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Abstract

Bi-cooperative games have been introduced by Bilbao as a generalization of classical cooperative games, where each player can participate positively to the game (defender), negatively (defeater), or do not participate (abstentionist). In a voting situation (simple games), they coincide with ternary voting game of Felsenthal and Machover, where each voter can vote in favor, against or abstain. In this paper, we propose a definition of value or solution concept for bi-cooperative games, close to the Shapley value, and we give an interpretation of this value in the framework of (ternary) simple games, in the spirit of Shapley-Shubik, using the notion of swing. Lastly, we compare our definition with the one of Felsenthal and Machover, based on the notion of ternary roll-call, and the Shapley value of multi-choice games proposed by Hsiao and Ragahavan.
Keywords: cooperative game theory, bi-cooperative games, power index, Shapley value.

1 Introduction

Cooperative game theory deals with set functions $v : 2^N \longrightarrow \mathbb{R}$, where N is the set of players (or voters, etc.), and $v(S)$ is the worth of coalition $S \subseteq N$. Binary voting games are examples of such games [16] in which the value of v is limited to $\{-1, 1\}$. For $S \subseteq N$, $v(S)$ is interpreted as the result of the vote ($+1$ means that the bill is accepted whereas -1 means that the bill is rejected) when S is the set of voters in favor, the other voters $N \setminus S$ being against. For general games, $v(S)$ can take any non-negative value and represents the asset that all players of S will win if they play together against $N \setminus S$.

The field of cooperative game theory has been enriched these recent years by many new kind of games generalizing previous concepts, trying to model in a more accurate way the behavior of players in a real situation. Let us cite for example, *ternary voting games* of Felsenthal and Machover [8], where *abstention* is an alternative option to the usual *yes* and *no* opinions, and *bi-cooperative games* proposed by Bilbao *et al.* [2], where the players can play in favor, play against or do not play. For these two examples, the game v is defined as a function of two arguments - the first one for the voters voting yes (resp. the participants playing in favor), and the second one for the voters voting no (resp. the participants playing against). The remaining set of voters or players corresponds to the abstentionists (resp. the players that do not participate). The following example gives an illustration of a bi-cooperative game.

Example 1 *Two companies A and B are the only companies that sell a specific product. This can happen for instance for military equipments or very large constructive works (construction of a metro,...). A and B are thus always in competition for the same calls for tenders. In such case, the project is won by either A or B. If the offers proposed by the two companies are quite similar, the project is split into two equal pieces - one piece for each company.*

Some traders N are currently working for company A and are linked with A by a temporary employment contract. Each trader has three options: stay in A, negotiate to break the contract and join another company than A and B, or negotiate to break the contract and join company B.

The efficiency of these traders is measured regarding the profit company A obtains thanks to them. More precisely, let $v(S, T)$ be the difference between company A's profit when S works for A, T works for B and $N \setminus (S \cup T)$ works for neither A nor B, and company A's profit when none of N works for either A or B.

One of the main concerns in game theory is to define the notion of value or power index $\phi_i(v)$ of a player i. For usual games, classical examples are the Shapley value, and the Shapley-Shubik power index. For ternary voting games, a power index has been defined by Felsenthal and Machover, based on the notion of roll-call [8]. We aim here at defining a value and a power index for bi-cooperative games in the spirit of Shapley and Shapley-Shubik.

In usual games, each player has only two options: either to join a coalition or to stay aside. For a player that chooses the first option, he or she is supposed to cooperate as much as possible in order to maximize the total worth that the coalition will get. Hsiao and Raghavan have introduced *multi-choice games* in which each player has several possible levels of participation (among a finite number of possible levels that are ordered from non-participation to complete participation) to the game [12]. Multi-choice games can be seen as an intermediate notion between classical games and fuzzy games [3] in which each player participates to some degree in $[0, 1]$. Clearly, the contribution of a player to a game depends on the investment he or she puts, that is on his level of participation to the game. A player shall be rewarded more if he or she participates more to a game. The importance index of a player depends naturally on his level of participation, leading to the introduction of an importance index $\phi_{ji}(v)$ of player i when he or she participates at level j [12].

There is a close link between bi-cooperative games and multi-choice games since there are three ordered levels in bi-cooperative games, which are *against*, *abstention* and *in favor*. However, the levels in the multi-choice games are of *unipolar* nature in the sense that one can only bring a non-negative value, whereas the levels in the bi-cooperative games are of *bipolar* nature in that one can bring a positive value (play in favor of the coalition), a negative value (play against) or null value (*status quo* when doing nothing). Moreover, the worth of a multi-choice game if all players choose the lowest level of participation is zero, and the worth of a bi-cooperative game if all players are against is non-positive. Hence, bi-cooperative games cannot be seen as a particular case of multi-choice games due to this difference in nature, and the value defined for multi-choice games is not à priori suited for bi-cooperative games. Anyhow, the idea of defining a value depending on the participation level can be applied to bi-cooperative games. For a player i, we will denote by $\phi_i^+(v)$ the payoff for being defender, and $-\phi_i^-(v)$ the payoff for being defeater (with $\phi_i^-(v) \geq 0$). The payoff for being abstentionist is 0.

Example 2 *Consider Example 1 with $N = \{1, 2\}$. Trader 1 (resp. trader 2) is very efficient in helping A (resp. B) to win but is completely inefficient when working for B (resp. A). Consider a call for tenders between A and B in which A and B propose basically the same quality of offer. When trader 1 works for A and 2 works for none, company A outranks B and wins. When 2 works for B and 1 works for none, company B wins. In the remaining cases, there is a draw. In particular, when 1 works for A and 2 works for B, the offers of A and B are of better quality but are still of the same level.*

From the standpoint of company A, trader 1 should work for A and trader 2 should not work for B in the future. It is indeed basically useless for trader 2 to stay in A. Hence, company A should pay $\phi_1^+(v)$ to trader 1 for staying in A, and trader 1 can leave company A for free whatever he/she does after. Moreover, trader 2 is not rewarded if he/she decides to stay in A, can break hie/her contract for free if he/she signs an engagement that he/she will not join B, but has to pay $\phi_2^-(v)$ otherwise.

After having given formal definitions of bi-cooperative games and related notions in Section 2, we introduce in Section 3 values and power indices in a general way. Then Section 4 gives two axiomatizations of what we call the Shapley value for bi-cooperative games, and its interpretation as a Shapley-Shubik-like power index. Lastly, Section 5 performs a comparison with the power index of Felsenthal and Machover, and with multi-choice games.

2 Ternary simple games and bi-cooperative games

Let $N := \{1, \ldots, n\}$ be a finite set of players. In classical cooperative game theory, the characteristic function of a game is a function $v : \mathcal{P}(N) \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$, where $\mathcal{P}(N)$ is the power set of N. For any coalition $S \subseteq N$, $v(S)$ is the *worth* of S. Depending on the context, the worth of S may represent the monetary value of the output produced by the cooperation between its members, the cost of a project designed solely for the members of S, or the abstract power of coalition S on a voting or decision system [4]. A game is monotonic whenever $S \subseteq T$ implies $v(S) \leq v(T)$. The set of all cooperative games on N is denoted $\mathcal{G}(N)$.

Let us first focus on this last interpretation, which is related to simple games. A *simple game* is a monotonic game v taking its value in $\{0, 1\}$, (it could be $\{-1, 1\}$ as well, in this case $v(\emptyset) = -1$) and models voting situations, i.e., players are voters. $v(S) = 1$ means that if players in S vote in favor for some bill and players in $N \setminus S$ vote against, then the bill will pass, and S is said to be a winning coalition. If on the contrary $v(S) = 0$, S is a losing coalition and the bill will be rejected.

This is a coarse modeling of real situations since it often happens that voters abstain, and it is not possible in the above model to distinguish between voting against and abstention. For this reason, Felsenthal and Machover [8] proposed *ternary voting games*, where each player can choose between voting in favor, against or abstaining. Formally, we introduce $\mathcal{Q}(N) := \{(S, T) \mid S, T \subseteq N, S \cap T = \emptyset\}$, the set of pairs of disjoint coalitions, and endow it with a partial order $\sqsubseteq$ defined by $(S, T) \sqsubseteq (S', T')$ iff $S \subseteq S'$ and $T \supseteq T'$. A ternary voting game is a function $v : \mathcal{Q}(N) \longrightarrow \{-1, 1\}$, where $v(S, T)$ is the result of the vote (1 if the bill is accepted, -1 if it is rejected) when voters in S vote in favor, voters in T vote against, and the remaining voters abstain. Moreover, v satisfies

(i) $v(\emptyset, N) = -1$, $v(N, \emptyset) = 1$

(ii) if $(S', T') \sqsubseteq (S, T)$, then $v(S', T') \leq v(S, T)$ (*monotonicity*).

The first condition says that if all voters vote in favor (against), then the bill is accepted (rejected). The second condition is monotonicity, in the sense that the more voters vote in favor and/or the less voters vote against, the more the bill will be accepted.

Let us slightly refine the proposal of Felsenthal and Machover by introducing a third possible value for v, which is 0, and which represents the "no decision" situation. A *ternary simple game* or *bi-cooperative simple game* is a function $v : \mathcal{Q}(N) \longrightarrow \{-1, 0, 1\}$ which is monotonic is the above sense, and satisfies $v(\emptyset, \emptyset) = 0$ in addition to $v(\emptyset, N) = -1$, $v(N, \emptyset) = 1$, these last two conditions being useful only for avoiding triviality. Condition $v(\emptyset, \emptyset) = 0$ means that when everybody abstains, no decision can be taken.

