Chapitre sur le pulvérisateur chez Bhâskara IChapter on the pulverizer in Bhâskara'I work

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A BAB.2.32-33: The pulverizer

Bhāskara has two general interpretations of the procedure given in verses 32-33 that describe a "pulverizer computation" (*kuṭṭākāragaṇita*). He reads in these verses a "pulverizer with remainder (*sāgrakuṭṭākāra*)" and a "pulverizer without remainder (*niragrakuṭṭākāra*)" . Having explained and illustrated these two different interpretations, he then gives a long list of solved examples which show how one or the other procedure is used in an astronomical context[1].

We will describe and comment on the two different procedures given by Bhāskara, and then we will explain the many astronomical situations in which he applies them. Descriptions, under the label "General comments", will use a symbolical algebraization of the problem.

A.1 Two different problems

The problems that a pulverizer "with remainder" and that a pulverizer "without remainder" solve, are different but nevertheless equivalent.

Indeed, the problem solved by a pulverizer "with remainder" is the following:

What is the natural number N that divided by a leaves R_1 for remainder and divided by b leaves R_2 for remainder?[2]

In a modern mathematical language:

$$\begin{aligned} N = ax + R_1 \quad 0 \leq R_1 < a \\ N = by + R_2 \quad 0 \leq R_2 < b \end{aligned}$$

The problem solved by a pulverizer "without remainder" is the following:

What is the integer x, that multiplied by a, increased or decreased by c and divided by b, produces an integer y?

In other words the problem consists of finding two integers (x, y) that verify:

$$y = \frac{ax \pm c}{b}$$

[1]For Āryabhaṭa's and Bhāskara's treatment of the pulverizer, see [Jain 1995; p. 422-447].

[2]Concerning the conditions under which this problem is solvable, please see the the section A.2 of this supplement.

Where a, b and c are known positive integers. x is called the pulverizer or the multiplier (*guṇaka*), y the quotient (*labdha*).

If we consider the problem solved by a pulverizer with remainder: $R_1 > R_2$, and $R_1 - R_2 = c$,

$$\begin{cases} N = & ax+ & R_1 \\ N = & by+ & R_2 \end{cases} \Leftrightarrow \; y = \; \frac{ax+c}{b}$$

What is called "the divisor of the greater remainder" (a) in the pulverizer with remainder process is called in the pulverizer without remainder "the divisor which is a large number" or "the dividend"; what is called "the divisor of the smaller remainder" in the procedure of the pulverizer with remainder is called here "the divisor"; and what is called the "difference of remainders" ($R_1 - R_2$) is called "the interior of a number".[3]

As we will see, the pulverizer with remainder transforms the problem it solves into a pulverizer without remainder problem. Both procedures, therefore, share common steps. However the two problems and their two procedures are separated in Bhāskara's commentary.

We will now describe the process followed for a pulverizer without remainder.

A.2 Procedure for the pulverizer "with remainder"

We will present here the different steps of this algorithm. We will then expose some of its variations as observed in solved examples, and finally present a mathematical analysis of it.

General case

Problem

The problem this procedure solves is the following:

What is the natural number N that divided by a leaves R_1 for remainder and divided by b leaves R_2 for remainder?[4]

In a modern mathematical language:

$$N = ax + R_1 \quad 0 \leq R_1 < a$$
$$N = by + R_2 \quad 0 \leq R_2 < b$$

[3]For a brief description of how Bhāskara proceeds to give two different interpretations of the same compound see [Keller 2000; Volume I, I] and in Volume I, Introduction.

[4]Concerning the conditions under which this problem is solvable, please see the last part of this section of the supplement BAB.2.32-33.

For $R_1 > R_2$ the "setting-down", in examples, follows this pattern:

R_2 R_1
b a

Step 1

Sanskrit *Ab. 2.32ab. adhikāgrabhāgahāraṃ chindyād ūnāgrabhāgahāreṇa*

English Ab. 2.32ab. One should divide the divisor of the greater remainder by the divisor of the smaller remainder.

General Comments Supposing $R_1 > R_2$, then a is "the divisor of the greater remainder", and b is "the divisor of the smaller remainder", the following computation is carried out:

$$\frac{a}{b} = q_1 + \frac{r_1}{b} \quad \Leftrightarrow \quad a = bq_1 + r_1$$

We can note that Bhāskara in examples describes the result as follows: "the remainder is r_1 above, b below". This is probably a way of describing the fractional part that the division produces.

Step 2

Sanskrit *Ab.2.32c. śeṣaparasparabhaktaṃ*

English Ab.2.32c. The mutual division ⟨of the previous divisor⟩ by the remainder ⟨is made continuously.⟩

General comments In other words, the following successive divisions are carried out:

$$\begin{array}{rcl}
\frac{b}{r_1} = q_2 + \frac{r_2}{r_1} & \Leftrightarrow & b = r_1 q_2 + r_2 \\
\frac{r_1}{r_2} = q_3 + \frac{r_3}{r_2} & & r_1 = r_2 q_3 + r_3 \\
\frac{r_2}{r_3} = q_4 + \frac{r_4}{r_3} & & r_2 = r_3 q_4 + r_4 \\
\vdots & & \\
\frac{r_{n-2}}{r_{n-1}} = q_n + \frac{r_n}{r_{n-1}} & & r_{n-2} = r_{n-1} q_n + r_n
\end{array}$$

No indication is given concerning how to end the process. "procedure" parts of solved examples suggest that it was stopped when the remainder obtained was considered sufficiently small, i.e. before zero was obtained as remainder. We do not know according to what criteria a quantity was considered to be small enough.

Step 3

Sanskrit *Ab.2.32cd matiguṇam agrāntare kṣiptam*

English Ab.2.32cd ⟨The last remainder⟩ having a clever ⟨thought⟩ for multiplier is added to the difference of the ⟨initial⟩ remainders ⟨and divided by the last divisor⟩.

General comments As we will see in the next step, Bhāskara indicates how the clever quantity should be placed in regard to the previously computed remainder. The placement presupposed, though not explicitly mentioned, would be:

$$\begin{array}{l} q_2 \\ q_3 \\ \vdots \\ q_n \end{array}$$

Bhāskara adds the following gloss which explains under what conditions and how the "clever ⟨thought⟩" is found[5]:

> *matiguṇaṃ, svabhuddhiguṇam ity arthaḥ| kathaṃ punaḥ svabuddhiguṇaḥ kriyate ? ayaṃ rāśiḥ kena guṇitedam ⟨edition reads guṇitam idam⟩ agrāntaraṃ prakṣipya viśodhya vā asya rāśeḥ śuddhaṃ bhāgaṃ dāsyatīti agrāntare kṣiptam| sameṣu kṣiptaṃ viṣameṣu śodhyam iti sampradāyāvicchedād vyākhyāyate|*
>
> ⟨As for⟩ "having a clever ⟨quantity⟩ for multiplier", the meaning is: having a multiplier according to one's own intelligence.
>
> ⟨Question⟩
> But how is the multiplier according to one's own intelligence?
>
> ⟨It should answer this question:⟩ Will this quantity (the remainder), multiplied by what ⟨is sought⟩ give an exact division, when one has added or subtracted this difference of remainders ⟨to the product⟩?

[5][Shukla 1976; p.132, lines 15 to 19].

> ⟨As for⟩ "Added to the difference of remainders"; ⟨it is⟩ added when ⟨the number of placed terms is⟩ even, subtracted when uneven, as it has been explained by an uninterrupted tradition.

From this remark, we can deduce the following computation.

If the number of placed terms is pair ($n = 2p + 1$, and, because the placement starts with the quotient q_2, the number of placed terms is $n-1 = 2p$) one should solve, the following equation having the following pair of integer unknowns: (k, l), where k is called "the clever ⟨thought⟩" (*mati*).

$$l = \frac{r_n k + c}{r_{n-1}} = \frac{r_{2p+1} k + c}{r_{2p}},$$

where $c = R_1 - R_2$.

If the number of placed terms is not pair ($n = 2p$, so that the number of placed terms is $n - 1 = 2p - 1$), the following equation should be solved:

$$l = \frac{r_n k - c}{r_{n-1}} = \frac{r_{2p} k - c}{r_{2p-1}}.$$

We do not know how these equations where solved. They have the same form as the problem solved by a pulverizer without remainder. However, only one solution is sought. It is not required that this solution is the smallest possible. The clever quantity, may have been found by trial and error.

Step 4

Sanskrit *Ab.2.33a. adhoparigunitam antyayug*

English Ab.2.33a. The one above is multiplied by the one below, and increased by the last.

Bhāskara furthermore adds:[6]:

[6] [Shukla 1976; p.132 lines 20 to 23].

> *evaṃ paraspareṇa labdhāni padāny āsthāpya, matiś cādhaḥ, paścimalabdhaś ca matyā adhaḥ| (...) evaṃ bhūyo bhūyaḥ karma yāvat karma parisamāptitam iti|*
>
> When one has placed in this way the terms obtained by the mutual ⟨division⟩, the clever ⟨quantity⟩ is placed below, and the last obtained below the clever ⟨thought⟩. (...) In this way, again and again the operation ⟨is repeated⟩ until the computation comes to an end.

General comments The placement, will then be:

$$\begin{array}{c} q_2 \\ q_3 \\ \vdots \\ q_n \\ k \\ l \end{array}$$

Then the operation:" The one above is multiplied by the one below, and increased by the last ", is repeated, for all rows, beginning from the bottom ($i = n, n-1, ..., 2$):

$$\begin{array}{c} q_i \\ q'_{i+1} \\ q'_{i+2} \end{array} \longrightarrow \begin{array}{c} q_i q'_{i+1} + q'_{i+2} \\ q'_{i+1} \\ \end{array}$$

The third element from the bottom of the column is replaced by the result of the computation prescribed, and the last element is deleted.

