Gödel on concepts

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This article is an attempt to present Gödel's discussion on concepts, from 1944 to the late 1970s, in particular relation to the thought of Frege and Russell. The discussion takes its point of departure from Gödel's claim in notes on Bernays's review of 'Russell's mathematical logic'. It then retraces the historical background of the notion of intension which both Russell and Gödel use, and offers some grounds for claiming that Gödel consistently considered logic as a free-type theory of concepts, called intensions, considered as the denotations of predicate-names.

Introduction

It is a well-known Gödelian thesis that a pure theory of concepts is the central part of logic (considered as the theoretical discipline of all *Grundlagenforschungen*) and that it cannot be reduced to set theory.[1] Mathematics, according to Gödel, has to do with objects, but logic (in the Leibnizian, Fregean sense, namely as the unifying framework of all knowledge) is more than mathematics; in fact, the notion of concept is treated by logic in its most general sense. For these reasons, as Gödel explicitly states at the end of his 1944 paper on Russell[2], even if axiomatic set theory and the simple theory of types have been successful, at least to the extent that they permit the derivation of modern mathematics, many indications show that their primitive concepts require further elucidation. Gödel's assertion that a solution for the Continuum Hypothesis should be found by way of a pure theory of concepts, although unusual, reveals the importance that he assigned to the notion of concept. But what does Gödel mean by concepts? The aim of this paper is to propose some preliminary reflections in order to clarify this question.

[1] Cf. ch. 8 of *Wang 1996*.
[2] 'Russell's mathematical logic', *Gödel 1944*. I will quote it from *Gödel 1990,* pages 119-141.

There are at least four elements that should be considered in such a clarification. 1) The interpretation of Gödel's specific claims in 'Russell's mathematical logic', which is the primary text about concepts among his published works. However, as Parsons notes in the introduction to this paper in the edition of the *Collected works*, it far from provides an univocal answer to our previous question: what does Gödel mean by concepts? 2) Gödel's realistic attitude about concepts, reiterated in 1951, 1953, 1954 and all the way through to the 70s, according to Hao Wang's account of his conversation with Gödel.[3] 3) The relation between Gödel's realistic interpretation of concepts and Frege's one, which is very often evoked,[4] albeit without a systematic analysis of the similarities and differences between them. 4) The Gödelian explicit interest in Husserl's phenomenology as a tool for a theory of concepts[5]; an interest that if it does not contrast with the evoked relation to Frege, at least asks for a deeper comprehension. In fact, one of the more profound differences between Frege and Husserl lies specifically in the analysis of concepts.

What I propose in this paper is a journey through all these problems, guided by the following working hypothesis: many of the enigmas surrounding Gödel's conception are tied to the interpretation and the terminological choices of the Gödelian use of some classical oppositions such as *Sinn*/*Bedeutung*, intension/extension, and content/meaning. In order to clarify Gödel's own claims, we have to compare Frege's and Gödel's position on these binaries, taking account of Gödel's commentaries and reactions to Bernays's, Russell's, and Husserl's claims on the same subjects. When Gödel's interpretation of these binaries and of their specific relations is clarified, we will gain a deeper insight to his thought. This journey is then the outline of a systematic exploration, stemming from our working hypothesis. It will

[3] *Wang 1996*. For Gödel's realistic attitude about concepts cf. for example *Gödel 1995,* pages 320-321, pages 360, and *Gödel 2003*, pages 503.
[4] See for example Parsons's introduction to *Gödel 1944*, *Consuegra 2001* and *Crocco 2003*.
[5] Cf. 'The modern development of the foundations of mathematics in the light of philosophy' *Gödel 1995,* pages 374-387.

point to a main thread in Gödel's works, which may shed light on his understanding of concepts.

After a short survey of some of Gödel's theses about concepts in his Russell's paper of 1944, we will begin our journey (section 1 and 2) from a modest point of departure, a small problem attending the interpretation of an annotation page from Gödel about Bernays's review of his 1944 paper.[6] This problem seems not to have received much attention. Nevertheless, as it essentially concerns the use of the binary oppositions *Sinn*/*Bedeutung*, intension/extension, it leads us directly to a main theme, the debate on the notion of intension, which took place from the end of the 19th through the beginning of the 20th century. This debate seems essential for clarifying Gödel's notion of concept and leads us on to an important crossroads. Should we understand Gödel's notion of concept in intension from the modern point of view, stemming from Carnap and his assimilation of Frege's analysis? Or should we instead use an older notion of intension, still very much in use in the 19th century debate in logic and mathematics, and used by Russell in the first edition of *Principia*? We will resolutely take the latter direction.[7] This decision furnishes new insights into some passages of the 1944 paper, and it will also solve our starting problem, that being the interpretation of Gödel's remarks on Bernays's analysis of his paper (section 4). Moreover, taking this direction brings into view (section 5) the amazing coherence between Gödel's 1944 and 1970 position, revealed by his assertions as reported in Wang's book of 1996. The journey concludes (section 6) with a tentative comparison between some of Frege's theses about concepts and their

[6] *Bernays 1946*.

[7] In Parsons's introduction to *Gödel 1944* in *Gödel 1990*, we find clear hints in this direction (for example page 102 and page 113). Nevertheless, such hints are not fully explored in the introduction and, combined with others, only deepen the mystery of Gödel's conception (see for example note *u*, where Gödel's concepts are compared with Frege's *objects* signified by such phrases as 'the concept horse'; note *i,* where an assimilation of concepts to classes is suggested on the basis of the second page of *Gödel 1940*, or note *m,* where it is affirmed, without any explanation, that Gödel clearly understands concepts to be 'objects'). On the contrary, what I propose here is to focus on the identification of concepts as references and intensions, to make it the object of my inquire and, with the help of the reconstruction of the historical background, to attempt a general analysis of Gödel's concepts up to his later claims in the 1970s.

‘corresponding’ ones in Gödel. This comparison will give us the opportunity to introduce some remarks about the Gödelian interest in Husserl.

1. ‘Russell's mathematical logic’ and Bernays's criticisms

In his paper of 1944 on Russell, Gödel says that the theory of simple types can be taken as the most complete attempt, at that time, to provide a theory of concepts avoiding all intensional paradoxes-that is paradoxes of concepts.[8]

Simple type theory is, for Gödel, the theory according to which:

> [...] the objects of thought (or in another interpretation, the symbolic expressions) are divided into types, namely: individuals, properties of individuals, relations between individuals, properties of such relations etc. (with a similar hierarchy for extensions) […][9].

Gödel claims that the simple type theory of concepts has the merit of bringing in a new idea for the solution of the paradoxes, especially suited for their intensional version:

> It consists in blaming the paradoxes not on the axiom that every propositional function defines a concept or class, but on the assumption that every concept gives a meaningful proposition, if asserted for any arbitrary object or objects as arguments.[10]

On this proposed solution the intuitive idea that for each propositional function there is a corresponding concept is saved. To preserve logic from paradoxes we only have to admit that there are limited ranges of significance for propositional functions; exceeding these ranges, predication becomes meaningless.

