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Decomposing Berge graphs

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Abstract

A hole in a graph is an induced cycle on at least four vertices. A graph is Berge if it has no odd hole and if its complement has no odd hole. In 2002, Chudnovsky, Robertson, Seymour and Thomas proved a decomposition theorem for Berge graphs saying that every Berge graph either is in a well understood basic class or has some kind of decomposition. Then, Chudnovsky proved a stronger theorem by restricting the allowed decompositions and another theorem where some decompositions were restricted while other decompositions were extended. We prove here a theorem stronger than all these previously known results. Our proof uses at an essential step one of the theorems of Chudnovsky.

AMS Mathematics Subject Classification: 05C17, 05C75

1 Definitions and known theorems

In this paper graphs are simple and finite. A *hole* in a graph is an induced cycle of length at least 4. An *antihole* is the complement of a hole. A graph is said to be Berge if it has no odd hole and no odd antihole. A graph G is said to be *perfect* if for every induced subgraph G' the chromatic number of G' is equal to the maximum size of a clique of G'. In 1961, Berge [1] conjectured that every Berge graph is perfect. This was known as the *Strong Perfect Graph Conjecture*, was the object of much research

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and was finally proved by Chudnovsky, Robertson, Seymour and Thomas in 2002 [4]. In fact, they proved a stronger result: a decomposition theorem, first conjectured by Conforti, Cornuéjols and Vušković [7], stating that every Berge graph is either in a well understood basic class of perfect graph, or has a structural fault that cannot occur in a minimum counter-example to Strong Perfect Graph Conjecture. Before stating this decomposition theorem, we need some definitions.

We call *path* any connected graph with at least a vertex of degree 1 and no vertex of degree greater than 2. A path has at most two vertices of degree 1 that are the *ends* of the path. If a, b are the ends of a path P we say that P is *from a to b*. The other vertices are the *interior* vertices of the path. We denote by $v_1 - \cdots - v_n$ the path whose edge set is $\{v_1v_2, \dots, v_{n-1}v_n\}$. When P is a path, we say that P is *a path of G* if P is an induced subgraph of G. If P is a path and if a, b are two vertices of P then we denote by $a - P - b$ the only induced subgraph of P that is path from a to b. The *length* of a path is the number of its edges. An *antipath* is the complement of a path. Let G be a graph and let A and B be two subsets of $V(G)$. A path of G is said to be *outgoing from A to B* if it has an end in A, an end in B, length at least 2, and no interior vertex in $A \cup B$.

If $X, Y \subset V(G)$ are disjoint, we say that X is *complete* to Y if every vertex in X is adjacent to every vertex in Y. We also say that (X, Y) is a *complete pair*. We say that X is *anticomplete* to Y if there are no edges between X and Y. We also say that (X, Y) is an *anticomplete pair*. We say that a graph G is anticonnected if its complement $\overline{G}$ is connected.

Skew partitions were first introduced by Chvátal [5]. A *skew partition* of a graph $G = (V, E)$ is a partition of V into two sets A and B such that A induces a graph that is not connected, and B induces a graph that is not anticonnected. When A_1, A_2, B_1, B_2 are non-empty sets such that (A_1, A_2) partitions A, (A_1, A_2), (B_1, B_2) partitions B, and (B_1, B_2) is complete, we say that (A_1, A_2, B_1, B_2) is a *split* of the skew partition (A, B). An *even skew partition* (first defined in [4]) is a skew partition (A, B) with the additional property that every induced path with ends in B, interior in A and every antipath with ends in A, interior in B have even length. If (A, B) is a skew partition, we say that B is a *skew cutset.* If (A, B) is even we say that the skew cutset B is *even.* Note that Chudnovosky et al. [4] proved that no minimum non-perfect graph have an even skew partition.

We call *double split graph* (first defined in [4]) any graph G that may be constructed as follows. Let $m, n \geq 2$ be integers. Let $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_m\}$, $C = \{c_1, \dots, c_n\}$, $D = \{d_1, \dots, d_n\}$ be four disjoint sets. Let G have vertex set $A \cup B \cup C \cup D$ and edges in such a way that:

- a_i is adjacent to b_i for $1 \leq i \leq m$. There are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq m$;
- c_j is non-adjacent to d_j for $1 \leq j \leq m$. There are all four edges between $\{c_j, d_j\}$ and $\{c_{j'}, b_{j'}\}$ for $1 \leq j < j' \leq n$;
- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ and these two edges are disjoint.

Note that $C \cup D$ is a non-even skew cutset of G and that $\overline{G}$ is a double split graph. Note that in a double split graph, the vertices in $A \cup B$ all have degree $n+1$ and vertices in $C \cup D$ all have degree $2n+m-2$. Since $n \geq 2, m \geq 2$ implies $2n-2+m > 1+n$, it is clear that given a double split graph it is relevant to consider the *matching edges*, that have an end in A and an end in B, independantly of the choice of the sets A, B, C, D. Figure 1 are depicted 2 examples of double split graphs.

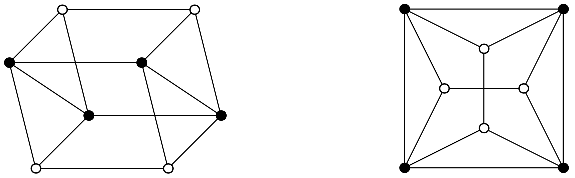

Figure 1: The double-diamond and $L(K_{3,3} \setminus e)$

A graph is said to be *basic* if one of $G, \overline{G}$ is either a bipartite graph, the line-graph of a bipartite graph or a double-split graph.

The 2-join was first defined by Cornuéjols and Cunningham [9]. We say that a partition (X_1, X_2) of the vertex set is a *2-join* when there exist disjoint non-empty $A_i, B_i \subseteq X_i$ $(i = 1, 2)$ satisfying:

- every vertex of A_1 is adjacent to every vertex of A_2 and every vertex of B_1 is adjacent to every vertex of B_2;
- there are no other edges between X_1 and X_2.

The sets X_1, X_2 are the two *sides* of the 2-join. When sets A_i's B_i's are like in the definition we say that $(X_1, X_2, A_1, B_1, A_2, B_2)$ is a *split* of (X_1, X_2). Implicitly, for $i = 1, 2$, we will denote by C_i the set $X_i \setminus (A_i \cup B_i)$.

A 2-join (X_1, X_2) in a graph G is said to be *connected* when for $i = 1, 2$, every component of $G[X_i]$ meets both A_i and B_i. A 2-join (X_1, X_2) in a graph G is said to be *proper* when it is connected and when for $i = 1, 2$, if

$|A_i| = |B_i| = 1$, and if X_i induces a path of G joining the vertex of A_i and the vertex of B_i, then it has length at least 3.

A 2-join is said to be a *path 2-join* if it has a split $(X_1, X_2, A_1, B_1, A_2, B_2)$ such that $G[X_1]$ is an outgoing path from A_1 to B_1. Implicitly we will then denote by a_1 the unique vertex in A_1 and by b_1 the unique vertex in B_1. We say that X_1 is the *path-side* of the 2-join. Note that when G is not a hole then this path-side is unique. A *non-path 2-join* is a 2-join that is not a path 2-join.

The *homogeneous pair* was first definied by Chvátal and Sbihi [6]. The definition that we give here is a slight variation used in [4]. An homogeneous pair is a partition of $V(G)$ into six non-empty sets (A, B, C, D, E, F) such that:

- every vertex in A has a neighbor in B and a non-neighbor in B, and vice versa;
- the pairs (C, A), (A, F), (F, B), (B, D) are complete;
- the pairs (D, A), (A, E), (E, B), (B, C) are anticomplete.

G is *path-cobipartite*[1] if it is a Berge graph obtained by subdivising an edge between the two cliques that partionned a cobipartite graph. More accurately, a graph is path-cobipartite if its vertex set can be partitionned into three sets A, B, P where A and B are non-empty cliques and P consist of vertices of degree 2, each of which belongs to the interior of a unique path of odd length with one end a in A, the other one b in B. Moreover, a has neighbors only in $A \cup P$ and b has neighbors only in $B \cup P$. Note that a path-cobipartite graph such that P is empty is the complement of bipartite graph.

A *cutset* is a graph G is a set $C \subset V(G)$ such that $G \setminus C$ is disconnected ($G \setminus C$ means $G[V(G) \setminus C]$). A *double star* in a graph is a subset D of the vertices such that there is an edge ab in $G[D]$ satisfaying: $D \subset N(a) \cup N(b)$.

Now we can state the known decomposition theorems of Berge graphs. The first decomposition theorem for Berge graph ever proved is the following:

Theorem 1.1 (Conforti, Cornuéjols and Vušković, 2001, [8]) *Every graph with no odd hole is either basic or has a proper 2-join or has a double star cutset.*

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[1] Our path-cobipartite graphs are simply the complement of the path-bipartite graphs definied by Chudnovsky in [2]. For convinience, we prefer to think about them in the complement as we do.

It could be thought that this theorem is useless to prove the Strong Perfect Graph Theorem since there are minimal imperfect graphs that have double star cutset: the odd antiholes of length at least 7. However, by the Strong Perfect Graph Theorem, we know that the following fact is true: for any minimal non-perfect graph G, one of $G, \overline{G}$ has no double star cutset. A direct proof of this — of which we have no idea — would yield together with Theorem 1.1 a new proof of the Strong Perfect Graph Theorem.

The following theorem was first conjectured in a slighly different form by Conforti, Cornuéjols and Vušković, who proved it in the particular case of square-free graphs [7]. A corollary of it is the Strong Perfect Graph Theorem.

Theorem 1.2 (Chudnovsky, Robertson, Seymour and Thomas, 2002, [4]) *Let G be a Berge graph. Then either G is basic or G has an homogeneous pair, of G has an even skew partition or one of $G, \overline{G}$ has a proper 2-join.*

The two theorems that we state now are due to Chudnovsky who proved them from scratch, that is without assuming Theorem 1.2. Her proof uses the notion of *trigraph*. The first theorem shows that homogeneous pairs are not necessary to decompose Berge graphs. Thus it is a result stronger than Theorem 1.2. The second one shows that path 2-joins are not necessary to decompose Berge graphs, but at the price of exending even skew partitions to general skew partitions and introducing a new basic class. Note that a third theorem can be obtained by viewing the second one in the complement of G.

Theorem 1.3 (Chudnovsky, 2003, [3, 2]) *Let G be a Berge graph. Then either G is basic, or one of $G, \overline{G}$ has a proper 2-join or G has an even skew partition.*

Theorem 1.4 (Chudnovsky, 2003, [3, 2]) *Let G be a Berge graph. Then either G is basic, or one of $G, \overline{G}$ is path-bipartite, or G has a proper 2-join that is not a path 2-join, or $\overline{G}$ has a proper 2-join or G has a skew partition.*

Main results and Motivation

Our main result is Theorem 1.5, that easilly implies Theorems 1.2, 1.3 and 1.4. We expect algorithmic applications that will be given in a work in preparation. Note that our proof of Theorem 1.5 is not a new proof of the previously known decomposition theorems for Berge graphs, since it uses at an essential step Theorem 1.3. Before going further we need more definitions.

We call *flat path of a graph G* any path whose interior vertices all have degree 2 in G and whose ends have no common neighbors outside of the path.

We call *path-double split graph* any graph obtained from a double split graph G by subdivising matching edges of G into paths of odd length. Note that a double split graph is a path-double split graph. More accurately, a path-double split graph is any graph G that may be constructed as follows. Let $m, n \geq 2$ be integers. Let $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_m\}$, $C = \{c_1, \dots, c_n\}$, $D = \{d_1, \dots, d_n\}$ be four disjoint sets. Let E be another possibly empty set. Let G have vertex set $A \cup B \cup C \cup D \cup E$ and edges in such a way that:

- for every vertex v in E, v has degree 2 and there exists $i \in \{1, \dots m\}$ such that v lies on path of odd length from a_i to b_i;
- for $1 \leq i \leq m$, there is a unique path of odd length (possibly 1) between a_i and b_i whose interior is in E. There are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq m$;
- c_j is non-adjacent to d_j for $1 \leq j \leq m$. There are all four edges between $\{c_j, d_j\}$ and $\{c_{j'}, b_{j'}\}$ for $1 \leq j < j' \leq n$;
- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ and these two edges are disjoint.

An *homogeneous 2-join* is a partition of $V(G)$ into six non-empty sets (A, B, C, D, E, F) such that:

- (A, B, C, D, E, F) is an homogeneous pair.
- Every vertex in E has degree 2 and belongs to a flat path of odd length with an end in C, an end in D and whose interior is in E.