The same as games generalize simple games, bi-cooperative games, as proposed by Bilbao [2], generalize ternary simple games. A *bi-cooperative game* is a function $v : \mathcal{Q}(N) \longrightarrow \mathbb{R}$ such that $v(\emptyset, \emptyset) = 0$. $v(S, T)$ represents the worth of the *bi-coalition* (S, T), where players in S play against players in T, the other players not taking part to the game. As for classical games, the exact meaning of "worth" of (S, T) depends on the context. It may be e.g. the monetary value that S will win (or lose, if the amount is negative). In a bi-coalition (S, T), we call S the *defender* part, while T is the *defeater* part. Bi-cooperative games which are monotonic in the sense of ternary simple games have been introduced independently by the authors under the name of *bi-capacities* [10, 9, 11].

We denote by $\mathcal{G}^{[2]}(N)$ the set of all bi-cooperative games on N.

For illustrative purpose, we present yet two other examples of bi-cooperative games.

Example 3 *The teacher of a classroom wants the pupils to collect some funds for a humanitarian association. So, he/she asks the pupils to join (with cans) another class in a big shopping mall to ask coins to people. Not all the children are really willing to do so eventhough they are obliged to go to the mall. Some of them are doing their best to get as much money as possible by trying to convince people, some others are not participating to the collection (going for instance in a toy store), and the remaining ones are disturbing the pupils that try to collect some funds, making them less efficient. So, some children are participating to the task, some others are acting against the task and the remaining ones are abstentionist. The bi-cooperative game measures the difference between the total amount of* *money obtained at the end of a day and what the other class would have got alone. The more children are participating, the more money the association gets, and the more children are acting against the collection, the less money the association gets.*

The teacher is used to sharing out candies to the pupils. More precisely, each child has a credit of candies that increases during the year. In order to motivate the children in the task and also to deter the most unruly of them from annoying the others, the teacher wishes to give candies (increment the credit) to the well-behaved pupils and take back (decrement the credit) some candies to the unruly pupils on the basis of their contribution.

Example 4 *A set N of farmers raise three kinds of plants called A, B and C (for instance colza, grass and reed) in a given area. Plant A (defeater) needs a lot of pesticide and chemical fertilizers so that it pollutes a lot the local river. Plant B (abstentionist) needs no special treatment and thus no pollution is caused by this plant. Plant C (defender) helps in reducing the pollution since it absorbs some chemicals. The Governor of this area wants to determine the tax for each farmer on the basis of the impact of the farming on the river pollution rate. The bi-cooperative game $v(S,T)$ measures the pollution rate in the river compared to the time when there were only meadows in the area, when farmers S raise plant C, farmers T raise plant A and farmers $N \setminus (S \cup T)$ raise plant B.*

Before ending this section, we introduce bi-unanimity games. Let $(S',T') \in \mathcal{Q}(N) \setminus \{(\emptyset,\emptyset)\}$. The *bi-unanimity game* $u_{(S',T')}$ is defined as:

$$u_{(S',T')}(S,T) = \begin{cases} 1 & \text{if } S \supseteq S' \text{ and } T \subseteq T' \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

As for classical games, they form a $3^n - 1$ dimensional basis of $\mathcal{G}^{[2]}(N)$, and coordinates of v in this basis are the Möbius transform of v (see [9, 11] for details).

3 Value and power index for bi-cooperative games

Let us try to define the notion of value and power index for bi-cooperative games and ternary simple games, in the spirit of what was done by Shapley for cooperative games [15], and Shapley and Shubik for simple games [17].

Let us first take the point of view of a value. In classical cooperative game theory, an *imputation* (more precisely, a *pre-imputation*) is a vector $x \in \mathbb{R}^n$ satisfying the *efficiency principle*, that is, $\sum_{i\in N} x_i = v(N)$ [4]. The imputation represents the share of the total worth of the game $v(N)$ among the players, assuming that all players have decided to join the grand coalition N. A *value* is a mapping $\phi : \mathcal{G}(N) \longrightarrow \mathbb{R}^n$ which assigns to every game an imputation. The well-known Shapley value [15] is defined by

$$\phi_i(v) := \sum_{S\subseteq N\setminus i} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)]. \tag{2}$$

For bi-cooperative games, the situation differs since apart of not participating to the game, each player has two possible actions, namely to play in the defender or the defeater part, while he/she has only one in the classical case. In order to generalize the notion of imputation, we introduce the concept of *reference action* or *level*. The reference action is the action such that if all players do this action, then the output of the game is 0. For classical games, the reference action is to "not participate", since $v(\emptyset) = 0$. For bi-cooperative games, it is also the non participation since $v(\emptyset, \emptyset) = 0$. An imputation is defined for each possible action (except the reference one) of a player *with respect to the reference action*, that is, it represents a kind of average contribution of the player for a given action, compared to the reference action (here, to do nothing). For bi-cooperative games, the possible actions are: to play in the defender part, or to play in the defeater part. Consequently, we need two imputations vectors, and thus two values, which we will denote by ϕ^+ (defender part) and ϕ^- (defeater part). For preserving the meaning of "contribution" (which has a positive sense) and for compatibility with previous works, we make the convention that ϕ^+ is the contribution of "playing in the defender part" instead of "doing nothing", and ϕ^- is the contribution of "doing nothing" instead of "playing in the defeater part", since this last action is supposed to be harmful. For ϕ^- the inverse convention could have been taken as well, the only change would be the sign of ϕ^-. This convention can be controlled by axiom **(I)**, as it will be shown later.

As explained above, a central notion for imputations is the efficiency principle. Sticking to the interpretation of the classical case, let us suppose that the defender part is the grand coalition N. Compared to the situation where all players would have been in the defeater part (the most unfavorable case), the gain is $v(N, \emptyset) - v(\emptyset, N)$. This amount is to be shared among players, rewarding them for their action to play in the defender part *instead of playing in the defeater part*. Hence, according to the above convention and assuming additivity of contributions, the *efficiency principle* writes:

$$\sum_{i \in N} \phi_i^+(v) + \sum_{i \in N} \phi_i^-(v) = v(N, \emptyset) - v(\emptyset, N). \tag{3}$$

Let us illustrate this with Ex. 1. Trader i gets bonus $\phi_i^+(v)$ if he/she stays at A, gets nothing to break his/her contract if he/she promise not to work for B and must pay $\phi_i^-(v)$ to break his/her contract if he/she is willing to join B. Hence the wealth discrepancy for trader i from joining B to staying in A is $\phi_i^+(v) + \phi_i^-(v)$. To determine the value of $\phi^\pm$, company A explains to the traders that when if they all join B, then A looses $v(\emptyset, N)$, but if they all stay at A, then A earns $v(N, \emptyset)$. This means that all traders of N brings an added worth of $v(N, \emptyset) - v(\emptyset, N)$ to company A, which has to be shared among all traders. Hence one shall have equation (3).

Another natural proposal for the efficiency principle could be

$$\sum_{i \in N} \phi_i^+(v) = v(N, \emptyset), \quad \sum_{i \in N} \phi_i^-(v) = -v(\emptyset, N). \tag{4}$$

As we will see in next Section, this does not lead to an interesting solution.

Let us turn now to the simple game point of view. In this context, we speak of *power index* to measure the decisive power of each voter. According to Shapley and Shubik [17], the power index is related to the number of times a voter makes a coalition winning. Specifically, let σ be a permutation on N, defining an ordering of the voters, and denote by Γ the set of all permutations on N. Let us consider the sequence of coalitions $\emptyset, \{\sigma(1)\}, \{\sigma(1), \sigma(2)\}, \ldots, \{\sigma(1), \ldots, \sigma(j)\}, \ldots, N$. Due to monotonicity, there exists one and only one index j such that $\{\sigma(1), \ldots, \sigma(j)\}$ is a losing coalition, while $\{\sigma(1), \ldots, \sigma(j+1)\}$ is winning. We say that player $I_\sigma := \sigma(j+1)$ *swings* coalition $\{\sigma(1), \ldots, \sigma(j)\}$. The Shapley-Shubik power index is defined as the relative number of times a voter swings:

$$\phi_i(v) := \frac{|\{\sigma \in \Gamma \mid i = I_\sigma\}|}{|\Gamma|} \tag{5}$$

where $|\Gamma| = n!$. It happens that the explicit expression of $\phi_i(v)$ is precisely (2). Note that here, no efficiency principle underlies the construction, even it happens that the above index satisfies it. Hence, other power indices have been defined, like the Banzhaf index [1], which do not satisfy the efficiency principle.

Let us generalize the notion of swing to ternary voting games. In the above, a coalition $\{\sigma(1),\dots,\sigma(j)\}$ defines a partition of N into the set of voters voting in favor and the set of voters voting against. Hence, considering that $\{\sigma(1),\dots,\sigma(j)\}$ is a coalition voting against would lead to the same definition, just by considering swings from winning to losing coalitions. Since now abstention is a third alternative, the situation is getting more complicated. A coalition $\{\sigma(1),\dots,\sigma(j)\}$ could be voting in favor (resp. against), the remaining voters $\{\sigma(j+1),\dots,\sigma(n)\}$ being partitioned into abstainers and those voting against (resp. abstainers and those voting in favor).