This procedure is repeated until only two elements remain.

$$\begin{array}{c} q'_2 \\ q'_3 \end{array}$$

(q'_2, q'_3) is a pair of integer solutions of the original problem[7], this is not mentioned in the text. The procedure continues, considering q'_2, from which another couple of solutions will be derived.

Step 5

[7]Please see the last part of this section of the supplement BAB.2.32-33.

Sanskrit *Ab.2.33b ūnāgracchedabhājite*

English Ab.2.33b. When ⟨the result of this procedure⟩ is divided by the divisor of the smaller remainder.

Bhāskara furthermore adds [8]:

> *ūnāgracchedhabhājite śeṣam, (...) pūrvagaṇitakarmaṇā niṣpannarāśer vibhaktaśeṣaṃ parigṛhyate|*
>
> ⟨As for⟩ "When ⟨the result of this procedure⟩ is divided by the divisor of the smaller remainder, the remainder". (...) The remainder of the division of, the quantity produced by means of the previous mathematical operation, by the divisor of the smaller remainder is understood.

General comments In other words, the solution, q'_2, is divided by b:

$$\frac{q'_2}{b} = t + \frac{s}{b} \Leftrightarrow \; q'_2 = bt + s \quad (o \leq s < b).$$

The remainder, s, is thereafter considered.

s is the least positive solution for x of the original problem[9], this is not mentioned in the text.

Step 6

Sanskrit *Ab.2.33bcd. śeṣam adhikāgracchedaguṇam dvicchedāgram adhikāgrayutam*

English Ab.2.33bcd. The remainder multiplied by the divisor of the greater remainder and increased by the greater remainder, is the ⟨quantity that has such⟩ remainders for two divisors.

Bhāskara furthermore adds[10]:

> *tad dvayor api chedayor bhājyarāśir bhavatīti|*
> ...That is the quantity to be divided for (i.e. by) both of these two divisors.

[8] [Shukla 1976; p. 132 lines 23 to 25].
[9] Please see the last part of this section of the supplement for BAB.2.32-33.
[10] [Shukla 1976; p.133, lines 2-3].

General comments

$$N_1 = as + R_1.$$

> N_1 is the least positive integer that satisfies the original problem, and, at the same time, it is regarded as the "remainder" (*agra*) corresponding to the two divisors, a and b, when there is another problem: Find the number N that when divided by ab leaves for remainder N_1, and when divided by another number leaves another given remainder.

Understanding the general case of the pulverizer with remainder

Let us recall that the problem treated ("What is the natural number N that divided by a leaves R_1 for remainder and divided by b leaves R_2 for remainder?"), can be summarized as follows:

$$\begin{array}{ll} N = ax + R_1 & 0 \leq R_1 < a \\ N = by + R_2 & 0 \leq R_2 < b \end{array}$$

Preliminary remarks

Conditions on a and b The original problem supposes that $a, b > 1$, since a division by 1 would leave no remainder, and that the problem if one of them were equal to zero would equally have no sense in this context.[11]

If $R_2 = R_1 = R$ when a and b are not coprime (that is their only common divisor is 1), as we can see in example 4, then the smallest integer solution N would be

$$N = LCM(a, b) + R,$$

where $LCM(a, b)$ is the least common multiple of a and b. This is the case of the 5 first quantities in example 4. We do not know, however, how Bhāskara proceeded in this case.

[11] If we consider however the set of equations written above, let us suppose that: either a or b are equal to zero. If say a would be equal to zero, then we would have a value for N, R_1, that would verify the original problem, if and only if:

$$R_1 = by + R_2$$

has an integer solution, that is if and only if, $R_1 - R_2$ is a multiple of b.

Conditions on the remainders Usually, in examples, $R_1 \neq R_2$ and $R_1 \neq 0, R_2 \neq 0$.

Let us remark here that the above system of equations has a solution if and only if $R_1 - R_2$ is a multiple of the Greatest Common divisor of a and b. Indeed, let (x_0, y_0) be a solution. Then:

$$R_1 - R_2 = by_0 - ax_0.$$

It is a common result of elementary number theory[12] that such a number is necessarily a multiple of the Greatest Common divisor of a and b.

So that there should always be a common multiple for a, b, and $R_1 - R_2$.

If a and b are coprime, then for any difference of remainders solutions can be found. Bhāskara in the case of this interpretation of the pulverizer problem does not make any such remark on a and b. However concerning a pulverizer without remainder, such a fact is stated rather clearly, as we have noted it in the section concerning this procedure below.

When $R_1 = R_2 = 0$, then N is a common multiple of both a and b. If (x_0, y_0) is the smallest solution of this set of equation then by definition, N is the Least Common Multiple of a and b.

Bhāskara at the beginning of example 14 writes:

> *kaścid rāśiḥ sūryasya nirapavartitabhūdivasair bhāgaṃ hriyamāṇaḥ śūnyāgraḥ, candrasyāpi śūnyāgraḥ eva saḥ|*
> Some quantity when divided by the reduced number of terrestrial days ⟨in a *yuga*⟩ for the sun, has a zero-remainder (*śūnyāgra*), just that ⟨same quantity when divided by the reduced number of civil days in a *yuga*⟩ for the moon too has a zero-remainder.

He later exhibits as such a quantity, the Lowest Common Multiple of both numbers.

Understanding the procedure In the following we will consider that $a, b > 1$ and that $R_1 > R_2$, $c = R_1 - R_2$.

The process is interrupted, it seems, when the remainder obtained is sufficiently small[13]. We can formalize the process in the following way (in

[12] See for instance, [Jones 1998; Proof of Theorem 1.8., p. 10].

[13] Bhāskara's contemporary, Brahmagupta, and all following known authors continue the process until zero is obtained as remainder, and therefore do not compute the "clever quantity".

exactly the same terms as in Step 1 and 2 of the procedure described in the commentary):

For an arbitrary n:

$$\begin{array}{rcl} \frac{a}{b} = q_1 + \frac{r_1}{b} & \Leftrightarrow & a = bq_1 + r_1 \\ \frac{b}{r_1} = q_2 + \frac{r_2}{r_1} & \Leftrightarrow & b = r_1 q_2 + r_2 \\ \frac{r_1}{r_2} = q_3 + \frac{r_3}{r_2} & & r_1 = r_2 q_3 + r_3 \\ \frac{r_2}{r_3} = q_4 + \frac{r_4}{r_3} & & r_2 = r_3 q_4 + r_4 \\ \vdots & & \\ \frac{r_{n-2}}{r_{n-1}} = q_n + \frac{r_n}{r_{n-1}} & & r_{n-2} = r_{n-1} q_n + r_n \end{array}$$

By using this set of equations, the equation (*) can be rewritten as a set of two equations, (A, i) and (B, i), for $i = 1, \ldots, n$.

$y = \frac{ax+c}{b} = \frac{(bq_1+r_1)x+c}{b} = q_1 x + y_1$ where

$$y_1 = \frac{r_1 x + c}{b} \quad (A, 1)$$

$x = \frac{by_1-c}{r_1} = \frac{(r_1q_2+r_2)y_1-c}{r_1} = q_2 y_1 + x_1$ where

$$x_1 = \frac{r_2 y_1 - c}{r_1} \quad (B, 1)$$

$y_1 = \frac{r_1x_1+c}{r_2} = \frac{(r_2q_3+r_3)x_1+c}{r_2} = q_3 x_1 + y_2$ where

$$y_2 = \frac{r_3 x_1 + c}{r_2} \quad (A, 2)$$

$x_1 = \frac{r_2y_2-c}{r_3} = \frac{(r_3q_4+r_4)y_2-c}{r_3} = q_4 y_2 + x_2$ where

$$x_2 = \frac{r_4 y_2 - c}{r_3} \quad (B, 2)$$

$\vdots$

$$\begin{cases} y_{p-1} = q_{2p-1} x_{p-1} + y_p \\ y_p = \frac{r_{2p-1}x_{p-1}+c}{r_{2p-2}} \quad (A, p) \end{cases}$$

$$\begin{cases} x_{p-1} = q_{2p}y_p + x_p \\ x_p = \frac{r_{2p}y_p - c}{r_{2p-1}} \quad (B,p) \end{cases}$$

$$\begin{cases} y_p = q_{2p+1}x_p + y_{p+1} \\ y_{p+1} = \frac{r_{2p+1}x_p + c}{r_{2p}} \quad (A,p+1) \end{cases}$$

etc.

Now, with the equation (B,p) is associated an even number of quotients (q_{2p}), and in the computation of x_p, c is subtracted.

With the equation $(A,p+1)$ is associated an uneven number of quotients (q_{2p+1}), and in the computation of y_{p+1}, c is added.

We can recognize here the computation of the clever quantity and the quotient that is associated to it, as in Step 3 of the algorithm.

If the number of quotients is uneven, the equation $(A,p+1)$ should be solved by trial and error; the solution, k, for x_p is called "the clever ⟨thought⟩" (*mati*).

$$l = \frac{r_n k + c}{r_{n-1}} = \frac{r_{2p+1}k + c}{r_{2p}}.$$

If the number of quotients is pair, the equation of (B,p) should be solved by trial and error; the solution, k , for y_p is called "the clever ⟨thought⟩" (*mati*).

$$l = \frac{r_n k - c}{r_{n-1}} = \frac{r_{2p}k - c}{r_{2p-1}}.$$

Once a couple of solutions is found, by working the solutions backwards, one arrives at a solution x for (*).

Indeed, by solving the second equation of $(A,p+1)$ (resp. of (B,p)), one obtains a numerical value for both (x_p, y_{p+1}) (resp. of (x_p, y_p)), which in turn gives a value for y_p (resp. for x_{p-1}). With this value of y_p (resp. of x_{p-1}) the value of x_p (resp. for y_{p-1}) can be computed and so forth until we have obtained a value for (x_1, y_1), which gives a value for x.