Our main result is the following:

Theorem 1.5 *Let G be a Berge graph. Then either G is basic, or one of $G, \overline{G}$ is a path-cobipartite graph, or one of $G, \overline{G}$ is a path-double split graph, or one of $G, \overline{G}$ has an homogeneous 2-join, or one of $G, \overline{G}$ has a non-path proper 2-join, or G has an even skew partition.*

This theorem is stronger than Theorems 1.2, 1.3 and 1.4 because path-cobipartite graphs may be seen either as graphs having a proper path 2-join (Theorems 1.2 and 1.3) or as a new basic class (Theorem 1.4). Path-double split graphs may be seen as graphs having a proper path 2-join (Theorems 1.2 and 1.3) or as graphs having a non-even skew partition (Theorem 1.4). And graphs having an homogeneous 2-join may be seen as graph having an homogeneous pair (Theorems 1.4 and perhaps 1.2) or as graphs having a proper path 2-join (Theorems 1.3 and perhaps 1.2). Formally all these remarks are not always true: it may happen in special cases that path-cobipartite graphs and path-double split graphs have no proper 2-join. But such graphs are established in Lemma 2.3 to be basic or to have an even skew partition.

2 Lemmas

The following fact is clear and useful:

Lemma 2.1 *If (A, B) is an even skew partition of a graph G then (B, A) is an even skew partition of $\overline{G}$. In particular, a graph G has an even skew partition if and only if $\overline{G}$ has an even skew partition.*

A *star* in a graph is a set of vertices B such that there is a vertex x in B, called a *center* of the star, seeing every vertices of $B \setminus x$. Note that a star cutset of size at least 2 is a skew cutset.

Lemma 2.2 *Let G be a Berge graph. If G has a star cutset then either G has an even skew partition or G has no edges or G has size 3 or G is the complement of C_4.*

PROOF — We may assume that G has size at least 4 and at least one edge. Let B be a star cutset of G. Let us suppose $|B|$ being maximum with that property. Let A_1, A_2 being such that A_1, A_2, B are pairwise disjoint, there are no edges between A_1, A_2, and $A_1 \cup A_2 \cup B = V(G)$.

Suppose first that B has size 1. Thus up to a symmetry $|A_1| \geq 2$ since G has at least 4 vertices. There is no edge between B and A_1 for otherwise such an edge would be a cutset contradicting $|B|$ being maximum. There is no edge in A_2 since such an edge would be a cutset of G. If there is no edge in A_1, any edge of G is a cutset of G. So, there is an edge e in A_1. So, $|A_1| = 2$ and B is complete to A_2 for otherwise, e is a cutset of G. So, $|A_2| = 1$ for otherwise, any edge between B and A_2 is a cutset edge of G. Now, we observe that G is the complement of C_4.

If B has size at least 2 then B is a skew cutset of G. Let x be a center of B. By maximality of B, every component of $G \setminus B$ has either size 1 or contains no neighbor of x. Thus, if P is a path that makes the skew cutset B non-even, then $P \cup x$ induces an odd hole of G. If Q is an antipath that makes the skew cutset B non-even, then $Q \cup x$ induces an odd antihole of G. □

The following lemma is useful to establish formally that Theorem 1.5 really implies Theorems 1.2, 1.3 and 1.4. But we also need it at several places in the next section.

Lemma 2.3 *Let G be a Berge graph. Then:*

- *If G has a flat path P of length at least 3 then either G is bipartite, or G has an even skew partition or P is the path-side of a proper path 2-join of G.*
- *If G is a path-cobipartite graph, a path-double split graph or has an homogeneous 2-join, then either G has a proper 2-join or G has an even skew partition or G is a bipartite graph, the complement of a bipartite graph, or a double-split graph.*

PROOF — Let us prove the first item. Let P be a flat path of G of length at least 3. So $(P, V(G)\setminus P)$ is a path 2-join of G. Let $(P, X_2, \{a_1\}, \{b_1\}, A_2, B_2)$ be a split of this 2-join. If (P, X_2) is not proper, then either there is a component of X_2 that does not meet one of A_2, B_2, or X_2 induces a path of length 1 or 2. In the last case, G is bipartite, and in the first one, we may assume that there is a component C of X_2 that does not meet B_2. But then, $\{a_1\} \cup (A_2 \setminus C)$ is a star cutset of G that separates C from B_2, and so by Lemma 2.2, G has an even skew partition.

The second item follows easilly: if G is a path-cobipartite graph, then we may assume that G is not the complement of a bipartite graph. If G is a path-double split graph then we may assume that G is not a double split graph. In both cases, G has a flat path of length at least 3. If G has an homogeneous 2-join then it also has a flat path of length at least 3. In every cases, the conclusion follows from the first item. □

Paths and antipaths overlapping 2-joins

Lemma 2.4 *Let G be a Berge graph with a connected 2-join (X_1, X_2). Then all the outgoing paths from A_1 to B_1 and all the outgoing paths from A_2 to*

B_2 have same parity.

PROOF — Note that since (X_1, X_2) is connected there actually exists in $G[X_1]$ an outgoing path P_1 from A_1 to B_1. Similarly, there exists in $G[X_2]$ an outgoing path P_2 from A_2 to B_2. The paths P_1, P_2 have same parity because $P_1 \cup P_2$ induces a hole. Let P be an outgoing path from A_1 to B_1 (the proof is the same for an outgoing path from A_2 to B_2). Let P^* be the interior of P. Then one of $P \cup P_2$, $P^* \cup P_2$ induces a hole. Hence, P, P_1, P_2 have same parity. □

Lemma 2.5 *Let G be a Berge graph with a 2-join (X_1, X_2). Let i be in $\{1, 2\}$. Then every outgoing path from A_i to A_i (resp. from B_i to B_i) has even length. Every antipath of length at least 2 whose interior is in A_i (resp. B_i) and whose ends are outside A_i (resp. B_i) has even length.*

PROOF — Note that we do not suppose (X_1, X_2) being connected, so Lemma 2.4 does not apply. Let P be an outgoing path from A_1 to A_1 (the other cases are similar). If P has a vertex in A_2, then P has length 2. Else, P must lie entirely in X_1 except possibly for one vertex in B_2. If P lies entirely in X_1, then $P \cup \{a_2\}$ where a_2 is any vertex in A_2 induces a hole, so P has even length. If P has a vertex $b_2 \in B_2$, then we must have $P = a - \cdots - b - b_2 - b' - \cdots - a'$ where $a - P - b$ and $b' - P - a'$ are outgoing paths from A_1 to B_1. Suppose that P has odd length. Let a_2 be a vertex of A_2. Then $V(P) \cup \{a_2\}$ induces an odd cycle of G whose only chord is $a_2 b_2$. So one of $V(a - P - b_2) \cup \{a_2\}$, $V(a' - P - b_2) \cup \{a_2\}$ induces an odd hole of G, a contradiction.

Let Q be an antipath of length at least 2 whose interior is in A_1 and whose ends are outside A_1 (the other cases are similar). If Q has length at least 3, then the ends of Q must have a neighbor in A_1 and a non-neighbor in A_1. Hence these ends are in X_1. Thus, $Q \cup \{a\}$, where a is any vertex of A_2 is an antihole of G. Thus, Q has even length. □

Lemma 2.6 *Let G be a graph with a 2-join (X_1, X_2). Let P be a path of G whose end-vertices are in X_2. Then either:*

1. *There are vertices $a \in A_1$, $b \in B_1$ such that $V(P) \subseteq X_2 \cup \{a, b\}$. Moreover, if a, b are both in $V(P)$, then they are non-adjacent.*
2. *$P = c - \cdots - a_2 - a - \cdots - b - b_2 - \cdots - c'$ where: $a \in A_1$, $b \in B_1$, $a_2 \in A_2$, $b_2 \in B_2$. Moreover $V(c - P - a_2) \subset X_2$, $V(b_2 - P - c') \subset X_2$, $V(a - P - b) \subset X_1$.*

PROOF — If P has no vertex in X_1, then for any $a \in A_1, b \in B_1$, the first outcome holds. Else let c, c' be the end-vertices of P. Starting from c, we may assume that first vertex of P in X_1 is $a \in A_1$. Note that a is the only vertex of P in A_1. If a has its two neighbors on P in X_2, then P has no other vertex in X_1, except possibly a single vertex $b \in B_1$ and the first outcome holds. If a has only one neighbor on P in X_2, then let a_2 be this neighbor. Note that P must have a single vertex b in B_1. Let b_2 be the neighbor of b in X_2 along P. Vertices a_2, a, b_1, b_2 show that the second outcome holds. $\Box$

Lemma 2.7 *Let G be a Berge graph with a 2-join (X_1, X_2). Let P be a path of G whose end-vertices are in $A_1 \cup X_2$ (resp. $B_1 \cup X_2$) and whose interior vertices are not in A_1 (resp. B_1). Then either:*

1. *P has even length.*
2. *There are vertices $a \in A_1$, $b \in B_1$ such that $V(P) \subseteq X_2 \cup \{a, b\}$. Moreover, if a, b are both in $V(P)$, then they are non-adjacent.*
3. *$P = a - \cdots - b - b_2 - \cdots - c$ where: $a \in A_1$, $b \in B_1$, $b_2 \in B_2$, $c \in X_2$.*

   Moreover $V(a - P - b) \subset X_1$ and $V(b_2 - P - c) \subset X_2$.

   (resp. $P = b - \cdots - a - a_2 - \cdots - c$ where: $b \in B_1$, $a \in A_1$, $a_2 \in A_2$, $c \in X_2$.

   Moreover $V(b - P - a) \subset X_1$ and $V(b_2 - P - c) \subset X_2$.)

PROOF — Note that we do not suppose (X_1, X_2) being proper. Suppose that the end-vertices of P are in $A_1 \cup X_2$ (the case when the end-vertices of P are all in $B_1 \cup X_2$ is similar).

If P has its two end-vertices in A_1, then by Lemma 2.5, P has even length and Output 1 of the lemma holds.

If P has exactly one end-vertex in A_1, let a be this vertex. Let $c \in X_2$ be the other end-vertex of P. Let a' be the neighbor of a along P. If a' is in A_2, then we may apply Lemma 2.6 to $a' - P - c$: Outcome 2 is impossible and Outcome 1 yields Outcome 2 of the lemma we are proving now since P has exactly one vertex in A_1. If a' is not in A_2, then let b be the last vertex of X_1 along P and b_2 the first vertex of X_2 along P. Outcome 3 of the lemma holds.

If P has no end-vertex in A_1 then Lemma 2.6 applies to P. The second outcome is impossible. The first outcome implies that there is a vertex $b \in B_1$ such that $V(P) \subseteq X_2 \cup \{b\}$ since no interior vertex of P is in A_1. So, Outcome 2 of the lemma we are proving now holds. $\Box$

Lemma 2.8 *Let G be a graph with a 2-join (X_1, X_2). Let Q be an antipath of G of length at least 4 whose interior vertices are all in X_2. Then there is a vertex a in $A_1 \cup B_1$ such that $V(Q) \subseteq X_2 \cup \{a\}$.*

PROOF — Let c, c' be the end-vertices of Q. Note that $N(c) \cap N(c') \cap X_2$ have to be non-empty and that $N(c) \cap X_2$ must be different of $N(c') \cap X_2$, because c, c' are the end-vertices of an antipath of length at least 4. No pair of vertices in X_1 satisfies these two properties, so at most one of c, c' is in $V(Q) \cap X_1$. If none of c, c' are in X_1, then let a be any vertex in A_1, else let a be the unique vertex in X_1 among c, c'. Since c, c' must have a neighbor in X_2, $a \in A_1 \cup B_1$ and clearly $V(Q) \subseteq X_2 \cup \{a\}$. $\square$

Lemma 2.9 *Let G be a Berge graph with a 2-join (X_1, X_2). Let Q be an antipath of G of length at least 5 whose interior vertices are all in $A_1 \cup X_2$ (resp. $B_1 \cup X_2$) and whose end-vertices are not in A_1 (resp. B_1). Then either:*

1. *Q has even length.*
2. *There is a vertex $a \in A_1 \cup B_1$ such that $V(Q) \subseteq X_2 \cup \{a\}$.*

PROOF — We suppose that the interior vertices of Q are all in $A_1 \cup X_2$. The case when the interior vertices of Q are all in $B_1 \cup X_2$ is similar.

If Q has at least 2 vertices in A_1, then let $a \neq a'$ be two of these vertices. Since the end-vertices of Q are not in A_1, a, a' may be chosen in such a way that there are vertices $c, c' \notin A_1$ such that $\overline{c - a - \overline{Q} - a' - c'}$ is an antipath of G. Since c must miss a while seeing a', c must be in $X_1 \setminus A_1$, and so is c'. But the interior vertices of Q cannot be in $X_1 \setminus A_1$, so c, c' are in fact the end-vertices of Q. Also, every interior vertex of Q must be adjacent to at least one of c, c'. Hence, either all the interior vertices of Q are in A_1 and by Lemma 2.5, Q has even length, or $c, c' \in B_1$ and Q has interior vertices in B_2. But in this last case, the interior of Q is an antipath of length at least 3 with vertices in both A_1, B_2, which are anticomplete to one another, a contradiction.

If Q has exactly one vertex a in A_1 then by assumption, a is an interior vertex of Q. Let c, c' be the ends of Q. Suppose $c \in X_1$. Since Q has length at least 5, c must have a neighbor in the interior Q that is different of a, hence $c \in B_1$. Since Q has length at least 5, a and c must have a common neighbor, that must be c' since it must be in X_1. Hence $c' \in X_1$, implying $c' \in B_1$. Now the non-neighbor of c' along Q is not a, so it must be a vertex of X_2 while seeing c and missing c', a contradiction. We proved $c \in X_2$, and similarly $c' \in X_2$. Hence $V(Q) \subset X_2 \cup \{a\}$.