Specifically, let us define positive and negative swings. We consider as before an increasing sequence of coalitions $\{\sigma(1),\dots,\sigma(j)\}$ voting in favor, and a particular coalition T voting against. We say that voter $I^+_{\sigma,T} := \sigma(j+1)$ makes a *positive swing* for (σ, T) if

$$v(\{\sigma(1),\dots,\sigma(j)\},\{\sigma(j+2),\dots,\sigma(n)\}\cap T) < v(\{\sigma(1),\dots,\sigma(j+1)\},\{\sigma(j+2),\dots,\sigma(n)\}\cap T). \quad (6)$$

Hence, when $\sigma(j+1)$ changes from abstention to voting in favor, v increases. Similarly, let us consider a coalition $\{\sigma(1),\dots,\sigma(j)\}$ voting against, and a particular coalition S voting in favor. Voter $I^-_{\sigma,S} := \sigma(j+1)$ makes a *negative swing* for (σ, S) if

$$v(\{\sigma(j+2),\dots,\sigma(n)\}\cap S,\{\sigma(1),\dots,\sigma(j)\}) > v(\{\sigma(j+2),\dots,\sigma(n)\}\cap S,\{\sigma(1),\dots,\sigma(j+1)\}). \quad (7)$$

We define positive and negative power indices as the relative number of positive and negative swings for a given voter i, possibly taking into account the size of the swing, denoted $\Delta^+_{\sigma,T}(i)$ and $\Delta^-_{\sigma,S}(i)$ and defined by:

$$\Delta^+_{\sigma,T}(i) = v(\{\sigma(1),\dots,\sigma(j+1)\},\{\sigma(j+2),\dots,\sigma(n)\}\cap T) - v(\{\sigma(1),\dots,\sigma(j)\},\{\sigma(j+2),\dots,\sigma(n)\}\cap T)$$

$$\Delta^-_{\sigma,S}(i) = v(\{\sigma(j+2),\dots,\sigma(n)\}\cap S,\{\sigma(1),\dots,\sigma(j)\}) - v(\{\sigma(j+2),\dots,\sigma(n)\}\cap S,\{\sigma(1),\dots,\sigma(j+1)\})$$

with $i = \sigma(j+1)$.

Yet we can think of different ways to define a "relative number". Three simple ways of defining it come up to mind, depending on the way we choose coalitions S, T, which are (we treat the case of positive swings only, the other case is similar):

- we may consider for a given permutation σ all possible coalitions T voting against. Let us denote this power index by ϕ^{2^N+}, we obtain

$$\phi_i^{2^N+}(v) := \frac{\sum_{(\sigma,T),\sigma\in\Gamma,T\subseteq N|i=I^+_{\sigma,T}} \Delta^+_{\sigma,T}(i)}{2^n n!} \quad (8)$$

- instead of considering all possible T, we choose only one, which is the most unfavorable (in terms of forming a winning coalition). This is achieved by always choosing $T = N$. We denote this power index by ϕ^{N+}, it writes

$$\phi_i^{N+}(v) := \frac{\sum_{\sigma\in\Gamma,|i=I^+_{\sigma,N}} \Delta^+_{\sigma,N}(i)}{n!}. \quad (9)$$

- we choose only one T, which is the most favorable, i.e., $T = \emptyset$. We denote this index by $\phi^{\emptyset +}$.

Formulas for $\phi^{2^N -}, \ldots$ are obtained by replacing $I^+_{\sigma,T}$ by $I^-_{\sigma,S}$, and $\Delta^+_{\sigma,T}(i)$ by $\Delta^-_{\sigma,S}(i)$.

The following result gives an explicit form of these power indices.

Proposition 1 *For any ternary simple game v, the above Shapley-Shubik-like power indices are given by:*

$$\phi_i^{2^N +}(v) = \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} \frac{s!(n-s-1)!}{n!2^{n-s}}[v(S\cup i, T) - v(S,T)]$$

$$\phi_i^{2^N -}(v) = \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} \frac{t!(n-t-1)!}{n!2^{n-t}}[v(S,T) - v(S,T\cup i)]$$

$$\phi_i^{N+}(v) = \sum_{S\in N\setminus i} \frac{(n-s-1)!s!}{n!}[v(S\cup i, N\setminus (S\cup i)) - v(S, N\setminus (S\cup i))]$$

$$\phi_i^{N-}(v) = \sum_{S\in N\setminus i} \frac{(n-s-1)!s!}{n!}[v(S, N\setminus (S\cup i)) - v(S, N\setminus S)]$$

$$\phi_i^{\emptyset +}(v) = \sum_{S\in N\setminus i} \frac{(n-s-1)!s!}{n!}[v(S\cup i, \emptyset) - v(S,\emptyset)]$$

$$\phi_i^{\emptyset -}(v) = \sum_{S\in N\setminus i} \frac{(n-s-1)!s!}{n!}[v(\emptyset, S)) - v(\emptyset, S\cup i)]$$

Proof: Let us prove the first formula. Using the definition of $\Delta^+_{\sigma,T}(i)$ and putting $S = \{\sigma(1),\ldots,\sigma(j)\}$ and $T' = \{\sigma(j+2),\ldots,\sigma(n)\}\cap T$, we get

$$\Delta^+_{\sigma,T}(i) = v(S\cup i, T') - v(S,T')$$

with $(S,T')\in\mathcal{Q}(N\setminus i)$, hence we get the term into brackets. It remains to find the number of pairs (σ, T) leading to the same (S,T'). For finding S, the number of possible permutations is $s!(n-s-1)!$. For finding T', the only possible T are of the form $T'\cup S'$, $S'\subseteq S$. This gives 2^s possibilities. Hence in total we have $s!(n-s-1)!2^s$, which proves the result.

Let us prove the third formula. Putting $S := \{\sigma(1),\ldots,\sigma(j)\}$, we get:

$$\Delta^+_{\sigma,N}(i) = v(S\cup i, N\setminus (S\cup i)) - v(S, N\setminus (S\cup i))$$

which is precisely the term into brackets in $\phi_i^{N+}(v)$. It remains to count the number of permutations such that $S = \{\sigma(1),\ldots,\sigma(j)\}$. S having s elements and the number of free elements being $n-s-1$, this number is $s!(n-s-1)!$, which proves the result.

Proofs for other formulas are similar. ■

We remark that indices $\phi^{\emptyset +}, \phi^{\emptyset -}$ are not very interesting. Indeed, by putting $v_1(S) := v(S,\emptyset)$ and $v_2(T) := v(\emptyset, T)$, we are back to the classical case, and the ternary simple game is equivalent to two classical simple games.

4 Axiomatization of the Shapley value for bi-cooperative games

We give in this section an axiomatization of values for bi-cooperative games, in the spirit of Shapley [15]. Our way of introducing axioms and proving results follow Weber [18]. Recall that we have a value for the defender part and for the defeater part, denoted by ϕ^+, ϕ^- respectively.

The first axiom that characterizes the Shapley value is *linearity* with respect to the game. This axiom states that if several games are combined linearly then the values of each individual game shall be combined in the same way to obtain the value of the resulting game. This axiom is trivially extended to the case of bi-cooperative games.

Linearity (L): ϕ^+, ϕ^- are linear on $\mathcal{G}^{[2]}(N)$.

Proposition 2 *ϕ^+, ϕ^- satisfy* **(L)** *if and only if for all $i \in N$, there exists real constants $a^{+i}_{S,T}, a^{-i}_{S,T}$ for all $(S,T) \in \mathcal{Q}(N)$ such that for every game $v \in \mathcal{G}^{[2]}$*

$$\phi^+_i(v) = \sum_{(S,T)\in\mathcal{Q}(N)} a^{+i}_{S,T} v(S,T)$$
$$\phi^-_i(v) = \sum_{(S,T)\in\mathcal{Q}(N)} a^{-i}_{S,T} v(S,T).$$

Proof: The *if* part of the proof is obvious and left to the reader.

Consider ϕ^+ satisfying **(L)**. Let $U_{S,T}$ be defined by $U_{S,T}(S',T')$ equals 1 if $S = S'$ and $T = T'$, 0 otherwise. We have $v = \sum_{(S,T)\in\mathcal{Q}(N)} v(S,T)U_{S,T}$. By **(L)**,

$$\phi^+_i(v) = \sum_{(S,T)\in\mathcal{Q}(N)} v(S,T)\, \phi^+_i(U_{S,T}).$$

Setting $a^{+i}_{S,T} := \phi^+_i(U_{S,T})$, we obtain the wished result. The proof for ϕ^- is much the same. ■

Player i is said to be *left-null* (resp. *right-null*) if $v(S\cup i, T) = v(S,T)$ (resp. $v(S, T\cup i) = v(S,T)$) for all $(S,T) \in \mathcal{Q}(N \setminus i)$. Hence, a left- (resp. right-) null player does not bring any contribution to the defender (resp. defeater) part. For example, trader 1 in Ex. 3 is right-null and trader 2 is left-null. The following axioms say that the imputation given to such players should be 0.

Left-null axiom (LN): $\forall v \in \mathcal{G}^{[2]}(N)$, for all $i \in N$, $\phi^+_i(v) = 0$ if i is left-null.

Right-null axiom (RN): $\forall v \in \mathcal{G}^{[2]}(N)$, for all $i \in N$, $\phi^-_i(v) = 0$ if i is right-null.

Proposition 3 *Under **(L)** and **(LN)**, for all $v \in \mathcal{G}^{[2]}(N)$,*

$$\phi^+_i(v) = \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p^{+i}_{S,T}[v(S\cup i, T) - v(S,T)].$$

Proof: It is easy to check that the formula satisfies the axioms. Conversely, by **(L)**, and assuming i is left-null, we have:

$$\begin{aligned}
\phi_i^+(v) &= \sum_{(S,T)\in\mathcal{Q}(N)} a_{S,T}^{+i} v(S,T) \\
&= \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} [a_{S,T}^{+i} v(S,T) + a_{S\cup i,T}^{+i} v(S\cup i,T) + a_{S,T\cup i}^{+i} v(S,T\cup i)] \\
&= \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} v(S,T)(a_{S,T}^{+i} + a_{S\cup i,T}^{+i}) + \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} a_{S,T\cup i}^{+i} v(S,T\cup i) \\
&= 0.
\end{aligned}$$

We consider v' in $\mathcal{G}^{[2]}(N\setminus i)$ and extend it as follows:

$$\begin{aligned}
v(S,T) &= v'(S,T), \quad \forall (S,T)\in\mathcal{Q}(N\setminus i) \\
v(S\cup i,T) &= v'(S,T), \quad \forall (S,T)\in\mathcal{Q}(N\setminus i) \\
v(S,T\cup i) &= 0, \quad \forall (S,T)\in\mathcal{Q}(N\setminus i).
\end{aligned}$$

Then i is left-null for v, so that the above formula applies, and reduces to:

$$\phi_i^+(v) = \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} v'(S,T)(a_{S,T}^{+i} + a_{S\cup i,T}^{+i}) = 0.$$

Since this must hold for any $v'\in\mathcal{G}^{[2]}(N\setminus i)$, we deduce that $a_{S,T}^{+i} + a_{S\cup i,T}^{+i} = 0$, for any $(S,T)\neq(\emptyset,\emptyset)$. Introducing this in the above, we have for any game v for which i is left-null:

$$\sum_{(S,T)\in\mathcal{Q}(N\setminus i)} a_{S,T\cup i}^{+i} v(S,T\cup i) = 0,$$

from which we deduce that $a_{S,T\cup i}^{+i} = 0$, for all $(S,T)\in\mathcal{Q}(N\setminus i)$. Letting $p_{S,T}^{+i} := a_{S\cup i,T}^{+i}$, the result is proved. ■

Similarly, one can show the following.