In other words, by using the succession of equations, for example in the case of an uneven number of quotients:

$$\begin{array}{cr} y_p = q_{2p+1}x_p + y_{p+1} & (A,p) \\ x_{p-1} = q_{2p}y_p + x_p & (B,p-1) \\ y_{p-1} = q_{2p-1}x_{p-1} + y_p & (A,p-1) \\ \vdots & \\ x_1 = q_4y_2 + x_2 & (B,1) \\ y_1 = q_3x_1 + y_2 & (A,1) \\ x = q_2y_1 + x_1 & \end{array}$$

one arrives at a solution for x.

Now in this succession of equations we can recognize the computations of Step 4, taking for example an even number of quotients:

$$\begin{array}{lcccl} q_2 & & q_2 & & \\ q_3 & & q_3 & & \\ \vdots & \longrightarrow & \vdots & & \longrightarrow \cdots \\ q_{2p-1} & & q_{2p-1} & & \\ q_{2p} & & q_{2p}y_p + x_p = x_{p-1} & & \\ k = y_p & & y_p & & \\ l = x_p & & & & \end{array}$$

$$\begin{array}{lcccr} q_2 & \longrightarrow & q_2 & \longrightarrow & q_2' = q_2y_1 + x_2 = x \\ q_3 & & q_3' = q_3x_1 + y_2 & & q_3' = y_1 \\ q_4' = x_1 & & x_1 & & \\ q_5' = y_2 & & & & \end{array}$$

As we can see, only q_2 is needed to compute x, this may explain why there is no need to "set down" q_1.

Step 5, by dividing that very value of x by the "smaller divisor", and thereafter considering the remainder of the division, assures that the value found for x is the smallest possible. Step 6 replaces the value for x in the first equation:

$$N = ax + R_1$$

So that N_1, the value obtained for N is such that

$$N_1 = as + R_1.$$

Procedure with more than two quantities and short cut N_1 satisfies the original problem, and, at the same time, it is regarded as the “remainder” (*agra*) corresponding to the two divisors, a and b, when there is another problem: Find a number N that when divided by ab leaves for remainder N_1. This can be formalized as:

$$N = (ab)u + N_1.$$

A solution, N, of this problem is also such that when divided by a, it has for remainder R_1. Likewise, when N is divided by b, it has R_2 for remainder. This property is used when the problem concerns more than two couples of divisors and remainders. This is the case for instance in examples 3 and 4. If one has to solve a problem with more than two couples of divisors and remainders, if all the remainders are equal an evident solution will be the LCM of all divisors increased by the remainder (this is the case of the solution the example of Ms. E would bear). If just a certain number of these integers have the same remainder, the problem will be equivalent to solving the pulverizer of the LCM of those integers with their common remainder, and the others.

In example 1, Bhāskara stops short of the ”Euclidian Algorithm”. The clever quantity he computes and the corresponding quotient, correspond, with our notations, to the computation of:

$$y_1 = \frac{r_1 x + c}{b} \, (A, 1)$$

The clever quantity is hence a value for x, which is then reduced to its smallest possible value by Step 5, and with which the value of N is computed in Step 6.

We will briefly expose here the steps followed by Bhāskara when he uses his short cut, and when considering more than two quantities.

Bhāskara’s short cut

In example 1, Bhāskara uses a “short-cut” whose steps we will now expose. The problem solved is the same and starts in the same way:

Step 1

“One should divide the divisor of the greater remainder by the divisor of the smaller remainder.”

Supposing $R_1 > R_2$, then a is "the divisor of the greater remainder", and b is "the divisor of the smaller remainder":

$$\frac{a}{b} = q_1 + \frac{r_1}{b} \quad \Leftrightarrow \quad a = bq_1 + r_1$$

However here r_1 is considered sufficiently "small" and step 2 is skipped

Step 3

The number of placed terms is considered to be pair.

One should solve the following equation having the following pair of integer unknowns: (k, l), where k is called "the clever ⟨thought⟩" (*mati*),

$$l = \frac{r_1 k + c}{b}.$$

Step 4 is skipped also but the "setting-down" would be: $\begin{matrix} k \\ l \end{matrix}$

Step 5

The upper element of this column, k is divided by b:

$$\frac{k}{b} = t + \frac{s}{b} \Leftrightarrow \; k = bt + s \quad (o \leq s < b).$$

The remainder, s, is thereafter considered.

Step 6

$$N_1 = as + R_1.$$

N_1 is the least positive integer that satisfies the original problem.

Procedure for problems with more than two couples of numbers

Problem

What is the integer N that when divided by a_1 has r_1 for remainder, that when divided by a_2 has r_2 for remainder, $\cdots$, that when divided by a_n has r_n for remainder?

procedure

A first pair of couples is chosen (say (a_1, r_1) and (a_2, r_2)) to which the pulverizer procedure is applied, and for which an integer N_1 is found. Then a following pair is taken (say, (a_3, r_3)) , to which the pulverizer procedure is applied together with the couple formed of the product of the previous divisors and the result found ((a_1a_2, N_1)). And so forth, until all the couples are used. The last pulverizer procedure applied gives the solution of the problem. If two remainders are the same, Bhāskara indicates in example 4:

> *atrecchayā 'dhikāgro rāśiḥ parikalpanīyaḥ|*
> In this case, the quantity which has the greater remainder should be chosen according to one's will.

We do not know if Bhāskara computed the largest common multiple of these divisors, in order to overcome the problem that occurs when two divisors are multiples of one another.

A.3 Procedure of the pulverizer without remainder

We will present here the different steps of this algorithm such as it is described in the general commentary. Then we will present two alternative procedures, solving the same problem, and found in the "procedure" part of solved astronomical examples.

General procedure

Problem

What is the integer x, that multiplied by a, increased or decreased by c and divided by b, produces an integer y?

In other words the problem consists of finding two integers (x, y) that verify:

$$y = \frac{ax \pm c}{b}$$

a, b and c are known positive integers. x is called the pulverizer or the multiplier (*guṇaka*), y the quotient (*labdha*).

In the "setting-down" part of examples, this is the pattern followed: $\begin{matrix} a & c \\ b & \end{matrix}$

Sometimes c is omitted.

At the beginning of example 22 Bhāskara writes[14]:

> *bhāgahārabhājyāgrāṇam ekena apavartanacchedena apavartanaṇ kṛtvā pūrvavat kuṭṭākāraḥ kriyate| atha punar etāni bhāgahārabhājyāgrāṇi chedenaikanāpavartanaṇ na prayacchati yathā tathā sāv uddeśakaḥ,*

[14][Shukla 1976; last paragraph p.149-150]

> *tādṛṣas' caiko rāṣir eva nāsty ato na ānīyate*|
>
> When one has performed the reduction, by a unique reducing divisor, of the divisor, dividend and remainder, as before, a pulverizer is performed. Now, on the other hand, ⟨if⟩ that example is such that these divisor, dividend and remainder do not allow such a reduction with a unique divisor, as there is no such one quantity ⟨that satisfies this equation⟩, ⟨such a quantity⟩ is not computed ⟨with a pulverizer⟩.

So that as we have noted above, Bhāskara suggest to reduce the numbers used in examples before starting the computation (these truly get to huge proportions in astronomical problems) but is also well aware that c should be a multiple of a and b in order for such a problem to have a solution.

Step 1

Sanskrit *adhikāgrabhāgahāraṃ chindyād ūnāgrabhāgahāreṇa*

English One should reduce the divisor which is a large number ⟨and the dividend⟩ by a divisor which is a small number.

General Comments In other words, one should discard common factors from a (the dividend) and b (the divisor), a new couple (a', b') is therefore considered; where a' and b' are coprime (that is there sole common divisor is 1). This step can be seen as a "short-cut" for the following process of the "Euclidian Algorithm". Practically, Bhāskara always discards their GCD.

Step 2-Step 4

As we have noted before, if we consider the problem solved by a pulverizer with remainder: $R_1 > R_2$, and $R_1 - R_2 = c$,

$$\begin{cases} N = & ax+ & R_1 \\ N = & by+ & R_2 \end{cases} \Leftrightarrow \; y = \; \frac{ax+c}{b}$$

Therefore, as noted by Bhāskara as well, these steps are similar to Step 2- Step 4 of the pulverizer with remainder.

Therefore here, the first division is that of the divisor *by* the dividend. In the end of this process we have two quantities, q'_2 and q'_3.

Step 5

Sanskrit *ūnāgracchedabhājite śeṣam*

English Ab. When ⟨the remaining upper quantity⟩ is divided by the divisor which is a small number, the remainder is ⟨the pulverizer. When the lower one remaining is divided by the dividend the quotient of the division is produced.⟩

Bhāskara further glosses[15]:

> *upari[rāśiḥ] bhāgahāreṇa bhaktaḥ [kāryaḥ], adhorāśir bhājya rāśinā bhājyaḥ*
>
> The upper [quantity should be made to be] divided by the divisor; the lower quantity should be divided by the dividend quantity. (...)

The two remainders are the pulverizer and the quotient of the division.

General With the same notation as before q_2' ("the upper quantity") is divided by b ("the divisor"):

$$q_2' = tb + u.$$

u is called the pulverizer.

q_3' ("the lower quantity") is divided by a ("the dividend"):

$$q_3' = va + w.$$

w is called the quotient.

The result is usually set down in a column: $\begin{array}{c}\text{u}\\ \text{w}\end{array}$

At the end of his resolution of example 9[16], Bhāskara indicates:

> *[athavā] yāvad abhirūcitaṃ pṛcchakāya*
>
> [Or else] until it pleases the inquirer (*pṛcchaka*), ⟨the values should be increased by multiples of the constants⟩.

This somewhat elliptic remark, may refer to the following rule, given in the *Mahābhāskarīya* [Shukla 1960; sk p. 8, eng. p. 40]:

[15] [Shukla 1976; p.135 lines 17 to 21].

[16] [Shukla 1976; p.139].

