomous objects and the construction of a comprehensive theory of these objects. However, according to eighteenth-century mathematicians, a function[7] was not an association between the elements of two given sets: it was a symbolic notation (which was termed "analytical expression", "formula" or "form") expressing a quantity in terms of another quantity[8]. It was not merely an expression, but the expression of a certain quantity, or else a function was a quantity as long as it was expressed, or could be in principle expressed, by a certain symbolic notation. During the century, mathematicians endeavoured to enlarge the set of known functions, however they always seemed to reason as if the set of functions was somehow fixed a priori by means of a genetic definition according to which a function had to derive from a finite number of elementary functions by applying a finite number of combination rules[9]. As long as it was conceived as an expression, a function was thought as a finitary composition of two sorts of atomic symbols: the atomic symbols for constant or variable quantities (*i. e.*, a, b,...; x, y,...; 0, 1,...; *etc.*) and the atomic symbols for the elementary operations on these quantities. As there was a finite number of elementary operations (*i. e.*, algebraic elementary operations, logarithm, exponential and trigonometric direct and inverse operations), a function was thus conceived as a composition of a finite number of elementary functions. It was conceived to be the expression of a quantity since these elementary functions were thought of as expressions of quantities and the rules of composition was conceived as conservative with respect to such a property of elementary functions.

This concept of a function implied that infinite series, as such, were "not themselves regarded as functions" [see Fraser (1989), 322]: they were instruments for facilitating the study of functions and for rendering them more intelligible [see Euler (1748), §.59]. During the eighteenth century, "infinite series were never introduced arbitrarily" [see Fraser (1989), 321): they always arose in some definite way in a particular mathematical problem, process or procedure.

Power series were conceived of as *quasi*-polynomial entities (that is mere infinitary extensions of polynomials). Even the symbolism was ambiguous and suggested this idea. Generally speaking, series were denoted by "$a + b + c + d + \&c.$" or "$a + bx + cxx + dx^3 + \&c.$", but the symbol "&c." was also used in some cases to denote a finite number of terms. The ambiguity of the notation depended on the fact that eighteenth-century mathematicians considered a series

[7]On the concept of a function, see Fraser (1989), Panza (1996) and Ferraro (2000). Here we limit ourselves to a short summary.

[8]By "quantity" eighteenth-century mathematicians meant what can be increased or decreased The most convenient means of representing a quantity in modern mathematical terms is by means of real values (and we shall also use this representation, for the sake of simplicity), however, we should not imagine a quantity as an element of a given well-defined set such as $\mathbb{R}$, since this notion was lacking in eighteenth-century mathematics.

[9]We shall use the term "function" even when we refer to authors, such as Newton or de Moivre, who never used this term. It seems to us that this terminological anachronism simplifies the exposition provided a "function" is intended in the previous terms.

as being known when one could explicitly exhibit its first terms and knew the law for deriving the following ones. Whether, starting from a certain point, these terms were all equal to zero or not could not be an essential difference in a lot of cases.

For instance, the product of two series was not openly defined: it seemed obvious that

$$(a + bx + cxx + dx^3 + \&c.) \cdot (A + Bx + Cxx + Dx^3 + \&c.)$$

was equal to

$$\begin{array}{r} aA+ \\ (aB + Ab)\,x+ \\ (aC + bB + cA)\,x^2+ \\ (aD + bC + cB + dA)\,x^3+ \\ \&c. \end{array}$$

independently of the meaning of "&c."' in such expressions: the rule of the ordinary multiplication between two polynomials was extended to infinite series without the difference between finite and infinite series was pointed out.

This approach could lead us to think that series were considered as entirely formal objects: but the matter is different. To make this clear, let us consider two examples.

In *De vera proportione*, Leibniz [see Leibniz (1682), 44] argued that $\frac{\pi}{4}$ is equal to $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$ and justified it by observing that if we take the first term of this series, then $\frac{\pi}{4}$ is approximated with an error less than $\frac{1}{3}$, if we take the first two terms of this series, the error is less than $\frac{1}{5}$, etc. If the series is continued, the error becomes less than any given quantity and thus the whole series contains all approximations and expresses the exact value.

In his famous *Epistola posterior* to Leibniz of October, 24, 1676 [see Newton (C), II, 110—161], Newton considered several applications of the binomial expansion, which he wrote in the form:

$$(P + PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m}{n}} Q + \frac{m\,(m-n)}{2n^2} P^{\frac{m}{n}} Q^2 + \&c. \tag{2}$$

In the case of the function $\sqrt[5]{c^5 + c^4x - x^5}$ he first put $P = c^5$ and $Q = \frac{c^4x - x^5}{c^5}$ and obtained

$$\sqrt[5]{c^5 + c^4x - x^5} = c + \frac{c^4x - x^5}{5c^4} - \frac{2c^8x^2 - 4c^4x^6 + 2x^{10}}{25c^9} + \&c.$$

then he put $P = -x^5$ and $Q = -\frac{c^4x + c^5}{x^5}$ and obtained

$$\sqrt[5]{c^5 + c^4x - x^5} = -x + \frac{c^4x + c^5}{5x^4} + \frac{2c^8x^2 + 4c^9x + 2c^{10}}{25x^9} + \&c.$$

Finally, he observed that the first procedure is preferable when x is very small, the second when it is very large.

This shows that mathematicians were concerned with convergence at the very origin of the theory of series. (We shall later argue that this was not contradictory with considering series as *quasi*-polynomial entities in the previous sense.) Eighteenth-century mathematicians also knew that the convergence of a power series $\sum_{i=0}^{\infty} a_i x^i$ depended on the value of its variable x. Of course, they did not possess the modern notion of interval of convergence, meanly because of the lacking of $\mathbb{R}$[10]. Nevertheless, it seems to us that the term "interval of convergence" could be conveniently used, provided one takes into account that when referring to the interval over which the power series $\sum_{i=0}^{\infty} a_i x^i$ converges, we are not referring to a subset of $\mathbb{R}$, but merely we mean that the series is convergent if the variable x varies from $-\delta$ to δ, where δ is an appropriate positive value. We shall also use the expression "non-null interval" to underline that the domain of variation of x does not reduce only one value and, specifically, does not reduce to the only value $x = 0$.

There are difficulties for the term "convergence", too. Even if, as far as we know, nobody of the most important mathematicians of the eighteenth century attributed a finite sum to a series that we today refer to as "divergent", they often used the terms "convergent" and "series" in an ambiguous way. Here we shall not classify and discuss the different meanings given to these terms in the eighteenth century by a textual analysis. Later we shall use the term "convergent series" to refer to the series that satisfy the following condition:

($\mathbf{C}_0$) A power series $\sum_{i=0}^{\infty} a_i x^i$ is said to be convergent to $f(x)$ on a non-null interval I_x of the values of x if and only if, for any value α of x belonging to I_x, the sequence $\left\{\sum_{i=0}^{j} a_i \alpha^i\right\}_{j=0}^{\infty}$ approaches $f(\alpha)$ indefinitely when j increases, and it is finally equal to $f(\alpha)$, when j is a infinite number.

At this juncture, some remarks are appropriate.

First, it is clear from the texts that eighteenth-century mathematicians considered this condition as salient and knew how to distinguish series depending on whether they satisfied ($\mathbf{C}_0$) or not. However it is certainly not a precise condition. and is not possible to formulate it in more precise terms without adding elements which were essentially alien to eighteenth-century analysis[11].

Second, a power series was considered as being the expression of a quantity (for any value of x belonging to I_x) if and only if it was considered to satisfy ($\mathbf{C}_0$). For instance, the series $\sum_{i=0}^{\infty} (-1)^i x^i$ was considered to express the ordinate of an hyperbola for certain values of x, as long as it was considered as being convergent to $\frac{1}{1+x}$ for these values of x.

Third, eighteenth-century mathematicians thought that even if the series $\sum_{i=0}^{\infty} a_i x^i$converged to a function $f(x)$ only on a non-null interval of values of x, the relation $f(x) = \sum_{i=0}^{\infty} a_i x^i$ could be manipulated without regard to the interval of convergence, namely they did not limit the validity of this equality

[10] Cf. the previous note (8).

[11] In particular the concept of limit was not been defined in mathematical terms and, even, there was ambiguity concerning the achievement of the limit. On the notions of "limit-achieving" and "limit-avoiding", see Grattan-Guiness (1969-70).

to the interval over which the series converged to the function[12]. For instance, though it was well known that the series $\sum_{i=0}^{\infty}(-1)^i x^i$ converges only for $|x| < 1$, the relation $\frac{1}{1+x} = \sum_{i=0}^{\infty}(-1)^i x^i$ was freely used in manipulations, without being restricted to the condition $|x| < 1$. Thus, the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ stated a general result, which was considered as concerned with the formal nature of the function $f(x)$ and not with the convergence of the series $\sum_{i=0}^{\infty} a_i x^i$.

3 The development of functions into series

At this juncture, a very natural question arises: What did the sign "=" mean in the equality (1)?

To answer to such a question, we, first, consider the simpler case of the equality $f(x) = g(x)$ between two finite analytical expressions $f(x)$ and $g(x)$.