Gödel adds that simple type theory is entirely independent of the theory of orders (the name he uses for the ramified type theory) because the latter conforms to the vicious circle principle (VCP), while the former ‘has nothing to do with it’.[11] Russell himself recognised this independence since he first motivated the theory of types by the theory of the ambiguity of the

[8] Gödel just mentions Tarski's and Ramsey's work on this effect.
[9] *Gödel 1944,* p. 126, note 17.
[10] *Ibidem* p. 137.
[11] *Ibidem* p. 126-7.

propositional functions, which bears striking resemblance to Frege's theory of *unsaturatedness*. [12] The result of this theory of the ambiguity of propositional functions is the addition of a new assumption to the 'limited ranges of significance' scheme: 'Whenever an object *x* can replace another object *y* in one meaningful proposition, it can do so in every meaningful proposition'.[13] This assumption has the effect of dividing entities into mutually exclusive ranges of significance; each range being made up by the objects that can mutually replace each other. Mixed types are thereby excluded.

This solution is particularly suspect to Gödel because a) the formulation of this restriction as a meaningful proposition is impossible, and b) whether an object *x* is (or is not) of a given type, cannot be expressed by a meaningful proposition. In addition, it also contains a disadvantage for a realistic interpretation of concepts:

> If one considers concepts as real objects, the theory of simple types is not very plausible since what one would expect to be a concept (such as, e.g. 'transitivity' or 'the number two') would seem to be something behind all its various 'realisations' on the different levels and therefore does not exist according to the theory of types.[14]

In his review of Gödel's paper, Bernays criticises this last assertion.[15] In general, Bernays stresses a fundamental ambiguity in the paper between the sense and reference of concept-terms. He affirms that many difficulties and misunderstandings arise from considering reference in cases where the question is about sense. He clearly interprets concepts as senses and classes as references of concept-terms. Based on this interpretation, Gödel's criticism of

[12] 'Accordingly, Russell also based the theory of simple types on entirely different reasons. The reason adduced (in adjunction with its 'consonance' with common sense) is very similar to Frege's one, who, in his system, already had assumed the theory of simple types for functions, but failed to avoid the paradoxes, because he operated with classes, (or rather functions in extension) without any restriction. The reason is that (owing to the variable it contains) a propositional function is something ambiguous (or, as Frege says, something unsaturated, wanting supplementation) and therefore can occur in a meaningful proposition only in such a way that this ambiguity is eliminated (e.g., by substituting a constant for the variable or applying quantification to it). The consequence is that a function cannot replace an individual in a proposition, because the latter has no ambiguity to be removed, and that functions with different kinds of arguments (i.e., different ambiguities) cannot replace each other; which is the essence of the theory of simple types.' *Gödel 1944*, page 136.
[13] *Gödel 1944*, page 138.
[14] *Ibidem* page 137.
[15] Bernays's criticism, in the last part of the review, primarily concerns the notion of analyticity defined by Gödel and the notion of meaning that it implies. Nevertheless, the criticism is extended, in the sequel, to Gödel's remark about simple type theory.

the viability of a realistic view of concepts in a simple types theory is inconclusive. As Bernays claims about Gödel's latter quotation:

> This objection directed against the theory of types can be applied, it seems, quite generally, against the extensional characterisation of concepts. In fact the extension of a concept includes properties which are not at all peculiar to the concept in itself but only in relation to a certain domain of individuals.[16]

That means: concepts in themselves, i.e. concepts in intension, as senses expressed by the corresponding concept-terms, are not split up in levels, because the way we grasp, form or unify a plurality of things in a given level is identical to the way we grasp, form or unify a plurality of things on any other level. That concepts in extension (i.e. classes) are split up into levels is not problematic, since extensions have properties that are not peculiar to their corresponding concepts. Bernays therefore suggests that Gödel's remark on the difficulty of a realistic point of view for a simple type theory of concepts is not acceptable.

2. An intriguing question: Bernays's misunderstandings

Gödel did not react to the last criticism, but the text of the *Nachlass,* presented by the editors as Reprint E in the second volume of the *Collected Works*, bearing the heading '*Bernays rev. meiner Arbeit über Russell*', begins with this assertion:

'Misunderstanding, in two places, of my interpretation of type theory for concepts'.[17]

In his introduction to *Gödel 1944*, (note j), Charles Parsons, admits to not being able to indicate what these misunderstandings are. Let us try to provide an answer.

2.1 The first misunderstanding: the theory of simple types and the VCP

The first misunderstanding seems easy to detect. It lies in Bernays's reconstruction of Gödel's interpretation of the theory of simple types. Bernays says that the theory of simple

[16] *Bernays 1946*, page 78.
[17] *Gödel 1990*, page 321. Note that the type theory for concepts is simple type theory, as Gödel always uses the name 'theory of orders' to indicate Russell's ramified version.

types can, according to Gödel, be considered a solution to paradoxes on the basis of two general ideas:

> 1. On the one hand there is the weaker form of the vicious circle principle, that no totality can contain members involving this totality […]
> 2. The other general idea realized in a special way is that we need not deny […] that every propositional function determines a concept or a class, provided that we admit the possibility that a concept may not be meaningfully applicable to certain objects.[18]

Does (1) really states a Gödelian assertion and what is the weaker form of the VCP mentioned in it ?

Gödel distinguishes three forms of the VCP in the paper. The stronger states that: ‘no totality can contain members definable only in terms of this totality’. The second weaker form of the VCP is obtained by replacing ‘definable only in terms of’ with ‘involving’ and the third form with ‘presupposing’. When the notion of ‘presupposing’ means ‘presupposing for the existence’, the third form, the weakest, becomes a rational exigency that every theory should satisfy, but the first and the second form can be denied, at least from a realistic point of view on the entities of logic and mathematics. Concerning propositional functions, in order to prevent intensional paradoxes, we have to add the principle that a propositional function presupposes the totality of its arguments and of its values (*Gödel 1944* page 126). Its seems clear from Gödel's claims that we have, as a consequence, three corresponding forms of VCP for propositional functions. The first states that ‘nothing defined in term of a propositional function can be a possible argument of this function’. The second form is obtained by replacing ‘defined in term of ’ with ‘involving’ and the third with the term ‘presupposing’.

The first form (in both extensional and intensional versions) makes impredicative definitions impossible. The theory of ramified types conforms to it. Zermelo-Fraenkel set theory does not satisfy the first form, but conforms to the second. Actually it allows impredicative definitions but forbids that a set could belong to itself. Concerning simple types

[18] *Bernays 1946* page 77.

theory, if it is interpreted intensionally (with quantifiers ranging over properties and relations considered as real entities) and if the universal quantifier is interpreted as an infinite conjunction (thus involving the properties on which it ranges over) then it violates the second form of the VCP in its version for propositional functions.[19] This fact motivated Gödel's assertion on page 127, according to which the theory of simple types ‘has nothing to do with the vicious circle principle’, which means that the theory of simple types contradicts the first form of the VCP and is independent from the second one. Actually the theory of simple types can conform to the second weaker form (when considered extensionally as in Zermelo-Fraenkel set theory, allowing mixed types) or not conform to it (when considered intensionally with universal quantification taken as an infinite conjunction and with or without mixed types) and so independently from the fact of allowing mixed types.[20]

Therefore, Gödel explicitly denies that the theory of simple types has something to do with the vicious circle principle. In addition to the principle of limited ranges of significance, he mentions Russell's theory of the ambiguity of propositional functions (and its predecessor, Frege's theory of concepts as unsaturated), as a source of the simple type theory. Bernays does not mention the latter, and with 1. states an assertion totally opposed to Gödel's thoughts and claims.