> *prakṣipya bhāgahāraṃ kuṭṭākāre punaḥ punaḥ prājñāiḥ|*
> *yojyaṃ ca bhāgalabdhaṃ bhājye prastārayuktyaiva||*
> Mbh.1.50. (To obtain the other solutions of a pulverizer) the intelligent (astronomer) should again and again add the divisor to the multiplier and the dividend to the quotient as in the process of *prastāra* ("representation of combinations").

In other words if (m, n) is a solution of

$$y = \frac{ax \pm c}{b},$$

where (x, y) are the unknowns, then, for any integer t

$$\begin{array}{l} m_t = m + tb \\ n_t = n + ta \end{array},$$

are also solutions of this problem.

Alternative procedures

The *sthirakuṭṭāka* In his commentary on example 7, and then systematically in all resolutions after this one, when solving

$$y = \frac{ax \pm c}{b},$$

Bhāskara, instead of the usual procedure, proposes as an alternative to solve first with the same procedure the following problem:

$$y' = \frac{ax' \pm 1}{b}.$$

The values found as solution are then used in a Rule of Three, with the following proportions:

$$\begin{array}{l} 1 : x' = c : x'' \\ 1 : y' = c : y'' \end{array}$$

The smallest values possible for x and y are found, by considering the remainders of the divisions of x'' by b, and of y'' by a.

This is known in later literature as the *sthirakuṭṭāka* (fixed-pulverizer).

The versified table that ends the *gaṇitapāda* gives the smallest possible solutions for problems of the type

$$y = \frac{ax - 1}{b},$$

using many different types of astronomical constants[17].

Solutions of

$$u = \frac{av + 1}{b},$$

may be easily derived from the type above, as

$$x = b - v$$
$$y = a - u$$

If no general rule is given by Bhāskara in his commentary, such a process is described in the *Mahābhāskarīya* [Shukla 1960; p. 32-33]:

> *Mbh.45. rūpaṃ ekam apāsyāpi kuṭṭākāraḥ prasādhyate|*
> *guṇakāro 'tha labdhaṃ ca rāśī syātām upary-adhaḥ||*
> *Mbh.46.ab. iṣṭena śeṣam abhihatya bhajed dṛḍhābhyāṃ śeṣam*
> *dināni bhagaṇādi ca kīrtyate 'tra|*

> Mbh.I.45-46ab. Alternatively, the pulverizer is solved by subtracting one (i.e., by assuming the residue to be unity). The upper and lower quantities (in the reduced chain) are the (corresponding) multiplier and quotient (respectively). By the multiplier and quotient (thus obtained) multiply the given residue, and then divide the respective products by the abraded divisor and dividend. The remainders obtained are here (in astronomy) the *ahargaṇa* and the revolutions (performed respectively).

This can be understood as follows:

If (m, n) is a solution of

$$y = \frac{ax \pm 1}{b},$$

[17] We have not translated this versified table. It is summarized, and all values given, in [Shukla 1976; Appendix ii, p.335-339].

where (x, y) are the unknowns. If (m_0, n_0) are respectively the remainders of the division of cm by b, and of cn by a.

$$\begin{array}{ll} m_0 = cm - bq & (0 \leq m_0 < b) \\ n_0 = cn - aq & (0 \leq n_0 < a) \end{array}$$

Then, (m_0, n_0) is a solution of

$$y = \frac{ax \pm c}{b}.$$

Another alternative In his resolution of example 11[18], Bhāskara describes an alternative procedure:

> *atra bhāgahāreṇa bhājyaṃ vibhajya labdhaṃ pṛthag-avinaṣṭaṃ sthāpayet| śeṣasya bhūdivasānāṃ ca kuṭṭākāraṃ kṛtvā labdhasyoparirāśiṃ kuṭṭākāram avinaṣṭasthāpitena pṛthak saṃguṇayya bhāgalabdhaṃ prakṣipet| bhāgalabdhaṃ bhavati |*
> In this case, having divided the dividend by the divisor, one should place the quotient separately ⟨and keep it⟩ unerased. When one has performed the pulverizer of the terrestrial days and the residue, when one has multiplied separately the higher quantity of the ⟨two⟩ obtained by the pulverizer of the ⟨quantity⟩ kept unerased, one should add the quotient of the division ⟨which stands below⟩. ⟨This⟩ produces the quotient of the division.

Which can be understood as follows. What is obtained at the end of the process which proceeds upwards is:

$$\begin{array}{c} q'_2 = x \\ y_1 \end{array}.$$

Where y_1 is defined as:

$$y = xq_1 + y_1.$$

Bhāskara, here indicates that one should set aside q_1 defined as the quotient of the division of a by b:

$$a = bq_1 + r_1.$$

Therefore the computation described here corresponds to a computation of y:

$$xq_1 + y_1 = y.$$

[18] [Shukla 1976; p.141, line 15-18].

A.4 Astronomical applications

The kind of astronomical problem solved by the procedure of the pulverizer without remainder is introduced in Bhāskara's commentary without an explanation relating that process to given astronomical problems. These relations, however, can be found in the *Mahābhāskarīya*.

The basic idea is that the number of revolutions of a given planet, during a certain time is not a round number, but has, in addition to an integral value, a fractional part, or residue (*śeṣa*). This is also true, if are considered not only the number of revolutions, but also the number of signs (*rāśi* or *bhagaṇa*), degrees (*bhāga*) or minutes (*liptā*), crossed by the planet during a given time. This time is usually evaluated in terms of civil days (*ahargaṇa*).

We will consider from now on, the following notations[19]:

Let A_y be the number of civil days in a *yuga*, G_y the number of revolutions performed by planet g in a *yuga*.

All the planet's revolutions in a *yuga* are given in Ab.1.3; the number of civil days in a *yuga* are deduced from both Ab.1.3 and Ab.3.3 and 5. This computation is described in the Appendix **??**, which shows how this value of A_y is obtained: $A_y = 1577917500$.

As A_y and G_y will respectively be the dividend and divisor of a pulverizer without remainder, they are systematically reduced by their greatest common divisor. This can be seen in Bhāskara's commentary, at the beginning of the section on *maṇḍalakuṭṭākāra* (p. 135-136):

> *etāv ūnāgracchedārthaṃ parasspareṇa bhājyau| śeṣam ūnāgracchedaḥ*
> These two should be divided by one another in order ⟨to obtain⟩ the divisor which is a smaller number. What remains is the divisor which is a smaller number...

Since the "divisor which is a smaller number" is, in this case, the greatest common divisor of the two first numbers, it appears that it was found by what is commonly called "the Euclidian Algorithm".

In the following, for the sake of conveniency, we will also call A_y and G_y the numbers obtained after reduction. (G_y is usually called in second-hand literature, the "revolution number" of the planet.)

Let A be the number of days elapsed since a given epoch (*ahargaṇa*). Here it is always the number of civil days elapsed since the beginning of the *Kaliyuga*.

[19] All the notations used in this supplement are summed up on a list, at the end of this supplement.

Let G be the number of revolutions performed by a planet g in A days. G can be decomposed as the integral number of revolutions (*maṇḍala*) performed, M, the integral number of signs (*rāśi*), R, degrees (*bhāga*), B, and minutes (*liptā*), L crossed.

All the procedures use the following ratio:

$$\frac{A}{A_y} = \frac{G}{G_y}.$$

The reasoning followed in all the problems is basically the same, involving different ratios, according to the units considered, and occasionally a difference of sign in the pulverizer to solve, whether the fractional part of the path of g is considered as a surplus of the integral number of revolutions, or the part missing to obtain an integral number of revolutions. For the sake of simplicity, we have set aside here both, the operations involving, the reduction of the numbers of days and revolutions in a *yuga*, and, those converting values given in examples in homogeneous units (that is the conversion of a latitude given in degrees and minutes into minutes, etc.).

Planet's pulverizer (*maṇḍalakuṭṭāka*)

This computation concerns the commentary on verses 32-33, p.136-138. The planet considered is the sun.

Planet's pulverizer with the residue of revolutions

Problem Let $A = x$, be the number of days elapsed since a given epoch (*ahargaṇa*), usually the beginning of the *Kaliyuga*. Let $M = y$ be the integral number of revolutions (*maṇḍala*) of a planet g during x days. These are the unknowns to be found, knowing:

-λ, the mean longitude of planet g in minutes after x days. ($\lambda = (30 \times 60)R + (60 \times B) + L$.)

- G_y, the reduced number of revolutions of planet g in a *yuga*.

- A_y, the reduced number of civil days in a *yuga*.

In the "setting down" part of examples, the disposition follows this pattern:

Integral number of signs crossed	Integral number of degrees crossed	Integral number of minutes crossed
R	B	L

or

Integral number of signs crossed	R
Integral number of degrees crossed	B
Integral number of minutes crossed	L

procedure with the mean longitude Let λ be the mean longitude of planet g in minutes. R_M the "residue of revolutions", is defined as follows:

$$R_M = \frac{\lambda \times A_y}{21600}.$$

In the *Mahābhāskarīya*, the following rule occurs ([Shukla 1960; p. 33][20]):

> *rāśyādayo nirapavartitav āsaraghnā rāśyādimānabhajitāḥ pravadanti śeṣam*
> Mbh.1.46cd. (In the case the longitude of a planet is given in terms of signs, etc.) the signs, etc. are multiplied by the abraded number of civil days (in a *yuga*) and the product is divided by the number of signs, etc., (in a circle). The quotient is stated to be the residue (of revolutions).

In this case here the mean longitude of g (λ) is reduced to minutes, so that the divisor is the number of minutes in a circle.

The residue of revolutions, R_M, can be understood as the number of civil days taken to accomplish that part of a revolution indicated by λ_g. Since 21600 is the number of minutes in a circle, we have:

$$\frac{R_M}{A_y} = \frac{\lambda}{21600}.$$

When computing R_M in his commentary, Bhāskara always considers an approximation of the quotient obtained, so that it maybe an integer.