Proposition 4 *Under **(L)** and **(RN)**, for all $v\in\mathcal{G}^{[2]}(N)$,*

$$\phi_i^-(v) = \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p_{S,T}^{-i}[v(S,T) - v(S,T\cup i)].$$

The classical symmetry axiom says that the numbering of the players has no importance. Let σ be a permutation on N. With some abuse of notation, we denote $\sigma(S) := \{\sigma(i)\}_{i\in S}$.

> **Symmetry axiom (S):** $\phi_{\sigma(i)}^+(v\circ\sigma^{-1}) = \phi_i^+(v)$, and $\phi_{\sigma(i)}^-(v\circ\sigma^{-1}) = \phi_i^-(v)$, for all $i\in N$, for all $v\in\mathcal{G}^{[2]}(N)$.

Proposition 5 *Under **(L)**, **(LN)**, **(RN)** and **(S)**,*

$$\begin{aligned}
\phi_i^+(v) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p_{s,t}^L[v(S\cup i,T) - v(S,T)] \\
\phi_i^-(v) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p_{s,t}^R[v(S,T) - v(S,T\cup i)].
\end{aligned}$$

Proof: By the linear and null axioms, we have

$$\phi_i^+(v) = \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p_{S,T}^{+i}[v(S\cup i,T) - v(S,T)]$$

$$\phi_{\sigma(i)}^+(v\circ\sigma^{-1}) = \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p_{S,T}^{+\sigma(i)}[v(\sigma^{-1}(S\cup\sigma(i)),\sigma^{-1}(T)) - v(\sigma^{-1}(S),\sigma^{-1}(T))].$$

Putting $(S',T') : (\sigma^{-1}(S),\sigma^{-1}(T))$, which gives $(S,T) = (\sigma(S'),\sigma(T'))$, we get:

$$\phi_{\sigma(i)}^+(v\circ\sigma^{-1}) = \sum_{(S',T')\in\mathcal{Q}(N\setminus i)} p_{\sigma(S'),\sigma(T')}^{+\sigma(i)}[v(S'\cup i,T') - v(S',T')].$$

Applying symmetry, the result is proved. The case of ϕ^- works similarly. ■

The following axiom is particular to bi-cooperative games.

Invariance axiom (I): Let us consider v_1, v_2 in $\mathcal{G}^{[2]}(N)$ such that the following holds for some $i\in N$:

$$v_1(S\cup i,T) = v_2(S,T), \quad v_1(S,T) = v_2(S,T\cup i) \quad \forall (S,T)\in\mathcal{Q}(N\setminus i).$$

Then $\phi_i^+(v_1) = \phi_i^-(v_2)$.

The axiom says that when a game v_2 is identical to v_1 up to the fact the action of player i is "shifted" upward for v_1 (i.e., if i is in the defeater part, then i becomes non participant, and if he/she is non participant, then he/she becomes defender), then the imputation for i being defeater in v_2 equals the imputation for i being defender in v_1. It implies that the way the computation is done for ϕ^+ and ϕ^- are the same, as shown in the next proposition.

REMARK 1: In axiom **(I)**, we could have written $\phi_i^+(v_1) = -\phi_i^-(v_2)$ as well. This has no influence on the rest, simply Shapley values ϕ^- will always be negative for monotonic games, instead of being positive. See Section 3.

Proposition 6 *Under axioms* **(L)**, **(LN)**, **(RN)** *and* **(I)**, $p_{S,T}^{+i} = p_{S,T}^{-i}$, *for all* $i\in N$ *and all* $(S,T)\in\mathcal{Q}(N\setminus i)$.

Proof: Under the above axioms, we have:

$$\begin{aligned}\phi_i^+(v_1) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p_{S,T}^{+i}[v_1(S\cup i,T) - v_1(S,T)]\\ &= \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p_{S,T}^{+i}[v_2(S,T) - v_2(S,T\cup i)]\\ \phi_i^-(v_2) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p_{S,T}^{-i}[v_2(S,T) - v_2(S,T\cup i)]\end{aligned}$$

which yields the result. ■

We finally introduce the efficiency axiom, as we explained it in Section 3.

Efficiency axiom (E): For every game in $\mathcal{G}^{[2]}$, $\sum_{i\in N}\big(\phi_i^+(v)+\phi_i^-(v)\big) = v(N,\emptyset) - v(\emptyset,N)$.

Proposition 7 *Under axioms **(L)**, **(LN)**, **(RN)**, **(S)** and **(E)**, the coefficients $p^L_{s,t}, p^R_{s,t}$ satisfy:*

$$
\begin{aligned}
p^L_{n-1,0} &= \frac{1}{n}\\
p^R_{0,n-1} &= \frac{1}{n}\\
sp^L_{s-1,t} - tp^R_{s,t-1} &= (n-s-t)(p^L_{s,t} - p^R_{s,t}), \quad \forall s,t \neq n.
\end{aligned}
$$

Proof: Under the above axioms,

$$
\begin{aligned}
\sum_{i\in N}\left(\phi_i^+(v) + \phi_i^-(v)\right) &= \sum_{i\in N}\sum_{(S,T)\in\mathcal{Q}(N\setminus i)}\left[p^L_{s,t}(v(S\cup i,T) - v(S,T)) + p^R_{s,t}v(S,T) - v(S,T\cup i))\right]\\
&= \sum_{(S,T)\in\mathcal{Q}(N)} v(S,T)\left[sp^L_{s-1,t} - tp^R_{s,t-1} + (n-s-t)(p^R_{s,t} - p^L_{s,t})\right].
\end{aligned}
$$

Axiom **(E)** implies that the coefficient of $v(N,\emptyset)$ and $v(\emptyset,N)$ should be 1 and -1 respectively, and all others being 0.This yields the result. ■

The final result is the following.

Theorem 1 *Under axioms **(L)**, **(LN)**, **(RN)**, **(S)**, **(I)** and **(E)**,*

$$
\begin{aligned}
\phi_i^+(v) &= \sum_{S\in N\setminus i}\frac{(n-s-1)!s!}{n!}[v(S\cup i, N\setminus(S\cup i)) - v(S, N\setminus(S\cup i))]\\
\phi_i^-(v) &= \sum_{S\in N\setminus i}\frac{(n-s-1)!s!}{n!}[v(S, N\setminus(S\cup i)) - v(S, N\setminus S)].
\end{aligned}
$$

Proof: If axiom **(I)** is added, then $p^L_{s,t} = p^R_{s,t}$, and the result of Prop. 7 becomes:

$$
\begin{aligned}
p_{n-1,0} &= \frac{1}{n}\\
p_{0,n-1} &= \frac{1}{n}\\
p_{s,0} &= p_{0,s} = 0, \quad \forall s \in \{1, n-2\}\\
sp_{s-1,t} &= tp_{s,t-1}, \quad \forall s,t \in \{1,\dots,n-1\}.
\end{aligned}
$$

For any $(s,t)\in\{1,n-1\}^2$ with $s+t \leq n-1$, we have

$$
p_{s,t} = \frac{t}{s+1}a_{s+1,t-1} = \cdots = \frac{t!}{(s+1)(s+2)\cdots(s+t)}p_{s+t,0}.
$$

For $s+t = n-1$ we obtain

$$
p_{s,t} = \frac{s!t!}{(n-1)!}p_{n-1,0} = \frac{s!t!}{n!}.
$$

When $s+t < n-1$, we get $p_{s,t} = 0$. The result is proved. ■

We call ϕ^+, ϕ^- as given above the *Shapley value* for bi-cooperative games.

An important remark is that, due to Prop. 1, the Shapley value for bi-cooperative games coincides with the Shapley-Shubik-like power indices ϕ^{N+}, ϕ^{N-} introduced in Section 3.

By using the same technique as in Th. 1, it is not difficult to prove that if the efficiency axiom is replaced by a stronger one:

Strong efficiency axiom (SE): For every game in $\mathcal{G}^{[2]}$, $\sum_{i\in N}\phi_i^+(v) = v(N,\emptyset)$, and $\sum_{i\in N}\phi_i^-(v) = -v(\emptyset, N)$.

then, under axioms **(L)**, **(LN)**, **(RN)**, **(S)**, and **(SE)**, formulas in Th. 1 become:

$$\phi_i^+(v) = \sum_{S\in N\setminus i}\frac{(n-s-1)!s!}{n!}[v(S\cup i,\emptyset) - v(S,\emptyset)]$$
$$\phi_i^-(v) = \sum_{S\in N\setminus i}\frac{(n-s-1)!s!}{n!}[v(\emptyset,S)) - v(\emptyset,S\cup i)].$$

Note that the above formulas do not satisfy **(I)**. By putting $v_1(\cdot) := v(\cdot,\emptyset)$ and $v_2(\cdot) := v(\emptyset,\cdot)$, it is clear that we are back to the classical case, and this solution has no interest. Indeed, Prop. 1 shows that these indices coincide with the Shapley-Shubik-like power indices $\phi^{\emptyset+}, \phi^{\emptyset-}$.