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[19] Cf. Gödel’s example in the beginning of page 130. Gödel considers a propositional function ϕ, of a given type, involving a general quantification on the totality of the properties of that type (to take Russell’s example, let ϕ be $\hat{x}$ *has all the properties of a great general* which contains the propositional function *ϕ is a property of a great general*). He denies any absurdity in the idea that ‘[…] ϕ(a) consists in a certain state of affairs involving all properties (including ϕ itself and properties defined in terms of ϕ) […]’ and adds ‘It is true that such properties ϕ (or such propositions ϕ(a)) will have to contain themselves as constituents of their content (or of their meaning), and in fact in many ways, because of the properties defined in terms of ϕ; but this only makes impossible to construct their meaning, (i.e., explain it as an assertion about sense perception or any other non-conceptual entities), which is no objection for one who takes the realistic standpoint.’ In note j (page 109) of his introduction to *Gödel 1944*, Parsons remarks that in the Reprint A (*Gödel 1990* pages 317), Gödel calls into question these assertions. It seems to me that he only calls into question the interpretation of the universal quantifier as an infinite conjunction. Nevertheless, he still seems to agree that under this interpretation the simple theory of types does not conform to the weaker form of the VCP. (For a discussion of Gödel's position see section 4 below).

[20] cf. *Gödel 1944*, page 136: ‘the former [the theory of orders] is quite independent of the latter [the theory of simple types] since mixed types evidently do not contradict the vicious circle principle in any way’.

2.2 The second misunderstanding: concepts, intensions and senses

The question concerning the second misunderstanding is more difficult. Bernays's criticisms, as presented in the first section, stress the fact that Gödel does not explicitly say how his own concepts have to be considered when compared with the 'traditional' extension/intension opposition and the Fregean sense/reference opposition. Bernays's suggested more careful distinction between the sense and the reference of a concept-term is a proposal for clarification, one that could, however, be based on a false interpretation of Gödel's thought. Such a suggestion calls for more careful explanation. We can graphically render Bernays's suggestion outlined in the previous section in the scheme of the figure 1.

[Insert figure 1 about here]

Of these levels, the first concerns language, the second the sense expressed by language, which is objective in so far as it is intersubjective (i.e. able to be grasped by everyone understanding the language or sharing some sort of intuition), and the third level, that of reference or signification. According to this interpretation, when Gödel speaks of concepts we should understand concepts in intension. That is, the senses expressed by concept-terms that allow us to determine the plurality of things satisfying the concept. This view suggests that concept-words can be considered alternatively from the point of view of the sense (the common nature of the plurality of things) or from the point of view of the signification (the plurality of things in itself) depending on the context in which they occur. This 'standard' interpretation of the notions of intension and concept was quite usual at this time, and we can find it, for example, in Carnap and in Church.[21] It opens the debate about the identity criteria of properties and relations in intension for all intensional contexts like modal contexts or contexts of propositional attitudes.

In Gödel's *Collected Works*, there is only one passage, as far as we know, which directly contradicts this interpretation. In the lecture from 1951, entitled 'Some basic theorems on the foundations of mathematics and their implications', positioning himself against the conventionalist conception of mathematics sustained by Carnap, Gödel says:

> It is correct that a mathematical proposition says nothing about the physical or psychical reality existing in space and time, because it is true already owing to the meaning of the terms occurring in it, irrespectively of the world of real things. What it is wrong, however, is **that the meaning of the terms (that is the concepts they denote)** is asserted to be something man-made and consisting mainly in semantical conventions.[22] (*Emphasis mine*)

This text was not published and it cannot be proved that this strong assertion and what it implies (i.e. concepts are denotations, that is references of terms, and not senses), is to be considered as a well-thoughtout and rigorous claim about the nature of concepts. Nothing can prove that this assertion, made in the context of a criticism of conventionalism, can be used to support the interpretation of other theses expressed by Gödel seven years earlier about the theory of simple types and Russell's mathematical logic.[23] Nevertheless, let us try and test whether we can give a coherent and *historically* justified account of Gödel's thoughts in 1944, with an interpretation of concepts in intension as references of concept-words, and at the same time solve the puzzle of Bernays's second misunderstanding pointed out by Gödel. In order to attempt such a strategy, a short historical investigation of the notion of intension is necessary. Only such an investigation can explain what is at stake in the interpretation of the words 'concept' and 'intension', from a logical point of view.

3. Brief remarks on the problem of intension in the 19th and 20th centuries

The first identification of intensions with senses goes back to Frege's reviews of Schröder's and Husserl's works. In his reaction to Husserl's position, Frege particularly identifies Husserl's *Inhalte* (the German word for intensions) with his own *Sinne*. Nevertheless, the late

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[21] *Carnap 1935*, especially §38, *Carnap 1946*, *Church 1944*.
[22] *Gödel 1995*, page 320.
[23] It could even be advanced that using Husserl's terminology, everything comes in order. Actually, Husserl uses *Bedeutung* where Frege and Carnap use *Sinn*, and then concepts as intensions, are the *Bedeutungen* even if not

19th and early 20th century witness a revival of the traditional notion of intension, independently of Frege's distinction between *Sinn* and *Bedeutung*. Intensions were generally used, in the sense of Leibniz's analysis, as just the sum of characteristic marks which constitutes a concept and which is necessary for the concept to be what it is. There are at least two important reasons for a renewal of intensions in the latter part of 19th century. The first is related to the discussion of the notion of function, and particularly to the reaction of many mathematicians (especially in France) to the 'set-theoretic' notion of function as a mere correspondence (formulated rigorously for the first time by Dirichlet in 1837). The second reason is related to the reaction of many German logicians to the extensionalist approach of the Algebra of logic arriving from the Anglo-Saxon world (Boole, Peirce, Jevons etc.) but having gathered followers in Germany (Schröder for example).[24]

Concerning the evolution of the concept of function, some seminal works of history of mathematics[25] have shown that instead of a linear evolution of the concept of function we must recognise a dialogue between two opposing conceptions of functions, a dialogue appearing at every turning point in the progress of analysis throughout the 18th, 19th and early 20th century. On the one hand, there was the exigency of defining functions as composed of primitive operations. A necessity enhanced by the fact that the traditional Eulerian notion of a function as an 'analytical expression' with variables was no longer

the references (i.e. the *Gegenstand*) of the concept-terms. Nevertheless this last objection seems to anticipate Gödel's interest in Husserl over least eight years, as he began to be interested in Husserl's work in 1959.