In the eighteenth century the equality $f(x) = g(x)$ meant that one of these expressions, say $f(x)$ resulted by a transformation of the other one, to say $g(x)$. In the chapters 2 and 3 of the *Introductio*, Euler investigated the transformation of functions. According to him, "Functions are transmuted into other forms either by introducing another variable quantity instead of initially used or retaining the same variable quantity[13]." For instance, the expression $2 - 3z + z^2$ becomes $(1 - z)(2 - z)$ by factorising and $\sqrt{a + bz}$ is transformed into bx by substituting z with $bx^2 - \frac{a}{b}$.

This is also the case for the equality (1). The sign "=" interposed between a function and a series meant that the series was derived by the function by means of certain rules of transformation. Thus, the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ meant[14] that the power series $\sum_{i=0}^{\infty} a_i x^i$ was associated with the function $f(x)$ and that such an association depended on the fact that this power series resulted from

[12]It is known that Euler dealt with series which power series like

$$x^m - px^{m+q} + p(m+q)x^{m+2q} - p(m+q)(m+2q)x^{m+3q} + \dots$$

or

$$1 - 2x + 3!x^2 - \dots$$

which does converge over no non-null interval [for instance, cf. Euler **1754-1755**]. The investigation of these series originated in the attempt of solving certain differential equations or certain integral by continued fractions. While a series convergent over a non-null interval was considered as the development of a certain function and was thus used to express or study quantities, totally divergent series were only considered as tools for relating integrals or differential equations with continuous fractions, that is as formal links between different expressions of a quantity.

[13]See Euler (1748), I, 32: "Functiones in alias formas transmutantur vel loco quantititas variabilis aliam introducendo vel eandem quantitatem variabilem retinendo"

[14]It should be clear that we use the symbols "$f(x)$" and "$\sum_{i=0}^{\infty} a_i x^i$" as common names in order to refer to any particular functions or power series. The relation expressed by (1) should be intended as a relation between a particular function and a particular power series, whatever these functions and series are.

operating on the expression $f(x)$ according to certain rules of transformation[15]. We shall express this fact by saying that the series $\sum_{i=0}^{\infty} a_i x^i$ is the development of $f(x)$.

It is appropriate to note that under such a condition, the equality (1) was not logically symmetrical since the two expressions $f(x)$ and $\sum_{i=0}^{\infty} a_i x^i$ played different roles in such an equality. The first directly expressed a quantity and had a meaning *per se*; it was the proper object[16] of eighteenth-century analysis. The second was simply the result of a transformation of the given function $f(x)$. A series $\sum_{i=0}^{\infty} a_i x^i$ expressed a certain quantity only indirectly, since it was associated to the function $f(x)$ expressing this quantity. Therefore, the left-hand side, $f(x)$, of the equality (1) established the real object to be investigated, while the right-hand side, $\sum_{i=0}^{\infty} a_i x^i$, merely exhibited the result of a transformation useful to investigate the function in the left-hand side.

By speaking of certain rules of transformation, we mean a number of explicitly stated rules or a finite composition of them. Thus, a power series was associated with a given function and indicated as being equal to it if and only if it derived from this function by means of the application of one of these rules or of a finite combination of them.

Eighteenth-century mathematicians presented the accepted procedures of development of a function in different ways. In the *De analysis*, composed in 1671 but only published in 1711[17], Newton presented two procedures to be used for expanding a given function in a power series. These procedures are generally known as Mercator's rules, since particular cases of them had already been used by N. Mercator in his *Logarithmotechnia*[18]. They consisted of the application of the arithmetic rules for dividing a number for another or for extracting a root of a given number to literal expressions (see below). In the *De methodis*, composed in 1671 but only published in an English translation in 1736[19], Newton

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[15]We note explicitly that, according to such a condition, the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ has a precise sense independently of its interpretation as an identity (one should have this case if "$f(x)$" and "$\sum_{i=0}^{\infty} a_i x^i$" were considered as two different notations of the same object), or even as an equivalence (this case should occur if "$f(x)$" and "$\sum_{i=0}^{\infty} a_i x^i$" were considered as two names for distinct objects belonging to a common class of equivalence). The essential reason for that is that (1) does not concern primarily the object denoted by the symbols "$f(x)$" or "$\sum_{i=0}^{\infty} a_i x^i$", but these symbols themselves. It states that the finitary expression "$f(x)$"—i. e. not the quantity that this expression expresses, but this expression itself—has a certain relation (i. e. the relation of being transformable in) with the expression "$\sum_{i=0}^{\infty} a_i x^i$". Only once this relation between these two expressions had been stated, one could interpret (1) as being concerned with the quantity expressed by $f(x)$ (if the condition of convergence were satisfied)

[16]For the notion of proper object, cf. Panza (1997*b*).

[17]See Newton (MP), II, 206-247, for the original version, and Newton (1711), for the published text. We refer here to Newton (MP), II, 210-219.

[18]See Mercator (1668).

[19]See Newton (MP), III, 3-372, for the original version, and Newton (1736), for the published text. We refer here to Newton (MP), III, 51-57.

presented another procedure, known as Newton's method of parallelogram, to be used for expressing by means of a series the solution of a given algebraic equation $P(x,y) = 0$[20]. The crucial idea of this procedure was the following: by substituting the indeterminate series $\sum_{k=0}^{\infty} b_k x^{\alpha_k}$ for y in $P(x,y)$ one should obtain a new polynomial $Q(x)$, where all the coefficients of the powers x^{α_k} have to be separately equal to zero. The method of parallelogram was a method to be used to determine the coefficients b_k $(k = 0, 1, \dots)$ in the series $\sum_{k=0}^{\infty} b_k x^{\alpha_k}$ under such a condition, supposing that the value of x is close to a certain given value (for example 0). In short, Newton reduced the given equation in such a way that the coefficients b_k could be determined step by step. He thus obtained a series convergent in the considered interval. What is important here is not the specific nature of this method (it is well known), but the general principle on which it is founded. This principle, generally known as the principle of indeterminate coefficients, states that a series $\sum_{k=0}^{\infty} b_k x^{\alpha_k}$ is equal to 0 for every x on a non-null interval (if and) only if all the coefficients b_k $(k = 0, 1, \dots)$ are separately equal to zero.

Generally speaking, we can classify the accepted procedures of development into two classes. The first class comprises:

($\mathcal{P}_1$) The Mercator's expansions of fractions and square roots of polynomials.

($\mathcal{P}_2$) The binomial expansion for any (rational or irrational) exponent.

($\mathcal{P}_3$) Any expansion following the method of indeterminate coefficients.

Consider first the Mercator's expansions. We have already said that they consisted in applying the usual rules of division and extraction of square root of numbers to literal expressions. Take, for instance, the fraction $\frac{a^2}{b+x}$ [see Newton (MP), II, 212]. By dividing a^2 by $b + x$, one obtains the quotient $\frac{a^2}{b}$ and the remainder $-\frac{a^2}{b}x$. By dividing such a remainder by $b+x$, one obtains the quotient $\frac{a^2}{b^2}x$ and the remainder $-\frac{a^2}{b^2}x^2$. By continuing *in infinitum* one obtains the series $\frac{a^2}{b} - \frac{a^2}{b^2}x + \frac{a^2}{b^3}x^2 - \dots$ and the equality

$$\frac{a^2}{b+x} = \frac{a^2}{b} - \frac{a^2}{b^2}x + \frac{a^2}{b^3}x^2 - \&c. \tag{3}$$

An analogous procedure can be used to extract a square root. For instance, if one is looking for the development $\sum_{i=0}^{\infty} a_i x^i$ of $\sqrt{p+q}$, one proceeds in the following way:

i) one calculates a_0 as the square root of p;

ii) one calculates the first remainder $R_1 = p + q - \left(\sqrt{p}\right)^2 = q$;

iii) one looks for an x such that $(a_0 + x)^2 = p + R_1 + x^2$, and gets $x = \frac{R_1}{2\sqrt{p}} = \frac{q}{2\sqrt{p}}$;

[20]A similar procedure has been already presented in the *De analysis*: see Newton (MP), II, 218-233.

iv) one puts $a_1 = \frac{R_1}{2\sqrt{p}} = \frac{q}{2\sqrt{p}}$;

v) one considers the binomial $a_0 + a_1$ and calculates its square without the first term, that is $\Gamma_1 = 2a_0 a_1 + a_1^2 = q + \frac{q^2}{4p}$;

vi) one calculates the second remainder R_2 by difference, that is $R_2 = R_1 - \Gamma_1 = -\frac{q^2}{4p}$;

vii) one repeats the step (*iii*) on R_2 in order to find a_2, that is one looks for an x such that $(a_0 + x)^2 = p + R_2 + x^2$, and gets $x = a_2 = \frac{R_2}{2\sqrt{p}} = -\frac{q^2}{8p\sqrt{p}}$;

viii) one continues in this way, by posing $\Gamma_i = 2a_i (a_0 + \ldots + a_{i-1}) + a_i^2$ and by considering only the first term of R_i in searching a_i.