Another axiomatization avoiding efficiency and **(I)** is possible, and is based on bi-unanimity games, in the spirit of the axiomatization of Hsiao and Raghavan [12].

Let us consider again Example 1. Let N be a set of consultants working for A. $S'\subseteq N$ are consultants that are decisive for A so that A cannot win without the presence of all of them. Consultants T' are quite bad and neither A nor B gains anything keeping or hiring some of them. The other consultants $N\setminus(S'\cup T')$ are also important but not for a positive asset to A (they do not bring anything directly to A) but in the negative for B in the sense that they can deter A from winning the project if any of them works for B. Let S be the set of consultants that stay at A and T the set of consultants quiting A and hired by B, the other consultants quiting A and remaining unemployed. To sum up, A wins over B iff $S\supseteq S'$ and $T\subseteq T'$, and there is a draw otherwise. Hence, the bi-unanimity game $u_{(S',T')}$ perfectly models this situation (see (1)).

We see that $N\setminus(S'\cup T')$ is in some sense as important as S' to the success of A since S' acts positively in the favor of A (this concerns $\phi_i^+\left(u_{(S',T')}\right)$), whereas $N\setminus(S'\cup T')$ acts negatively against B (this concerns $\phi_i^-\left(u_{(S',T')}\right)$). Company A pays a certain amount of money $\phi_i^+\left(u_{(S',T')}\right)$ for consultants of S' to continue working for it, and let them go for free if they want to quit the company whoever they will work for after. Company A pays nothing to consultants of T' if they want to stay in A and let them break their contract for free whatever they do after leaving. Consultants $N\setminus(S'\cup T')$ can break their contract for free if they sign an engagement that they will not join B, but have to pay $\phi_i^-\left(u_{(S',T')}\right)$ otherwise. All consultants of $N\setminus T'$ have the same decisiveness since the project is not won if any consultant of S' leaves A or any of $N\setminus(S'\cup T')$ joins B. Thus, one should have $\phi_i^+\left(u_{(S',T')}\right) = \phi_j^-\left(u_{(S',T')}\right)$ for $i\in S'$ and $j\in N\setminus(S'\cup T')$. All the consultants of $N\setminus T'$ can "share" the total asset for A, that is 1. Hence

Unanimity Game axiom (UG). For any bi-unanimity game $u_{(S',T')}$

$$\phi_i^+\left(u_{(S',T')}\right) = \begin{cases}\frac{1}{n-t'} & \text{if } i\in S'\\ 0 & \text{if } i\notin S'\end{cases}$$

$$\phi_i^-\left(u_{(S',T')}\right) = \begin{cases}\frac{1}{n-t'} & \text{if } i\in N\setminus(S'\cup T')\\ 0 & \text{if } i\in S'\cup T'.\end{cases}$$

Theorem 2 *Under* **(L)**, **(LN)**, **(RN)**, **(S)** *and* **(UG)**,

$$\begin{aligned}
\phi_i^+(v) &= \sum_{S\subseteq N\setminus i} \frac{s!(n-s-1)!}{n!}\left[v(S\cup i, N\setminus(S\cup i)) - v(S, N\setminus(S\cup i))\right] \\
\phi_i^-(v) &= \sum_{S\subseteq N\setminus i} \frac{s!(n-s-1)!}{n!}\left[v(S, N\setminus(S\cup i)) - v(S, N\setminus S)\right].
\end{aligned}$$

Proof: We consider only the case of $\phi_i^+(v)$. The proof is similar for $\phi_i^-(v)$. By Prop. 5,

$$\phi_i^+(v) = \sum_{(S,T)\in\mathcal{Q}(N\setminus i)} p^L_{s,t}\left[v(S\cup i, T) - v(S,T)\right].$$

For $(S,T)\in\mathcal{Q}(N\setminus i)$, we have $u_{(S',T')}(S\cup i,T) - u_{(S',T')}(S,T) = 1$ iff $S\cup i\supseteq S'$, $S\not\supseteq S'$ and $T\subseteq T'$, that is $i\in S'$, $S\supseteq S'\setminus i$ and $T\subseteq T'$. Hence for $i\in S'$,

$$\begin{aligned}
\phi_i^+\left(u_{(S',T')}\right) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus i), S\supseteq S'\setminus i, T\subseteq T'} p^L_{s,t} \\
&= \sum_{T\subseteq T'} \sum_{S=(S'\setminus i)\cup S'',\ S''\subseteq N\setminus(S'\cup T)} p^L_{s,t} \\
&= \sum_{t=0}^{t'} \sum_{s''=0}^{n-s'-t} \frac{t'!}{t!(t'-t)!}\frac{(n-s'-t)!}{s''!(n-s'-t-s'')!} p^L_{s'+s''-1,t} \qquad (10)
\end{aligned}$$

Let us show by induction on s' that for all $s\in\{s'-1,\ldots,n-1\}$

$$p^L_{s,t} = \begin{cases} 0 & \text{if } s+t<n-1 \\ \frac{t!(n-t-1)!}{n!} & \text{if } s+t=n-1 \end{cases} \qquad (11)$$

- For $s'=n$, we have for any $i\in N$,

  $$\phi_i^+\left(u_{(N,\emptyset)}\right) = p^L_{n-1,0}.$$

  By **(UG)**, we get that $p^L_{n-1,0} = \frac{1}{n}$. Hence, the induction assumption holds for $s'=n$.

- Assume the induction hypothesis holds for n, $n-1$, ..., $s'+1$. Let us show by induction on $t'\in\{0,\ldots,n-s'+1\}$ that

  $$p^L_{s'-1,t'} = 0. \qquad (12)$$

  - For $t'=0$, we have for $i\in S'$,

    $$\phi_i^+\left(u_{(S',\emptyset)}\right) = \sum_{s''=0}^{n-s'} \frac{(n-s')!}{s''!(n-s'-s'')!} p^L_{s'+s''-1,0}$$

    The terms $p^L_{s'+s''-1,0}$ are known for $s''>0$ by the induction assumption on s'. Hence

    $$\phi_i^+\left(u_{(S',\emptyset)}\right) = p^L_{s'-1,0} + p^L_{n-1,0} = p^L_{s'-1,0} + \frac{1}{n}.$$

    By **(UG)**, we get $\phi_i^+\left(u_{(S',\emptyset)}\right) = \frac{1}{n}$ so that $p^L_{s'-1,0} = 0$.

– Assume (12) holds for $t' = 0, \ldots, t'' - 1$. From (10) applied to $\phi_i^+ \left(u_{(S',T'')}\right)$, the term $p^L_{s'+s''-1,t}$ is known from (11) by the induction assumption on s' for $t \in \{0, \ldots, t''\}$, $s'' \in \{1, \ldots, n - s' - t\}$, and the term $p^L_{s'+s''-1,t}$ is known from (12) by the induction assumption on t' for $t \in \{0, \ldots, t'' - 1\}$, $s'' = 0$. The term $p^L_{s'-1,t''}$ is not already known. Hence

$$\begin{aligned}\phi_i^+ \left(u_{(S',T'')}\right) &= p^L_{s'-1,t''} + \sum_{t=0}^{t''} \frac{t''!}{t!(t''-t)!} p^L_{n-t-1,t} \\ &= p^L_{s'-1,t''} + \sum_{t=0}^{t''} \frac{t''!(n-t-1)!}{(t''-t)!n!}\end{aligned}$$

It can be shown by induction that [11]

$$\sum_{t=0}^{a} \frac{a!(n-t-1)!}{(a-t)!n!} = \frac{1}{n-a}. \tag{13}$$

Hence

$$\phi_i^+ \left(u_{(S',T'')}\right) = p^L_{s'-1,t''} + \frac{1}{n-t''}.$$

By **(UG)**, we have $\phi_i^+ \left(u_{(S',T'')}\right) = \frac{1}{n-t''}$, so that $p^L_{s'-1,t''} = 0$. Hence (12) holds for any $t' = 0, \ldots, t'' - 1$.

Hence (12) holds for any $t' \in \{0, \ldots, n - s' - 1\}$.

From (10) applied to $\phi_i^+ \left(u_{(S',N\setminus S')}\right)$, the term $p^L_{s'+s''-1,t}$ is known from (11) by the induction assumption on s' for $t \in \{0, \ldots, n - s'\}$, $s'' \in \{1, \ldots, n - s' - t\}$, that is for $t \in \{0, \ldots, n - s' - 1\}$, $s'' \in \{1, \ldots, n - s' - t\}$. Moreover, the term $p^L_{s'+s''-1,t}$ is known from (12) for $t \in \{0, \ldots, n - s' - 1\}$, $s'' = 0$. The term $p^L_{s'-1,n-s'}$ is not already known. Hence

$$\begin{aligned}\phi_i^+ \left(u_{(S',N\setminus S')}\right) &= p^L_{s'-1,n-s'} + \sum_{t=0}^{n-s'-1} \frac{(n-s')!}{t!(n-s'-t)!} p^L_{n-t-1,t} \\ &= p^L_{s'-1,n-s'} + \sum_{t=0}^{n-s'-1} \frac{(n-s')!(n-t-1)!}{(n-s'-t)!n!}\end{aligned}$$

By (13),

$$\phi_i^+ \left(u_{(S',N\setminus S')}\right) = p^L_{s'-1,n-s'} + \frac{1}{s'} - \frac{(n-s')!(s'-1)!}{n!}.$$

By **(UG)**, we have $\phi_i^+ \left(u_{(S',N\setminus S')}\right) = \frac{1}{s'}$, so that

$$p^L_{s'-1,n-s'} = \frac{(n-s')!(s'-1)!}{n!}.$$

Hence, (11) also holds for $s = s'$.

■

We end this section by illustrating the computation of ϕ^+, ϕ^- on two examples.