If Q has no vertex in A_1 then Lemma 2.8 applies: there is a vertex $a \in A_1 \cup B_1$ such that $V(Q) \subseteq X_2 \cup \{a\}$. □

Even skew partitions overlapping 2-joins

It is convenient to consider a degenerated kind of 2-join that implies the existence of an even skew partition. A 2-join (X_1, X_2) is said to be *degenerate* if either:

- there exists $i \in \{1, 2\}$ and a vertex v in A_i that has no neighbor in $X_i \setminus (A_i \setminus \{v\})$;
- there exists $i \in \{1, 2\}$ and a vertex v in B_i that has no neighbor in $X_i \setminus (B_i \setminus \{v\})$;
- one of $A_1 \cup A_2$, $B_1 \cup B_2$ is a skew cutset of G;
- there exists $i \in \{1, 2\}$ and a vertex in A_i that is complete to B_i or a vertex in B_i that is complete to A_i;
- there exists $i \in \{1, 2\}$ and a vertex in C_i that is complete to $A_i \cup B_i$.

Lemma 2.10 *Let G be a Berge graph and (X_1, X_2) be a degenerate proper 2-join of G. Then G has an even skew partition.*

PROOF — Let us look at the possible reasons why (X_1, X_2) is degenerate.

If there is a vertex v in A_1 that has no neighbor in $X_1 \setminus (A_1 \setminus \{v\})$, then note that $|A_1| > 1$ since every component of X_1 meets A_1. So $(A_1 \setminus \{v\}) \cup A_2$ is a skew cutset separating v from the rest of the graph. Hence, in $\overline{G}$ there is a star cutset of center v, and by Lemma 2.2 and 2.1, G has an even skew partition. The cases with A_2, B_1, B_2 are similar.

If $A_1 \cup A_2$ is a skew cutset of G then let us check that this skew cutset is even (the case when $B_1 \cup B_2$ is a skew cutset is similar). Since A_1 is complete to A_2, any outgoing path from $A_1 \cup A_2$ to $A_1 \cup A_2$ is either outgoing from A_1 to A_1 or outgoing from A_2 to A_2. Thus, such a path has even length by Lemma 2.5. If there is an antipath Q of length at least 5 with its interior in $A_1 \cup A_2$ and its ends in the rest of the graph, then it must lie entirely in X_1 or X_2, say X_1 up to symmetry. Thus, such an antipath has even length by Lemma 2.5. The case with $B_1 \cup B_2$ is similar.

If there is a vertex $a \in A_1$ that is complete to B_1 (the other cases are symmetric) then suppose first $|A_1| > 1$. Consider $a' \neq a$ in A_1. Hence $(\{a\} \cup N(a)) \setminus a'$ is a star cutset of G separating a' from B_2. So, by Lemma 2.2, we may assume $A_1 = \{a\}$. If $|B_1| > 1$, consider $b \neq b'$ in B_1. Hence, $(\{b\} \cup N(b)) \setminus b'$ is a star cutset of G separating b' from A_2. So again we may assume $B_1 = \{b\}$. Since (X_1, X_2) is proper, $|X_1| \geq 3$, and there is a vertex c in $V(G) \setminus (A_1 \cup B_1)$. Now, $\{a, b\}$ is a star cutset separating c from X_2.

If there is a vertex c complete to $A_i \cup B_i$ then we may assume $C_i = \{c\}$ for otherwise there is another vertex c' in C_i and $\{c\} \cup A_i \cup B_i$ is a star cutset separating c' from the rest of the graph. By the preceding paragraph, we may assume that there is a vertex $a \in A_1$ and a vertex $b \in B_1$ missing a. Then $a{-}c{-}b$ is an outgoing path of even length from A_i to B_i. Thus by Lemma 2.4, there is no edge between A_i and B_i. If there are two vertices $a \neq a' \in A_i$ then $\{a\} \cup N(a) \setminus \{a'\}$ is a star cutset of G separating a' from B_{3-i}. Thus may assume $|A_i| = 1$, and similarly $|B_i| = 1$. Thus, X_i is an outgoing path of length 2 from A_i to B_i contradicting (X_1, X_2) being proper. □

Lemma 2.11 *Let G be a graph with a non-degenerate connected 2-join (X_1, X_2). Let i be in $\{1, 2\}$. Then for every vertex $v \in X_i$ there is a path $P_a = a{-}\cdots{-}v$ and a path $P_b = b{-}\cdots{-}v$ such that:*

- *$a \in A_i$, $b \in B_i$;*
- *Every interior vertex of P_a, P_b is in $X_i \setminus (A_i \cup B_i)$.*

PROOF — Suppose first $v \in X_i \setminus (A_i \cup B_i)$. By the definition of the connected 2-join, every connected component of X_i must meet both A_i and B_i. So X_v, the connected component of v in $G[X_i]$, meets both A_i, B_i and there is at least one path from v to a vertex of B_i in $G[X_i]$. If every path of $G[X_i]$ from v to B_i goes through A_i, then A_i is a cutset of $G[X_i]$ that separates v from B_i. Thus $A_1 \cup A_2$ is a skew cutset of G, so (X_1, X_2) is degenerate, a contradiction. So there is a path P_b as desired, and by the same way, P_a exists.

If $v \in A_i$, then P_a exists and have length 0: put $P_a = v$. The vertex v has a neighbor w in $X_i \setminus A_i$ otherwise (X_1, X_2) is degenerate. By the preceding paragraph, there is a path Q from w to $b \in B_i$ whose interior vertices lie in $X_i \setminus (A_i \cup B_i)$. So P_b exists: consider a shortest path from v to b in $G[V(Q) \cup \{b\}]$. □

Lemma 2.12 *Let G be a Berge graph with a non-degenerate and connected 2-join (X_1, X_2). Let F be an even skew cutset of G. Then for some $i \in \{1, 2\}$ either:*

- *$F \subsetneq X_i$;*
- *$F \cap X_i \subsetneq X_i$ and one of $(F \cap X_i) \cup A_{3-i}$, $(F \cap X_i) \cup B_{3-i}$ is an even skew cutset of G;*

PROOF — We consider three cases:

Case 1: $F \cap A_1$, $F \cap A_2$, $F \cap B_1$, $F \cap B_2$ are all non-empty.

If there is a vertex $a \in A_1 \cap F$ non-adjacent to a vertex $b \in B_1 \cap F$ then there is an antipath of length at most 3 between any vertex of F and a, contradicting $\overline{G}[F]$ being disconnected. Thus $A_1 \cap F$ is complete to $B_1 \cap F$, and similarly $A_2 \cap F$ is complete to $B_2 \cap F$. Similarly, we prove $F \cap C_1 = F \cap C_2 = \emptyset$. If $A_1 \subset F$ then there is a vertex in B_1 that is complete to A_1, contradicting (X_1, X_2) being non-degenerate. Thus $A_1 \setminus F \neq \emptyset$, and similarly $A_2 \setminus F \neq \emptyset$, $B_1 \setminus F \neq \emptyset$, $B_2 \setminus F \neq \emptyset$.

Let E_1 be the component of $G \setminus F$ that contains $(A_1 \setminus F) \cup (A_2 \setminus F)$. Let E_2 be another component of $G \setminus F$. Up to a symmetry we assume $E_2 \cap X_2 \neq \emptyset$. We claim that $F' = (F \cap X_2) \cup A_1$ is a skew cutset of G that separates $E_1 \cap X_2$ from $E_2 \cap X_2$. For suppose not. This means that there is a path P of $G \setminus F'$ with an end in $E_1 \cap X_2$ and an end in $E_2 \cap X_2$. If P has no vertex in X_1 then $P \subset G \setminus F$ and P contradicts E_1, E_2 being components of $G \setminus F$. If P has a vertex in X_1 then this vertex b is unique and is in B_1 because $A_1 \subset F'$. By replacing b by any vertex of $B_1 \setminus F$, we obtain again a path that contradicts E_1, E_2 being components of $G \setminus F$. Thus F' is a skew cutset of G. Note that this skew cutset is included in $A_1 \cup A_2 \cup B_2$. Let us prove that this skew cutset is even.

Let P be an outgoing path from F' to F'. Let us apply Lemma 2.7 to P. If Outcome 1 of the lemma holds then P has even length. If Outcome 2 of the lemma holds then $V(P) \subset X_2 \cup \{a, b\}$. Let a_1 be a vertex of $A_1 \cap F$ and b_1 be a vertex of $B_1 \setminus F$ such that a_1 misses b_1. Note that b_1 exists for otherwise (X_1, X_2) is a degenerate 2-join of G. After possibly replacing a by a_1 and b by b_1, we obtain an outgoing path from F to F that has same length than P. Thus, P has even length since F is an even skew cutset. If Outcome 3 of the lemma holds then P has one end in A_1 and one end in B_2 and P is an outgoing path from A_1 to B_1 plus one edge. Note that there is an edge between A_2 and B_2 so by Lemma 2.4 every outgoing path from A_1 to B_1 has odd length. Hence in every cases P has even length.

Let Q be an antipath with both ends in $G \setminus F'$ and interior in F'. If Q has length 3 then Q may be seen as an outgoing path from F' to F', so we may assume that Q has length at least 5. By Lemma 2.9 applied to Q, either Q has even length or $V(Q) \subset X_2 \cup \{a\}$. If $a \in A_1$ let us replace a by a vertex of $F \cap A_1$ and if $a \in B_1$ let us replace a by a vertex of $B_1 \setminus F$. We obtain an antipath that have same length than Q, that has both ends outside of F and interior in F. Thus Q has even length because F is an even skew cutset.

Case 2: one of $F \cap A_1$, $F \cap A_2$, $F \cap B_1$, $F \cap B_2$ is empty and $F \cap X_1$, $F \cap X_2$ are both non-empty.

We assume up to a symmetry that one of $B_1 \cap F$, $B_2 \cap F$ is empty. Since $F \cap X_1$ and $F \cap X_2$ are both non-empty, there is a least an edge between $F \cap X_1$ and $F \cap X_2$ because $\overline{G}[F]$ is disconnected. Thus we know that $F \cap A_1$ and $F \cap A_2$ are both non-empty. If $(F \cap X_1) \setminus A_1$ and $(F \cap X_2) \setminus A_2$ are both non-empty then there is a vertex of F in one of C_1, C_2 since one of $B_1 \cap F$, $B_2 \cap F$ is empty. Up to a symmetry, suppose $C_1 \cap F \neq \emptyset$. Then $\overline{G}[F]$ is connected since every vertex in it can be linked to a vertex of C_1 by an antipath of length at most 2, a contradiction. Hence one of $(F \cap X_1) \setminus A_1$ and $(F \cap X_2) \setminus A_2$ is empty. Thus we may assume $F \subset X_2 \cup A_1$. Suppose $B_2 \subset F$. Then B_2 and $F \cap A_1$ are in the same component of $\overline{G}[F]$, thus there must be a vertex v in F that is complete to $B_2 \cup (F \cap A_1)$. So, v is in A_2, and v is complete to B_2, contradicting (X_1, X_2) being non-degenerate. We proved that there is at least a vertex u in $B_2 \setminus F$. In particular, $F \cap X_2 \subsetneq X_2$. By Lemma 2.11 there is a path from every vertex of $X_1 \setminus F$ to u, thus there is a component E_1 of $G \setminus F$ that contains $X_1 \setminus F$ and u. There is another component E_2 included in X_2. Thus $(F \cap X_2) \cup A_1$ is a skew cutset of G that separates B_1 from E_2. We still have to prove that the skew cutset $(F \cap X_2) \cup A_1$ is even.

Let P be an outgoing path from $(F \cap X_2) \cup A_1$ to $(F \cap X_2) \cup A_1$. Let us apply Lemma 2.7 to P. If Outcome 1 of the lemma holds then P has even length. If Outcome 2 of the lemma holds then $V(P) \subset X_2 \cup \{a, b\}$. Let a_1 be a vertex of $A_1 \cap F$ and b_1 be a vertex of B_1 such that a_1 misses b_1. Note that b_1 exists for otherwise (X_1, X_2) is a degenerate 2-join of G. After possibly replacing a by a_1 and b by b_1 then we obtain an outgoing path from F to F that has the same length than P. Thus, P has even length since F is an even skew cutset. If Outcome 3 of the lemma holds then $P = a \text{—} \cdots \text{—} b \text{—} b_2 \text{—} \cdots \text{—} c$. Let a_1 be in $A_1 \cap F$. By Lemma 2.11 there is a path P_1 of $G[X_1]$ from a_1 to a vertex $b_1 \in B_1$. Moreover, P_1 is outgoing from A_1 to B_1. Note that by Lemma 2.4, P_1 and $a - P - b$ have same partity. Thus $a_1 - P_1 - b_1 - b_2 - P - c$ is an outgoing path from F to F that has the same parity that P. Thus P

has even length.

If Q is an antipath with both ends in $G \setminus ((F \cap X_2) \cup A_1)$ and its interior in $(F \cap X_2) \cup A_1$, we prove that Q has even length like in Case 1.