[24] We can find a clear and explicit connection between the discussion about the nature of function and that about intensional/extensional interpretation of concepts in Hermann Weyl's work *Das Kontinuum*. Certainly, Weyl is concerned with a constructive reconstruction of analysis, and he writes in 1918, but the way he describes the problem stems on the one hand from the discussion on the nature of functions, and on the other hand from the discussion on the priority between extensions and intensions of concepts. In *Das Kontinuum,* Weyl grounds classes (as 'relations in extension') on relations in intension, and considers that 'The notion that an infinite set is a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical' *Weyl 1988*, page 23. He then continues 'So our view is that the transition from "property" to "set" (of those things which have the property) signifies merely that one brings to bear the *objective* [sachlich] rather than the purely logical point of view, i.e., one regards *objective* [sachlich] correspondence (that is "relation in extension" as logicians say) established entirely on the bases of acquaintance with the relevant objects as decisive rather than logical equivalence. Therefore I contrast the concept of set and function formulated here in an exact way with the completely vague concept of function which has become canonical in analysis since Dirichlet and, together with it, the prevailing concept of set.' (*Ibidem*) See also page 46 for the necessity to base the notion of function on logical operations.

sufficient for the needs of analysis. On the other hand, since it was not easy to find a satisfying characterisation of the new 'analytical operations' able to cover all the new domains, the followers of the 'extensionalist' approach pointed out the generality and usefulness of their own characterisation.[26] This debate, incidentally, opened up the reflection on the nature of definitions in which logicians and mathematicians actively participate. In a sense, the question was how to characterise a function as a complex of clear and distinct operations, where the term operation was often used as an alternative to that of function in intension. This approach does not presuppose that a function in intension, as a complex of operations, is considered as the result of our acts of definition or construction. Actually, the idea of the mathematician who creates his own objects from intuition was quite alien at that time and does not appear before the intuitionistic proposals of Brouwer.[27] This is especially clear in the controversy between the French mathematicians (Lebesgue, Borel, and Baire); even if it has been noted later by Brouwer to bear on pre-intuitionism, it was at that time independent from the controversy of realism *versus* intuitionism.

The German debate, involving Husserl, Schröder, Voigt and Frege around 1891-1895, focused, on the one hand, on the question of the primacy of the notion of *Inhalt* over the notion of *Umfang*, and, on the other, on the question of the nature and aims of logic. Leibniz is very often evoked, in this discussion, because he was the first to raise the question of a priority between *Inhalt* and *Umfang* and because of his very strong views on the aims of

[25] *Youschkevitch 1977*, *Monna 1972*.

[26] A text from the French mathematician Baire 'Sur les fonctions de variables réelles' (Baire1899) resumes in these terms part of the French debate, involving Borel, Lebesgue, Boutroux and many others: 'Le mot *fonction*, qui a servi primitivement à désigner les différentes puissances d'une même quantité, a pris une signification de plus en plus étendue, jusqu'à ce que Dirichlet ait donné à ce mot le sens qu'on lui attribue aujourd'hui. Il y a *fonction*, dès qu'on imagine une correspondance entre les nombres, qu'on convient de considérer comme les états de grandeur d'une même variable Y, avec d'autres nombres, tous distincts, qu'on convient de considérer comme les états de grandeur d'une même variable x. On ne s'occupe pas, dans cette définition, de rechercher par quels moyens la correspondance peut être effectivement établie; on ne recherche même pas s'il est possible de l'établir. La notion de fonction, entendue de cette manière, est entièrement contenue dans la notion de *détermination*; ce point de vue s'oppose à celui qui consiste à partir de certaines fonctions simples, et à considérer des expressions composées avec ces fonctions simples, en réservant le mot fonction aux expressions ainsi obtenues'.

logic.[28] Frege, as we have noted, enters into this debate after having distinguished the notions of *Sinn* and *Bedeutung,* and his position is clear.

He is against Schröder and, in general, against the *Umfangs*-logicians (the extensionalists), who, following the new current of the algebra of logic stemming from Boole and Peirce, wanted to reduce logic to a domain-calculus, without considering how classes (that is extensions) are formed.[29] For Frege, extensions are grounded in concepts. To overlook this fact means one considers classes as collections of individuals, thereby inheriting all the difficulties implied by such an assumption, e.g. the definition of the void class.

At the same time Frege opposes the *Inhalts*-logicians and in particular Husserl's analysis of concepts, as stressed in his correspondence with Husserl.[30] The famous diagram explaining Frege's own theory of concepts (see figure 2) stands in stark contrast to what Frege considers to be Husserl's theory (see figure 3).

[Insert here figures 2 and 3 about here]

In the latter scheme, said Frege, the only difference between concept-words and proper names would be that the latter could be related only to one object and the former to more than one object. A concept-word whose extension is empty should then be rejected from science exactly like a proper name to which there is no corresponding object.

[27] See *Monna 1972*, page 81 '[…] mathematicians felt they had to seek definitions or descriptions of objects which already existed in a natural way and that all other objects were artificial or pathological. The idea that the mathematician creates his objects by means of definitions was alien to the mathematicians of the early times'.
[28] We will not go into the detail of the German controversy, especially that between Husserl and Voigt, as it has been largely discussed in recent works in Germany: for example *Hamacher-Hermes 1994*, or *Walther-Klaus 1987*. *Peckhaus 2004* also offers an interesting reconstruction of the Frege/Schröder debate, even if from other analytic perspectives.
[29] To be more specific, for Schröder the most general calculus of domains does not contemplate any specific formation principle. The only contraint in the formation of a class asks that the elements of domain have to be distinguished from one another. Formation principles are, however, considered for the other more specific calculi, such as the calculus of classes, propositions, and relations. Nevertheless Frege was totally unsatisfied with a piecemeal system of logic made up of heterogeneous calculi, where the primacy of intensions over extensions is not a general principle.
[30] *Frege1980*, pages 60-66.

As a consequence of these two rejections, and of the implicit identification of Husserl's *Inhalte* with his own senses, Frege condemns, in the *Ausführungen über Sinn und Bedeutung,* both the *Inhalts-* and *Umfangs*-logicians. The former for only considering senses, thereby neglecting that what is important in logic are not the *Sinne* but the *Bedeutungen,* and the latter because they consider extensions as primitives without basing them on concepts,

Without going into the analysis of the real differences between Husserl's *Inhalte* and Frege's senses, what I want to stress here is just that Frege's identification of senses and intensions, *Sinne* and *Inhalte,* is essentially driven by a *polemic* motivation. For Frege, concepts have the property of extensionality i.e. two concepts are identical if and only if their courses of values (i.e. their extensions) are identical. Traditional and Husserlian concepts lack this property and cannot, therefore, be identified with Fregean concepts (the *Bedeutungen* of the concept-words). Another possibility is still available and Frege took it without hesitation: Husserlian intensions are nothing but Fregean senses of concept-words, as stated by the famous passage in the *Ausführungen* :

> The *Inhalts*-logicians only remain too happily with the *Sinn*, for what they call '*Inhalt'* if is not quite the same as *Vorstellung*, is certainly the *Sinn*.[31]

In the German logical context, after Frege and with the exception of the Husserlian tradition, the notion of *Inhalt* was generally attached to the notion of sense. This is how Carnap treats it, and this is also how Bernays considered it in his criticism of Gödel. Nevertheless, outside of the German tradition, the possibility of using the word 'intension' to signify the reference of a function word (when the function word is not intended to indicate a determined value with its argument) was left open.

Peano does not use the name 'intension', but rather the word 'operation' for functions used in intensions[32] as opposed to the class of their values. Russell refers to Peano's distinction in

[31] The previous passage is from *Frege 1969*. We quote it from Mohanty's translation of *Mohanty 1982*. Cf. his interpretation of the passage especially on pages 8-9.

some parts of his work as a distinction between the intension and extension of a function and generally recognises the interest and usefulness of this distinction.[33]

Furthermore, Russell avoids senses[34] but speaks about intensions in the introduction of the first edition of *Principia mathematica*. The most explicit text on intensions is in chapter III, page 72 of the Introduction, where Russell presents classes as incomplete symbols, mere symbolic or linguistic conveniences, 'not genuine objects as their members are if they are individuals'. Then he goes on to claim that:

> It is an old dispute whether formal logic should concern itself mainly with intensions or with extensions. In general logicians whose training was mainly philosophical have decided for intensions, while those whose training was mainly mathematical have decided for extensions. The fact seems to be that, while mathematical logic requires extensions, philosophical logic refuses to supply anything except intensions. Our theory of classes recognises and reconciles these two apparently opposite facts, by showing that an extension (which is the same as a class) is an incomplete symbol, whose use always acquires its meaning through a reference to intension.