On obtain thus:

$$\sqrt{p+q} = \sqrt{p} + \frac{q}{2\sqrt{p}} - \frac{q^2}{8p\sqrt{p}} + \frac{q^3}{16p^2\sqrt{p}} - \&c. \tag{4}$$

The equalities (3) and (4) could also be easily obtained by applying the binomial expansion (2), or better its simplified form:

$$(p+q)^r = p^r + rp^{r-1}q + \frac{r(r-1)}{2!}p^{r-2}q^2 + \frac{r(r-1)(r-2)}{3!}p^{r-3}q^3 + \&c. \tag{5}$$

where r is any rational exponent. When obtained by means of Mercator's procedures, the equalities (3) and (4) are however directly extracted by the given expressions $\frac{a^2}{b+x}$ or $\sqrt{p+q}$ by operating on such expressions, while, when obtained by means of binomial expansions, they result from a particularisation of a general equality as (5) which has, at its turn, to be proved. Hence, as long as they were obtained by means of Mercator's procedures, the equalities (3) and (4) were viewed as particular confirmations of such a general equality, rather than as a consequence of it. After Newton, nobody really doubted the validity of (5) or were reluctant to apply it in order to get the development of particular functions. Nevertheless, many efforts were made in order to provide this equality with a proof more satisfying than Newton's argument in favor of it (which finally relied on an *a priori* assumption of the same extension of algebraic rules that (5) seem to guarantee), or to prove its generalization to irrational exponents. A simply way to do that would have been to draw such an equality from the "Taylor's theorem":

$$f(x+\xi) = f(x) + \frac{df}{dx}\xi + \frac{1}{2!}\frac{d^2f}{dx^2}\xi^2 + \frac{1}{3!}\frac{d^3f}{dx^3}\xi^3 + \&c. \tag{6}$$

However, this was not considered as acceptable since (5) and its particular consequences were thought to be independent from differential calculus and the rules of differentiation of the elementary functions depended on (5).

An other way for obtaining many development of particular functions—including the equalities (3) and (4)—by operating directly on these functions was to resort to the principle of indeterminate coefficients. Unlike Mercator's procedures, this principle allowed to determine a development in power series whose existence were previously supposed: one started from the hypothesis that the given function $f(x)$ could be developed in a power series and rely on such a principle in order to determine (or construct) this series. An example of such a procedures is found in Stirling's *Methodus differentialis*, where it is used in order to develop the function $\frac{1}{A+Bx+Cx^2}$ [see Stirling (1730), 2]. Stirling supposed that

$$\frac{1}{A+Bx+Cx^2} = \sum_{i=0}^{\infty} a_i x^i$$

and, then, that the equality

$$\left[\sum_{i=0}^{\infty} a_i x^i\right]\left[A+Bx+Cx^2\right] - 1 = 0$$

should hold for any x in a non-null interval. By multiplying and rearranging he derived

$$(Aa_0-1)+(Aa_1+Ba_0)x+(Aa_2+Ba_1+Ca_0)x^2+(Aa_3+Ba_2+Ca_1)x^3+\&c. = 0$$

Finally, by applying the principle of indeterminate coefficients, he obtained the equations

$$\begin{aligned} Aa_0 - 1 &= 0 \\ Aa_1 + Ba_0 &= 0 \\ Aa_2 + Ba_1 + Ca_0 &= 0 \\ Aa_3 + Ba_2 + Ca_1 &= 0 \\ &\&c. \end{aligned}$$

which allowed him to determine the coefficients:

$$a_0 = \frac{1}{A} \quad ; \quad a_1 = -\frac{B}{A^2} \quad ; \quad a_2 = \frac{B^2 - AC}{A^3} \quad ; \quad a_3 = \frac{2ABC - B^3}{A^4} \quad ; \quad \&c.$$

The principle of indeterminate coefficients is here employed in order to look for a development of the given function in the family of power series. One supposes that this function has a development in such a family and, by means of such a procedure, explicitly constructs it. In order to justify this procedure as a simply extension of algebraic rules, one has to assume that the indeterminate series $\sum\limits_{i=0}^{\infty} a_i x^i$ which is initially supposed to be equal to the given function converge to such a function in a non-null interval of values of x. Under this hypothesis, it is simple to show that the development is unique. A proof of it is found in the first volume of Euler's *Introductio* [see Euler (1748), I, 230-231]. It works as follows. From the supposition

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} a_i x^i \\ f(x) &= \sum_{i=0}^{\infty} b_i x^i \end{aligned}$$

it follows

$$\sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} b_i x^i$$

It is thus enough to put $x = 0$ to draw $a_0 = b_0$. If the series converge in a non-null interval, it is then possible to divide the last equality by x, and then put again $x = 0$, in order to draw $a_1 = b_1$, and so on [see also Euler (1740), 471].

The procedures $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$ are all concerned with an infinitary extension of algebraic rules and, therefore, we term it "*quasi*-algebraic" procedures. A second class of procedures was composed of:

$(\mathcal{P}_4)$ Any expansion deriving from contemporary differentiation or integration both of a certain function $f(x)$ and a certain determinate power series $\sum_{i=0}^{\infty} a_i x^i$ already associated to $f(x)$, or of a function $f(x)$ and a certain indeterminate power series $\sum_{i=0}^{\infty} a_i x^i$ supposed to be associated to$f(x)$—the operations on the power series being perfomed term by term.

Like $(\mathcal{P}_3)$, these procedures also depend on the supposition that the series which is associated to the given function converge to it on a non-null interval. Moreover, the procedures $(\mathcal{P}_4)$ also depend on an infinitary extension of the properties of linearity of differentiation and integration (today we know that they do not follow from simple convergence). In chapter II of the second part of the *Institutiones calculi differentialis* [see Euler (1755), 235], Euler justified this supposition for differentiation by asserting that from $f(x) = \sum_{i=0}^{\infty} a_i x^i$ it follows

$df(x) = f(x+dx) - f(x) = \sum_{i=0}^{\infty} a_i (x+dx)^i - \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} a_i \left[(x+dx)^i - x^i\right] = \sum_{i=0}^{\infty} a_i i x^{i-1} dx.$

In chapter III of the *Institutiones calculi differentialis* [see Euler (1768-1670), **1, 76-85**], he relies on an analogous rule for integration in order to state that the integral of a function whose development is $\sum_{i=0}^{\infty} a_i x^{m+in}$ is equal to $\sum_{i=0}^{\infty} \frac{a_i}{m+in+1} x^{m+in+1}$.

The first of these rules was used by Newton in a preliminary version of the *De quadratura curvarum*, in order to obtain the first version on Taylor's development of any given function [cf. Newton (MP), VII, 96-98]. If one puts

$$f(x) = \sum_{i=0}^{\infty} A_i (x-a)^i$$

by reiterating the differentiation term by term, one obtained

$$\begin{aligned}
\frac{df}{dx} &= \sum_{i=0}^{\infty} A_i i (x-a)^{i-1} \\
\frac{d^2 f}{dx^2} &= \sum_{i=0}^{\infty} A_i i\,(i-1)\,(x-a)^{i-2} \\
\frac{d^3 f}{dx^3} &= \sum_{i=0}^{\infty} A_i i\,(i-1)\,(i-2)\,(x-a)^{i-3} \\
&\&c.
\end{aligned}$$

and then, by posing $x = a$:

$$\begin{aligned}
A_1 &= \left.\frac{df}{dx}\right|_{x=a} \\
A_2 &= \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=a} \\
A_3 &= \frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=a} \\
&\&c.
\end{aligned}$$

In eighteenth century, the compositions of the procedures ($\mathcal{P}_1$)-($\mathcal{P}_4$) was also used; namely, if the power series $\sum_{i=0}^{\infty} a_i x^i$, $\sum_{i=0}^{\infty} b_i x^i$, &c. were respectively associated to the functions $f(x)$, $g(x)$, &c., and a single function $F(x)$ was constructed by composing $f(x)$, $g(x)$, &c., then the power series constructed by composing the series $\sum_{i=0}^{\infty} a_i x^i$, $\sum_{i=0}^{\infty} b_i x^i$, &c. in the same manner was considered as being the development of $F(x)$.

In general, the accepted procedures for the development of a given function were reducible to the procedures ($\mathcal{P}_1$)-($\mathcal{P}_4$) or a composition of them[21]. This does not mean that these procedures were the only elementary procedures capable of providing power series developments of given functions. Mathematicians were open to the possibility of finding other procedures and in effect other more particular procedures were applied in some particular cases.

Now, let us consider the question

($\mathbf{Q_1}$) Under what conditions was a particular power series $\sum_{i=0}^{\infty} a_i x^i$ associated with a certain function $f(x)$ in eighteenth-century analysis?

Of course, a particular power series $\sum_{i=0}^{\infty} a_i x^i$ associated with a certain function $f(x)$ if it was the result of a transformation. However, as long as a function was considered not merely as an expression but rather as the expression of a quantity, not all transformations can be acceptated. Therefore, we cannot answer to ($\mathbf{Q_1}$) simply by listing a finite list of procedures of transformation like ($\mathcal{P}_1$)-($\mathcal{P}_4$) (even though such an answer is factually correct); indeed we had to complete this answer by examining another question:

($\mathbf{Q_2}$) For what reason was a certain procedure $\mathcal{P}$ transforming a function $f(x)$ into a series $\sum_{i=0}^{\infty} a_i x^i$ considered as an acceptable rule of development[22] in eighteenth-century analysis?

We have previously observed that the equality $f(x) = g(x)$ between two function meant that the expression $g(x)$ was derived by a transformation of the expression $f(x)$. However the given expression $f(x)$ was taken into account insofar as it expressed a certain quantity: thus mathematicians thought that

[21] As we have already observed, they are not completely independent from each other.

[22] By an acceptable rule of development, we intend a rule that, when applied to a given function, generated a power series, which can be considered as the development of the function

the result of the transformation had to express the same quantity, too[23]. The twofold nature of functions was thus transmitted to the equality $f(x) = g(x):$ on one side, this equality stated that $g(x)$ is the result of a certain transformation of $f(x)$; on the other side, it stated that $f(x)$ and $g(x)$ expressed the same quantity[24].