Case 3: One of $F \cap X_1, F \cap X_2$ is empty.

Since $F \subsetneq X_2$ is an output of the lemma, we may assume up to a symmetry $F = X_2$. If there is an outgoing path of odd length from A_2 to B_2, then there is by Lemma 2.4 an outgoing path P from A_1 to B_1 of odd length. Hence A_2 is complete to B_2 because a pair of non-adjacent vertices yields together with P an outgoing path of odd length from F to F, contradicting F being an even skew cutset. In particular, there is a vertex of A_2 that is complete to B_2, implying (X_1, X_2) being degenerate, a contradiction. If there is an outgoing path of even length from A_2 to B_2 then by Lemma 2.4 there are no edges between A_2 and B_2. Since $X_2 = F$ is not anticonnected, there is a vertex in C_2 that is complete to $A_2 \cup B_2$, implying again (X_1, X_2) being degenerate, a contradiction. □

Now we turn our attention to types of 2-join whose contraction may create even skew partitions:

- A 2-join (X_1, X_2) is said to be *cutting of type 1* if it has a split $(X_1, X_2, A_1, B_1, A_2, B_2)$ such that:
  1. $G[X_1]$ is an outgoing path from A_1 to B_1.
  2. $G[X_2 \setminus A_2]$ is disconnected.
- A 2-join is said to be *cutting of type 2* if it has a split $(X_1, X_2, A_1, B_1, A_2, B_2)$ such that there exist sets A_3, B_3 satisfaying:
  1. $G[X_1]$ is an outgoing path from A_1 to B_1.
  2. $A_3 \neq \emptyset$, $B_3 \neq \emptyset$, $A_3 \subset A_2$, $B_3 \subset B_2$;
  3. A_3 is complete to B_3;
  4. every outgoing path from $B_3 \cup \{a_1\}$ to $B_3 \cup \{a_1\}$ (resp. from $A_3 \cup \{b_1\}$ to $A_3 \cup \{b_1\}$) has even length;
  5. every antipath with its ends outside of $B_3 \cup \{a_1\}$ (resp. $A_3 \cup \{b_1\}$) and its interior in $B_3 \cup \{a_1\}$ (resp. $A_3 \cup \{b_1\}$) has even length.
  6. $G \setminus (X_1 \cup A_3 \cup B_3)$ is disconnected;
- A 2-join is said to be *cutting* if it is either cutting of type 1 or cutting of type 2.

Let G be a Berge graph and $(X_1, X_2, A_1, B_1, A_2, B_2)$ be a split of a proper 2-join of G. The *pieces* of G with respect to $(X_1, X_2, A_1, B_1, A_2, B_2)$ are the two graphs G_1, G_2 that we describe now. We obtain G_1 by replacing X_2 by a flat path P_2 from a vertex a_2 complete to A_1, to a vertex b_2 complete to B_1. This path has the same parity than an outgoing path from A_1 to B_1. The length of P is decided as follow: if (X_1, X_2) is a path 2-join then P has length 1 or 2, else it has length 3 or 4. The piece G_2 is obtained similarly by replacing X_1 by a flat path.

Lemma 2.13 *Let G be a Berge graph and (X_1, X_2) be a non-cutting, non-degenerate and proper 2-join of G. Then G has an even skew partition if and only if one of the pieces of G has an even skew partition.*

PROOF — Suppose first that G has an even skew partition (E, F). By Lemma 2.12 and up to a symmetry either $F \subsetneq X_2$, or $(F \cap X_2) \subsetneq X_2$ and $A_1 \subset F$ (after possibly replacing F by $(F \cap X_2) \cup A_1$).

If $F \subsetneq X_2$ then we claim that F is an even skew cutset of G_2. Note that there is at least a component E of $G \setminus F$ that has some vertex in X_2 but no vertex in $A_2 \cup B_2$. Else every component of $G \setminus F$ has neighbors in A_1 and B_1 (because (X_1, X_2) is proper) or in $A_2 \cup B_2$, implying $G \setminus F$ being connected, a contradiction. Thus, F is a skew cutset of G_2 that separates E from $V(G_2) \setminus X_2$. Let P be an outgoing path of G_2 from F to F. Let us apply Lemma 2.6 to P. If Outcome 1 of the Lemma holds then after possibly replacing a be a_1 and b by b_1, P may be viewed as an outgoing of G from F to F, thus P has even length. If Outcome 2 of the lemma holds, then $P = c - \cdots - a_2 - a_1 - \cdots - b_1 - b_2 - \cdots - c'$. Let P' be any outgoing path from A_1 to B_1 whose interior is in X_1. Then $c - \cdots - a_2 - P' - b_2 - \cdots - c'$ is an outgoing path of G from F to F that has same parity than P by Lemma 2.4. Thus P has even length. Let Q be an antipath of G_2 with its ends out of F and its interior in F. Let us apply Lemma 2.8 to Q: $V(Q) \subseteq X_2 \cup \{a\}$. Thus, after possibly replacing a by a vertex in $A_1 \cup B_1$, Q may be seen as an antipath of G that has same length than Q. Thus Q has even length.

If $(F \cap X_2) \subsetneq X_2$ and $A_1 \subset F$ then we put $F' = (F \cap X_2) \cup \{a_1\}$. We claim that F' is an even skew cutset of G_2. Exactly as above, we prove that F' is a skew cutset of G_2 that separates b_1 from a component of $G \setminus F$ that has vertices in X_2 but no vertex in B_2. Let P be an outgoing path from F' to F'. As above we prove that P has even length by Lemma 2.7. Let Q be an antipath of G_2 with its ends out of F' and its interior in F'. As above, we prove that Q has even length by Lemma 2.9.

Let us suppose converly that one of G_1, G_2 (say G_2 up to a symmetry) has an even skew cutset F'. We denote by $P_1 = a_1 - \dots b_1$ the path induced by $V(G_2) \setminus X_2$. Note that G_2 has an obvious connected path 2-join: (P_1, X_2).

(1) Either:

- $F' \subsetneq X_2$*;*
- $F' \cap X_2 \subsetneq X_2$ *and one of* $(F' \cap X_2) \cup \{a_1\}$*,* $(F \cap X_2) \cup \{b_1\}$ *is an even skew cutset of* G*;*

If P_1 has length 3 or 4, then (P_1, X_2) is proper. It is non-degenerate because (X_1, X_2) is non-degenerate. Let us apply Lemma 2.12. The conclusion $F' \subsetneq X_1$, is impossible by Lemma 2.11. Also $(F' \cap P_1) \cup A_2$ and $(F' \cap P_1) \cup B_2$ cannot be skew cutsets of G_2, because a_1, b_1 cannot be both in a skew cutset of G_2 since they are non adjacent with no common neighbors. Hence, Lemma 2.11 proves that $(F' \cap P_1) \cup A_2$ and $(F' \cap P_1) \cup B_2$ are not cutsets of G_2. Thus (1) is simply the only possible conclusion of Lemma 2.12.

If P_1 has length 2 then $P_1 = a_1 - c_1 - b_1$. If a_1, b_1 are both in F', then $F' = \{a_1, c_1, b_1\}$ because c_1 is the only common neighbor of a_1, b_1 in G_2. This means that $G_2[X_2] = G[X_2]$ is disconnected, implying that (X_1, X_2) is a cutting 2-join of type 1, a contradiction. By Lemma 2.11 applied to $G_2[X_2] = G[X_2]$, none of a_1, b_1 can be the center of a star cutset of G. Hence, $c_1 \notin F'$. Thus, $F \cap X_2 \subsetneq X_2$ because any induced subgraph of P_1 containing c_1 is connected. We proved (1) when P_1 has length 2.

We are left with the case when $P_1 = a_1 - b_1$. If a_1, b_1 are both in F' then $F' \subset \{a_1, b_1\} \cup A_2 \cup B_2$. If $F' \cap A_2 \neq \emptyset$ and $F' \cap B_2 \neq \emptyset$ then putting $A_3 = F' \cap A_2$ and $B_3 = F' \cap B_2$ then we see that (X_1, X_2) is a cutting 2-join of type 2 of G. If at least one of $F' \cap A_2$ and $F' \cap B_2$ is empty then we see that (X_1, X_2) is a cutting 2-join of type 1. Both cases contradict (X_1, X_2) being non-cutting. Thus we know that at most one of a_1, b_1 is in F. Also $F' \cap X_2 \subsetneq X_2$ because every induced subgraph of P_1 is connected. This proves (1).

By (1), we may assume that not both a_1, b_1 are in F'. Up to a symmetry, we assume $b_1 \notin F'$. If $a_1 \in F'$, put $A'_1 = A_1$, else put $A'_1 = \emptyset$. Now $F = (F' \cap X_2) \cup A'_1$ is a skew cutset of G that separates a vertex of X_2 from $X_1 \setminus A'_1$. The proof that F' is an even skew cutset of G is entirely similar to the similar proofs above: we consider a an outgoing path of G from F' to F'. Lemma 2.6 or Lemma 2.7 shows that P has the same parity than an outgoing path of G_2 from F' to F'. We consider an antipath Q of G of length at least 2 with all its interior vertices in N and with its end-vertices outside of N. Lemma 2.8 or Lemma 2.9 shows that Q has the same parity than an outgoing path of G_2 from F' to F'. $\square$

Even skew partitions overlapping homogeneous 2-joins

Lemma 2.14 *Let G be a Berge graph with an homogeneous 2-join (A, B, C, D, E, F). Let $c \in C, d \in D$ be two vertices such that there is path whose interior is in E between them. Then $F \subset N(c) \cup N(d)$. Moreover, if c, d are not adjacent then $N(c) \cap N(d) \cap F = \emptyset$.*

PROOF — Let P be a path whose interior is in E joinning c, d. If a vertex $f \in F$ misses both c, d, then consider a pair $a \in A, b \in B$ of non-adjacent vertices. Then $\{a, b, f\} \cup P$ induces an odd hole. Thus $F \subset N(c) \cup N(d)$. If c, d are not adjacent, suppose that a vertex $f \in F$ sees both c, d. Since there is at least an edge $a'b'$ with $a' \in A$, $b' \in B$, P has odd length for otherwise $P \cup \{a', b'\}$ induces an odd hole. But since there is a non-edge ab with $a \in A$, $b \in B$, $P \cup \{a, b, f\}$ is an odd hole. Hence $N(c) \cap N(d) \cap F = \emptyset$. $\square$

An homogeneous 2-join (A, B, C, D, E, F) is said to be *degenerate* if either:

- there is a vertex $x \in C$ such that $N(x) \subset A \cup D \cup E$ or a vertex $y \in D$ such that $N(y) \subset B \cup C \cup E$;
- there is a vertex $x \in C$ with no neighbor in $E \cup D$ or a vertex $y \in D$ with no neighbor in $E \cup C$;

Lemma 2.15 *Let G be a Berge graph with a degenerate homogeneous 2-join. Then G has a proper non-path 2-join or G has an even skew partition.*

PROOF — Supppose first that there exists $x \in C$ be such that $N(x) \subset A \cup D \cup E$ (when there exists $y \in D$ such that $N(y) \subset B \cup C \cup E$, the proof is similar). Let N_x be the set containning x plus the vertices of E that lie on a path from x to a vertex of D. Note that by Lemma 2.14, for every $d \in D$ that is the end of such a path, d is complete to F since x has no neighbor in F. Thus, for any $f \in F$, $\{f\} \cup N(F) \setminus B$ is a star cutset of G that separates N_x from B. Thus, by Lemma 2.2, G has an even skew partition.

Suppose now that there exists a vertex $x \in C$ with no neighbor in $E \cup D$. Then, $(A \cup C \cup F) \setminus \{x\}$ is a skew cutset that separates x from the rest of the graph. Thus, $\overline{G}$ has a star cutset centered at x. By Lemma 2.2, $\overline{G}$ has an even skew partition and by Lemma 2.1 so is G. $\square$

The following is needed twice in the next section:

Lemma 2.16 *Let G be a Berge graph. Suppose that G has a vertex u of degree 3 whose neighborhood induces a stable set. Moreover, G has a stable set $\{x, y, z\}$ such that x, y, z all have degree at least 3. Then G is not a path-cobipartite graph, not a path-double split graph and G has no non-degenerate homogeneous 2-join.*

PROOF — In a path-cobipartite graph the vertices of degree at least 3 partition into 2 cliques. Since $\{x, y, z\}$ contradicts this property, G is not a path-cobipartite

In a path-double split graph, every vertex of degree exactly 3 must have an edge in his neighborhood. Since u contradicts this property, G is not a path-double split graph

If G has a non-degenerate homogeneous 2-join (A, B, C, D, E, F), then every vertex in F has degree at least 4. Every vertex in A, B has an edge in his neighborhood. Every vertex in C has a neighbor in C or F for otherwise, (A, B, C, D, E, F) is degenerated. Thus, every vertex in C, and by the same way every vertex in D, has an edge in his neighborhood. Every vertex in E has degree 2. Hence, u is in none of A, B, C, D, E, F, a contradiction. □

3 Proof of Theorem 1.5

For any graph G that is not a hole, let $f(G)$ be the number of maximal flat paths of G. Let us consider G, a counter-example to Theorem 1.5 such that $f(G) + f(\overline{G})$ is minimal.
Since G is a counter-example and since G is Berge, by Theorem 1.3 and up to a complementation of G, we may assume that:

a. G is not basic;

b. None of $G, \overline{G}$ is a path-cobipartite graph;

c. None of $G, \overline{G}$ is a path-double split graph;

d. G has no even skew partition;

e. None of $G, \overline{G}$ has a non-path proper 2-join;

f. None of $G, \overline{G}$ has an homogeneous 2-join;

g. G has a path proper 2-join.