Example 5 *The gain for a classroom of* 5 *pupils named* A, B, C, D, E *in Example 3 are given by :*

$$v(S,T) = \max\left[0, \left(\sum_{i\in S}\delta_i^+ - \sum_{i\in T}\delta_i^-\right)\right]$$

where δ_i^+ *is the amount of money collected by pupil* i *if he/she contributes to the collection, and* δ_i^- *is the maximal amount of money that is lost due to his/her disturbing the other pupils. That* $v(S,T) \geq 0$ *means that the pupils do not achieve in disturbing the children of the other classroom.*

	pupil A	*pupil B*	*pupil C*	*pupil D*	*pupil E*
δ_i^-	2	2	4	1	0
δ_i^+	5	3	1	1	0

Pupil E is left- and right-null, and pupil D is almost null. The two pupils B and C provide the same added value (i.e. $\delta_i^+ + \delta_i^-$*) from being defeater to being defender.*

We have $v(\emptyset, N) = 0$*. Formula (4) yields* $\phi_i^-(v) = 0$ *for all children* i*. This is off course not desirable since some pupils (for instance pupil C) are quite good in annoying the other pupils of their classroom, and deserve thus to be punished (high value of* $\phi_i^-(v)$*) if they are defeaters. The sum* $\sum_{i\in N}(\phi_i^+(v) + \phi_i^-(v))$ *represents the discrepancy between the maximum amount of possible candies that can be shared and the maximum amount of candies that can be taken back. This quantity corresponds to the importance the teacher puts in the collection activity. He/She decides to assign* 60 *to this value. The number of candies taken back to* i *if he/she chooses defeater is* $\frac{60}{v(N,\emptyset)-v(\emptyset,N)}\phi_i^-(v)$*, and the number of candies given to* i *if he/she chooses defender is* $\frac{60}{v(N,\emptyset)-v(\emptyset,N)}\phi_i^+(v)$*. No candy is neither given nor taken back to the abstentionist children.*

	A	*B*	*C*	*D*	*E*
# *candies taken back when choosing defeater*	4	5	12	3	0
# *candies given when choosing defender*	18	11	4	3	0

There are 36 *candies distributed when all children participate positively to the collection.*

Example 6 *We take Ex. 4, and consider that basically, two fields of plant C are necessary to balance the pollution caused by one field of plant A. We have* $v(S,T) = 0$ *if* $-2 \leq |S| - 2|T| \leq 5$*. When* $|S| - 2|T| \leq -3$*, the pollution rate increases and* $v(S,T) = |S| - 2|T| + 2$*. When* $|S| - 2|T| \geq 6$*, plant C helps in reducing the pollution coming from the cities and* $v(S,T) = 1$*.*

Farmers that raise plant A should have a tax increase proportional to $\phi_i^-(v)$ *and farmers that raise plant C should have a tax decrease proportional to* $\phi_i^+(v)$*. We obtain* $\phi_i^-(v) = 1.3$ *and* $\phi_i^+(v) = 0.6$ *for* $n = 10$*.*

5 Comparison with related indices and values

We consider here the power index proposed by Felsenthal and Machover [8] for ternary voting games and the value proposed by Hsiao and Raghavan [12] for multi-choice games. To perform this comparison, we set $\phi_i(v) = \phi_i^-(v) + \phi_i^+(v)$ and we first characterize $\phi_i(v)$.

5.1 Axiomatization of ϕ_i

From the axioms satisfied by $\phi_i^-(v)$ and $\phi_i^+(v)$, index $\phi_i(v)$ fulfills the following ones.

Linearity (l): ϕ is linear on $\mathcal{G}^{[2]}(N)$.

Efficiency (e): $\sum_{i\in N}\phi_i(v) = v(N,\emptyset) - v(\emptyset,N)$ for all $v \in \mathcal{G}^{[2]}(N)$.

Let σ be a permutation on N. Using previous notations

Symmetry (s): $\phi_{\sigma(i)}(v\circ\sigma^{-1}) = \phi_i(v)$, for all $i \in N$ and for all $v \in \mathcal{G}^{[2]}(N)$.

A player is said to be *null* if the asset is exactly the same when joining the defenders or the defeaters.

Definition 1 *The player i is said to be* null *for the bi-cooperative game v if* $v(S, T\cup\{i\}) = v(S,T) = v(S\cup\{i\},T)$ *for any* $(S,T) \in \mathcal{Q}(N\setminus\{i\})$.

This definition implies that i is also left- and right-null. By **(LN)** and **(RN)**, we obtain that $\phi_i(v) = \phi_i^-(v) + \phi_i^+(v) = 0$.

Null player (n): If a player i is null for the bi-cooperative game $v \in \mathcal{G}^{[2]}(N)$ then $\phi_i(v) = 0$.

For $i \in N$, consider two games for which the result is exactly the same when i belongs either to the defeater or the defender part. These two games differ when i is abstentionist. The case when i is abstentionist shall not be considered in the computation of the value so that these two games shall have the same value for player i.

Abstentionist Player (ap): Let $i \in N$, and v_1, v_2 in $\mathcal{G}^{[2]}(N)$ such that for any $(S,T) \in \mathcal{Q}(N\setminus\{i\})$

$$v_1(S\cup\{i\},T) = v_2(S\cup\{i\},T),\ v_1(S,T\cup\{i\}) = v_2(S,T\cup\{i\}),$$

and

$$v_1(S,T) = v_1(S\cup\{i\},T),\ v_2(S,T) = v_2(S,T\cup\{i\}).$$

Then $\phi_i(v_1) = \phi_i(v_2)$.

The following can be shown (see [13, 14]).

Theorem 3 $\{\phi_i(v)\}_{i\in N}$ *satisfies* **(l)**, **(n)**, **(ap)**, **(s)** *and* **(e)** *if and only if*

$$\phi_i(v) = \sum_{S\subset N\setminus\{i\}} \frac{s!\,(n-s-1)!}{n!}\left[v(S\cup\{i\}, N\setminus(S\cup\{i\})) - v(S, N\setminus S)\right].$$

Let us end up this section by giving an interpretation of this index in the framework of ternary simple games. In Section 3, we defined positive and negative swings as change in decision whenever a voter was changing from abstention to either voting in favor or against. Let us here define a swing *without* taking into consideration the abstention option. We say that voter $I_\sigma := \sigma(j+1)$ makes a swing in permutation σ if

$$v(\{\sigma(1),\dots,\sigma(j)\},\{\sigma(j+1),\dots,\sigma(n)\}) < v(\{\sigma(1),\dots,\sigma(j+1)\},\{\sigma(j+2),\dots,\sigma(n)\}).$$

Denoting as before the size of the swing as $\Delta_\sigma(i)$, we define the power index as the relative number of time a voter makes a swing, taking into account the size of the swing:

$$\phi_i(v) := \frac{1}{n!}\sum_{\sigma\in\Gamma | i=I_\sigma}\Delta_\sigma(i)$$

It is easy to see that this expression coincides with the one of Th. 3. Clearly, the abstention option is not considered in the expression of $\phi_i(v)$.

5.2 The Felsenthal-Machover power index

For binary voting games, the definition of the power index of a voter $i \in N$ can be also based on the notion of *binary roll-call* [5, 6]. A binary roll-call R is composed of an ordering σ_R of the voters and a coalition D_R which contains all voters that are in favor of the bill. Roll-calls are interpreted as follows. The voters are called in the order given by σ_R: $\sigma_R(1), \ldots, \sigma_R(n)$. When a voter i is called, he or she tells his or her opinion, that is to say *in favor* if $i \in D_R$ or *against* otherwise. Let j be the smallest index for which the result of the vote remains the same whatever voters called at position $j+1, \ldots, n$ say. The *pivot* $Piv(v,R)$ for game v and roll-call R is the voter $\sigma_R(j)$ called at position j. The opinion of the voters called after $Piv(v,R)$ does not count. Voter $Piv(v,R)$ is decisive in the result of the vote since if the bill is accepted (i.e. $v(D_R)=1$) then $Piv(v,R)$ is necessarily in favor of the bill (i.e. $Piv(v,R) \in D_R$), whereas if the bill is rejected ($v(D_R)=-1$) then $Piv(v,R)$ is necessarily against of the bill (i.e. $Piv(v,R) \notin D_R$). So, it seems natural to define the power index $\phi_i^{\mathrm{FM}}(v)$ of voter i as the percentage of times i is the pivot in a binary roll-call [7, 6] :

$$\phi_i^{\mathrm{FM}}(v) = \frac{|\{R \in \mathcal{B}_N\ ,\ i = Piv(v,R)\}|}{|\mathcal{B}_N|}\ ,$$

where $\mathcal{B}_N$ is the set of all roll-calls, and $|\mathcal{B}_N| = 2^n\, n!$. The importance index $\phi_i^{\mathrm{FM}}(v)$ is exactly the classical Shapley value [7].

In a similar spirit, Felsenthal and Machover have proposed a power index for ternary voting games based on the notion of ternary roll-call [8]. A *ternary roll-call* R is a triplet $R = (\sigma_R, D_R, E_R)$ composed of an ordering σ_R of the voters, a coalition D_R which contains all voters that are in favor of the bill, and a coalition E_R which contains all voters that are against the bill. The voters in $N \setminus (D_R \cup E_R)$ are abstentionist. The set of all ternary roll-calls for the voters N is denoted by $\mathcal{T}_N$. As previously, when a voter i is called he or she tells his or her opinion, that is to say *in favor* if $i \in D_R$, *against* if $i \in E_R$ or *abstention* otherwise. The *pivot* $Piv(v,R)$ is defined as previously. The following definition is then proposed [8] :

$$\phi_i^{\mathrm{FM}}(v) = \frac{|\{R \in \mathcal{T}_N\ ,\ i = Piv(v,R)\}|}{|\mathcal{T}_N|}\ ,$$

where $|\mathcal{T}_N| = 3^n\, n!$. This index will be referred to as the *F-M power index*.