These intensions are nothing but the propositional functions, which are the constituent parts of propositions.

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[32] Cf. for example: G. Peano: *Notations de logique mathematiques*, Turin 1894, reprinted in *Peano 1958* pages 123-176, especially page 146.

[33] At least two passages are relevant in this sense. The first concerns Couturat's criticism of Leibniz's 'confused' account of the notion of intension. In 'Recent works on the philosophy of Leibniz' (1903) in *Russell 1984*, pages 537-563, Russell's writes: '[…] the logical calculus undoubtedly requires a point of view more akin to that of extension than to that of intension. But it would seem that the truth lies somewhere between the two, in a theory not yet developed. This results from the considerations on infinite classes. Take e.g. the proposition "Every prime is an integer". It is impossible to interpret such a proposition as stating the results of an enumeration, which would be the standpoint of pure extension. And yet it is essentially concerned with the terms that are primes, not, as the intensional view would have us believe, with the concept prime. This appears to be here a logical problem, as yet unsolved and almost unconsidered; and in any case the matter is less simple than M. Couturat represents it as being.' Even more striking is the second passage from 'On the relation of Mathematics to Symbolic logic' (1905) (*Ibidem* pages 524-532) where Russell, arguing against Boutroux's 'Mathematical correspondences and Logical relations', which appeared in the *Revue de metaphysique et de Morale* in July 1905, says: 'There is no ambiguity in having two different notions which are clearly distinguished, as Peano's two kinds of function are. These two kinds are in fact the function in *intension* and the function in *extension:* the *u* in *ux* may be viewed as an *operation*, which is conceived intensionally; while the "definite function" F is in fact a correlation between x and ux for a given class of values of x, which is extensional, and may be identified with "my relation in extension". It is not true to say that "an infinity of conditions" […] enter into the definition of a correlation: the single condition expressed by the propositional function $\phi!(x,y)$, which x and y must satisfy, is sufficient, since the function determines a corresponding extension.' The point is that according to Russell, Boutroux confuses the logically determined relation between the function in intension and its extension, and the epistemological or psychological possibility for us to practically determine this extension.

[34] The Correspondence with Frege is so explicit on this point that Russell's own identification in the Appendix of *Principles* has mostly been considered a misunderstanding. Cf. especially the famous passage about Mont Blanc in Russell's letter of 12.12. 1904, *Frege1980* page 166-170.

Can Russell's intensions in *Principia* be considered as senses? The answer is negative since they do not share any of the central features of senses. Russell's propositional functions denote their values (i.e. propositions) indeterminately, but they are not ways presenting aspects of the reference, and they are not what we have to grasp to identify reference. Russell explicitly considers this second point, when he affirms that a value of a function does not presuppose the function:

> Its is sufficiently obvious, in any particular cases, that a value of a function does not presuppose the function. Thus for example the proposition 'Socrates is human' can be perfectly apprehended without regarding it as a value of the function 'x is human'.[35]

Russell's propositional functions are then real entities even if they are organised by a dependency relation, a relation of ontological presupposition between propositional functions themselves and between propositional functions and propositions.[36]

4. Bernays's second misunderstanding and Gödel's concepts in 1944

Having travelled a little on the intricate road of the intension/extension debate, let us now come back to the problem of Bernays's misunderstanding, and to the analysis of Gödel's claims in 1944.

Bernays's second misunderstanding can now be very quickly identified. If we assume that in 'Russell's mathematical logic' Gödel uses the words 'concept' and 'intension' as Russell uses the word 'intension', i.e. to signify the reference of a propositional-function name, Bernays's criticism no longer applies to Gödel, and is in fact based on a misunderstanding of

[35] *Whitehead Russell 1910*. Introduction, to the first edition, chapter II, page 39.
[36] Cf. *Linsky 1999,* chap. 2, and *Hylton 1990,* chap. 7. In this paper, we will not discuss the question of Russell's reasons for ramification. Nor is it our concern to evaluate Gödel's thesis according to which it was Russell's constructivism and antirealistic attitude that prompted him to accept the VCP in the first form. Nevertheless, behind Gödel's criticism of constructivism, that could or could not pertain to Russell's position, there is an aspect of Gödel's analysis that requires further consideration. In sum, even if Russell is concerned by intensions as references and propositional functions as real entities, treating them with the ramified theory of types transforms them into linguistic entities or, more precisely, into entities inextricably tied to their definitions. The real problem seems to be the analysis of the ontological dependency. Cf. section 4 below, and Gödel analysis of abstract structures.

the notion of intension. If we deny that propositional functions can enter in propositions only through their values, then even concepts (as references) are split up into types. So it is true that concepts in themselves (as references) only exist through their realisations, though this is not the case for concepts as senses, if the distinction between *Sinn* and *Bedeutung* is also to be used for concepts.

As to Gödel's claims in 1944, matters are complicated by the question concerning senses. It is very likely that Gödel did not agree with Russell's denial of the 'curious shadowy'[37] entities that senses should be, and there are some texts[38] which can be used to sustain Gödel's deep appreciation of Frege's distinction. Nevertheless, in the paper on Russell there is no reference to senses of concepts, and it would be strange that Gödel, expressing his thought on Russell's work, used the term intension in a rather different way without mentioning the fact.

It must be stressed that in the paper on Russell Gödel's own claims about concepts are coherent with our identification of 'intensions' and 'concepts' as references. Stronger still, our identification is necessary to understand Gödel's claims. Three main Gödelian theses in 1944 are important for our purposes. The first concerns the difference between concepts and some linguistic expressions such as predicates and notions. The second concerns the difference between concepts and classes and the third concerns the modes of resolving the intensional paradoxes. Understanding these theses presupposes the above mentioned identification.

1) Gödel says that concepts are objective entities, i.e. properties of and relations between objects (*Gödel 1944* page 128). As real entities they have to be distinguished from predicates (which are linguistic entities) and from notions, where the term notion means a symbol of a predicate together with a rule of translation of sentences containing this symbol into sentences

[37] *Gödel 1944*, page 122, note 7.
[38] For example the Reprint E, *Gödel 1990*, page 321, point 4, or in version III of the unpublished paper on Carnap in *Gödel 1995* page 350, note 40. In this latter Gödel, having criticised Carnap's definition of 'content' as presupposing the notion of logical consequence, claims that, on the contrary, we should consider the 'conceptual content' and pursues 'The neglect of the conceptual content of sentences (i.e. the 'sense' according to Frege) also is responsible for the wrong view that the conclusion in logical inference, objectively contains no information

not containing it. Two different definitions determine different notions. On the contrary, says Gödel, and this is the important point, the same concept can be expressed in different ways and it is also possible that the principle of extensionality holds for concepts (that is, that concepts corresponding to the same classes can be identified) (*ibidem*). This last assertion would be totally absurd if Gödel did not consider concepts as references of concept-words, since the essential characteristic of the notion of concept as sense is the fact that it is not extensional. Here Gödel has in mind a solution *à la Frege*, as Frege's concepts are indeed extensional even if distinct from their extensions. Nevertheless, Gödel considers such a solution dubious.[39]