Since G has a path proper 2-join, G has flat path of length at least 3, implying $f(G) \geq 1$. We choose such a flat path X_1 inclusion-wise maximal. Note that by Lemma 2.3, $(X_1, V(G) \setminus X_1)$ is a proper 2-join of G since G is not basic and has no even skew partition. Let us consider $(X_1, X_2, A_1, B_1, A_2, B_2)$ a split of this 2-join. Note that $G[X_2]$ is not a path since G is not bipartite. We denote by a_1 the only vertex in A_1 and by b_1 the only vertex in B_1. We put $C_1 = X_1 \setminus \{a_1, b_1\}$, and $C_2 = X_2 \setminus (A_2 \cup B_2)$. Since X_1 is a maximal flat path we know:

h. a_1, b_1 both have degree at least 3 in G.

If one of G, $\overline{G}$ has a degenerate proper 2-join, a degenerate homogeneous 2-join or a star cutset then one of $G, \overline{G}$ has an even skew partition by Lemma 2.10, Lemma 2.15 or Lemma 2.2. So G has an even skew partition by Lemma 2.1. This contradicts G being a counter-example. Thus:

i. G and $\overline{G}$ have no degenerate proper 2-join, no degenerate homogeneous 2-join and no star cutset.

Let us study the connectivity of G. If $G[X_2]$ is disconnected, then let X_2' be any component of $G[X_2]$. Since (X_1, X_2) is proper, the sets $A_2 \cap X_2'$ and $B_2 \cap X_2'$ are not empty. So $(V(G) \setminus X_2', X_2')$ is a 2-join of G. Let us suppose that X_2' is not an outgoing path length 1 or 2 from A_2 to B_2. This implies that $(V(G) \setminus X_2', X_2')$ is a proper 2-join. So since G is a counter example, we know that $(V(G) \setminus X_2', X_2')$ is a path 2-join of G. Since X_1 is a maximal flat path of G, $V(G) \setminus X_2'$ cannot be the path side of this 2-join. Thus $G[X_2']$ is the path side of this 2-join. Hence we know that every component of X_2 is an outgoing path from A_2 to B_2. This implies that G is bipartite contradicting G being a counter example. Hence:

j. $G[X_2]$ is connected.

Since by property i, (X_1, X_2) is non-degenerate, the following is a direct consequence of Lemma 2.11:

k. In $G[X_2]$, there exists an outgoing path from A_2 to B_2. Moreover, for every $A_2' \subseteq A_2$, $B_2' \subseteq B_2$ the graphs $G[A_2' \cup C_2 \cup B_2 \cup \{b_1\}]$, $G[B_2' \cup C_2 \cup A_2 \cup \{a_1\}]$ are connected.

The eleven properties listed above will be refered as the *properties of* G in the rest of proof. We denote by $\varepsilon \in \{0, 1\}$ the parity of the length of

$G[X_1]$. We now consider three cases according to the properties of (X_1, X_2). In every case, we will consider a graph G' obtained from G by detroying the path 2-join (X_1, X_2), and we will show that G' is a counter-example that contradicts $f(G) + f(\overline{G})$ being minimal.

Case 1: (X_1, X_2) is cutting of type 1.

Up to a symmetry we assume that $G[X_2 \setminus A_2]$ is disconnected. Let X be a component of $G[X_2 \setminus A_2]$. If X is disjoint from B_2 then $\{a_1\} \cup A_2$ is a star cutset of G separating X from $X_2 \setminus X$, contradicting the properties of G. Thus X intersects B_2, and by the same proof so is any component of $X_2 \setminus X$. Hence, there are two non-empty sets $B_3 = B_2 \cap X$ and $B_4 = B_2 \setminus X$. Also we put $C_3 = C_2 \cap X$, $C_4 = C_2 \setminus X$. Possibly, C_3, C_4 are empty. There are no edges between $B_3 \cup C_3$ and $B_4 \cup C_4$.

We consider the graph G' obtained from G by deleting $X_1 \setminus \{a_1, b_1\}$. Moreover, we add new vertices: c_1, c_2, b_3, b_4. Then we add every possible edge between b_3 and B_3, between b_4 and B_4. We also add edges a_1c_1, c_2b_3, c_2b_4. If $\varepsilon = 0$, we consider for convenience $c_1 = c_2$, so that c_1 is always a vertex of G'. Else we consider $c_1 \neq c_2$ and we add an edge between c_1 and c_2. Note that in G', b_1 has neighbors only in B_2. Here are seven claims about the parity of various kinds of paths and antipaths in G'.

(1) Every outgoing path of G' from B_2 to A_2 has length of parity ε.

If such a path contains one of a_1, b_3, b_4, c_1, c_2 then it has length $4 + \varepsilon$. Else such a path may be viewed as an outgoing path of G from B_2 to A_2. By Lemma 2.4 it has parity ε. This proves (1).

(2) Every outgoing path of G' from B_2 to B_2 has even length.

For suppose there is such a path $P = b - \cdots - b'$, $b, b' \in B_2$. If P goes through b_1 then it has length 2. If P goes through b_3 and b_4 it has length 4. If P goes through only one of b_3, b_4 then either P has length 2 or we may assume up to a symmetry that $P = b - b_3 - c_2 - c_1 - a_1 - a - \cdots - b'$ where $a \in A_2$. So, $a - P - b'$ is an outgoing path from A_2 to B_2 and by (1) it has parity ε. So, P has even length. If P goes through c_2 or c_1 then it must goes through at least one of b_3, b_4, and by the discussion above it must have even length. So we may assume that P goes through none of c_1, c_2, b_1, b_3, b_4. Hence P may be viewed as a path of G. Thus, P has even length by Lemma 2.5. In every cases, P has even length. This proves (2).

(3) Every outgoing path of G' from A_2 to A_2 has even length.

For suppose there is such a path $P = a - \cdots - a'$, where $a, a' \in A_2$. If P

goes through a_1 then it has length 2. So we may assume that P does not go through a_1. Note that if $c_1 \neq c_2$ then P does not go through c_1.

If P goes through c_2 or through both b_3, b_4 then we may assume $P = a\text{--}\cdots\text{--}b\text{--}b_3\text{--}c_2\text{--}b_4\text{--}b'\text{--}\cdots\text{--}a'$ where $b \in B_3$ and $b' \in B_4$. By (1) $b\text{--}P\text{--}a$ and $a'-P-b'$ have both parity ε. Thus, P has even length. If P goes through B_3, b_1 and B_4 then we prove that it has even length by the same way. So we may assume that P neither goes through c_2 nor through both b_3, b_4 nor through B_3, b_1 and B_4.

If P goes through exactly one of b_3, b_4, say b_3 up to a symmetry, then just like above $P = a\text{--}\cdots\text{--}b\text{--}b_3\text{--}b'\text{--}\cdots\text{--}a'$, where both $b\text{--}P\text{--}a$ and $a'\text{--}P\text{--}b'$ are outgoing paths from B_2 to A_2. So by (1), they both have parity ε. Thus, P has even length. If P goes through b_1 and exactly one of B_3, B_4, then we prove that it has even length by the same way. So we may assume that P goes though none of b_1, b_3, b_4.

Now P goes through none of $a_1, c_1, c_2, b_1, b_3, b_4$, so P may be viewed as an outgoing path of G from A_2 to A_2. It has even length by Lemma 2.5.

In every cases, P has even length. This proves (3).

(4) Every outgoing path from B_3 to B_3 (resp. from B_4 to B_4) has even length.

Suppose that there is an outgoing path $P = b - \cdots - b'$ from B_3 to B_3 (the case with B_4 is similar). Note that P may have interior vertices in B_4, so (2) does not apply to P. If P goes through b_1 it has length 2. So we may assume that P does not go through B_1. If P has no vertex in A_2, then P has no interior vertices in B_4 since B_3 and B_4 are in distinct components of $G \setminus (B_1 \cup A_2)$. So (2) applies and P has even length.

So we may assume that P has at least a vertex in A_2. Let us then call *B-segment of P* every subpath of P whose end vertices are in B_2 and whose interior vertices are not in B_2. Note that P is edgewise partitioned into its B-segment. Similarly, let us call *A-segment of P* every subpath of P whose end-vertices are in A_2 and whose interior vertices are not in A_2. By (3), every A-segment has even length or has length 1. An A-segment of length 1 is called an *A-edge*. Suppose that P has odd length. Let $b, b' \in B_2$ be the end-vertices of P. Along P from b to b', let us call a the first vertex in A_2 after b, and a' the last vertex in A_2 before b'. So $b-P-a$ and $a'-P-b'$ are both outgoing paths from B_2 to A_2, and by (1) they have same parity. So $a-P-a'$ is a path of odd length that is edgewise partitioned into its A-segment, and that contains all the A-segments of P. Thus P has an odd number of A-edges. Since P is edgewise partitioned into into its B-segments, there is a B-segment P' of P with an odd number of A-edges. Let β, β' be the end-vertices of P'. Along P' from β to β', let us call α the first vertex in A_2 after β, and α' the last vertex in A_2 before β'. So $P'' = \alpha - P' - \alpha'$ is a path that is edgewise partitioned into its A-segment with an odd number of A-edge. Thus P'' has odd length. Since $\beta - P - \alpha$ and $\alpha' - P - \beta'$ are both outgoing paths from B_2 to A_2, they have same parity by (1). Finally, P' is of odd length, outgoing from B_2 to B_2, and contradicts (2). Thus P has even length. This proves (4).

(5) *Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus A_2$, and all its interior vertices in A_2 has even length.*

Let Q be such an antipath. We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in A_2 and a non-neighbor in A_2. So none of $a_1, c_1, c_2, b_1, b_3, b_4$ can be an end-vertex of Q, and Q may be viewed as an antipath of G. So Q has even length by Lemma 2.5. This proves (5).

(6) *Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus B_2$, and all its interior vertices in B_2 has even length.*

Let Q be such an antipath. We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in B_2 and a non-neighbor in B_2. So none of a_1, b_1, c_1, c_2 can be an end-vertex of Q. If b_3 is an end-vertex of Q, then the other end-vertex must be adjacent to b_3 while not being in $B_2 \cup \{a_1, b_1, c_1, c_2\}$, a contradiction. So b_3 is not an end-vertex of Q and by a similar proof, neither b_4 is. So none of $a_1, c_1, c_2, b_1, b_3, b_4$ is in Q and Q may be viewed as an antipath of G. So Q has even length by Lemma 2.5. This proves (6).

(7) *Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus B_3$ (resp. $V(G') \setminus B_4$), and all its interior vertices in B_3 (resp. B_4) has even length.*

Let Q be such an antipath whose interior is in B_3 (the case with B_4 is similar). We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in B_3. So no vertex of B_4 can be an end-vertex of Q. Thus (6) applies and Q has even length. This proves (7).

(8) *Let Q be an antipath of G' of length at least 4. Then Q does not go through c_1, c_2. Moreover Q goes through at most one of a_1, b_1, b_3, b_4.*

In an antipath of length at least 4, each vertex either is in a square of the antipath or in a triangle of the antipath. So, c_1, c_2 are not in Q since they are not in any triangle or square of G'. In an antipath of length at least 4, for any pair x, y of non-adjacent vertices, there must be a third vertex adjacent to both x, y. Thus, Q goes through at most one vertex among a_1, b_3, b_4. Suppose now that Q also goes through b_1. Then it does not go through a_1 since a_1, b_1 have no common neighbours. So, up to a symmetry we may assume that Q goes through b_3 and b_1. There is no vertex in $G' \setminus c_2$ seeing b_3 and missing b_1. So b_1 is an end of Q. Along Q, after b_1 we meet b_3. The next vertex along Q must be in B_4. The next one, in B_3. The next one must see b_3 and must have a neighbor in B_4, a contradiction. This proves (8).

(9) G' is Berge.

Let H be a hole of G'. Suppose first that H goes through a_1. If H does not go through c_1, then $H \setminus a_1$ is a path of even length by (3), so H has even length. If H goes through c_1 then H goes though exactly one of b_3, b_4, say b_3 up to symmetry, and $H \setminus \{a_1, c_1, c_2, b_3\}$ is a path P. If P does not go through b_1 then it has parity ε by (1). If P goes through b_1, then $P = b–b_1–b'–\dots–a$ where $b'-P-a$ is outgoing from B_4 to A_2. So, again P has parity ε by (1). So H has even length and we may assume that H does not go through a_1. If $c_1 \neq c_2$ then H does not go through c_1. If H goes through c_2 then the path $H \setminus \{b_3, c_2, b_4\}$ has even length by (2), so H is even. If H goes through b_1 then the path $H \setminus \{b_1\}$ has even length by (2), so H is even. So we may assume that H does not go through b_1, c_2. If H goes through both b_3, b_4 then $H \setminus \{b_3, b_4\}$ is partitionned into two outgoing paths from B_2 to B_2 that both have even length by (2). Thus H has even length. If H goes through b_3 and not through b_4, then $H \setminus b_3$ is an outgoing path from B_3 to B_3. By (4) it has even length, so H is even. If H goes through b_4 and not through b_3 then H is even by a similar proof. So we may assume that H goes through none of b_3, b_4. Now, H goes through none of $a_1, c_1, c_2, b_1, b_3, b_4$. So H may be viewed as a hole of G, and so it is even. So every hole of G' is even.