Thus, the rules of transformation of a function into another one had to preserve the expressed quantity in order to be rightful. This means that a rule of transformation $\mathcal{R}$ was considered as rightful only if it was ascertained or supposed that one of the following conditions was satisfied:

$\mathbf{C}_1$) for any function F, F and $\mathcal{R}(F)$ express the quantity[25];

$\mathbf{C}_2$) for any two functions F and G, if F and G express the same quantity, then if $\mathcal{R}(F)$ and $\mathcal{R}(G)$ also express the same quantity.

If $\mathcal{R}$ was considered to satisfy ($\mathbf{C}_1$), then the equality $\mathcal{R}(F) = F$ was considered as to be rightful. If R was considered to satisfy ($\mathbf{C}_2$), then the equality $\mathcal{R}(F) = \mathcal{R}(G)$ was considered as to be rightful.

It was precisely because the usual algebraic rules satisfied the condition ($\mathbf{C}_1$) that they were considered as rightful rules of transformation; and it was precisely because the contemporary differentiation or integration of two finite functions $f(x)$ and $g(x)$ satisfied the condition ($\mathbf{C}_2$), that this rule was considered as a rightful rule of transformation.

In order to extend this approach to the rules of transformation of a function into a series, a preliminary problem should be solved: under what condition could a power series considered to be the expression of a quantity?

An initial answer to such a question could rely on the notion of convergence. We saw that a power series expressed a quantity if and only if it was convergent to this quantity and that if this quantity was analytically expressed by the function $f(x)$, then the series had to converge to $f(x)$. We could then answer ($\mathbf{Q_2}$) in the following way:

($\mathbf{A_2}$) In eighteenth-century analysis, a certain procedure $\mathcal{P}$ transforming a function $f(x)$ into a series $\sum\limits_{i=0}^{\infty} a_i x^i$ was acceptable if and only if the power series $\sum\limits_{i=0}^{\infty} a_i x^i$ was convergent to $f(x)$ on a non-null interval I_x of the values of x.

By composing ($\mathbf{A_2}$) with ($\mathbf{C}_0$), we would obtain the following condition:

($\mathbf{C}_3$) In eighteenth-century analysis, a certain procedure $\mathcal{P}$ that transformed a function $f(x)$ into a series $\sum\limits_{i=0}^{\infty} a_i x^i$ was an acceptable rule of development

[23]See Euler (1748), I, 159: "Si fuerit $y = \frac{1-zz}{1+zz}$ atque ponatur $z = \frac{1-x}{1+x}$, hoc valore loco z substituto erit $y = \frac{2x}{1+xx}$. Sumpto ergo pro x valore quocunque determinato ex eo reperientur valores determinati pro z et y sicque invenitur valor ipsius y respondens illi valori ipsius z, qui simul prodiit. Uti, si sit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$ et $y = \frac{4}{5}$; reperitur autem quoque $y = \frac{4}{5}$, si in $\frac{1-zz}{1+zz}$, cui expressioni y aequatur, ponatur $z = \frac{1}{3}$."

[24]See Euler (1748), I, 38: "Omnis transformatio consistit in alio modo eandem functionem exprimendi, quemadmodum ex Algebra constat eandem quantitatem per plures diversas formas exprimi posse."

[25]Of course $\mathcal{R}(F)$ denotes the expression that is obtained by applying the rule $\mathcal{R}$ to F.

of such a function if and only if for any value α of x belonging to a certain non-null interval I_x, the sequence $\left\{\sum_{i=0}^{j} a_i\alpha^i\right\}_{j=0}^{\infty}$ approached $f(\alpha)$ indefinitely when j increases, and it was finally equal to $f(\alpha)$, when j is a infinite number.

Since ($\mathbf{C}_0$) is not a precise condition, ($\mathbf{C}_3$) is not a precise condition too or, at least, that it does not provide a sufficiently clear criterion for deciding whether a certain procedure $\mathcal{P}$ is an acceptable rule of development. However, this did not mean that ($\mathbf{A}_2$) was not taken into account, but only that, in order to decide if a particular series was convergent, eighteenth century mathematicians rely upon a criterion that did not depend on the intrinsic nature of the series but on the procedure of development generating the series.

Indeed a procedure of transformation were acceptable if and only if it was an infinitary extension of the rules of transformation of finite expressions into finite expressions and satisfied one of the conditions ($\mathbf{C}_1$) or ($\mathbf{C}_2$) regarding the conservation of the equality of the expressed quantities. In other terms, what guaranteed the convergence of the development of a function $f(x)$ to this function on a non-null interval of values of x was the formal nature of the procedure of the development –the fact that it was an infinitary extension of finitary rules satisfying ($\mathbf{C}_1$) or ($\mathbf{C}_2$)– and not an analysis of the nature of the resulting series, in accordance with the definition ($\mathbf{C}_0$). It is precisely this the reason that made the procedures ($\mathcal{P}_1$)-($\mathcal{P}_4$) and their compositions acceptable in eighteenth century.

Therefore a satisfactory answer to ($\mathbf{Q_1}$) is the following:

($\mathbf{A_1}$) In eighteenth-century analysis, a certain power series $\sum_{i=0}^{\infty} a_i x^i$ was associated with a certain function$f(x)$ if it appeared as the result of the application to $f(x)$ of one of the accepted procedures ($\mathcal{P}_1$)-($\mathcal{P}_4$), of any finite combination of them, or of any other particular infinitary extension of the rules of transformation of finitary expressions satisfying one of the conditions ($\mathbf{C_1}$), ($\mathbf{C_2}$), and operating on a given, determinate or indeterminate, power series term by term.

This should be a satisfactory formulation of a sufficient condition for the truth of (1), in eighteenth-century analysis. Certainly, this condition is not necessary. However, it is not necessary only in the following sense. A certain power series $\sum_{i=0}^{\infty} a_i x^i$ could be intended as convergent to $f(x)$ on a non-null interval I_x of values of x, according to ($\mathbf{C_0}$), without appearing as the result of the application to $f(x)$ of one of the accepted procedure. However it was implicit in eighteenth-century that a certain power series $\sum_{i=0}^{\infty} a_i x^i$ could converge to $f(x)$ on a non-null interval I_x of values of x, according to ($\mathbf{C_0}$), only if it could result from the application to $f(x)$ of one of the accepted procedures, even if one did know how this was performed.

In conclusion to the previous section 2, we have observed that the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ was conceived in eighteenth-century as concerned with the formal nature of the function $f(x)$ and not with the convergence of the series $\sum_{i=0}^{\infty} a_i x^i$ and that this equality was considered as valid independently of the value of x. This would seem to be contradicted by our last conclusion, namely that the validity of such en equality depends on the convergence of the series on a certain non-null interval. However, there is no contradiction. Simply, eighteenth-century mathematicians considered the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ to be valid if and only if the series $\sum_{i=0}^{\infty} a_i x^i$ was considered as convergent to the function $f(x)$ on a non-null interval, but they did not think that the validity of such an equality had to be restricted to the values of x belonging to such an interval.

4 Direct and converse problems in power series theory

At this juncture, it should be evident that the very heart of the eighteenth-century theory of series was constituted by the following pair of problems:

P.1.a To develop a given function into a power series;

P.1.b To return from a given power series to the function to which this power series is the development.

These problems should not be confounded with the following ones, with which modern real analysis is concerned:

P.2.a To look for a power series which converges to a given function of a real variable;

P.2.b To sum a given (convergent) power series of a real variable.

Clearly, both the pair (**P.1**) and the pair (**P.2**) consist of a direct and a converse problem. By "converse problem", we mean a problem that can be only formulated by referring to another problem, namely the direct one.

In modern analysis, the direct problem is (**P.2.b**), i.e., the problem of summing a given series, which is solved by seeking the limit of the n-th partial sums. The converse problem is (**P.2.a**), the problem of expressing a given function by means of a power series. For instance, we say that $\frac{1}{1+x}$ is the sum of $\sum_{i=0}^{\infty}(-1)^i x^i$ for $|x| < 1$, because $\lim_{n\to\infty} \sum_{i=0}^{\infty}(-1)^i x^i = \lim_{n\to\infty} \frac{1+(-1)^n x^{n+1}}{1+x} = \frac{1}{1+x}$ for $|x| < 1$; *viceversa*, we say that $\sum_{i=0}^{\infty}(-1)^i x^i$ is the development of the function $\frac{1}{1+x}$ for$|x| < 1$ because $\frac{1}{1+x}$ is the sum of $\sum_{i=0}^{\infty}(-1)^i x^i$.

Instead, in eighteenth-century analysis (**P.1.a**), the problem of developing a function into a series, was the direct problem and (**P.1.b**) was the converse one[26]. The path providing a solution to the problem (**P.1.a**) was a progressive path, since it progressed from the function, which was a proper object of eighteenth-century mathematics, to the series, which was a particular expression associated with the function: in other terms, it was a *synthetic* path. The path providing a solution to the problem (**P.1.b**) regressed from the series to the function: it led from a particular expression associated with an unknown object to this object (this is exactly what the verb "to return" indicates). It was a regressive path, that is an *analytical* path. For instance, given the function $\frac{1}{1+x}$, the direct problem was to develop such a function and the solution of this problem was given by the series $\sum_{i=0}^{\infty}(-1)^i x^i$. *Viceversa*, the converse problem was to find the function whose development is given by the series $\sum_{i=0}^{\infty}(-1)^i x^i$. And thus the solution of this problem was given by the function $\frac{1}{1+x}$ just because $\sum_{i=0}^{\infty}(-1)^i x^i$ was considered as the development $\frac{1}{1+x}$.