2) Gödel in some way contrasts concepts with classes (as pluralities of objects). He never mentions propositional attitudes or modal contexts as a ground for such an opposition. On the contrary, the difference between concepts and classes is grounded in self-reflexivity. It is perfectly coherent to think that concepts are able to support self-reflexivity, and that therefore the VCP in its second form does not also apply to concepts.[40] Of concepts (as opposed to notions) it is possible to refer to their totality, to claim that some of them can be described only by reference to all of them (or at least all of a given type), and to say that a property ϕ can contain itself as constituent of its meaning or content (page 130). Gödel adds that an approximation of this kind of self-reflexivity is in fact given in constructivistic logic. It is given in his theorems of incompleteness, where a proposition contains, as part of its meaning, the assertion of its own demonstrability, and where the demonstrability of a proposition (in the case where the axioms and the rules of inference are correct) implies the proposition in question. Concerning classes, on the contrary, we can consider the fact that the VCP applies in the second and third form as a plausible assumption, sufficient for the development of all

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beyond that contained in the premises'. It is noteworthy that, in general, Gödel contrasts content and meaning in a way quite similar to the Fregean *Sinn*/*Bedeutung* distinction.

[39] 'It might even be that the axiom of extensionality or at least some near to it holds for concepts' *Gödel 1944* page 129. In the 'Reprint E', *Gödel 1990* page 321, Gödel denies such a possibility.

contemporary mathematics. Impredicative definitions are allowed for classes, but it is impossible to say that $x \in y$ when x is not less than y. One is then led, says Gödel, to something like Zermelo's theory of sets. Zermelo's theory is based on an iterative notion of set, where sets are split up into levels, obtained by the relation 'set of' on a specific level and where mixture and transfinite types are allowed. Nevertheless, contrary to the iterative notion of set, it is also possible to think of a class as a certain kind of structure to which the second form of the VCP does not apply. As an example of such an idea, Gödel mentions the work of Mirimanoff. However, Gödel clearly considers Zermelo's solution sufficient for the needs of mathematics.

The last thesis clearly proposes that we seek for different solutions for paradoxes of concepts and those of classes. This difference has not to do with intensional restrictions on identity (because Gödel, as we saw, even takes a property of extensionality for concepts into consideration). On the contrary, it is justified by opposing the needs of logic and the needs of mathematics, that is, by opposing the notion of structure in the abstract sense and the iterative notion of set. Gödel stresses that abstract structures can *contain* elements without *presupposing* them ontologically, just as a sentence or a phrase belongs to a language, without having ontological primacy over it. On the contrary, sentences *presuppose* a language, even if they are *involved* in (belong to) it.[41] In the same way concepts (which are abstract structures) can contain concepts (can have other concepts as parts of them) without ontologically presupposing them.

3) Gödel envisages a formal system conforming to the idea of a logic of concepts that does not satisfy the second form of the VCP and allows mixed types. In fact in the note 28 he says that such a system should, in addition to the substitution rules for functions considered in Hilbert–Bernays *Grundlagen der Mathematik*, also contain some axioms of intensionality

[40] Cf. section 2.1.

required by the concept of property. These axioms should not treat concepts as notions, as did those of Chwistek. That is: if we abandon Frege's idea, the identity criteria for concepts would no longer state that two concepts with the same extensions are to be considered identical. At the same time they should be weaker than those of Chwistek, i.e. they cannot be based on the idea that to different definitions correspond different concepts. In such a system, a new solution of intensional paradoxes, first and foremost of the paradox of concept, must be sought in the direction of Church's work. The main problem for Gödel was finding a type-free theory of concepts in which self-reference would be allowed. Gödel cites Quine's theory of stratification and Church's work of 1932-1933 as attempts in this direction. Considering the latter, 'in spite of the strong inconsistency theorems of Kleene and Rosser', Gödel appreciates Church's analysis as the most valuable attempt to work out the hypothesis of the limited ranges of significance. Applied to concepts such a hypothesis would make paradoxes appear 'as something analogous to divide by zero'.

5. Gödel's concepts in the 1970s

This analysis of concepts, except for two points[42], is plainly coherent with Gödel's analysis in the 1970s as reported by *Wang 1996*.

Concerning the thesis of the reality of concepts and the distinction between concepts and sets, and that between logic and mathematics, Gödel's position is as strong as in 1944:

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[41] Cf. Gödel's discussion on page 130 about the sentence 'Every sentences (of a given language) contains at least a relational word'.

[42] The points concern: a) the possible admission of extensionality for concepts, which was rejected by Gödel very early, as the *Addendum E* proves; b) the conception of classes as real entities, which is replaced by the idea that only sets are real entities. Classes are just extensions of concepts and therefore, even if useful for mathematics, are only convenient ways to speak about concepts. Cf. *Wang 1996* sentence 8.6.6 'Classes are introduced by contextual definitions –definitions in use–construed in an objective sense. [For instance something x belongs to a class K if there is a concept C such that K is the range of C and C applies to x]. They are nothing in themselves, and we do not understand what are introduced only by contextual definitions, which merely tell us how to deal with them according to certain rules. Classes appear to us so much like sets that we tend to forget the line of thought which leads from concepts to classes; if, however, we leave out such considerations, the talk about classes become a matter of make-believe, arbitrarily treating classes as if they were sets again'. We will follow Wang's numbering of Gödel's quotations.

> 8.6.1 The subject matter of logic is intensions (concepts); that of mathematics is extensions (sets).
> 8.6.2 Mathematicians are primarily interested in extensions and we have a systematic study of extensions in set theory, which remains a mathematical subject except in its foundations. Mathematicians form and use concepts, but they do not investigate generally how concepts are formed, and that is to be done in logic. We do not have an equally well-developed theory of concepts comparable to set theory.
> 8.6.15 For a long time there has been confusion between logic and mathematics. Once we make and use a sharp distinction between sets and concepts, we have made several advances. We have a reasonably convincing foundation for ordinary mathematics according to the iterative concept of set. Going beyond sets becomes an understandable and, in fact, a necessary step for a comprehensive conception of logic. We come back to the program of developing a *grand* logic, except that we are no longer troubled by the consequences of the confusion between sets and concepts. For example, we are no longer frustrated by wanting to say contradictory things about classes , and can now say both that no set can belong to itself and that a concept –and therewith a class-can apply (or belong) to itself.

Concepts, that is intensions, are, as in 1944, abstract structures of reality that apply (with only few exceptions) universally to whatever, including themselves (cf. Wang 4.4.7, 8.4.7, 9.4.5). They are also independent from our constructions and intuitions (cf. Wang 8.5.19, 8.5.20).

Some of the problems that Gödel faces in the conversation with Wang are a natural extension of this framework advocated in 1944. In the Russell paper he rejected the theory of the primacy of propositions over their components[43] (concepts and arguments). As a consequence he rejected Frege's theory of concepts as unsaturated and Russell's theory of ambiguity. Therefore, Gödel has, first, to reassess the opposition between concepts and objects on new grounds and, second, to give an explicit analysis of predication (or application of a concept).