Let us now consider an antihole H of G'. Since the antihole on 5 vertices is isomorphic to C_5, we may assume that H has at least 7 vertices. Let v be a vertex of H that is not in $\{a_1, c_1, c_2, b_1, b_3, b_4\}$. By (8) applied to $H \setminus \{x\}$, H does not go through c_1, c_2 and goes through at most one vertex of $\{a_1, b_1, b_3, b_4\}$. If H goes through a_1, the antipath $H \setminus a_1$ has all its interior vertices in A_2 and by (5), $H \setminus a_1$ has even length, thus H is even. If H goes through b_1 then the antipath $H \setminus b_1$ has all its interior vertices in B_2 and by (6), $H \setminus b_1$ has even length, thus H is even. If H goes through one of

b_3, b_4, say b_3 up to a symmetry, the antipath $H \setminus b_3$ has all its interior vertices in B_3 and by (7), $H \setminus b_3$ has even length, thus H is even. If H goes through none of $a_1, c_1, c_2, b_1, b_3, b_4$ then H may be viewed as an antihole of G. So every antihole of G' has even length. This proves (9).

(10) G' has no even skew partition.

Let (F', E') be an even skew partition of G' with a split (E'_1, E'_2, F'_1, F'_2). Starting from F', we shall build an even skew cutset F of G which contradicts the properties of G.

Let us first suppose $c_1 \neq c_2$ and $c_1 \in F'$. Then, F' must contains at least a neighbor of c_1. If F' contains a_1 and not c_2, then F' is a star cutset of G' centered at a_1. But this contradicts the property k of G. If F' contains c_2 and not a_1, then F' is a star cutset of G' centered at c_2. But this again contradicts the property k of G. So, F' must contain a_1 and c_2. Since a_1, c_2 have no common neighbors we have $F' = \{a_1, c_1, c_2\}$. This is a contradiction since $G' \setminus \{a_1, c_1, c_2\}$ is connected by the property k of G. So if $c_1 \neq c_2$ then $c_1 \notin F'$.

Suppose $c_2 \in F'$. By the property k of G, no subset of $\{c_2, b_3, b_4\}$ can be a cutset of G. So, F' must be a star cutset centered at one of b_3, b_4. This again contradicts the property k of G. So $c_2 \notin F'$. Not both b_3, b_4 can be in F' since they have no common neighbors in F'. So we assume $b_4 \notin F'$

Up to a symmetry, we may assume $\{c_1, c_2, b_4\} \subset E'_1$. Also, $\{a_1, b_3\} \cap E' \subset E'_1$. We claim that $\{b_1\} \cap E' \subset E'_1$. Else, F' separates b_1 from c_2. Since F' separates b_1 from c_2 we must have $B_4 \subset F'$. Now $b_3 \in F'$ is impossible since there is no vertex seeing b_3 and having a neighbor in B_4. So, $B_3 \subset F'$. Since there is no edge between B_3 and B_4, there must be a vertex in F' that is complete to $B_3 \cup B_4 = B_2$. The only place to find such a vertex is in A_2. But this implyies (X_1, X_2) being degenerate, contradicting the properties of G.

We proved $\{c_1, c_2, b_4\} \subset E'_1$ and $\{a_1, b_1, b_3\} \cap E' \subset E'_1$. Let v be any vertex of E'_2. Since $\{a_1, c_1, c_2, b_1, b_3, b_4\} \cap E' \subset E'_1$, we have $v \in X_2$. If b_3 is in F, put $B'_1 = \{b_1\}$, else put $B'_1 = \emptyset$. Now $F = (F' \setminus \{b_3\}) \cup B'_1$ is a skew cutset of G that separates v from the interior vertices of the path induced by X_1. Indeed, either $F = F'$, or F' is obtained by deleting b_3 and adding b_1. Since $N(b_3) \cap X_2 \subset N(b_1) \cap X_2$, F is not anticonnected and is a cutset. It suffices now to prove that F is an even skew cutset of G.

Let P be an outgoing path of G from F to F. We shall prove that P has even length.

If $a_1, b_1 \notin F$, then $F \subset X_2$ and the end-vertices of P are both in X_2. So

Lemma 2.6 applies to P. Suppose that the first outcome of Lemma 2.6 is satisfied: $V(P) \subseteq X_2 \cup \{a_1, b_1\}$. Note that by the definition of F, $b_1 \notin F$ implies $b_1 \notin F'$. Hence, P may be viewed as an outgoing path from F' to F', so P has even length since F' is an even skew cutset of G'. Suppose now that the second outcome of Lemma 2.6 is satisfied: $P = c - \cdots - a_2 - a_1 - X_1 - b_1 - b_2 - \cdots - c'$. Put $i = 3$ if $b_2 \in B_3$ and $i = 4$ if $b_2 \in B_4$. Put $P' = c - P - a_2 - a_1 - c_1 - c_2 - b_i - b_2 - P - c'$. Note that by the definition of F, $b_1 \notin F$ implies $b_3 \notin F'$. The paths P and P' have same parity and P' is an outgoing path of G' from F' to F'. So P' and P has even length since F' is an even skew cutset of G'.

If $a_1 \in F$, note that $b_1 \notin F$ since a_1, b_1 are non-adjacent with no common neighbors (in both G, G'). We have $F' = F \subset X_2 \cup \{a_1\}$, the end-vertices of P are both in $X_2 \cup \{a_1\}$ and no interior vertex of P is in $\{a_1\}$ since $a_1 \in F$. So Lemma 2.7 applies. If Outcome 1 of the lemma holds, then P has even length. If Outcome 2 of the lemma holds, then just like in the preceding paragraph, we can build a path P' of G' that is outgoing from F to F and that has a length with the same parity than P. So P has even length. If Outcome 3 of the lemma holds, the proof is again similar to the preceding paragraph.

If $b_1 \in F$ then $a_1 \notin F$, $F \subset X_2 \cup \{b_1\}$, and Lemma 2.7 applies. If Outcome 1 of the lemma holds, then P has even length. If Outcome 2 of the lemma holds, we may assume that b_1 that is in $F \setminus F'$ and that b_1 is an end of P, for otherwise the proof works like in the paragraph above. Then we build a path P' of G' that is outgoing from F' to F' and that has a length with same parity than P, by replacing $\{b_1\}$ by $\{b_3\}$ (if P goes through B_3) or by $\{b_3, c_2, b_4\}$ (if P goes through b_4). So P has even length. If Outcome 3 of the lemma holds then $P = b_1 - X_1 - a_1 - a_2 - \cdots - c$ where $a_2 \in A_2$, $c \in X_2$. Note that one of b_1, b_3 is in F'. If $b_3 \in F'$, then we put $P' = b_3 - c_2 - c_1 - a_1 - a_2 - P - c$. If $b_3 \notin F'$ then up to a symmetry, we assume $V(a - P - c) \subset A_2 \cup C_3$. Note that $b_1 \in F'$. We put $P' = b_1 - b - b_4 - c_2 - c_1 - a_1 - a_2 - P - c$ where b is any vertex in B_4. It may happen that P' is not a path of G' because of the chord $a_2 b$. But then we put $P' = b_1 - b - a_2 - P - c$. In every cases, P' is outgoing from F' to F', and has same parity than P. Hence, P has even length.

Now, let Q be an antipath of G of length at least 2 with all its interior vertices in F and with its end-vertices outside of F. We shall prove that Q has even length. Note that we may assume that Q has length at least 5, because if Q has length 3, it may be viewed as an outgoing path from F to F, that have even length by the discussion above on paths.

If both $a_1, b_1 \notin F$, then $F \subset X_2$ and the interior vertices of Q are all in X_2. So Lemma 2.8 applies: $V(Q) \subseteq X_2 \cup \{a\}$ where $a \in \{a_1, b_1\}$. So Q may be viewed as an antipath of G' that has even length because F' is an even skew cutset of G'.

If $a_1 \in F$, let us remind that $b_1 \notin F$. We have $F \subset X_2 \cup \{a_1\}$, the interior vertices of Q are in $X_2 \cup \{a_1\}$ and the end-vertices of Q are not in $\{a_1\}$ since $a_1 \in F$. So Lemma 2.9 applies. We may assume that Outcome 2 holds. Once again, Q may be viewed as an outgoing path of G' that has even length because F' is even.

If $b_1 \in F$, we have to consider the case when $b_1 \notin F'$ (else the proof is like in the paragraph above). Since $b_1 \notin F'$, we have $b_3 \in F'$. Note that $B_4 \cap F' = B_4 \cap F = \emptyset$ since there are no edges between b_3, B_4 and no vertex seeing b_3 while having a neighbor in B_4. So, if Q is an antipath whose interior is in F, then Q does not go through B_4. Hence, if we replace b_1 by b_3, we obtain an antipath Q' whose interior is in F' and whose ends are not. Hence, Q has even length.

In every cases, Q has even length. This proves (10).

(11) G' and $\overline{G'}$ have no degenerate proper 2-join, no degenerate homogeneous 2-join and no star cutset.

If one of G', $\overline{G'}$ has a degenerate proper 2-join, a degenerate homogeneous 2-join or a star cutset then one of G', $\overline{G'}$ has an even skew partition by Lemma 2.10, 2.15 or 2.2. This contradicts (10). This proves (11).

(12) G' is not basic, not a path-cobipartite graph, not a path-double split graph and has no homogeneous 2-join.

If G' is bipartite then all the vertices of A_2 are of the same color because of a_1. Because of b_1 all the vertices of B_2 have the same color. By the property k of G, there is an outgoing path from A_2 to B_2 that has partity ε by (1). So, the number of colors in $A_2 \cup B_2$ is equal to $1+\varepsilon$, implying that G is bipartite and contradicting the properties of G. Hence G' is not bipartite.

One of the graphs $G'[c_2, c_1, b_3, b_4]$, $G'[a_1, c_1, b_3, b_4]$ is a claw, so G' is not the line-graph of a bipartite graph.

Let us choose $b \in B_3, b' \in B_4$. The graph $\overline{G'}[a_1, c_1, b, b']$ is a diamond, so $\overline{G'}$ is not the line-graph of a bipartite graph. Note that b, b' both have degree at least 3 in G' because since (X_1, X_2) is not degenerate, b, b' have neighbors in $A_2 \cup C_2$. Also a_1 has degree at least 3 in G' by the property h of G. So, there exist in G' a stable set of size 3 containning vertices of degree at least 3 ($\{a_1, b, b'\}$), and a vertex of degree 3 whose neighborhood induces a stable set (c_1). Hence, by Lemma 2.16, G' is not a path-cobipartite graph (and in particular, it is not the complement of a bipartite graph), not a path-double split graph (and in particular, it is not a double split graph) and G' has no non-degenerate homogeneous 2-join. Hence by (11), G' has no homogeneous 2-join. This proves (12).

(13) There exist no sets Y_1, Z_1, Y_2, Z_2 *such that:*

- Y_1, Z_1, Y_2, Z_2 *are pairwise disjoint and* $Y_1 \cup Z_1 \cup Y_2 \cup Z_2 = X_2$*;*
- *There are every possible edges between* Y_1 *and* Y_2*, and these edges are the only edges between* $Y_1 \cup Z_1$ *and* $Y_2 \cup Z_2$*;*
- $A_2 \subset Y_1 \cup Z_1$ *and* $B_2 \subset Y_2 \cup Z_2$*.*

Suppose such sets exist. Note that $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$ since by the property j of G, $G[X_2]$ is connected. Note that Z_1, Z_2 can be empty. Suppose $Y_2 \cap B_2 \neq \emptyset$ and pick a vertex $b \in Y_2 \cap B_2$. Up to a symmetry we assume $b \in B_3$ and we pick a vertex $b' \in B_4$. Since $B_2 \subset Y_2 \cup Z_2$ we have $b' \in Y_2 \cup Z_2$. Now $\{b\} \cup N(b)$ is a star cutset of G that separates a_1 from b', contradicting the properties of G. Thus $Y_2 \cap B_2 = \emptyset$. Hence $(Y_2 \cup Z_2, V(G) \setminus (Y_2 \cup Z_2))$ is a 2-join of G. This 2-join is proper (the check of connectivity relies on the fact that (X_1, X_2) is connected and on Lemma 2.11). By the properties of G, this 2-join has to be a path 2-join. Since X_1 is a maximal flat path of G, $Y_2 \cup Z_2$ is the path-side of the 2-join. This is impossible because $|B_2| \geq 2$. This proves (13).