Two alternative methods are proposed having obtained this"residue of revolution", to solve the above problem:

[20]The first example given on this topic in Bhāskaraś commentary is explained in the following pages of this book, p. 34-35.

procedure 1 Find a couple solution of:

$$y = \frac{G_y x - R_M}{A_y}.$$

$x = A$ is the number of days elapsed since a given epoch and $y = M$ is the integral number of revolutions of a planet g during x days.

We can understand the process used here as the follows. We have the following ratio, where $\frac{\lambda_g}{21600}$ as the residual mean longitude in terms of revolutions is the non integer part of the number of revolutions performed by G:

$$\frac{x}{A_y} = \frac{y + \frac{\lambda}{21600}}{G_y}.$$

This equivalent to:

$$y = \frac{G_y x - R_M}{A_y},$$

where $R_M = \frac{\lambda \times A_y}{21600}$.

procedure 2 Uses a "*sthirakuṭṭāka*" process[21], that is

Find a couple solution of:

$$y' = \frac{G_y x' - 1}{A_y}.$$

The values obtained for this pulverizer are tabulated by Bhāskara at the end of the *gaṇitapāda*[22].

Then using the following ratios, x'' and y'' are computed :

$$\begin{array}{l} 1 : x' = R_M : x'' \\ 1 : y' = R_M : y'' \end{array},$$

the smallest values possible for x and y are found, by considering the remainders of the divisions of x'' by A_y, and of y'' by G_y.

[21]This process is explained in the section on the pulverizer without remainder.

[22]We have not translated this versified table. This table is summarized in [Shukla 1976; Appendix ii, p.335-339].

Planet's pulverizer with the revolutions to be accomplished A similar procedure is found, when considering the complementary part of the partial revolution accomplished. In this case, the part of the revolution to be crossed is added, when considering the pulverizer to solve.

Problem Let $A = x$, be the number of days elapsed since the beginning of the *Kaliyuga* (*ahargaṇa*). Let $M = y$ be the integral number of revolutions of a planet g during x days. These are the unknowns to be found, knowing:

-Δ, the part of a revolution to be accomplished by g so that the number of revolutions would be integer ($\lambda + \Delta = 1$ revolution).

- G_y, the reduced number of revolutions of planet g in a *yuga*.

- A_y, the reduced number of civil days in a *yuga*.

In the "setting down" part of examples, the disposition follows this pattern-

Integral number of signs to be crossed	R
Integral number of degrees to be crossed	B
Integral number of minutes to be crossed	L

A rule is given for this problem in the *Mahābhāskarīya*[23]:

> *gantavyam iṣṭaṃ yadi kasyacit syād gantavyayogād idam eva karma|*
> *rūpeṇa vā yojya vidhir vacintyaḥ sarvaṃ samānaṃ khalu lakṣaṇena||*
> Mbh.1.51. When the part (of the revolution) to be traversed by some (planet) is the given quantity, then (also) the same process should be applied, treating the part to be traversed as the additive, or taking unity as the additive. All details of procedure are the same (as before).

Finding the part of a revolution to be accomplished The computation is exactly the same as the one described above. That is, if Δ is the part of a revolution to be accomplished by g in minutes, since 21600 is the number of minutes in a circle, then the " part of a revolution to be accomplished", R'_M, is:

$$R'_M = \frac{\Delta \times A_y}{21600}.$$

[23] [Shukla 1960; sk p. 8-9, eng. p. 41].

Two alternative methods are proposed having obtained this value, to solve the above problem:

procedure 1 Find the smallest couple solution of:

$$y = \frac{G_y x + R'_M}{A_y}.$$

procedure 2 Find the smallest couple solution of:

$$y' = \frac{G_y x' + 1}{A_y}.$$

The values of

$$u' = \frac{G_y v' - 1}{A_y},$$

are tabulated by Bhāskara at the end of the *gaṇitapāda*. From these, x' and y' are obtained:

$$x' = A_y - v'$$
$$y' = G_y - u'$$

Then, using the same following ratios:

$$\begin{array}{l} 1 : x' = R'_M : x'' \\ 1 : y' = R'_M : y'' \end{array},$$

the smallest values possible for x and y are found, by considering the remainders of the division of x'' by A_y, and of y'' by G_y.

Pulverizer with the residue of signs

Here, both the integral number of revolutions performed by g, M, and the following number of signs crossed by this planet, R, are unknown.

Problem Let $A = x$, be the number of days elapsed since the beginning of the *Kaliyuga* (*ahargaṇa*). Let $12 \times M + R = y$ be the integral number of signs crossed by g during x days. These are the unknowns to be found, knowing:

-λ', the remaining degrees and minutes crossed by g after x days in minutes ($\lambda' = 60 \times B + L$).

- G'_y, the reduced number of signs crossed by planet g in a *yuga*.

$$G'y = Gy \times 12,$$

as there are 12 signs in a revolution.

- A_y, the reduced number of civil days in a *yuga*.

In the "setting down" part of examples, the disposition follows this pattern, where the "0" indicates what is unknown or an empty space-

Integral number of revolutions crossed	0
Integral number of signs crossed	0
Integral number of degrees crossed	B
Integral number of minutes crossed	L

Finding the"residue of signs" A similar ratio to the one used in the cases above, gives us the residue of signs (R_R), from λ', 1800 being the number of minutes in a sign:

$$\frac{R_R}{A_y} = \frac{\lambda'}{1800}.$$

In other words

$$R_R = \frac{\lambda' \times A_y}{1800}.$$

Three alternative methods are proposed having obtained the residue of signs, to solve the above problem:

procedure 1 Find the smallest couple solution of:

$$y = \frac{G'_y x - R_R}{A_y}.$$

The value found for y is the number of signs crossed crossed by g during x days. The remainder of the division of y by 12 will give the number of revolutions performed by g in x days.

procedure 2 Find a couple solution of:

$$y' = \frac{G'_y x' - 1}{A_y}.$$

These values are tabulated by Bhāskara at the end of the *gaṇitapāda*. Performing a Rule of Three with 1 and R_R, and dividing the results respectively by A_y and G'_y will give the results.

procedure 3 Find a couple solution of:

$$v' = \frac{12u' - 1}{A_y}.$$

The following procedure is not given by Bhāskara, thought he indicates that a Rule of Three should be used. We can consider the following, thought this is just a hypothetical construction in order to understand why this pulverizer is computed:

We have the following ratio:

$$\frac{\lambda}{21600} = \frac{R_M}{A_y},$$

where, as in section C.3.1., R_M is the residue of revolutions and $\lambda = (30 \times 60)R + (60 \times B) + L = (30 \times 60)R + \lambda'$. So this equivalent to:

$$\frac{(30 \times 60)R + \lambda'}{21600} = \frac{R_M}{A_y}.$$

Now if we consider, this residual part of revolutions accomplished, not in terms of minutes, but in terms of signs (or if we reduce the left-hand fraction by $30 \times 60 = 1800$) we have:

$$\frac{R + \frac{\lambda'}{30\times60}}{12} = \frac{R_M}{A_y}.$$

Let $v = R$ and $u = R_M$ and we recognize here:

$$v = \frac{12u - \frac{\lambda' \times A_y}{1800}}{A_y} = \frac{12u - R_R}{A_y},$$

Bhāskara would thus solve this problem by a *sthirakuṭṭāka*.

u being the residue of revolutions, the following problem:

$$y' = \frac{G_y x - u}{A_y},$$

when solved gives with x the number of days elapsed since a given epoch, and with y' the number of revolutions accomplished in x days. Together with the value found for v, we can find the total number of signs crossed by g in x days.

Pulverizer for the residue of degrees The process follows the same pattern as before, the difference being that one seeks the total number of degrees crossed by g in x days, that is that, M, R and B are unknown.

Problem Let $A = x$, be the number of days elapsed since a given epoch (*ahargaṇa*). Let $12 \times 30M + 30 \times R + B = y$ be the integral number of degrees crossed by g during x days. These are the unknowns to be found, knowing:

-$\lambda''_g = L$, the remaining minutes crossed by g after x days.

- G''_y, the reduced number of degrees crossed by planet g in a *yuga*.

$$G''y = Gy \times 360,$$

as there are 360 degrees in a revolution.

- A_y, the reduced number of civil days in a *yuga*.

In the "setting down" part of examples, the disposition follows this pattern, where the "0" indicates what is unknown or an empty space-

Integral number of revolutions crossed	0
Integral number of signs crossed	0
Integral number of degrees crossed	0
Integral number of minutes crossed	L

Finding the"residue of degrees" A similar ratio to the one used in the cases above, gives us the residue of degrees (R_B), from λ''_g, 60 being the number of minutes in a degree:

$$\frac{R_B}{A_y} = \frac{\lambda''_g}{60}.$$

In other words

$$R_B = \frac{\lambda''_g \times A_y}{60}.$$

Three alternative methods are proposed having obtained the residue of degrees, to solve the above problem:

procedure 1 Find the smallest couple solution of:

$$y = \frac{G''_y x - R_B}{A_y}.$$

The value found for y is the number of degrees crossed by g during x days. The remainder of the division of y by 360 will give the number of revolutions performed by g in x days.

procedure 2 Find a couple solution of:

$$y = \frac{G''_y x - 1}{A_y}.$$

These values are tabulated by Bhāskara at the end of the *gaṇitapāda*.