Eighteenth-century mathematicians used different terms to refer to the return from a power series to the original function (the function which this power series expresses). They, at times, used the term "regressus" [see., for example, Leibniz (GMS), III, 351 and de Moivre (1730), 123]; more often they preferred the term "sum". The sense in which this term was employed was made explicit by Euler: "As series in analysis arise from the expansion of fractions or irrational quantities or even of transcendental, it will in turn be permissible in calculation to substitute in place of such a series that quantity out of whose development it is produced. For this reason [...]we employ this definition of sum, that is to say, the sum of a series is that quantity which generates the series" [see Euler (1754-55), 593-594; translation in Barbeau and Leah's (1976), 144].

In order to use a clear and uniform language, we shall use the verb "to envelop" to refer to the passage from a given power series to the function of

[26]Of course (**P.1.a**) and (**P.1.b**) were not the only pair of direct and converse problems in eighteenth-century analysis. Generally speaking, given an operation O transforming an object α, belonging to a certain set $\mathbf{S}$ of objects, into an object $\beta = O(\alpha)$ belonging to a set $\mathbf{T}$, we can look for a converse operation O' such that $O'(\beta) = \alpha$. A problem arises when O, $\mathbf{S}$ and $\mathbf{T}$ are such that, for some β in $\mathbf{T}$, there is no object α in $\mathbf{S}$, such that $O'(\beta) = \alpha$. Today this problem is solved by defining a new set $\mathbf{S}^*$ (which can be thought of as an enlargement of $\mathbf{S}$) whose objects are defined as the images of the objects of $\mathbf{T}$ under the operation O'. In this way, the problems of existence are settled *a priori*, by fixing the domain and range of the operations O and O' once and for all. In eighteenth-century, mathematicians viewed the matter differently. They did not define a set of objects $\mathbf{S}^*$ *a priori*, so that it is always possible to find an image of every object β of the set $\mathbf{T}$, under the operation O'; instead, for every specific object β of $\mathbf{T}$, they tried to construct a new object, somehow similar to the objects of $\mathbf{S}$, so that one could arrive at such an object by applying the operation O' to β.

The difference between the modern and eighteenth-century approach is crucial. Since an *a priori* definition is lacking, the object is constructed as the result $O'(\beta)$ of the application of the operation O' to the object β; for this reason, the nature of the object $O'(\beta)$ could only be understood by means of an implicit reference to objects that were already given outside the theory where the operations O and O' were initially defined. For example, the operation O and its converse O' might have a geometric interpretation providing an explanation of the nature of new objects. This is the case for differentiation and integration. In his (1768-70) [vol.1, def. 2, p. 7], Euler defined the "integral" $\int g(x)dx$ of a function $g(x)$ as a function $f(x)$ such that $d[f(x)] = g(x)dx$. If for some $g(x)$, no known function $f(x)$ were such that $d[f(x)] = g(x)dx$, the symbol "$\int g(x)dx$" was used formally to denote an unknown function (but subject to certain general conditions) such that $d[\int g(x)dx] = g(x)dx$, and to which one could give a geometric meaning (area, length, ...).

which the series is the development. Of course, by "envelopment" we shall denote the function that results from enveloping a series. Thus the problem (**P.1.b**) can be rephrased as follows:

(**P.1.b'**) To envelop a given power series into a function.

By using this terminology, we can state that, in eighteenth-century mathematicians, to sum a series meant to envelop it. Thus, for such mathematicians, the problem of summing a given power series was essentially different from our problem (**P.2.b**) and does not properly concern numerical series: there is no sense in speaking about the development or envelopment of a number. Numerical series can not be enveloped but only summed. Eighteenth-century mathematicians had a perfect knowledge of the fact that certain series can be used to express numbers (in particular, irrational numbers); however they usually considered a series like $\sum_{i=0}^{\infty} a_i$ as a particular case of the power series $\sum_{i=0}^{\infty} a_i x^i$ for the position $x = 1$, and thought that the most natural way to sum $\sum_{i=0}^{\infty} a_i$ was to determine the envelopment $f(x)$ of $\sum_{i=0}^{\infty} a_i x^i$ and then take $\sum_{i=0}^{\infty} a_i = f(1)$. This should make clear that eighteenth-century analysis, unlike modern real analysis, was not a theory of real numbers. It was rather a theory of (continuous) quantities, insofar as they were expressed by means of a convenient expression.

To end this section, we have to make explicit a general condition concerning the problems (**P.1.a**) and (**P.1.b**), which was only implicitly in the previous remarks. The generic symbol "$f(x)$", which indicates a function in the equality (1), is nothing but a written convention and cannot therefore support any formal procedure; the generic symbol "$\sum_{i=0}^{\infty} a_i x^i$", which indicates a power series in such an equality, is instead an explicit exhibition of a particular type of series and could therefore support some formal procedures. Therefore, in order to go from the first symbol to a complete determination of a particular object, one had to make different steps which are part of a process of progressive determination that necessarily include the determination of the particular form of the function. Only once these different steps are performed can a formal procedure be applied to $f(x)$. In order to proceed from the second symbol to a complete determination of a particular object, only one step has to be made, i.e., the determination of the coefficients occurring in it. And, even if this step is not performed, a formal procedure can be applied to the series. In other words: no formal procedure can be applied to a (completely) generic function, while certain formal procedures can be applied to (completely) generic power series.

5 Direct and converse strategies to develop a function and envelop a series

We are now ready to consider different strategies to solve the problems (**P.1**).

Let us imagine, first, that a particular function $f(x)$ is given. An initial obvious way for solving the problem (**P.1.a**) with respect to this function is:

($\mathbf{S_aD}$) To apply directly to $f(x)$ one of the procedures $(\mathcal{P}_1)$-$(\mathcal{P}_4)$, an appropriate composition of them, or any other particular accepted procedure of development.

Let us now imagine, instead, that a particular power series $\sum_{i=0}^{\infty} A_i x^i$ is given. An obvious strategy to solve the problem (**P.1.b**) with respect to this series is:

($\mathbf{S_bD}$) To operate directly on the given series $\sum_{i=0}^{\infty} A_i x^i$ and transform it, by suitable manipulations and/or substitutions in a finitary expression $f(x)$, which is supposed to be the sum of the series.

A successive application of strategy ($\mathbf{S_aD}$) to $f(x)$ can successively confirm that this function is precisely the solution to the problem (**P.1.b**) with respect to $\sum_{i=0}^{\infty} A_i x^i$.

Two examples of ($\mathbf{S_aD}$) are Euler's and Lagrange's development of the exponential function $y = a^x$, respectively in Euler (1748, pp. 85-86) and in Lagrange (1797, pp. 18-20 and 1813, pp. 31-33). Let us consider the second of these examples. Lagrange started from the identical equation

$$a^x = [(1 + (a-1))^n]^{\frac{x}{n}} \tag{7}$$

and, by applying the binomial expansion, he obtained

$$a^x = \left[1 + n(a-1) + \frac{n(n-1)}{2}(a-1)^2 + \frac{n(n-1)(n-2)}{3!}(a-1)^3 + \&c.\right]^{\frac{x}{n}}$$

By rearranging this equality, it is possible to give it the form

$$a^x = \left[1 + H_1 n + H_2 n^2 + H_3 n^3 + \&c.\right]^{\frac{x}{n}} \tag{8}$$

where the first coefficient H_1 is the series

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{2}{3!}(a-1)^3 + \&c.$$

By applying the binomial expansion to (8) Lagrange obtained

$$\begin{aligned} a^x = \ & 1 + x\left(H_1 + H_2 n + H_3 n^2 + \&c.\right) \\ & + \frac{x\,(x-n)}{2}\left(H_1 + H_2 n + H_3 n^2 + \&c.\right)^2 \\ & + \frac{x\,(x-n)\,(x-2n)}{3!}\left(H_1 + H_2 n + H_3 n^2 + \&c.\right)^3 \end{aligned}$$

By observing that "n is a entirely arbitrary" and that its elimination in (7) leads to the identity $a^x = a^x$, he finally concluded that in this last equality all the terms where n occurs have to simplify each other. Then

$$a^x = 1 + Ax + \frac{A^2}{2}x^2 + \frac{A^3}{3!}x^3 + \tag{9}$$

where

$$A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{2}{3!}(a-1)^3 + \&c.$$

As a first example of $(\mathbf{S_b D})$ let us consider the sum of the geometric series $\sum\limits_{i=0}^{\infty} x^{\alpha+i\beta}$. In his (1732-33, pp. 44-45), Euler set

$$S = \sum_{i=0}^{m} x^{\alpha+i\beta}$$

and obtained

$$S - x^{\alpha} = \sum_{i=1}^{m} x^{\alpha+i\beta} \tag{10}$$

By adding $x^{\alpha+(m+1)\beta}$ to both the sides of (10) and dividing them by x^{β}, he obtained

$$\frac{S - x^{\alpha} + x^{\alpha+(m+1)\beta}}{x^{\beta}} = \sum_{i=0}^{m} x^{\alpha+i\beta} = S$$

and then:

$$S = \sum_{i=0}^{m} x^{\alpha+i\beta} = \frac{x^{\alpha} - x^{\alpha+(m+1)\beta}}{1 - x^{\beta}} \tag{11}$$