We now give four claims describing the proper 2-joins of G'. Implicitly, when (X'_1, X'_2) is a 2-join, we consider a split $(X'_1, X'_2, A'_1, B'_1, A'_2, B'_2)$. We also put $C'_1 = X'_1 \setminus (A'_1 \cup B'_1)$ and $C'_2 = X'_2 \setminus (A'_2 \cup B'_2)$.

(14) If G' *has a proper 2-join* (X'_1, X'_2) *then either* $\{c_1, c_2\} \subset X'_1$ *or* $\{c_1, c_2\} \subset X'_2$*.*

Suppose not. We may assume that there is a 2-join (X'_1, X'_2) such that $c_1 \in X'_2$ and $c_2 \in X'_1$. In particular, $c_1 \neq c_2$. Up to a symmetry, we assume $c_1 \in A'_2$ and $c_2 \in A'_1$. Then, $a_1 \in X'_2$ for otherwise c_1 is isolated in X'_2, contradicting (X'_1, X'_2) being proper. Also one of b_3, b_4 must be in X'_1 for otherwise c_2 is isolated in X'_1. Up to a symmetry we assume $b_3 \in X'_1$.

By the property k of G there is an outgoing path $P = h_1 - \cdots - h_k$ from A_2 to B_3 with $h_1 \in A_2$, $h_k \in B_3$. We denote by H the hole induced by $V(P) \cup \{a_1, c_1, c_2, b_3\}$. Note that H has an edge whose ends are both in X'_1 (it is c_2b_3) and an edge whose ends are both in X'_2 (it is a_1c_1). So H is vertex-wise partitionned into an outgoing path from A'_1 to B'_1 whose interior is in X'_1 and outgoing path from B'_2 to A'_2 whose interior is in X'_2. Hence, starting from c_1, then going to a_1 and continuing along H, one will first stay in X'_2, will meet a vertex in B'_2, immediatley after that, a vertex in B'_1, and after that will stay in X'_1 and reach c_2. We now discuss several cases according to the unique vertex x in $H \cap B'_2$.

If $x = a_1$ then $a_1 \in B'_2$. So $b_3 \in C'_1$. This implies step by step $B_3 \subset X'_1$, $B_3 \subset C'_1$, $b_1 \in X'_1$, $b_1 \in C'_1$, $B_4 \subset X'_1$, $B_4 \subset C'_1$, $b_4 \in X'_1$. Let v a vertex in C_2 (if any). Then by the property k of G there is a path Q from v to B_2 with no vertex in A_2. If $v \in X'_2$, then Q must contain a vertex in $A'_1 \cup B'_1$. This is impossible since no vertex in $C_2 \cup B_2$ sees a_1 or c_1. So, $C_2 \subset C'_1$. Let v be a vertex in A_2. Note that by the property k of G, v must have a neighbor in $C_2 \cup B_2$. So, $v \in X'_1$ since $C_2 \cup B_2 \subset C'_1$. Finally, we proved $X'_2 = \{a_1, c_1\}$. This is impossible since (X'_1, X'_2) is proper.

If $x = h_i$ with $1 \leq i < k$, then $h_i \in B'_2 \cap (A_2 \cup C_2)$ and $h_{i+1} \in B'_1$. Note that $b_3 \in C'_1$ since b_3 misses c_1 and h_1. So, $B_3 \subset X'_1$. By the definition of x, we know that $a_1 \in C'_2$. So, $A_2 \subset X'_2$. We consider now two cases.

First case: $b_4 \in X'_1$. Since there are no edges between $\{b_3, b_4\}$ and $\{c_1, h_1\}$ we know that $\{b_3, b_4\} \subset C'_1$. This implies $B_3 \cup B_4 \subset X'_1$. Also, $b_1 \in X'_1$ for otherwise b_1 is isolated in X'_2. Now, $A'_1 \cup B'_1 \subset (B_2 \cup C_2)$. Let us put: $Y_1 = B'_2$, $Z_1 = (X'_2 \cap X_2) \setminus Y_1$, $Y_2 = B'_1$, $Z_2 = (X'_1 \cap X_2) \setminus Y_2$. These four sets yield a contradiction to (13).

Second case: $b_4 \in X'_2$. Then $b_4 \in A'_2$ and $A'_1 = \{c_2\}$. If there is a vertex v of X'_1 in B_4 then $v \in A'_1$. This is impossible since v misses $c_1 \in A'_2$. So, $B_4 \subset X'_2$. Hence, if $b_1 \in X'_1$ then $b_1 \in A'_1 \cup B'_1$. But this is impossible since b_1 misses c_1 and h_1. So, $b_1 \in X'_2$. Since $B_3 \subset X'_1$, we know $B_3 = B'_1$ and $b_1 \in B'_2$. So b_3 is a vertex of C'_1 complete to $A'_1 \cup B'_1$, implying (X'_1, X'_2) being degenerate, a contradiction.

If $x = h_k$ then $a_1 \in C'_2$ and $A_2 \subset X'_2$. Let v be a vertex of $C_2 \cup B_3 \cup B_4 \cup \{b_1, b_4\}$. By the property k of G there is a path Q from v to A_2 with no interior vertex in $B_3 \cup A_2$. If $v \in X'_1$, then Q must have a vertex $u \neq v$ in $A'_2 \cup B'_2$. Note $u \notin B_3$.This is impossible because u misses c_2 and b_3. So, $v \in X'_2$. Hence, $X'_1 = \{c_2, b_3\}$ contradicting (X'_1, X'_2) being proper. This proves (14).

(15) If G' has a 2-join (X'_1, X'_2) then either $\{c_1, c_2, b_3, b_4\} \subset X'_1$ or $\{c_1, c_2, b_3, b_4\} \subset X'_2$.

Suppose not. By (14), we may assume that there is a 2-join (X'_1, X'_2) such that $c_1, c_2 \in X'_1$ and $b_3 \in X'_2$. Up to a symmetry, we assume $c_2 \in A'_1$ and $b_3 \in A'_2$. At least one vertex of B_3 is in X'_2 for otherwise b_3 is isolated in X'_2. So let b be a vertex of $X'_2 \cap B_3$. We claim that there is a hole H that goes through b_3, c_2, c_1, a_1, $h_1 \in A_2$, $\dots h_k = b$, with at least an edge in X'_1 and at least an edge in X'_2. If $c_1 \neq c_2$ then our claim hold trivially: $c_1c_2 \in X'_1$ and $b_3b \in X'_2$. If $c_1 = c_2$, suppose that our claim fails. Then $a_1 \in X'_2$, implying $A'_1 = \{c_2\}$ and $a_1 \in A'_2$. We have $b_4 \in X'_1$ for othewise c_2 is isolated in X'_1. If $b_4 \in B'_1$ then (X'_1, X'_2) is degenerate since b_4 is complete to A'_1. So, $b_4 \in C'_1$ implying $B_4 \subset X'_1$. If $b_1 \in X'_2$ then $b \in B'_1$ since $b \in X'_2$. So $B'_2 \subset B_3$ and b_3 is a vertex of A'_2 that is complete to B'_2, implying (X'_1, X'_2) being degenerate, a contradiction. So $b_1 \in X'_2$. Hence $B'_1 = B_4$ because no vertex of B'_1 can be in B_3 since $b_3 \in A'_2$. So $b_4 \in C'_1$ is complete to $A'_1 \cup B'_1$, implying (X'_1, X'_2) being degenerate, a contradiction. Thus our claim holds: H has an edge in X'_1 and an edge in X'_2. So there is a unique vertex x in $H \cap B'_2$. We now discuss according to the place of x.

If $x = a_1$ then by the discussion above $c_1 \neq c_2$. Also, $a_1 \in B'_2$ and $c_1 \in B'_1$. Suppose that $X'_1 \cap X_2$ and $X'_2 \cap X_2$ are both non-empty. The vertices of $A'_2 \cup B'_2$ are not in X_2 because they have to see either c_1 or c_2. So there are no edges between $X'_1 \cap X_2$ and $X'_2 \cap X_2$. Hence, $G'[X_2]$ is not connected, contradicting the property j of G. So either $X_2 \subset X'_1$ or $X_2 \subset X'_2$. If $X_2 \subset X'_1$ then $X'_2 \subset \{a_1, b_1, b_3, b_4\}$, so X'_2 is a stable set, contradicting (X'_1, X'_2) being proper. If $X_2 \subset X'_2$ then b_1 is in X'_2 for otherwise it is isolated in X'_1. So, $X'_1 \subset \{c_1, c_2, b_4\}$. This is a contradiction since by checking every cases, we see that no subset of $\{c_1, c_2, b_4\}$ can be a side of a proper 2-join of G'.

If $x = h_1$ then $h_1 \in B'_2$ and $a_1 \in B'_1$. If $b_4 \in X'_1$ then $b_4 \in C'_1$ because of b_3 and h_1. So, $B_4 \subset X'_1$. But in fact, by the same way, $B_4 \subset C'_1$, and $b_1 \in C'_1$. So, $B_3 \subset X'_1$, contradicting $h_k \in X'_2$. We proved $b_4 \in X'_2$ implying $A'_1 = \{c_2\}$. If a vertex v of $X_2 \cup \{b_1\}$ is in X'_1, then by Lemma 2.11 applied to (X'_1, X'_2) there is a path of X'_1 from v to $A'_1 = \{c_2\}$ with no interior vertex in B'_1, a contradiction. So $X_2 \cup \{b_1\} \subset X'_2$. We proved $X'_1 = \{a_1, c_1, c_2\}$ contradicting (X'_1, X'_2) being proper.

If $x = h_i$, $2 \leq i \leq k$ then $h_i \in B'_2$, $h_{i-1} \in B'_1$. Since $a_1 \in C'_1$ we have $A_2 \subset X'_1$. If $b_4 \in X'_1$ then $b_4 \in C'_1$ implying $B_4 \subset X'_1$. If $b_1 \in X'_2$ then b_1 must be in $A'_2 \cup B'_2$, a contradiction since b_1 misses c_2 and h_{i-1}. So, $b_1 \in X'_1$. Since $h_k \in X'_2$, we know $b_1 \in B'_1$. Thus $B'_2 \subset B_3$. Hence b_3 is a vertex of A'_2 that is complete to B'_2, implying (X'_1, X'_2) being degenerate, a contradiction. We proved $b_4 \in X'_2$. Now $A'_2 = \{b_3, b_4\}$. Suppose that there is a vertex v of X'_1 in $B_3 \cup B_4$. Then v must be in A'_1 since v sees one of b_3, b_4. But this is a contradiction since v misses one of b_3, b_4. We proved $B_3 \cup B_4 \subset X'_2$. Also, $b_1 \in X'_2$ for otherwise, b_1 is isolated in X'_1. Let us put: $Y_1 = B'_1$, $Z_1 = (X'_1 \cap X_2) \setminus Y_1$, $Y_2 = B'_2$, $Z_2 = (X'_2 \cap X_2) \setminus Y_2$. These four sets yield a contradiction to (13). This proves (15).

(16) If G' has a proper 2-join (X'_1, X'_2) then either $\{c_1, c_2, b_1, b_3, b_4\} \subset X'_1$ or $\{c_1, c_2, b_1, b_3, b_4\} \subset X'_2$.

Suppose not. By (15), we may assume that there is a 2-join (X'_1, X'_2) of G' such that $c_1, c_2, b_3, b_4 \in X'_1$ and $b_1 \in X'_2$. If $\{b_3, b_4\} \cap (A'_1 \cup B'_1) = \emptyset$ then $\{b_3, b_4\} \subset C'_1$, so $B_3 \cup B_4 \subset X'_1$. Hence b_1 is isolated in X'_2, a contradiction.

If $|\{b_3, b_4\} \cap (A'_1 \cup B'_1)| = 1$, then up to a symmetry we may assume $b_3 \in A'_1$ and $b_4 \in C'_1$. Thus $B_4 \subset X'_1$. Since $b_2 \in X'_2$, we have $B_4 \subset A'_1 \cup B'_1$. But no vertex x of B_4 can be in A'_1 because x and b_3 have no common neighbors, so $B_4 \subset B'_1$. Thus $b_1 \in B'_2$. Because of b_3, $A'_2 \subset B_3$. So b_1 is a vertex of B'_2 that is complete to A'_2, implying (X'_1, X'_2) being degenerate, a contradiction. We proved $\{b_3, b_4\} \subset (A'_1 \cup B'_1)$.

Since b_3, b_4 have no common neighbors in X'_2, we may assume up to a symmetry that $b_3 \in A'_1$ and $b_4 \in B'_1$. So b_2 have non-neighbors in both A'_1, B'_1. This implies $b_2 \in C'_2$, and $B_3 \cup B_4 \subset X'_2$. Hence $A'_2 = B_3$ and $B'_2 = B_4$. Now, $b_1 \in C'_2$ is complete to $A'_2 \cup B'_2$, implying (X'_1, X'_2) being degenerate, a contradiction. This proves (16).

(17) G' has no proper non-path 2-join.