Let us make the expression of the F-M power index more explicit. Let $L_k := \{\sigma_R(1), \ldots, \sigma_R(k)\}$, and suppose that the pivot is voter $\sigma_R(j)$. Then, by monotonicity of v, the result of the vote remains the same whatever voters called at position $j+1, \ldots, n$ say if and only if $v(L_j \cap D_R, (L_j \cap E_R) \cup (N \setminus L_j)) = v((L_j \cap D_R) \cup (N \setminus L_j), (L_j \cap E_R))$. As a consequence, the pivot $\sigma_R(j) = Piv(v,R)$ satisfies

$$\begin{cases} v(L_j \cap D_R, (L_j \cap E_R) \cup (N \setminus L_j)) = v((L_j \cap D_R) \cup (N \setminus L_j), L_j \cap E_R) \\ v(L_{j-1} \cap D_R, (L_{j-1} \cap E_R) \cup (N \setminus L_{j-1})) \\ \qquad \neq v((L_{j-1} \cap D_R) \cup (N \setminus L_{j-1}), L_{j-1} \cap E_R) \end{cases} \tag{14}$$

Since

$$\begin{aligned} (L_{j-1} \cap D_R, (L_{j-1} \cap E_R) \cup (N \setminus L_{j-1})) &\sqsubseteq (L_j \cap D_R, (L_j \cap E_R) \cup (N \setminus L_j)) \\ &\sqsubseteq ((L_j \cap D_R) \cup (N \setminus L_j), L_j \cap E_R) \\ &\sqsubseteq ((L_{j-1} \cap D_R) \cup (N \setminus L_{j-1}), L_{j-1} \cap E_R) \end{aligned}$$

we have

$$\begin{aligned} v\left(L_{j-1}\cap D_R, (L_{j-1}\cap E_R)\cup(N\setminus L_{j-1})\right) &\le v\left(L_j\cap D_R, (L_j\cap E_R)\cup(N\setminus L_j)\right) \\ &\le v\left((L_j\cap D_R)\cup(N\setminus L_j), L_j\cap E_R\right) \\ &\le v\left((L_{j-1}\cap D_R)\cup(N\setminus L_{j-1}), L_{j-1}\cap E_R\right). \end{aligned}$$

Since the range of v is $\{-1,1\}$ and due to (14), $\sigma_R(j)$ is the pivot $Piv(v,R)$ if and only if we are in one of the following two cases :

- $v(L_{j-1}\cap D_R, (L_{j-1}\cap E_R)\cup(N\setminus L_{j-1})) = -1$ and $v(L_j\cap D_R, (L_j\cap E_R)\cup(N\setminus L_j)) = 1$.
- $v((L_j\cap D_R)\cup(N\setminus L_j), L_j\cap E_R) = -1$ and $v((L_{j-1}\cap D_R)\cup(N\setminus L_{j-1}), L_{j-1}\cap E_R) = 1$.

It is easy to see that the first case cannot happen when $\sigma_R(j)\in E_R$. If $\sigma_R(j)\in D_R$, the first case occurs if and only if $v(S\cup\{\sigma_R(j)\},T) = 1$ and $v(S,T\cup\{\sigma_R(j)\}) = -1$ with $S = L_{j-1}\cap D_R$, $T = (L_{j-1}\cap E_R)\cup(N\setminus L_j)$, that is to say if and only if $v(S\cup\{\sigma_R(j)\},T) - v(S,T\cup\{\sigma_R(j)\}) = 2$. If $\sigma_R(j)\notin D_R\cup E_R$, the first case occurs if and only if $v(S,T) = 1$ and $v(S,T\cup\{\sigma_R(j)\}) = -1$ with the same definition for S and T, that is to say if and only if $v(S,T) - v(S,T\cup\{\sigma_R(j)\}) = 2$. In this first case, the final decision is *yes* since $v(L_j\cap D_R, (L_j\cap E_R)\cup(N\setminus L_j)) = 1$. It is impossible that the *pivot* says *no* (i.e. $\sigma_R(j)\notin E_R$). Hence, if the bill is accepted then $Piv(v,R)$ is necessarily either in favor of the bill or abstentionist.

The second case cannot happen when $\sigma_R(j)\in D_R$. If $\sigma_R(j)\in E_R$, the second case occurs if and only if $v(S\cup\{\sigma_R(j)\},T) = 1$ and $v(S,T\cup\{\sigma_R(j)\}) = -1$ with $S = (L_{j-1}\cap D_R)\cup(N\setminus L_j)$, $T = L_{j-1}\cap E_R$, that is to say if and only if $v(S\cup\{\sigma_R(j)\},T) - v(S,T\cup\{\sigma_R(j)\}) = 2$. If $\sigma_R(j)\notin D_R\cup E_R$, the second case occurs if and only if $v(S\cup\{\sigma_R(j)\},T) = 1$ and $v(S,T) = -1$ with the same definition for S and T, that is to say if and only if $v(S\cup\{\sigma_R(j)\},T) - v(S,T) = 2$. In this second case, the final decision is *no* since $v((L_j\cap D_R)\cup(N\setminus L_j), L_j\cap E_R) = -1$. It is impossible that the *pivot* says *yes* (i.e. $\sigma_R(j)\notin D_R$). If the bill is rejected then $Piv(v,R)$ is necessarily either against the bill or abstentionist.

From the previous cases, we obtain

$$\begin{aligned} \phi_i^{\mathrm{FM}}(v) = \frac{1}{3^n\, n!} \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} \{ \\ \frac{v(S\cup\{i\},T) - v(S,T\cup\{i\})}{2} \sum_{K\subset N\setminus\{i\}} k!(n-k-1)! \sum_{\substack{(D,E)\in\mathcal{Q}(N) \\ i\in D\ ,\ S=D\cap K \\ T=(E\cap K)\cup(N\setminus(K\cup\{i\}))}} 1 \\ +\frac{v(S,T) - v(S,T\cup\{i\})}{2} \sum_{K\subset N\setminus\{i\}} k!(n-k-1)! \sum_{\substack{(D,E)\in\mathcal{Q}(N) \\ i\notin D\cup E\ ,\ S=D\cap K \\ T=(E\cap K)\cup(N\setminus(K\cup\{i\}))}} 1 \\ +\frac{v(S\cup\{i\},T) - v(S,T\cup\{i\})}{2} \sum_{K\subset N\setminus\{i\}} k!(n-k-1)! \sum_{\substack{(D,E)\in\mathcal{Q}(N) \\ i\in E\ ,\ T=E\cap K \\ S=(D\cap K)\cup(N\setminus(K\cup\{i\}))}} 1 \\ +\frac{v(S\cup\{i\},T) - v(S,T)}{2} \sum_{K\subset N\setminus\{i\}} k!(n-k-1)! \sum_{\substack{(D,E)\in\mathcal{Q}(N) \\ i\notin D\cup E\ ,\ T=E\cap K \\ S=(D\cap K)\cup(N\setminus(K\cup\{i\}))}} 1\} \end{aligned}$$

Consider the first term in the right hand side of previous relation. One has

$$\begin{aligned} \mathcal{T}_1 &:= \sum_{K\subset N\setminus\{i\}} k!(n-k-1)! \sum_{\substack{(D,E)\in\mathcal{Q}(N) \\ i\in D\ ,\ S=D\cap K \\ T=(E\cap K)\cup(N\setminus(K\cup\{i\}))}} 1 \\ &= \sum_{K\subset N\setminus\{i\}} k!(n-k-1)! \sum_{\substack{(D',E)\in\mathcal{Q}(N\setminus\{i\}) \\ D=D'\cup\{i\}\ ,\ S=D'\cap K \\ T=(E\cap K)\cup(N\setminus(K\cup\{i\}))}} 1 \end{aligned}$$

We write $D' = S \cup D''$, $E = T' \cup E''$ with $T' \subset K$ and $(E'', D'') \in \mathcal{Q}(N \setminus K)$. Then $K = (N \setminus (T \cup \{i\})) \cup T'$. Previous term is equal to

$$\mathcal{T}_1 = \sum\nolimits_{K=(N\setminus(T\cup\{i\}))\cup T'}^{T'\subset T} k!(n-k-1)! \sum_{(E'',D'')\in\mathcal{Q}(N\setminus K)} 1.$$

Since $\sum_{(E'',D'')\in\mathcal{Q}(N\setminus K)} 1 = 3^{n-k-1}$, we get

$$\mathcal{T}_1 = \sum_{T'\subset T} (n-1-t+t')!(t-t')!\, 3^{t-t'}.$$

Set

$$\eta_t = \frac{1}{3^n n!} \sum_{t'=0}^{t} \frac{t'!(t-t')!}{t!} (n-1-t+t')!(t-t')!\, 3^{t-t'}.$$

Then $\mathcal{T}_1 = 3^n\, n!\, \eta_s$. The other terms in the expression of $\phi_i^{\mathrm{FM}}(v)$ are computed similarly. We obtain

$$\begin{aligned}\phi_i^{\mathrm{FM}}(v) \quad = \quad & \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} \left[\left(\eta_s + \frac{\eta_t}{2}\right)(v(S\cup\{i\}, T) - v(S,T))\right. \\ & \left. - \left(\eta_t + \frac{\eta_s}{2}\right)(v(S, T\cup\{i\}) - v(S,T))\right].\end{aligned}$$

From the expression of $\phi_i(v)$, the power index $\phi_i^{\mathrm{FM}}(v)$ satisfies the axioms **(l)**, **(n)**, **(s)**. Moreover, since for any ternary roll-call R there is one and only one pivot, one clearly has

$$\sum_{i\in N} \phi_i^{\mathrm{FM}}(v) = 1.$$

Hence, since $v(N, \emptyset) = 1$ and $v(\emptyset, N) = -1$, $2\phi_i^{\mathrm{FM}}(v)$ satisfies the axioms **(l)**, **(n)**, **(s)** and **(e)**.

From the study of the two cases explained earlier, we see that $Piv(v, R)$ is not always so decisive in the final decision made since it can be abstentionist.