Taking $m = \infty$ in (11) and supposing that $|x| < 1$, Euler concluded that

$$\sum_{i=0}^{\infty} x^{\alpha+i\beta} = \frac{x^{\alpha}}{1 - x^{\beta}} \tag{12}$$

As another example, consider the series $\sum\limits_{i=1}^{\infty} (2i-1)\,\frac{x^i}{i!}$. In his (1732-33, pp. 70-71), Euler firstly set

$$\sum_{i=1}^{m} (2i-1)\,\frac{x^i}{i!} = S(x,m)$$

and, by integrating term by term (and supposing that the constant of integration is null), derived from it the equality

$$\frac{1}{2}x^{\frac{1}{2}}\left[\int x^{-\frac{3}{2}}\left[S\left(x,m\right)\right]dx\right] = \sum_{i=1}^{m} \frac{x^i}{i!} \tag{13}$$

He differentiated (13) and obtained

$$\frac{\int x^{-\frac{3}{2}}\left[S\left(x,m\right)\right]dx}{4x^{\frac{1}{2}}} + \frac{S\left(x,m\right)}{2x} = 1 + \sum_{i=1}^{m} \frac{x^i}{i!} - \frac{x^m}{m!}$$

By comparison with (13) he derived

$$(1-2x)\int x^{-\frac{3}{2}}\left[S\left(x,m\right)\right]dx = 4x^{\frac{1}{2}} - \frac{2S\left(x,m\right)}{x^{\frac{1}{2}}} - \frac{4x^{m+\frac{1}{2}}}{m!}$$

Supposing that $m = \infty$, this equality reduces to the following

$$\int x^{-\frac{3}{2}} \left[S(x)\right] dx = \frac{4x - 2S(x)}{(1-2x)\, x^{\frac{1}{2}}} \tag{14}$$

where $S(x) = S(x, \infty)$. By differentiating (14), and considering $S(x)$ as an indipendent variable S, Euler had

$$\frac{Sdx}{x\sqrt{x}} = \frac{2xdx + 4x^2dx + Sdx - 6Sxdx - 2xdS + 4x^2dS}{(1-2x)^2 x\sqrt{x}}$$

Hence

$$dx + 2xdx - Sdx - 2xSdx - dS + 2xdS = 0$$

and

$$dS + \frac{S(1+2x)dx}{1-2x} = \frac{(1+2x)dx}{1-2x}$$

Then Euler multiplied this equality by $\frac{e^{-x}}{1-2x}$ and noted that the left-hand side becomes equal to the differential of the function $\frac{e^{-x}S}{1-2x}$ of two variables x and S. Thus he obtained

$$\frac{e^{-x}S}{1-2x} = \int \frac{e^{-x}(1+2x)}{(1-2x)^2} dx = \frac{e^{-x}}{1-2x} - 1$$

(the constant -1 being determined under the condition $S(0) = 0$) and finally

$$S = \sum_{i=1}^{\infty} (2i-1)\frac{x^i}{i!} = 1 - e^x(1-2x). \tag{15}$$

A third example is taken from the *Institutiones calculi differentialis* (1755, 2, pp. 217-218). Supposing that a power series $\sum\limits_{i=1}^{\infty} A_i x^i$ is given, Euler transformed it by substitution

$$x = \frac{y}{1+y}$$

Since, for any integer i, we have

$$x^i = \left(\frac{y}{1+y}\right)^i = \sum_{k=0}^{\infty} \binom{-i}{k} y^{i+k},$$

he obtained

$$\begin{aligned}
\sum_{i=1}^{\infty} A_i x^i &= \sum_{i=1}^{\infty} A_i \left[\sum_{k=0}^{\infty} \binom{-i}{k} y^{i+k}\right] = \sum_{i=1}^{\infty} \left[\sum_{k=0}^{i-1} \binom{k-i}{k} A_{i-k}\right] y^i \\
&= \sum_{i=1}^{\infty} \left[\sum_{k=0}^{i-1} \binom{k-i}{k} A_{i-k}\right] \left(\frac{x}{1-x}\right)^i
\end{aligned}$$

and, since

$$\sum_{k=0}^{i-1} \binom{k-i}{k} A_{i-k} = \Delta^{i-1} A_1$$

he derived

$$\sum_{i=1}^{\infty} A_i x^i = \sum_{i=1}^{\infty} \Delta^{i-1} A_1 \left(\frac{x}{1-x} \right)^i \tag{16}$$

If the differences $\Delta^j A_1$ are equal to zero for large enough j, then $\sum_{i=1}^{\infty} \Delta^{i-1} A_1 \left(\frac{x}{1-x} \right)^i$ reduces to a finite expression which Euler assumed to be the envelopment of the given series $\sum_{i=1}^{\infty} A_i x^i$. For instance, by applying (16) to the series $\sum_{i=1}^{\infty} i^2 x^i$, one has:

$$\begin{aligned} \sum_{i=1}^{\infty} i^2 x^i &= \sum_{i=1}^{\infty} \Delta^{i-1} \left[\left(i^2\right)_{i=1} \right] \left(\frac{x}{1-x} \right)^i \\ &= \frac{x}{1-x} + 3 \left(\frac{x}{1-x} \right)^2 + 2 \left(\frac{x}{1-x} \right)^3 \\ &= \frac{x+x^2}{(1-x)^2} \end{aligned}$$

because

$$\begin{aligned} \Delta^0 \left[\left(i^2\right)_{i=1} \right] &= 1 \\ \Delta^1 \left[\left(i^2\right)_{i=1} \right] &= 3 \\ \Delta^2 \left[\left(i^2\right)_{i=1} \right] &= 2 \end{aligned}$$

and

$$\Delta^r \left[\left(i^2\right)_{i=1} \right] = 0$$

for any $r > 2$.

Shortly afterwards, (*ibid.*, 2, pp. 240-242) Euler considered a series $\sum_{i=0}^{\infty} A_i x^i$, such that $A_i = u_i v_i$, where the envelopment of $\sum_{i=0}^{\infty} v_i x^i$ is a known function $f(x)$, and $\{u_i\}_{i=0}^{\infty}$ is a suitable sequence. To envelop these series he put them in the form

$$\sum_{i=0}^{\infty} A_i x^i = \sum_{i=0}^{\infty} C_i \frac{x^i}{i!} \frac{d^i f(x)}{dx^i}$$

where the coefficients C_i had to be determined. To determine these coefficients, Euler remarked that from the supposed equality

$$f(x) = \sum_{i=0}^{\infty} v_i x^i$$

the other equalities

$$C_i \frac{x^i}{i!} \frac{d^i f(x)}{dx^i} = \sum_{j=1}^{\infty} C_j \begin{pmatrix} j \\ i \end{pmatrix} v_j x^j$$

follow. Thus

$$\sum_{i=0}^{\infty} u_i v_i x^i = \sum_{i=0}^{j} \left(\sum_{j=1}^{\infty} C_j \begin{pmatrix} j \\ i \end{pmatrix} v_j x^j \right)$$

and then, for the method of indeterminate coefficients,

$$u_j = \sum_{i=0}^{j} C_j \left(\begin{array}{c} j \\ i \end{array} \right)$$

that is

$$C_i = \Delta^i u_0$$

and therefore:

$$\sum_{i=0}^{\infty} A_i x^i = \sum_{i=0}^{\infty} u_i v_i x^i = \sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i f(x)}{dx^i} \Delta^i u_0$$

Once again, if the differences $\Delta^i u_0$ are equal to zero for large enough i, then $\sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i f(x)}{dx^i} \Delta^i u_0$ reduces to a finite expression, which Euler assumed to be the envelopment of the given series $\sum_{i=0}^{\infty} A_i x^i$. As an example, Euler considered the series $\sum_{i=0}^{\infty} \frac{(i+1)^2+1}{i!} x^i$. As it can be written in the form $\sum_{i=0}^{\infty} (i+1)^2 \frac{1}{i!}$ and we know that $\sum_{i=0}^{\infty} \frac{1}{i!} x^i = e^x$, one has:

$$\begin{array}{rcl} \sum_{i=0}^{\infty} \frac{(i+1)^2+1}{i!} x^i & = & \sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i (e^x)}{dx^i} \Delta^i \left[\left((i+1)^2 + 1 \right)_{i=0} \right] \\ & = & e^x \sum_{i=0}^{\infty} \frac{x^i}{i!} \Delta^i \left[\left((i+1)^2 + 1 \right)_{i=0} \right] \\ & = & e^x \left(2 + 3x + x^2 \right) \end{array}$$

since

$$\begin{array}{rcl} \Delta^0 \left[\left((i+1)^2 + 1 \right)_{i=0} \right] & = & 2 \\ \Delta^1 \left[\left((i+1)^2 + 1 \right)_{i=0} \right] & = & 3 \\ \Delta^2 \left[\left((i+1)^2 + 1 \right)_{i=0} \right] & = & 2 \end{array}$$

and

$$\Delta^r \left[\left((i+1)^2 + 1 \right)_{i=0} \right] = 0$$

for any $r > 2$.