Performing a Rule of Three with 1 and R_B, and dividing the results respectively by A_y and G''_y will give the required results. The remainder of the division of y by 360 (i.e. the number of degrees in a revolution) will give the number of revolutions performed by g in x days.

procedure 3 Find a couple solution of:

$$v' = \frac{30u' - 1}{A_y}.$$

The following procedure is not given by Bhāskara, thought he indicates that a Rule of Three should be used. We can consider the following:

We have the following ratio,

$$\frac{\lambda'_g}{1800} = \frac{R_R}{A_y}.$$

This is equivalent to:

$$\frac{(60 \times B) + L}{1800} = \frac{R_R}{A_y}.$$

Now if we consider, this residual part of signs crossed, not in terms of signs but in terms of degrees (or if we simplify the left hand fraction by 60):

$$\frac{B + \frac{L}{60}}{30} = \frac{R_R}{A_y}.$$

Let $v = B$ and $u = R_R$, then:

$$v = \frac{30u - \frac{\lambda''_g \times A_y}{60}}{A_y} = \frac{30u - R_B}{A_y},$$

Since u is residue of signs, the following problem:

$$y' = \frac{G'_y x - u}{A_y},$$

when solved, gives with x the number of days elapsed since the beginning of the *Kaliyuga*, and with y' the number of revolutions accomplished and the number of signed crossed in x days. Together with the value found for v, we can find the total number of degrees crossed by g in x days.

Pulverizer for the residue of minutes The procedure follows the same pattern, considering residual seconds, crossed by G.

Week-day pulverizer

Problem A planet g, has a given mean longitude, λ, on a week day V. After a certain number of weeks (w) and a couple of days (a), g has the same longitude on an other week-day, V_a.

Let a be the number of week-days seperating V from V_a (V excluded, V_a included; $a \leq 7$).

Let A_V be the number of days elapsed in the *Kaliyuga* when the sun is in V.

Let A_{V_a} be the number of civil days elapsed in the *Kaliyuga* for which the sun on V_a has the given mean longitude in V.

A_V and A_{V_a} are to be found, knowing λ on V; A_y and G_y.

Resolution The computation of A_V corresponds to a usual "planet-pulverizer": If $A_V = x$ and y=M then by solving with a pulverizer, the following problem:

$$y = \frac{G_y x - R_M}{A_y},$$

the required value for A_V is found. Let x_0 be such a value.

In the *Mahābhāskarīya* there is the following rule[24]:

> *apavartitav āsarādiśeṣāṭ kramaśastān apanīya rūpapūrvam|*
> *kuṭṭākalabdharāśim eṣāṃ guṇakāraṃ samuśanti vārahetoḥ||*
> MBh.1.48. Divide the abraded number of civil days (in a *yuga*) by 7. Take the remainder as the dividend and 7 as the divisor. Also take the excess 1,2, etc., of the required day over the given day as the residue. Whatever number (i.e. multiplier) results on solving this pulverizer is the multiplier of the abraded number of civil days. The product of these added to the *ahargaṇa* calculated (for the given day) gives the *ahargaṇa* for the required day.

And in his introduction to example 12 of the commentary to verses 32-33, Bhāskara writes:

> *nirapavartitabhūdineṣu saptahṛtāvaśiṣṭeṣu kuṭṭākāraḥ kriyate|*
> *grahavāro yo nirdiṣṭas tasmād y[ad u]ttaro grahavāras tataḥ prabhṛti ekottarayā vṛddhyāpacayaṃ parikalpya evaṃ labdhaṃ kuṭṭākāro nirapavartitabhūdinānāṃ guṇakāras tena guṇiteṣu nirapavartitabhūdineṣu nirdiṣṭasūryeṇānītam ahargaṇaṃ prakṣipya jātadivasatulyaḥ kāla ādeṣṭavyaḥ*
> A pulverizer should be performed for the residue of the division by seven of the reduced terrestrial days. When one has chosen a subtractive ⟨term for the pulverizer⟩ by means of a one-by-one increase beginning with the weekday which is immediately after the indicated week-day, what is obtained in this way is the pulverizer which is the multiplier of the reduced terrestrial days; when one has added the passed number of days ⟨in the *Kaliyuga*, obtained with⟩ the indicated sun, to the reduced terrestrial days multiplied by

[24] [Shukla 1960; sk p. 8, eng. p.36-37(this is an adaptation- see note 1, p.37)] .

> that ⟨pulverizer⟩, the time equal to what has been produced should be announced ⟨as the answer.

In this case, the pulverizer considered is, if A'_y is the residue of the division of A_y by seven ($A'_y = A_y - 7q$), a corresponding to the "one-by-one increase beginning with the weekday which is immediately after the indicated week-days":

$$w' = \frac{A'_y v' - a}{7}.$$

If (v'_0, w'_0) is a solution, then

$$A_{V_a} = A_y v'_0 + x_0.$$

This can be understood as follows: if A_y is the reduced number of civil days in a *yuga*, so that the number of weeks in a *yuga* is $\frac{A_y}{7}$, then we have the following proportion:

$$\frac{A_{\Delta V}}{A_y} = \frac{w + \frac{a}{7}}{\frac{A_y}{7}},$$

where $A_{\Delta V}$ is the number of civil days after which the sun, having had that given longitude in V, has the same longitude in V_a, and w is the number of weeks in $A_{\Delta V}$, so that $A_{\Delta V} = 7w + a$

If[25] $v = \frac{A_{\Delta V}}{A_y}$ then we have

$$\frac{v}{7} = \frac{w + \frac{a}{7}}{A_y}.$$

From this proportion we can deduce the following problem solved by a pulverizer:

[25] There seems to be a paradox here, as $A_{\Delta V}$ is thus defined as a multiple of A_y, therefore $A_{\Delta V} > A_y$. This assumption without any comment is also made by K.S. Shukla, when he solves example 12. [Shukla 1976; p.317] (A being what we note $A_{\Delta V}$, 210389 being the reduced number of civil days in a *yuga* for the sun).

We can, nonetheless, remark that A_y is, here, the *reduced* number of terrestrial days in a *yuga* and not the total number, so that this is not as absurd as it may seem. However, just why should this be presupposed and whether this is the exact rending of the computation described by Bhāskara, remains to be investigated.

$$w = \frac{A_y v - a}{7}.$$

Let (v_0, w_0) be a solution of that problem.

Since A_V is the number of days elapsed in the *Kaliyuga* when the sun is in V, A_{V_a} the number of civil days elapsed in the *Kaliyuga* for which the sun on V_a has the given mean longitude, and $A_{\Delta V}$ the number of civil days after which the sun, having had that given longitude in V, has the same longitude in V_a then

$$A_{V_a} = A_V + A_{\Delta V}.$$

By definition of v_0, and x_0:

$$A_{V_a} = x_0 + A_y v_0.$$

Now the particular solution, v'_0, for v' makes also the quotient

$$\frac{A_y v - a}{7}$$

integer because

$$w' + 7q = \frac{A_y v' - a}{7}.$$

Particular pulverizers

Some of the examples proposed by Bhāskara combine several of the problems and procedures exposed above.

A particular planet's pulverizer The problem here considers the remaining part of a degree to be crossed by a planet, combining thus a "pulverizer for a revolution to be accomplished" and "a pulverizer with the residue of degrees". In example 13 [Shukla 1976; p.143] is exposed a problem and resolution of this type.

Problem Let $A = x$, be the number of days elapsed since a given epoch (*ahargaṇa*). Let $(12 \times 30)M + 30R + B = y$ be the integral number of degrees crossed by g during x days. These are the unknowns to be found, knowing:

-Δ'', the part of a degree to be crossed by g so that the number of degrees crossed since the beginning of the *Kaliyuga* would be integer.

- G''_y, the reduced number of degrees crossed by planet g in a *yuga*.

$$G''y = 360 \times Gy,$$

as there are 360 degrees in a revolution.

- A_y, the reduced number of civil days in a *yuga*.

procedure After having computed the residue of degrees to be crossed,

$$R'_B = \frac{\Delta'' \times A_y}{60},$$

the following problem is to be solved directly by a pulverizer procedure, or by using a *sthirakuṭṭāka*:

$$y = \frac{G''_y x + R'_B}{A_y}.$$

The value found for $y-1$, when divided by 360 gives the integral number of revolutions performed by g in x days.

A particular week-day pulverizer

Problem In this case, the mean longitude of planet g_1 (λ_1), and the mean longitude of planet g_2 (λ_2) are known, for a given week-day (V); the number of days until they will both be of the same longitude again on another week-day (V_a) is what is sought.

Finding the LCM Let A_1 be the reduced number of days in a *yuga* for g_1; A_2 the reduced number of days in a *yuga* for g_2. The Lowest Common Multiple of these two numbers ($LCM(A_1, A_2)$), can be defined as:

$$LCM(A_1, A_2) = \frac{A_1 \times A_2}{GCD(A_1, A_2)}.$$

It is found by the following process:

-The Greatest Common divisor ($GCD(A_1, A_2)$) is found, probably by a “Euclidian algorithm”.

In the case of the preliminary part of example 14, it is defined as the quantity which leaves a zero remainder (*śūnyāgra*), when divided or by A_1 or by A_2. It bears the name “⟨quantity⟩ having such remainder for two divisors.” (*dvicchedāgra*).

-The quotient of the division of A_1 (resp. A_2) by $GCD(A_1, A_2)$ (q_1) (resp. q_2) is considered.

Then

$$LCM(A_1, A_2) = A_1 \times q_2 = A_2 \times q_1.$$

This is expressed quite elliptically in the preliminary part of example 14, but corresponds to the computations carried out:

> *dvicchedāgrasaṃvargo hi nāma sadṛśikaraṇaṃ*
>
> the product of ⟨one reduced day by the quotient of the other by the quantity⟩ having such remainder for two divisors (*dvicchedāgrasaṃvargo*) has the name “procedure of equalizing (*sadṛśīkaraṇaṃ*) for two quantities”.