Comparing the expressions of ϕ^{FM} and ϕ^{2^N+}, ϕ^{2^N-}, we see that ϕ^{FM} contains the same terms as $\phi^{2^N+} + \phi^{2^N-}$, but combinatorial coefficients are different. It remains an open problem to know whether the F-M index could be recovered as a particular case of a Shapley-Shubik-like index.

5.3 The multi-choice games

Hsiao and Raghavan have introduced multi-choice games, where each player has several ordered levels of participation $\sigma_0, \sigma_1, \ldots, \sigma_m$ at disposal [12], action σ_0 corresponding to doing nothing. Let $\beta := \{0, 1, \ldots, m\}$, any $x \in \beta^n$ describes the level of participation of each player in N. Each action σ_j has a weight $w(j)$ representing the importance of this action, and it is assumed that $0 = w(0) \le w(1) \le \cdots \le w(m)$.

A *multi-choice* game is a function $V : \beta^n \longrightarrow \mathbb{R}$ satisfying $V(0, \ldots, 0) = 0$. Hsiao and Raghavan define a power index or value as a matrix whose terms are denoted by $\phi_{ij}(V)$, representing the power index or value of player i when he takes action σ_j in game V. They propose the following definition, coming from axiomatic considerations.

$$\begin{aligned}\phi_{ji}(V) \quad = \quad & \sum_{k=1}^{j} \sum_{x\in\beta^n\setminus 0,\, x_i=k} \left[\sum_{L\subset M_i(x)} (-1)^{|L|} \frac{w(x_i)}{\|x\|_w + \sum_{r\in L}(w(x_r+1) - w(x_r))}\right] \\ & \times (V(x) - V(x - b(\{i\})))\end{aligned}$$

where $\|x\|_w = \sum_{r=1}^m w(x_r)$, $M_i(x) = \{k \neq i \ , \ x_k \neq m\}$ $b(\{i\}) = (1_i, 0_{N\setminus\{i\}})$.

Theorem 4 *The Shapley values for a bi-cooperative game v are related to the value of multi-choice games as follows:*

$$\phi_i^-(v) = \phi_{1i}(V) \quad \text{and} \quad \phi_i^-(v) + \phi_i^+(v) = \phi_{2i}(V) \tag{15}$$

with $m = 2$, $w(0) = 0$, $w(1) = w(2) = 1$, and $v(S,T) = V(2_S, 1_{N\setminus(S\cup T)}, 0_T)$, $\forall (S,T) \in \mathcal{Q}(N)$.

Proof: For simplicity, we denote $V(2_S, 1_T, 0_{N\setminus(S\cup T)})$ by $v(S,T)$.

Let $S = \{k, \ x_k = 1\}$ and $T = \{k, \ x_k = 2\}$. Since $\|x\|_w = |S| + |T|$, $M_i(x) = N \setminus (T \cup i)$ and $\sum_{r\in T} (w(x_r+1) - w(x_r)) = |L \cap (N \setminus (S \cup T))|$, we have

$$\phi_{1i}(V) = \sum_{(S,T)\in\mathcal{Q}(N)\,,\,i\in S} \left(\sum_{L\subset N\setminus(T\cup i)} \frac{(-1)^{|L|}}{|S|+|T|+|L\cap(N\setminus(S\cup T))|} \right) (V(S,T) - V(S\setminus i, T))$$

$$\begin{aligned}
&\sum_{L\subset N\setminus(T\cup i)} \frac{(-1)^{|L|}}{|S|+|T|+|L\cap(N\setminus(S\cup T))|} \\
&= \sum_{L_1\subset N\setminus(S\cup T)\ ,\ L_2\subset S\setminus i} \frac{(-1)^{l_1+l_2}}{s+t+l_1} = \sum_{l_1=0}^{n-s-t}\sum_{l_2=0}^{s-1} (-1)^{l_1+l_2} \binom{n-s-t}{l_1}\binom{s-1}{l_2} \frac{1}{s+t+l_1} \\
&= \sum_{l_1=0}^{n-s-t}\sum_{l_2=0}^{s-1} (-1)^{l_1+l_2} \binom{n-s-t}{l_1}\binom{s-1}{l_2} \int_0^1 u^{s+t+l_1-1}\,du \\
&= \int_0^1 \left(\sum_{l_1=0}^{n-s-t} (-1)^{l_1} \binom{n-s-t}{l_1} u^{l_1} \right) \times \left(\sum_{l_2=0}^{s-1} (-1)^{l_2} \binom{s-1}{l_2} \right) u^{s+t-1}\,du \\
&= \begin{cases} \int_0^1 (1-u)^{n-s-t} u^{s+t-1}\,du = \frac{(s+t-1)!(n-s-t)!}{n!} & \text{if } s = 1 \\ 0 \ \ \text{otherwise} \end{cases}
\end{aligned}$$

Hence

$$\phi_{1i}(V) = \sum_{T\subset N\setminus i} \frac{t!(n-t-1)!}{n!} (V(i,T) - V(\emptyset, T))$$

We have

$$\begin{aligned}
\phi_{2i}(V) &= \phi_{1i}(V) \\
&\quad + \sum_{(S,T)\in\mathcal{Q}(N)\,,\,i\in T} \left(\sum_{L\subset N\setminus T} \frac{(-1)^{|L|}}{|S|+|T|+|L\cap(N\setminus(S\cup T))|} \right) (V(S,T) - V(S\cup i, T\setminus i)) \\
&= \phi_{1i}(V) + \sum_{(S,T)\in\mathcal{Q}(N)\,,\,i\in T} \left(\sum_{L_1\subset N\setminus(S\cup T)\ ,\ L_2\subset S} \frac{(-1)^{l_1+l_2}}{s+t+l_1} \right) (V(S,T) - V(S\cup i, T\setminus i))
\end{aligned}$$

One can show as previously that

$$\sum_{L_1\subset N\setminus(S\cup T)\ ,\ L_2\subset S} \frac{(-1)^{l_1+l_2}}{s+t+l_1} = \begin{cases} \frac{(s+t-1)!(n-s-t)!}{n!} & \text{if } s = 0 \\ 0 \ \ \text{otherwise} \end{cases}$$

Hence

$$\phi_{2i}(V) = \sum_{T\subset N\setminus i} \frac{t!(n-t-1)!}{n!} \left(V(\emptyset, T\cup i) - V(\emptyset, T)\right)$$

Let v be the bi-cooperative game associated to V :

$$V(S,T) = v(T, N\setminus (S\cup T)) \ .$$

It is easy to see that $\phi_i^-(v) = \phi_{1i}(V)$ and $\phi_i^-(v) + \phi_i^+(v) = \phi_{2i}(V)$. ■

For a multi-choice game with $m = 2$, the payoff for each player if they all participate at the highest level depends of their contribution at the intermediate level of participation. We see that our definition corresponds to a multi-choice game in which the best two alternatives (abstention and defender) have exactly the same weight. This is a degenerate multi-choice game since the payoff is exactly the same when choosing abstention or defender. For such degenerate multi-choice game, one feels that the contribution of the players at the intermediate level do not count any more to compute the payoff for each player if they all participate at the highest level. As a consequence, it is useless to consider two separate options if we are only interested in the overall importance index $\phi_i^-(v) + \phi_i^+(v)$. This gives an interpretation of the indices we came up with in the framework of multi-choice games. Our indices are based on an equity between the defender and the defeater parts (see **(UG)** and **(I)**).

References

[1] J.F. Banzhaf. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.

[2] J.M. Bilbao. *Cooperative games on combinatorial structures.* Kluwer Academic Publishers, 2000.

[3] D. Butnariu and E.P. Klement. *Triangular norm-based measures and games with fuzzy coalitions.* Kluwer Academic Publishers, 1993.

[4] T. Driessen. *Cooperative Games, Solutions and Applications.* Kluwer Academic Publishers, 1988.

[5] P. Dubey and L.S. Shapley. Mathematical properties of the banzhaf power index. *Maths. of Operations Research*, 4:99–131, 1979.

[6] D. Felsenthal and M. Machover. Postulates and paradoxes of relative voting power: a critical re-appraisal. *Theory and Decision*, 38:195–229, 1995.

[7] D. Felsenthal and M. Machover. Alternative forms of the shapley value and the shapley-shubik index. *Public Choice*, 87:315–318, 1996.

[8] D. Felsenthal and M. Machover. Ternary voting games. *Int. J. of Game Theory*, 26:335–351, 1997.

[9] M. Grabisch and Ch. Labreuche. Bi-capacities. In *Joint Int. Conf. on Soft Computing and Intelligent Systems and 3d Int. Symp. on Advanced Intelligent Systems*, Tsukuba, Japan, October 2002.

[10] M. Grabisch and Ch. Labreuche. Bi-capacities for decision making on bipolar scales. In *EUROFUSE Workshop on Informations Systems*, pages 185–190, Varenna, Italy, September 2002.

[11] M. Grabisch and Ch. Labreuche. Bi-capacities. Part I: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, to appear.

[12] C.R Hsiao and T.E.S. Raghavan. Shapley value for multichoice cooperative games, I. *Games and Economic Behavior*, 5:240–256, 1993.

[13] Ch. Labreuche and M. Grabisch. Bi-cooperative games and their importance and interaction indices. In *14th Mini-EURO Conference on Human Centered Processes (HCP'2003)*, pages 287–291, Luxembourg, May 2003.

[14] Ch. Labreuche and M. Grabisch. The Shapley value and the interaction indices for bi-cooperative games. Technical Report 2003/001, LIP6, UPMC, http://www.lip6.fr/reports/lip6.2003.001.html, 2003.

[15] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[16] L.S. Shapley. Simple games: an outline of the descriptive theory. *Behavioral Science*, 7:59–66, 1962.

[17] L.S. Shapley and M. Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48:787–792, 1954.

[18] R.J. Weber. Probabilistic values for games. In A.E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 101–119. Cambridge University Press, 1988.