These examples show that the strategy ($\mathbf{S_b D}$) could have different forms. The previous examples rely on the following versions of it:

($\mathbf{S_b D}_1$) To construct the sequence $\left\{ \sum_{i=0}^{j} A_i x^i \right\}_{j=0}^{\infty}$ of the partial sums of the given series $\sum_{i=0}^{\infty} A_i x^i$, and to search for a (recursive or direct) rule of formation of the terms of this sequence giving the expression of its generic term $\sum_{i=0}^{m} A_i x^i$; if this expression reduces to another finitary expression $f(x)$ for the position $m = \infty$ this latter expression can be supposed to be the envelopment of the given series.

($\mathbf{S_bD_2}$) Imagine that, using some known developments, it is possible to transform a given power series $\sum_{i=0}^{\infty} A_i x^i$ into another series $\sum_{i=0}^{\infty} B_i x^i$, the terms of which are equal to 0 when i is greater than an appropriate m; then, the series $\sum_{i=0}^{\infty} B_i x^i$ is reduced to a finitary expression which is supposed to be the envelopment of the given series.

Both ($\mathbf{S_aD}$) and ($\mathbf{S_bD}$) are direct strategies for solving the problems (**P.1.a**) and (**P.1.b**), respectively: indeed they lead us to the desired result by manipulating the given object. In the case of ($\mathbf{S_aD}$), manipulating a known function, one derives a series which is its development; in the case of ($\mathbf{S_bD}$), manipulating a given series, one derives a function which is its envelopment. To use a classic expression, we can say that they are synthetic procedure, since they consist in operating on the known object to find the object which was considered as be unknown in the formulation of the problem.

Direct strategies are very natural, but they are not the only possible ones, and, as a matter of fact, they are not the only ones that have been followed in eighteenth-century analysis. In fact, although the problems (**P.1.a**) and (**P.1.b**) were conceived as such as essentially distinct from each other, it is clear that the solution to one of them also provides the solution of the other, supposing that in this latter problem it is considered as given what is sought in the former and *viceversa*. As an example, consider the equality (15). It has been obtained following a direct strategy and states that $1 - e^x(1 - 2x)$ is the envelopment of the given series

$$x + 3\frac{x^2}{2!} + 5\frac{x^3}{3!} + 7\frac{x^4}{4!} + \&c. \tag{17}$$

It is clear that, once this equality has been stated, and the function $1 - e^x(1 - 2x)$ is supposed to be given, one can easily conclude that the series (17) is its development. This is a simple example of the following converse strategy to solve the problem (**P.1.a**):

($\mathbf{S_aI_I}$) If a particular function $f(x)$ is given and it is possible to recognize it as a known envelopment of a known series $\sum_{i=0}^{\infty} A_i x^i$, then it can be immediately concluded that $\sum_{i=0}^{\infty} A_i x^i$ is the development of $f(x)$[27]

A similar strategy can be followed in order to solve the problem (**P.1.b**). Let us imagine, for example that the series

$$2 + x - 6x^2 - 3x^3 + 18x^4 + 9x^5 - 54x^6 - 27x^7 + \&c.$$

[27] An interesting example of this procedure can be found in Euler (1730-31) and (1732-33). We prefer not to present it here because it has some difficulties concerning with the nature of integrals which are independent of the scope of our paper.

is given and one recognizes it as the development of the function $\frac{x+2}{1+3x^2}$. One can thus immediately conclude that $\frac{x+2}{1+3x^2}$ is the envelopment of this series.

This is the strategy Newton used in a sketch of a treatise on quadratures and binomial developments composed in the summer of 1665, in order to express the area of the hyperbola of equation $y = \frac{a^2}{(b+x)^2}$ by means of a finitary expression (cf. Newton (MP), I, p. 129). By using the development of y into a power series, he firstly found that this area could be expressed by the power series

$$\frac{a^2}{b^2}x - \frac{a^2}{b^3}x^2 + \frac{a^2}{b^4}x^3 - etc.$$

Then he compared this series with the development of $\frac{a^2}{b+x}$, that is

$$\frac{a^2}{b}x - \frac{a^2}{b^2}x + \frac{a^2}{b^3}x^2 - etc.$$

and concluded that the area he was looking is was equal to

$$\frac{a^2}{b} - \frac{a^2}{b+x}$$

Generally speaking such a strategy is the following:

($\mathbf{S_b I_I}$) If a particular series $\sum\limits_{i=0}^{\infty} A_i x^i$ is given and it is possible to recognize it as a known development of a known function, then, it can be immediately concluded that $f(x)$ is the envelopment of $\sum\limits_{i=0}^{\infty} A_i x^i$.

A more sophisticated way for applying such a strategy is the following. Being

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \&c. = \sum_{j=0}^{\infty} \left[\sum_{k=0}^{j} \begin{pmatrix} j-k \\ k \end{pmatrix} \right] x^j$$

and thus, according to the binomial expansion for positive integers,

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \&c = \sum_{i=0}^{\infty} \left(x + x^2\right)^i$$

But, by posing $x + x^2 = y$ and applying the previous procedure to sum a geometric series, we have:

$$\sum_{i=0}^{n} \left(x + x^2\right)^i = \sum_{i=0}^{n} y^i = \frac{1 - y^n}{1 - y}$$

and thus

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \&c. = \sum_{i=0}^{\infty} \left(x + x^2\right)^i = \frac{1}{1-y} = \frac{1}{1 - x - x^2}$$

and $\frac{1}{1-x-x^2}$ is thus the envelopment of the given series $1+x+2x^2+3x^3+5x^4+8x^5 + \&c$, this result being obtained by observing that this latter series is the sum of an infinite number of finite developments.

The strategies ($\mathbf{S_aI_I}$) and ($\mathbf{S_bI_I}$) are two inverted strategies for solving (**P.1.a**) and (**P.1.b**), by immediately referring to an already given solution to the opposite problem. Both cases allow to obtain the desired result by using an already known result which has been derived by operating on the object that is considered as unknown in the given problem. Using more classic language, we can say that they are analytic procedures.

It is interesting to note that these procedures are analytic though they do not solve the problem which is proposed by manipulating a generic function or a generic series, but by handling a function or a series guessed at in some way, i. e. by following a synthetic path. This shows that an analytic procedure, consisting in working on an unknown object K which is to be found as it were given, can sometimes be performed by following a synthetic path: one works on a determinate object K and verifies that the given object can be derived from K. This is not the same as operating on a indeterminate object X which is precisely what has to be found. In this latter case the path also is analytic[28].

Let us now imagine that a particular function $f(x)$ is given and that it is possible to recognize it as the result of the application of a certain operation (also applicable to power series, term by term) to a known envelopment $g(x)$ of a known series $\sum_{i=0}^{\infty} A_i x^i$. This provides another strategy to solve the problem (**P.1.a**):

($\mathbf{S_aI_T}$) The operation, which leads from $g(x)$ to $f(x)$, can be applied to $\sum_{i=0}^{\infty} A_i x^i$; this produces a new series $\sum_{i=0}^{\infty} B_i x^i$, which is supposed to be the development of the function $f(x)$.

The successive application of one of the strategies ($\mathbf{S_aD}$) or ($\mathbf{S_bD}$) to $f(x)$ or to $\sum_{i=0}^{\infty} B_i x^i$, respectively, can successively confirm such a result.

A similar strategy for solving (**P.1.b**) is then obvious. Let us imagine then that a particular series $\sum_{i=0}^{\infty} A_i x^i$ is given and that it is possible to recognize it as the result of the application of a certain operation (also applicable to finitary analytic forms) to a known development $\sum_{i=0}^{\infty} B_i x^i$ of a known function $f(x)$, then:

($\mathbf{S_bI_T}$) The operation, which leads from $\sum_{i=0}^{\infty} B_i x^i$ to $\sum_{i=0}^{\infty} A_i x^i$, can be applied to $f(x)$; this produces a new function $g(x)$, which is supposed to be the envelopment of the series $\sum_{i=0}^{\infty} A_i x^i$.

[28]This remark should justify Euler's use of the term "synthetic" to characterize the procedure of development he adopted in his (1732-33). Here Euler termed as "synthetic a procedure he described as consisting in wondering "what the series could be whose sums are expressed" by a certain formulas [cf. Euler (1732-33), p. 42]. By asserting that this is a synthetic procedure, Euler seems to insist on the logical nature of the path rather that on the logical nature of the argument. He underlines in effect that, even if his procedure is regressive, it does not consist in operating on an unknown object since it concerns knowledge that is already available. A scheme based upon this point of view is presented in Ferraro (1998).

The successive application of one of the strategies ($\mathbf{S_a D}$) or ($\mathbf{S_b D}$) to $g(x)$ or to $\sum\limits_{i=0}^{\infty} A_i x^i$, respectively, can successively confirm such a result.

($\mathbf{S_a I_T}$) and ($\mathbf{S_b I_T}$) are two inverse strategies for solving (**P.1.a**) and (**P.2.b**). Indeed, in this case as well, one arrives at the result by operating on an object that is not given in the proposed problem. However, now, this object is not the unknown object of the problem but an object that is connected to the unknown object by means of a certain transformation. We can say that these strategies are not only inverted (*i. e.* analytic) but also indirect.