Finding the number of days elapsed in the *Kaliyuga* when g_1 and g_2 are in V This involves a usual planet-pulverizer: The smallest integral solution found for x (x_0) in any of these equations gives the desired value

$$\begin{cases} y = & \frac{G_1 x - R_{M_1}}{A_1} \\ y = & \frac{G_2 x - R_{M_2}}{A_2} \end{cases}$$

A week-day pulverizer The following problem is solved by a pulverizer:

$$w = \frac{LCM(A_1, A_2)v - a}{7}.$$

Let v_0 be the smallest integral value found. Then

$$A_{\Delta_V} = LCM(A_1, A_2)v_0 + x_0.$$

Thus , the following equality explains this formulation of the problem:

$$\frac{w + \frac{a}{7}}{LCM(A_1, A_2)} = \frac{v}{7},$$

where

$$v = \frac{A_{\Delta V}}{LCM(A_1, A_2)}.$$

A pulverizer using the sum of the longitudes of planets

Problem Let $A = x$ be the number of days elapsed in the *Kaliyuga*. This is the unknown to be found, knowing:

-$\Sigma\lambda$, the sums of the mean longitudes of n planets, in minutes, after x days. ($\Sigma\lambda = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n}(30 \times 60)R_i + (60 \times B_i) + L_i,\ n \leq 7$[26])

- ΣG_y, the reduced sum of the the number of revolutions performed by each planet in a *yuga*.

- A_y, the reduced number of civil days in a *yuga*.

procedure The procedure, with these constants, is the same as in a regular planet's pulverizer. Having computed the residue of revolution of the sum,

$$\Sigma R_M = \frac{\Sigma\lambda \times A_y}{21600},$$

the problem to be solved by a pulverizer or by a *sthirakuṭṭāka* is:

$$y = \frac{\Sigma G_y x - \Sigma R_M}{A_y}.$$

The smallest solution found for y is the sum of the revolutions performed by n planets in x days.

As before, the constant ratio behind this problem is:

$$\frac{A}{A_y} = \frac{G_1}{G_{g_1}} = \ldots = \frac{G_n}{G_{g_n}}.$$

So that

$$\frac{A}{A_y} = \frac{\Sigma G_y}{\Sigma G}.$$

[26] A list of the planets is given in Ab.3.15.

This procedure is described in example 15 [Shukla 1976; p.144sqq]; where only two planets are considered, the sun and the moon. However Bhāskara adds:

> *evam anyeṣām api samāsapraśneṣu kuṭṭākāraḥ kalpanīyaḥ, rāśibhāgaliptāśeṣvapi|*
> *evam eva tricatuḥsamaseṣvapi vistareṇa vyākhyeyam|*

> In this way, in questions concerning the sums of other ⟨planets⟩ too, a pulverizer is to be performed (*kalpanīya*), and also ⟨in questions⟩ concerning residues of signs, degrees and minutes. In this very way, in the case of the sums of three or four ⟨planets⟩ also an explanation should be given in detail ⟨if necessary⟩.

Knowing the number of revolutions performed by two planets

Problem The number of revolutions performed since the beginning of the *Kaliyuga* by g_1 (y) and the integral number of revolution performed by g_2 (z) are sought, knowing:

-λ_2, the mean longitude of g_2 in minutes, known when g_1 completes a revolution.

-G_1 and G_2, (previously reduced by their greatest common divisor), the reduced of revolutions performed by g_1 and g_2 in a *yuga*.

Resolution The problem to be solved by a pulverizer without remainder or by a *sthirakuṭṭāka* is:

$$z = \frac{G_2 y - R_{M_2}}{G_1}$$

This is understood by the following reasoning: If A is the number of civil days elapsed at a given time, A_y the number of civil days in a *yuga*, then we have:

$$\begin{cases} \dfrac{A}{A_y} = \dfrac{M_1 + \frac{\lambda_1}{21600}}{G_1} \\ \\ \dfrac{A}{A_y} = \dfrac{M_2 + \frac{\lambda_2}{21600}}{G_2} \end{cases}$$

And therefore

$$\frac{M_1 + \frac{\lambda_1}{21600}}{G_1} = \frac{M_2 + \frac{\lambda_2}{21600}}{G_2}$$

With the notation adopted above, that is:

$$\frac{y}{G_1} = \frac{z + \frac{\lambda_2}{21600}}{G_2}.$$

From this equality the problem to be solved by a pulverizer is readily deduced.

Similarly, if the ratio considered for g_1 is measured in minutes then:

$$\frac{21600M_1 + \lambda_1}{G_1''} = \frac{M_2 + \frac{\lambda_2}{21600}}{G_2}$$

And the problem to be solved by a pulverizer would then be:

$$z = \frac{G_2 Y - R_{M_2}}{G_1''},$$

where $Y = 21600y$, is the number of minutes crossed by g_1 since the beginning of the *Kaliyuga*.

The problem and method to solve such a pulverizer is described in general terms by Bhāskara in this way[27]:

> *atha kaścid divasakaramaṇḍalaśeṣaparisamāptikāle janitaṃ divicaramuddiśya divasakaraṃ divicarabhagaṇān pṛcchati, tasyāyam upāyaḥ nirdiṣṭadivicaraṃ ravibhagaṇāṃścāpavartya kuṭṭākāro yojyaḥ|*
> Now, when pointing at ⟨the longitude of⟩ a planet produced at the time when the sun completes what remains of a revolution, someone asks the ⟨number of⟩ revolutions ⟨performed⟩ by ⟨that planet⟩, this is a method for that ⟨question⟩ -When one has reduced the ⟨number of⟩ revolutions ⟨performed⟩ by a planet ⟨in a *yuga*⟩ and the ⟨number of⟩ revolutions ⟨performed⟩ by the sun ⟨in a *yuga*⟩, a pulverizer should be applied.

He then proceeds to solve the problem given in example 16, and concludes by the following statement[28]:

[27] [Shukla 1976; p.145, line 16 sqq].
[28] [Shukla 1976; p.146, line 13 sqq].

> *athavā graham uddiśya graham evānyaṃ [pṛcchati tatr]āpi bhāgahārabhājyaparikalpa*
> *kuṭṭākāraḥ kalpanīyaḥ*|
>
> Or else when ⟨someone⟩ pointing at a planet asks ⟨the number of passed revolutions⟩ of another planet only, then again a pulverizer should be performed by choosing ⟨an appropriate⟩ divisor and dividend.

Here therefore Bhāskara does not stress the unit in which the number of elapsed revolutions are obtained.

Mbh.1.10 gives the following procedure[29]

> *niśākaraṃ vā graham uccam eva vā kalīkṛtaṃ tat saha yātamaṇḍalaiḥ*|
> *yatheṣṭanakṣatragaṇair hataṃ haret tadīyanakṣatraganais tataḥ kalāḥ*||
>
> 10. The (mean) longitude of the moon, the planet, or the *ucca* (whichever is known) together with the revolutions performed should be reduced to minutes. The resulting minutes should then be multiplied by the revolution-number of the desired planet and (the product obtained should be) divided by the revolution-number of that (known) planet. The result is (the mean longitude of the desired planet) in terms of minutes.

In fact example 16 of BAB.2.32-33 follows a computation in term of revolutions whereas example 17 follows the above rule given in the *Māhabhāskarīya*.

Time-pulverizer (*velākuṭṭākāra*)

In this case, the number of days elapsed since the beginning of the *Kaliyuga* is not integral: the longitude of planet g is not given at sunrise- a day is defined from one sunrise to another in this treatise- but at another time of the day: midnight, noon, or sunset[30].

Problem The integral number of days elapsed since the beginning of the *Kaliyuga* (x) and the number of revolutions performed by g in that time (y) are sought, knowing λ the mean longitude of g at a fractional part of the day (day$\pm\frac{1}{m}$, $2 \leq m \leq 4$), G_y and A_y.

[29] [Shukla 1960; p.2-3 skt, p. 7 eng.].

[30] Other subdivisions of the days can be also considered: this is indicated by Bhāskara in the part just before example 21. example 21 considers a fractional part of a day in *nāḍīs* (1/60th of a day).

procedure The problem to be solved by a pulverizer without remainder or a *sthirakuṭṭāka* is:

$$y = \frac{\frac{G_y}{m} \times X - R_M}{A_y},$$

where y is the number of revolutions performed by g in $x \pm \frac{1}{m}$ days and $X = mx \pm 1$.

If $\frac{1}{m}$ is subtractive ($\frac{X}{m} = x - \frac{1}{m} \Leftrightarrow X = mx - 1$), then the integral value of days elapsed since the beginning of the *Kaliyuga* is $x - 1$. Therefore the value sought is $x - 1 = \frac{X+1}{m} - 1$.

If $\frac{1}{m}$ is additive $\frac{X}{m} = x + \frac{1}{m} \Leftrightarrow X = mx + 1$) then $x = \frac{X-1}{m}$ should be computed to obtain a solution.

The problem exposed in words here can be algebrised, in regard to a regular planet-pulverizer in this way:

$$y = \frac{G_y(x \pm \frac{1}{m}) - R_M}{A_y} \Leftrightarrow y = \frac{\frac{G_y}{m}(mx \pm 1) - R_M}{A_y}.$$

Bhāskara does not in fact describe exactly such a computation, concerning the passing first, from the pulverizer considering x to the one considering X and then from the result obtained for X to the one giving x.

In the part preceding example 19, Bhāskara writes[31]:

> *kaścit graham udayakālād anyakālajanitaṃ pradaśyaṃ divasagaṇaṃ pṛcchati, tasyāyam ānayanopāyaḥ: iṣṭakālacchedaguṇitān nirapavartitabhūdivasān kṛtvā pūrvavat kuṭṭākāraṃ niṣpādya iṣṭakālachedhabhakto 'hargaṇaḥ|*
>
> When someone pointing at ⟨the mean longitude of⟩ a planet produced at a time different from sunrise, asks the number of days ⟨elapsed in the *Kaliyuga*⟩, this is a method of computation for that ⟨question⟩:When one has multiplied the reduced ⟨number of⟩ days ⟨in a *yuga*, for that planet⟩ by the denominator of the desired time, and brought about a pulverizer, as before, ⟨the pulverizer⟩ is divided by the denominator of the desired time is the number of days ⟨elapsed in the *Kaliyuga*⟩.