There are several hints for new solutions of the problem of the object/concept opposition, in the conversations with Wang. The first lies on a new way of considering the classical concrete/abstract distinction (Wang 4.3.9). Concepts are abstract in the sense that they can be represented or exemplified in different ways. Objects are concrete in the sense that they cannot be exemplified in different ways. For example, while numbers are abstract[44], because a number can be represented by different sets, sets are concrete entities, according to the iterative conception.[45]

The second hint of the difference between objects and concepts is based on a general analysis between wholes and unities, which extends from the analysis of the differences between abstract structures and classes outlined by Gödel in 1944. It will be helpful to reconstruct a metaphysical scheme on the basis of Wang 91.25-26.

[Insert here figure 4]

Every entity is either a concept or an object. Every whole is a unity, and every unity (which is divisible) is a whole (8.2.4 and 9.1.26). Sets are a limiting case of unities inasmuch as the interconnection of the parts plays no role. Concepts are wholes in a stronger sense than sets; they are more organic wholes, just as a human body is an organic whole composed of its parts. Although complex abstract structures, as complex concepts, are wholes containing parts, they are not, however reducible to those parts, because they are unities where the interconnection of parts plays a role. Complex concepts cannot be explained away by enumerating their components even if they are finite complexes of simple entities. This explains on the one hand why self-reflexivity can be ascribed to them (the whole contains parts but does not presuppose them ontologically and cannot be reduced to them). On the other hand, it explains why extensionality cannot be supposed for concepts, because, while

[43] Cf. note 47, page 139 of *Gödel 1944*.
[44] *Wang 1996*, 4.3.9 Cf. the passage quoted before from the paper of 1944, in which the number two is said to be a concept.
[45] In 4.3.9 Gödel says that the relationship between concepts and objects can be seen as the relation between an axiomatic theory and its models. Axioms are conceptual, while their models are concrete representations of the concepts they contain.

the structure of the whole is the characteristic feature of complex concepts, classes and sets are always pluralities of things without a structure.

Concerning the second problem of a universal notion of predication (or application) free from paradoxes, Gödel provides two clues in the conversations that seem to justify the assertion that he is still searching for a type-free theory of concepts.

Firstly, he still recognises that the main open question of modern logic is how to solve the paradoxes of concepts.[46] He thus restates his assessment of 1944, adding only a more negative accent on Quine's solution:

> 8.4.19 The older search for a satisfactory set theory gives way to similar search for a satisfactory theory of concepts, which will, among other things, resolve the intensional paradoxes. For this purpose, Quine's idea of stratification is arbitrary, and Church's idea along the line of limited ranges of significance is inconsistent in its original form and has not been worked out.

The fact that he only considers these two theories (both of which are type-free) is a clear indication of the direction in which he, from his realistic point of view, thought that a satisfying solution for intensional paradoxes has to be sought.

Secondly, it seems that Gödel is now trying to work out Church's ideas of 1932-33. In the conversations with Wang, he presented what he calls Church's paradox, thereby revealing that he was reflecting on Church's system through to 1972. As Wang says, Gödel suggested that it would be helpful, for the purpose of developing a theory of concepts, to reflect on this paradox.[47] Here is the paradox:

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[46] On paradoxes, Gödel also repeats that intensional paradoxes should be distinguished from both set-theoretic paradoxes and semantic paradoxes. Gödel considered the mathematical notion of set free from paradoxes, since the iterative notion of set, which is the only one really used in mathematics, has never suffered from paradoxes. Self-reference must be banished in the case of sets, as sets are quasi-objects and no object can belong to itself. Concerning semantic paradoxes, Gödel, according to Wang, repeatedly emphasized that he himself had long ago resolved the semantic paradoxes as a by-product of his incompleteness theorem. Semantic paradoxes say nothing because with a semantic paradox we always are in a definite language, with its many countable symbols. We can never have 'true' in the same language; we can never have a complete epistemological description of a language inside the language. In contrast, intensional paradoxes have nothing to do with language because they only involve logical concepts. Semantic and intensional paradoxes are often mixed together because, without Platonism, concepts appear more like linguistic entities. Semantic paradoxes are therefore not problematic. Intensional paradoxes, on the other hand, remain a serious problem for logic, as concept theory is the major component of logic.

[47] *Wang 1996*, page 279.

> A function is said to be regular if it can be applied to every entity [which can be an object or a function (a concept)]. Consider now the following regular function of two arguments:
> 1) $d(\mathrm{F},x)=\mathrm{F}(x)$ if F is regular
> $=0$ otherwise
> Introduce now another regular function:
> 2) $E(x)=0$ if $x\neq 0$
> $=1$ if $x=0$
> We see immediately:
> 3) $\mathrm{E}(x)\neq x$
> Let $\mathrm{H}(x)$ be $\mathrm{E}(d(x,x))$ which is regular. By (1) we have:
> 4) $d(H,x)=H(x)=E(d(x, x))$
> Substituting H for x we get:
> 5) $d(H,H)=E(d(H,H))$, contradicting (3).

Gödel says that:

> 8.6.24 […] it may be called Church's paradox because it is most easily set up in Church's system (1932-33). It is particularly striking that this paradox is not well known. It makes clear that the intensional paradoxes have no simple solution. An interesting problem is to find a theory in which the classical paradoxes are not derivable but this one is.

Although Gödel presented the paradox as an intensional paradox, it is not intensional in any 'modern' sense of the word, as it does not concern modalities or propositional attitudes. It can concern intensions only if by intensions we mean the reference of a concept-word as distinguished from its extension. In Gödel's view, the paradox is a simpler version of the familiar paradox of the 'concept not applying to itself' in a system where we don't use any distinction of types (the paradox will disappear in any typed system as d(x, x) will be ill-formed).[48]

[48] In Church's system, concepts (as denotations of concept-words) can be formed without any restriction through the procedure of lambda abstraction and lambda application. The main idea is that such concepts are not supposed to apply to everything, but have limited ranges of significance. For example, the critical concept: $\lambda\phi.\neg\phi(\phi)$, is a concept of the system. It is true for some arguments, and false for others, but in no way we can derive, in the system, that it is significant for the argument $\lambda\phi.\neg\phi(\phi)$. So $\{\lambda\phi.\neg\phi(\phi)\}\ (\lambda\phi.\neg\phi(\phi))$ is just a mere sequence of symbols without any truth-value, i.e. it is not a proposition. As the law of *reductio ad absurdum* only applies to propositions, we cannot derive a contradiction from the statement $\{\lambda\phi.\neg\phi(\phi)\}\ (\lambda\phi.\neg\phi(\phi))$. To take into account concepts with limited ranges of significance, Church introduced a restricted rule of generalization Π (M, N) expressing the fact that N is true for all the values for which M is true. An introduction rule is then derived in the system for such a restricted generalization, and the first 13 axioms of the system are needed to prove this result. With the paradox presented before, Gödel is affirming that in addition to the Richard's Paradox constructed by Kleene and Rosser, it is also possible to construct the classical paradox of the concept not applying to itself in Church's system. That is, even if $\lambda\phi.\neg\phi(\phi)$ has a limited range of significance, it is possible

The fact that Gödel is still reflecting on a type-free system offering a tentative application of the principle of limited ranges of significance and the fact that he assigned such an importance to what he calls Church's paradox seem to us proof that Gödel did not change his mind on the question.

All this shows that, with only two exceptions, which are broadly understandable as a coherent evolution of Gödel's position, Gödel's thought is remarkable stable from 1944 to 1976. The analysis of the notion of concept, the diagnosis of the remaining difficulties, and the preference for a type-free a solution remain the same. This conclusion is made possible by our proposed identification between intensions and references of concept-words, identification that simultaneously solves the mysterious question about Bernays's second misunderstanding.