As an example of the strategy ($\mathbf{S_a I_T}$) let us consider Euler's development of the function $y = \log(1 + x + x^2 + x^3)$ in his (1768-70, vol. 1, p. 83). Euler started from the equality

$$y = \log(1 + x + x^2 + x^3) = \log \frac{1 - x^4}{1 - x} = \log(1 - x^4) - \log(1 - x)$$

and observed that

$$\frac{d}{dx}(y) = \frac{d}{dx}\big(\log(1 - x^4) - \log(1 - x)\big) = -\frac{4x^3}{1 - x^4} + \frac{1}{1 - x}$$

and, according to (12),

$$\frac{x^3}{1 - x^4} = \sum_{i=0}^{\infty} x^{3+4i}; \qquad \frac{1}{1 - x} = \sum_{i=0}^{\infty} x^i$$

He concluded from here that integrating term by term the series

$$\sum_{i=0}^{\infty} x^i - \sum_{i=0}^{\infty} 4x^{3+4i}$$

(and supposing that the constant of integration is null), one should have the development of $\log(1 + x + x^2 + x^3)$ which was searched:

$$\log(1 + x + x^2 + x^3) = \sum_{i=0}^{\infty} \frac{x^{i+1}}{i + 1} - \sum_{i=0}^{\infty} \frac{x^{4+4i}}{i + 1} = \sum_{i=0}^{\infty} (-3)^{\left(\left[\frac{i+1}{4}\right] - \left[\frac{i}{4}\right]\right)} \frac{x^{i+1}}{i + 1}$$

where the symbol $[q]$ denotes the integral part of the number q.

To have an example of ($\mathbf{S_b I_T}$) consider the series $1 - 2x + 3x^2 - 4x^3 + \&c.$ It is easy to recognize that this series can be obtained by differentiating the power series $\sum\limits_{i=0}^{\infty} (-)^{i+1} x^i$ term by term, and dividing the result by the differential dx. Since $\sum\limits_{i=0}^{\infty} (-)^{i+1} x^i$ is the known development of $-\frac{1}{1+x}$, it is then sufficient to calculate the differential ratio of the last function in order to derive the sum of the given series:

$$1 - 2x + 3x^2 - 4x^3 + \&c. = \frac{1}{(1 + x)^2}$$

This is the procedure by which Euler found in his (1761, pp. 71-72) the sum of the series

$$1 - 2^n x + 3^n x^2 - 4^n x^3 + \dots$$

for $n = 2, 3, ..., 6$, which are thus

$$\frac{1-x}{(1+x)^3}$$
$$\frac{1-4x+x^2}{(1+x)^4}$$
$$\frac{1-11x+11x^2-x^3}{(1+x)^5}$$
$$\frac{1-26x+66x^2-26x^3+x^4}{(1+x)^6}$$
$$\frac{1-57x+302x^2-302x^3+57x^4+x^5}{(1+x)^7}$$

respectively.

Let us finally imagine that a particular function $f(x)$ is given and that it is possible to recognize it as significantly similar to a known envelopment $g(x)$ of a known series $\sum_{i=0}^{\infty} A_i x^i$. One could try to transform this latter series in a new series $\sum_{i=0}^{\infty} B_i x^i$ as similar to it as $f(x)$ is similar to $g(x)$. It this is possible, one could guess that $\sum_{i=0}^{\infty} B_i x^i$ is the development of the given function $f(x)$, and then try to verify this conjecture in same way, for instance by applying the strategy ($\mathbf{S_b D}$) to $\sum_{i=0}^{\infty} B_i x^i$. This a further strategy to solve the problem (**P.1.a**):

($\mathbf{S_a I_A}$) If a particular function $f(x)$ is given and it is possible to recognize it as significantly similar to a known envelopment $g(x)$ of a known series $\sum_{i=0}^{\infty} A_i x^i$, and it is possible to transform $\sum_{i=0}^{\infty} A_i x^i$ into a new series $\sum_{i=0}^{\infty} B_i x^i$, which is as similar to $\sum_{i=0}^{\infty} A_i x^i$ as $f(x)$ is similar to $g(x)$, one could guess that $\sum_{i=0}^{\infty} B_i x^i$ is the development of $f(x)$, and try to verify it in some way.

The converse strategy to solve (**P.1.b**) is obvious, in this case as well :

($\mathbf{S_b I_A}$) If a particular function $\sum_{i=0}^{\infty} A_i x^i$ is given and it is possible to recognize it as significantly similar to a known development $\sum_{i=0}^{\infty} B_i x^i$ of a known series $f(x)$, and it is possible to transform $f(x)$ in a new function $g(x)$ as similar to $f(x)$ as $\sum_{i=0}^{\infty} A_i x^i$ is similar to $\sum_{i=0}^{\infty} B_i x^i$, one could guess that $g(x)$ is the envelopment of $\sum_{i=0}^{\infty} A_i x^i$, and try to verify it in some way

To perform this latter verification, one can apply for instance the strategy ($\mathbf{S_a D}$) to $g(x)$.

($\mathbf{S_a I_A}$) and ($\mathbf{S_b I_A}$) are also inverse and indirect strategies for solving (**P.1.a**) and (**P.2.b**), respectively. Indeed, in these case we reach the result by operating on an object that is neither given in the proposed problem nor is the unknown object of this problem. But, in this case, the object we operate on is connected to the unknown object not by means of a certain transformation, but by a sort of analogy. We can therefore say that not only are these strategies inverted (*i. e.* analytic) and indirect, but also analogical, rather than deductive.

An example of an analogical procedure is the transition from the discrete to the continuous, known as "Wallis's interpolation". Following Newton, Jakob Bernoulli (1713, p. 294) considered the development of $\left(\frac{l}{m-n}\right)^r$ for any natural integer r, and derived from it the development of $\left(\frac{l}{m-n}\right)^\alpha$ for any real number α. In general, such a procedure is performed by examining the known developments of a sequence of functions $F_n(x)$ according to the scheme:

$$\begin{array}{rcl} F_0(x) & = & A_{0,0} + A_{1,0}x + A_{2,0}x^2 + A_{3,0}x^3 + A_{4,0}x^4 + A_{5,0}x^5 + \dots \\ F_1(x) & = & A_{0,1} + A_{1,1}x + A_{2,1}x^2 + A_{3,1}x^3 + A_{4,1}x^4 + A_{5,1}x^5 + \dots \\ F_2(x) & = & A_{0,2} + A_{1,2}x + A_{2,2}x^2 + A_{3,2}x^3 + A_{4,2}x^4 + A_{5,2}x^5 + \dots \\ & & \dots \end{array}$$

If the law of coefficients $A_{n,m}$ is known, one has

$$F_\alpha(x) = A_{0,\alpha} + A_{1,\alpha}x + A_{2,\alpha}x^2 + A_{3,\alpha}x^3 + A_{4,\alpha}x^4 + A_{5,\alpha}x^5 + \dots$$

where α is any real number. If the function is given, this procedure is an example of ($\mathbf{S_a I_A}$); instead, if the series is given, it is an example of ($\mathbf{S_b I_A}$).

To resume, we have four different strategies to solve (**P.1.a**) and four converse strategies to solve (**P.1.b**), which we could arrange in the following scheme:

Strategies		To solve **P.1.a**	To solve **P.1.b**
Directed		$\mathbf{S_a D}$	$\mathbf{S_b D}$
Inverted	Immediate	$\mathbf{S_a I_I}$	$\mathbf{S_b I_I}$
	By Transformation	$\mathbf{S_a I_T}$	$\mathbf{S_b I_T}$
	Analogic	$\mathbf{S_a I_A}$	$\mathbf{S_b I_A}$

Of course, we do not claim that this exhausts all the possible analytic or synthetic procedures used by eighteenth-century mathematicians to solve the problems (**P.1**). For instance, we can hypothesize that, if a relation of the form (1) between a certain function and a determinate power series is already given, it is then possible to move on from this relation to determining the development of other functions or the envelopments of other series. For instance, in his (1730), de Moivre considered the development

$$\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \&c.$$

and observed that

$$\frac{1}{1-x-x^2} = \frac{1-x^2}{1-3x^2+x^4} + \frac{x}{1-3x^2+x^4}$$

$\frac{1-x^2}{1-3x^2+x^4}$ and $\frac{x}{1-3x^2+x^4}$ being respectively an even function and odd function. From here he concluded that

$$\begin{aligned}\frac{1-x^2}{1-3x^2+x^4} &= 1+2x^2+5x^4+\&c.\\ \frac{x}{1-3x^2+x^4} &= x+3x^3+8x^5+\&c.\end{aligned}$$

Given the distinction we have introduced, it would seem moreover that further and subtler distinctions can be identified. For example, the strategy ($\mathbf{S_aD}$) can be performed in different ways according to the procedure chosen among the procedures ($\mathcal{P}_1$)-($\mathcal{P}_4$) and their possible combinations. Another possible distinction has already mentioned briefly: the distinction between the procedures that follow a synthetic path, like ($\mathrm{S}_a\mathrm{I}_I$) and ($\mathrm{S}_b\mathrm{I}_I$), and the procedure that follow an analytic path. An analogous distinction could be made between the procedures of development of a function $f(x)$ which merely operate upon this given function, and the procedures which operate on this function and upon the generic form of a power series, for example by following the method of indeterminate coefficients. According to such a method, the coefficients of the development participate in the procedure before being determinate and we could say, also in this case, that such an analytical procedure follows a synthetic path.

Of course, other different classifications can be made, based upon different pairs than the pairs analytic/synthetic or direct/indirect. However, it is not our aim to investigate this possibility, here.

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