Bhāskara, quite typically since he is summing up a general case, is elliptic concerning the computation of the integral number of days elapsed since the

[31] [Shukla 1976; p. 147, line 15-17]

beginning of the *Kaliyuga*. The first step he describes, that of multiplying by m a "reduced number of days" has remained ununderstood. He states this again in the "procedure" part of solved examples, but with no numerical illustration. This may be referring to the computation $X = mx \pm 1$, however why then would x bear such a name remains unclear. Secondly, repeatedly the passing from the pulverizer obtained to the result sought (the integral number of days elapsed since the beginning of the *Kaliyuga*) is stated as a simple "division by the denominator of the desired time", no other computation being stated. We can note also that the integral part of $\frac{X}{m}$ will give the value of $x - 1$ if m is subtractive, and the value of x if m is additive. Therefore, this may have been the computation carried out here.

To sum it up, probably the computation we have algebrised in this case does not render the exact steps followed by Bhāskara.

Finding the Residue of revolutions and a certain number of days, for two planets

This problem combines two pulverizers. Such a procedure may be seen in example 23, where the two planets considered are the sun and Mars.

Problem Two planets g_1 and g_2 are considered. A certain amount of days, N is sought, knowing that divided by A_1 (the reduced number of days in a *yuga* for g_1) it leaves a remainder r_1 whose value is unknown, and divided by A_2, it leaves a remainder r_2 whose value is unknown.

We can recognize here a problem that can be solved by a "pulverizer with remainder" procedure, when r_1 and r_2 are known:

$$N = A_1 q_1 + r_1$$

$$N = A_2 q_2 + r_2$$

The values of r'_1 and r'_2 are known, and defined as:

$$\frac{G_1 r_1}{A_1} = q'_1 + \frac{r'_1}{A_1}$$

$$\frac{G_2 r_2}{A_2} = q'_2 + \frac{r'_2}{A_2}$$

Where G_1 and G_2 respectively are the reduced number of revolutions performed in a *yuga* by g_1 and g_2.

procedure The last problem is equivalent to this one:

$$q'_1 = \frac{G_1 r_1 - r'_1}{A_1}$$

$$q'_2 = \frac{G_2 r_2 - r'_2}{A_2}$$

So that values of r_1 and r_2 may be found by means of one of the procedures for a "pulverizer without remainder".

r_1 (resp. r_2) is interpreted as the number of days elapsed since the beginning of the *Kaliyuga*; q'_1 (resp. q'_2) as the integral number of revolutions performed by g_1 (resp. g_2) during that time, and r'_1 (resp. r'_2) as the residue of revolutions, R_{M_1} (resp. R_{M_2}).

Having obtained r_1 and r_2, N is found by applying a second pulverizer.

Planetary pulverizer with several planets using orbital computations

This is the last type of problem illustrated by Bhāskara (in examples 24-26), it combines a planetary pulverizer and the computations linking the length of the orbit of a planet to its mean longitude for a given number of elapsed days since the beginning of the *Kaliyuga*.

Residues in respect to a planet's orbit Let λ be the mean longitude of a given planet g.

$$\lambda = (M, R, B, L, S),$$

where M is the integer number of revolutions (*maṇḍala*) performed by the planet since the beginning of the *Kaliyuga*; R the remaining integer number of signs (*rāśi*) crossed, B the remaining integer number of degrees (*bhāga*) crossed, L the remaining integer number of minutes (*liptā*) crossed, and S, the remaining (*śeṣa*) fractional part of minutes crossed by that planet.

In terms of revolutions,

$$\lambda = M + \frac{R}{12} + \frac{B}{12 \times 30} + \frac{L}{12 \times 30 \times 60} + \frac{S}{12 \times 30 \times 60 \times (K \times A_y)}.$$

The residue of revolutions in respect to the planet's orbit is:

$$Rk_M = \frac{R}{12} + \frac{B}{12 \times 30} + \frac{L}{12 \times 30 \times 60} + \frac{S}{12 \times 30 \times 60 \times (K \times A_y)}.$$

The residue of signs in respect to the planet's orbit is:

$$Rk_M = \frac{B}{30} + \frac{L}{30 \times 60} + \frac{S}{30 \times 60 \times (K \times A_y)}.$$

The residue of degrees in respect to the planet's orbit is:

$$Rk_M = \frac{L}{60} + \frac{S}{60 \times (K \times A_y)}.$$

Case with two planets using a Residue of revolutions in respect to the planet's orbits

Problem The number of days elapsed since the beginning of the *Kaliyuga* and the mean longitudes, at that time, of 2 planets: λ_1, λ_2, are sought knowing:

-K_k the length in *yojanas* of the "orbit of the sky" (*khakakṣyā*)-the circumference of a great circle of the celestial sphere),

-K_1, K_2 the length in *yojanas* of the "orbit of the planets",

-A_y, the number of terrestrial days in a *yuga*,

-Rk_{M_1}, Rk_{M_2} the residue of revolutions of each planets at that time, in respect to the planet's orbit.

orbital computations In the resolution of example 24, Bhāskara quotes the following rule:

> *kakṣyābhir grahānayane khakaṣyāyā ahargaṇo guṇakākraḥ, svakaṣyābhūdinasaṃvar*
> *bhāgahāra iti*
> In a computation of ⟨the mean longitude of⟩ planets by means of the orbits, the number of days ⟨elapsed in the *Kaliyuga*⟩ is a multiplier of the orbit of the sky, the divisor is the product of the terrestrial days ⟨in a *yuga*⟩ with its (the planet's) own orbit

In other words, for any planet:

$$\lambda_i = \frac{Kx}{A_y \times K_i}.$$

So that for our two planets we have:

$$Kx = A_y \times K_1 \times \lambda_1 = A_y \times K_2 \times \lambda_2 = N$$

procedure Bearing the above equality in mind, for any planet:

$$A_y \lambda_i K_i = A_y K_i M_i + Rk_{M_i}.$$

In this problem M_i is sought and Rk_{M_i} is known.

The above equality may be written as a system of equations:

$$\begin{cases} N = A_y K_1 y + Rk_{M_1} \\ \\ N = A_y K_2 z + Rk_{M_2} \end{cases} \leftrightarrow \quad z = \frac{A_y K_1 y - (Rk_{M_2} - Rk_{M_1})}{A_y K_2}$$

where y is the integral number of revolutions performed by the first planet and z the integral number of revolutions performed by the second planet.

This problem may be solved by a "pulverizer with remainder" procedure. Any one value found for y or z thus gives a value for N.

As Bhāskara states in the resolution of example 24:

> *pūrva likhitadvicchedāgrarāśir apavartitakhakakṣyāharganasaṃvarga ity ataḥ svabhāgahārābhyāṃ vibhajya labdhaṃ sūryācandramasor yātabhāgaṇāḥ*
> Since the previously written quantity that has ⟨such⟩ remainders for two divisors is the product of the number of days ⟨elapsed in the *Kaliyuga*⟩ and the reduced orbit of sky, therefore, having divided ⟨it⟩ by their own divisors, the quotient is the passed revolutions of the sun and the moon.

In other words, since

$$N = A_y \times K_1 \times \lambda_1 = A_y \times K_2 \times \lambda_2,$$

then

$$\lambda_1 = \frac{N}{A_y \times K_1} \lambda_2 = \frac{N}{A_y \times K_2}$$

And, as Bhāskara adds:

> *asminn eva dvicchedagre apavartitakhakakṣyayā vibhakte labdham ahargaṇaḥ*
> When that which has ⟨such⟩ remainders is divided by the reduced orbit of the sky, the quotient is the number of days ⟨elapsed in the *Kaliyuga*⟩

In other words,

$$x = \frac{N}{K}.$$

Case with two planets and the residue of minutes in respect to the planet's orbits The problem is the same as before, only instead of the residue of revolutions in terms of the planet's orbits the residue of minutes Rk_{L_1} and Rk_{L_2} are given.

Two procedures are given to find the integral number of revolutions, signs, degrees and minutes crossed by both planets since the beginning of the *Kaliyuga*:

procedure 1 If y (resp. z) is the integral number of revolutions, signs, degrees and minutes crossed, in terms of revolutions by the first planet (resp. the second planet), then, the problem may be formalized as:

$$z \times 21600 = \frac{A_y K_1 \times 21600y - (Rk_{M_2} - Rk_{M_1})}{A_y K_2}.$$

And can be solved by any of the two methods used for this type of problem (a pulverizer without remainder or a *sthirakuṭṭāka*).

procedure 2 In this case the residue of degrees in terms of the planet's orbit (Rk_B) is found by solving the following problem:

$$y_B = \frac{60 \times x_B - Rk_L}{A_y \times K},$$

where x_B is the residue of degrees in terms of the planet's orbit, and y_B the integral number of degrees crossed by that planet.

Then the residue of signs in terms of the planet's orbit (Rk_R) is found by solving the following problem:

$$y_R = \frac{30 \times x_R - Rk_B}{A_y \times K},$$

where x_R is the residue of signs in terms of the planet's orbit, and y_R the integral number of signs crossed by that planet.

From this the residue of revolutions in terms of the planet's orbit (Rk_M) is found by solving the following problem:

$$y_M = \frac{30 \times x_M - Rk_R}{A_y \times K},$$

where x_M is the residue of revolutions in terms of the planet's orbit, and y_M the integral number of revolutions crossed by that planet.

Case with more than two planets This combines the above described procedures, with the case of the problems where what is sought is an integer N having given remainders for n different divisors.

For a first two couple of planets, g_1 and g_2, N_1 is found as described above, for the couple of divisors and remainders $(A_yK_1, Rk_{M_1}; A_yK_2, Rk_{M_2})$, if the residue of revolutions in terms of the planet's orbits is given . Then for a third planet, g_3, the same procedure is applied to the couple $(A_y^2K_1K_2, N_1; A_yK_3, Rk_{M_3})$. And so forth.