6. Conclusion: Gödel and Frege

If we now compare Frege's position with Gödel's, we find interesting similarities and differences. Let us recall Frege's theses and compare them with Gödel's assertions.

1) According to Frege, concepts are *Bedeutungen* of concept-words, as objects are *Bedeutungen* of object-names. For Gödel, while concepts are *Bedeutungen* of concept-words, they are not extracted from propositions. On the contrary the notion of proposition is dependent on that of object and concept.

2) For Frege there is a categorial difference between objects and concepts, based on the question of unsaturatedness. *Unsaturatedness* implies that concepts are divided into types. For Gödel there is a categorial difference between them, but it does not consist in unsaturatedness. Unsaturatedness seems to be unacceptable for Gödel (see section 1). For him, concepts,

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to construct in the system a concept (Ed(x,x)) applying to every argument and having the same meaning of $\lambda\phi.\neg\phi(\phi)$. Through such a concept the system is inconsistent.

functions, procedures, mappings and operations are all abstract entities that, whatever their nature may be, can be applied to themselves and can be predicated of themselves. This is the substance of their profound difference from objects; unsaturatedness has nothing to do with concepts. As a result, they are not divided into types. Objects and concepts are essentially different entities in their behaviour in the face of self-reference.

3) For Frege, concepts can be represented by their extension. This implies applying a principle of extensionality to concepts. According to Gödel, concepts can be represented by their extension, but only in the realm of mathematics. Classes and sets are ways of speaking about concepts, but in this way of speaking we loose one of the characteristics of concepts, i.e. their structure. For this reason Gödel can sustain the idea that there are simple concepts that form complex ones. He affirms this in 1970, but had expressed the same thesis in 1953.[49] At work here in some way is the Leibnizian notion of intension whereas simple ideas compose complex ones. Does the principle of extensionality, holding for Frege's concepts, make them compatible with such an idea? The answer is controversial. Nevertheless, there are grounds to answer this question positively, if we consider Frege's claims in 'Über Begriff und Gegenstand' and the assertion in the first section (Exposition of the *Begriffsschrift*) in the *Grundgesetze der Arithmetik* that a concept is made up of its characteristic marks.

4) According to Frege, we cannot talk about concepts without transforming them into something else, namely objects. For Gödel we can perfectly well speak about concepts, but must then face the unresolved problem of paradoxes. For both, concepts are prior to their extensions.

5) For both, Frege and Gödel, concepts as denotations have to be distinguished from their senses. For Frege, senses are those double-faced entities, looking both toward language and toward reference. Frege is Universalist at least in the sense that language is for him the way

[49] 'Is mathematics syntax of language?' Version V, *Gödel 1995* page 360.

we approach reality, thereby senses of explicit definitions are the essential tools for grasping entities. Gödel is an Universalist in a different sense. For him, while the axiomatic method (as opposed to explicit definitions) is the essential way to clearly and precisely define objects and concepts, it is not the only method to grasp them. Gödel clearly thought it necessary to provide a definition of concepts in a way independent from language. The concept of calculability (through the notion of a Turing Machine and its *simple operations*) is the most striking example of a definition independent of language. Gödel hoped that in the future the same kind of definition would be found for concepts. To arrive at such a definition of concepts we should try to grasp simple and primitive concepts and clarify what their knowledge consists of. Husserl's phenomenology appeared to him to be the right tool for clarifying the content of a concept, in other words for achieving the right perspective on it. Following this interpretation, what Gödel seeks in Husserl is a method with which to grasp concepts, that is, a way to clarify senses *independent from language*. Nevertheless, the sense of a concept, the acts we have to perform to grasp it, need not necessarily be identified with the concept itself. Incidentally, there is no mention in Husserl's works of a possible self-reflexivity for concepts and it could be said that it does not make sense in an Husserlian context.[50]

6) Concerning the relationship between logic and mathematics, there is a sense in which Gödel renewed Frege's logicism. In *Wang 96* Gödel conjectures that (8.6.4) even if it is not in the ideas of *set* and *concept* that every set is the extension of a concept, the proposition ‘for every set, there is a [defining] concept’ is true despite being in need of a proof. Such an assertion, while it implies a form of conjectural dependency of mathematics on logic, cannot be intended as the assertion of a reduction of the former to the latter. Gödel seems to claim more a question of priority than one of reducibility, as reduction implies an explicit definition

[50] Type-freeness is against the spirit of Husserl's Fourth logical investigation, where he clarifies the notion of semantical category, first time introduced in §67 of the Prolegomena to the Logical Investigations.

of mathematical concepts in terms of logical ones, and everything seems to exclude the possibility of such an explicit definition.

In conclusion, Gödel's analysis of concepts has it own *raison d'être* different from Frege's one; it is extremely coherent from 1944 to the 1970s, it is very deeply influenced by the Leibnizian notion of intension, but it has, for the moment no consistent solution. Nevertheless the question of it possibility should be left open.

Acknowledgements

This work was accomplished thanks to a secondment to the CNRS, accorded to me for 2003-2004 and to a stay at the University of Konstanz supported by the Alexander von Humboldt Foundation. I would like to thank Per Martin-Löf for some points on Husserl's and the anonymous referees for their useful suggestions.
I would also like to thank my colleagues of the Ceperc (especially E. Audureau, Alain Michel and Philippe Minh) for helpful discussions, and Philippe de Rouilhan to whom I have had the occasion to present this work at the IHPST in Paris.

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Concept-word

$\downarrow$

Sense of the concept-word
(concept in itself, intension)

$\downarrow$

Objects falling under the concept
(extension)

Figure 1

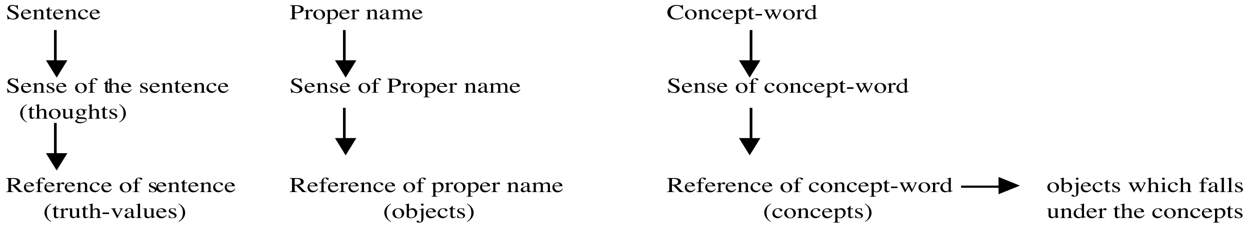
Sentence
Sense of the sentence (thoughts)
Reference of sentence (truth-values)
Proper name
Sense of Proper name
Reference of proper name (objects)
Concept-word
Sense of concept-word
Reference of concept-word (concepts)
objects which falls under the concepts

Figure 2

Concept-word

$\downarrow$

Sense of the concept-word
(concept)

$\downarrow$

objects which fall under the concept

Figure 3

Objects		Concepts	
Simple objects (monads, unit set, empty set)	Sets	Simple concepts (existence, generality, negation, to be a set, to be a concept...)	Compound concepts
unities, not wholes	wholes and unities	unities, not wholes	wholes and unities

Figure 4