Let (X'_1, X'_2) be a proper 2-join of G'. By (16), we may assume $\{c_1, c_2, b_1, b_3, b_4\} \subset X'_2$. If $b_3 \notin C'_2$ and $b_4 \notin C'_2$ then up to a symmetry we may assume $b_3 \in A'_2$, $b_4 \in B'_2$ since b_3, b_4 have no common neighbors in X'_1. So, there is a vertex of A'_1 in B_3 and a vertex of B'_1 in B_4 implying $b_1 \in A'_2 \cap B'_2$, a contradiction. We proved $b_3 \in C'_2$ or $b_4 \in C'_2$. Up to a symmetry we assume $b_3 \in C'_2$, implying $B_3 \subset X'_2$. Note that X'_1 is a subset of $V(G)$. If $A'_1 \cap B_4, B'_1 \cap B_4$ are both non-empty then b_1 must be in $A'_2 \cap B'_2$, a contradiction. Thus we may assume $A'_1 \cap B_4 = \emptyset$. If $a_1 \in X'_1$ and $B'_1 \cap B_4 \neq \emptyset$ then $a_1 \notin B'_1$ since a_1 misses b_1. Thus we may assume $B'_1 \cap \{a_1\} = \emptyset$.

Let us now put: $X''_1 = X'_1$, $X''_2 = V(G) \setminus X''_1$, $A''_1 = A'_1$, $B''_1 = B'_1$, $B''_2 = B'_2 \setminus \{b_4\}$. If $a_1 \in A'_1$ then $A''_2 = (A'_2 \cap X_2) \cup (N_G(a_1) \cap X_1)$ else $A''_2 = A'_2$. Note that $A''_2 \cap B''_2 = \emptyset$. Also, if $b_4 \in B'_2$ then $b_1 \in B'_2$ and $b_1 \in B''_2$. From the definitions it follows that (X''_1, X''_2) is a partition of $V(G)$, that $A''_1, B''_1 \subset X''_1$, $A''_2, B''_2 \subset X''_2$, that A''_1 is complete to A''_2, that B''_1 is complete to B''_2 and that there are no other edges between X''_1 and X''_2. So, (X''_1, X''_2) is a 2-join of G.

Let us put $D = B_3 \cup X_1 \setminus \{a_1\}$. By the properties above, $D \subset X''_2 \subset V(G)$. Since b_1 is complete to B_3, $G[D]$ is connected. We claim that (X''_1, X''_2) is a proper 2-join of G. Every component of X''_1 meets A''_1, B''_1: this follows from $A''_1 = A'_1$, $B''_1 = B'_1$ and from the fact that (X'_1, X'_2) is a proper 2-join of G'.

Let E be a connected component of X''_2. If $E \cap D = \emptyset$ then E is a component of $G[(X_2 \cup \{a_1\}) \cap X''_2] = G'[(X_2 \cup \{a_1\}) \cap X''_2]$, so E meets $A''_2 \cap A'_2$ and $B''_2 \cap B'_2$ because (X'_1, X'_2) is a proper 2-join of G'. If $E \cap D \neq \emptyset$ then $D \subset E$ since $G[D]$ is connected. We put $E' = (E \setminus D) \cup \{c_1, c_2, b_1, b_3, b_4\} \cup B_3$. Since E' is a component of X'_2 it meets A'_2, B'_2 because (X'_1, X'_2) is proper. This implies that E meets A''_2 and B''_2 . Note that $G[X''_1]$ is not an outgoing path of length 2 or 3 from A''_1 to B''_1, because (X'_1, X'_2) is a proper 2-join of G'. Also $G[X''_2]$ is not an outgoing path from A''_2 to B''_2 because b_1 has at least 2 neighbors in X''_2 (one in X_1, one in B_3) while having degree at least 3 because of B_4. This proves our claim.

Since (X''_1, X''_2) is proper, we know by the properties of G that (X''_1, X''_2) is a path 2-join of G. If X''_2 is the path-side of (X''_1, X''_2) then b_1 is an interior vertex of this path while having degree at least 3 by the properties of G, a contradiction. Hence, X''_1 is the path-side of (X''_1, X''_2). Since $X''_1 = X'_1$, (X'_1, X'_2) is a path 2-join of G'. This proves (17).

(18) $\overline{G'}$ has no proper 2-join.

In the proof of (18), the word "neighbor" refers to the neighborhood in $\overline{G'}$.

Suppose $c_1 \neq c_2$. In $\overline{G'}$, c_1 has degree $n - 3$, so up to a symmetry we may assume $c_1 \in A'_1$. In B'_2 there must be a non-neighbor of c_1. Also, since (X'_1, X'_2) cannot be a degenerate 2-join of $\overline{G'}$, vertex c_1 must have a non-neighbor in B'_1. So we have two cases to consider. Case 1: $a_1 \in B'_1$, $c_2 \in B'_2$. Then c_2 must have a non-neighbor in B'_2 for otherwise (X'_1, X'_2) is degenerate. This non-neighbor must be one of b_3, b_4. But this is imposible since b_3, b_4 both see a_1 in $\overline{G'}$. Case 2: $a_1 \in B'_2$, $c_2 \in B'_1$. Then $A'_2 \subset \{b_3, b_4\}$. So, $a_1 \in B'_2$ is complete to A'_2. Again, (X'_1, X'_2) is degenerate.

Suppose $c_1 = c_2$. Up to a symmetry we assume $c_1 \in X'_1$. If $c_1 \in C'_1$ then the only possible vertices in X'_2 are a_1, b_3, b_4, so $\overline{G'}[X'_2]$ induces a triangle. So, any vertex of A'_2 is complete to B'_2 and (X'_1, X'_2) is degenerate, a contradiction. So, $c_1 \notin C'_1$. Up to a symmetry, we assume $c_1 \in A'_1$. So, $B'_2 \subset \{a_1, b_3, b_4\}$. Thus, at least one of a_1, b_3, b_4 (say x) must be in B'_2. Since (X'_1, X'_2) is not degenerate, c_1 must have a non-neighbor in B'_1. So, one of a_1, b_3, b_4 (say y) must be in B'_1. Since (X'_1, X'_2) is not degenerate, x must have a non-neighbor z in A'_2. But z must also be a non-neighbor of y. This is imposible because in $G' \setminus c_1$, $N(a_1), N(b_3), N(b_4)$ are disjoint. This proves (18).

(19) $\overline{G'}$ is not path-cobipartite, not a path-double split graph, has no homogeneous 2-join and has no flat path of length at least 3.

Else, by Lemma 2.3 there is a contradiction with one of (12), (10) or (18). This proves (19).

(20) $f(G') + f(\overline{G'}) < f(G) + f(\overline{G})$.

Every vertex in $\{a_1\} \cup B_3 \cup B_4$ has degree at least 3 in G'. For a_1, this is a property of G and for vertices in $B_3 \cup B_4$, this is because (X_1, X_2) is not degenerate. Hence no vertex in $\{a_1\} \cup B_3 \cup B_4$ can be an interior vertex of a flat path of G', and no vertex in $\{c_1, c_2, b_3, b_4, b_1\}$ can be in a maximal flat path of G' of length at least 3. Hence, every maximal flat path of G' of length at least 3 is a maximal flat path of G, implying $f(G') \leq f(G)$. But in fact $f(G') < f(G)$ because X_1 is a flat path of G that is no more a flat path in G'. By (19) we know $0 = f(\overline{G'}) \leq f(\overline{G})$. We add these two inequalities. This proves (20).

Let us now finish the case. By (9), G' is Berge. By (12), G' is not basic, not path-cobipartite, not a path-double split graph, and has no homogeneous 2-join. By (10), G' has no even skew partition. By (17), G' has no proper non-path 2-join. By (18) $\overline{G'}$ has no proper 2-join. By (19), $\overline{G'}$ is not a path-cobipartite graph, a path-double split graph and has no homogeneous 2-join. So, G' is a counter-example to the theorem we are proving now. Hence there is a contradiction between the initial choice of G and (20). This completes the proof in Case 1.

Case 2: There are sets A_3, B_3 satisfaying the items 1–5 of the definition of cutting 2-joins of type 2.

The frame of the proof is very much like in Case 1, but the details differ. We consider the graph G' obtained from G by deleting $X_1 \setminus \{a_1, b_1\}$. Moreover, we add new vertices: c_1, c_2, a_3, b_3. Then we add every possible edge between a_3 and A_3, between b_3 and B_3. We also add edges a_1c_1, c_1c_2, c_2b_1, a_3b_3, c_1a_3, c_2b_3. Here are two claims about the connectivity of G and G'.

Here are six claims about the parity of various kinds of paths and antipaths in G'.

(21) Every outgoing path of G' from B_2 to A_2 has odd length.

If such a path contains one of $a_1, b_1, a_3, b_3, c_1, c_2$ then it has length 3 or 5. Else such a path may be viewed as an outgoing path of G from B_2 to A_2. By Lemma 2.4 it has odd length. This proves (21).

(22) Every outgoing path of G' from A_2 to A_2 (resp. from B_2 to B_2) has even length.

For suppose there is such a path P from A_2 to A_2 (the case with B_2 is similar). If P goes through a_1 then it has length 2. If P goes through at least one of c_1, c_2, a_3, b_3, b_1 then P is the union of two edge-wise-disjoint outgoing paths from A_2 to B_2. Thus P has even length by (21). Else, P may be viewed as an outgoing path of G from A_2 to A_2, that has even length by Lemma 2.5. In every cases, P has even length. This proves (22).

(23) Every outgoing path of G' from A_3 to A_3 (resp. from B_3 to B_3) has even length.

For suppose there is such a path P from A_3 to A_3 (the case with B_3 is similar). If P goes through a_1 or a_3 then it has length 2. Also, P cannot go through c_1. From now on, we assume that P goes through none of a_1, a_3, c_1.

If P goes through c_2 then $P = a\text{–}b\text{–}b_3\text{–}c_2\text{–}b_1\text{–}b'\text{–}\cdots a'$ where $a, a' \in A_3$, $b \in B_3$ and $b' \in B_2 \setminus B_3$. Also, $b_1 - P - a'$ may be viewed as an outgoing path of G from $A_3 \cup \{b_1\}$ to $A_3 \cup \{b_1\}$. By the definition of cutting 2-joins of type 2, this path has even length, thus P has length. From now on we assume that P does not go through c_2.

If P goes through b_3 it has length 4. If P goes through b_1 then P is the edge-wise-disjoint union of two outgoing paths of G from $A_3 \cup \{b_1\}$ to $A_3 \cup \{b_1\}$. Thus P has even length by the definition of cutting 2-joins of type 2. Thus we may assume that P goes through none of b_3, b_1.

Now P may be viewed as an outgoing path of G from A_3 to A_3, that does not go through b_1. Thus P is outgoing from $A_3 \cup \{b_1\}$ to $A_3 \cup \{b_1\}$, it has even length by the definition of cutting 2-joins of type 2. This proves (23).

(24) Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus A_2$ (resp. $V(G') \setminus B_2$), and all its interior vertices in A_2 (resp. B_2) has even length.

Let Q be such an antipath whose interior is in A_2 (the case with B_2 is similar). We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in A_2 and a non-neighbor in A_2. So none of a_1, c_1, c_2, b_1, b_3 can be an end-vertex of Q. If a_3 is an end of Q then the other end of Q must be a neighbor of a_3, a contradicition. Thus Q may be viewed as an antipath of G. By Lemma 2.5. So Q has even length. This proves (24).

(25) Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus A_3$ (resp. $V(G') \setminus B_3$), and all its interior vertices in A_3 (resp. B_3) has even length.

Let Q be such an antipath whose interior is in A_3 (the case with B_3 is similar). We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in A_3 and a non-neighbor in A_3. So none of $a_1, a_3, c_1, c_2, b_1, b_3$ can be an end-vertex of Q. Thus Q may be viewed as an antipath of G. It has even length by the definition of cutting 2-joins of type 2. This proves (25).

(26) Let Q be an antipath of G' of length at least 5. Then Q does not go through c_1, c_2. Moreover one of $V(Q) \cap \{a_1, a_3\}$, $V(Q) \cap \{b_1, b_3\}$ is empty.

Let Q be such an antipath. In an antipath of length at least 5, each vertex is in a triangle of the antipath. So, c_1, c_2 are not in Q since they are not in any triangle of G'.

Suppose $V(Q) \cap \{a_1, a_3\}$, $V(Q) \cap \{b_1, b_3\}$ are both non-empty. In an antipath of length at least 6, for every pair u, v of vertices, there is a vertex x seing both u, v. Thus Q has length 5 because no vertex of G' have neighbors in both $\{a_1, a_3\}$, $\{b_1, b_3\}$. Let $q_1, \dots, q_6$ be the vertices of Q in there natural order. Since $V(Q) \cap \{a_1, a_3\}$, $V(Q) \cap \{b_1, b_3\}$ are both non-empty there are two vertices of Q that have no common neighbors in G'. These vertices must be q_2 and q_5, and up to a symmetry we must have $q_2 = a_3$, $q_5 = b_3$. Thus q_3 must be a vertex of B_3 and q_4 must be a vertex of A_3. There is a contradiction since by the definition of cutting 2-joins of type 2, A_3 is complete to B_3. This proves (26).

(27) G' is Berge.

Let H be a hole of G'.

If H goes through both c_1, c_2 then H has length 4 or it must contains one of $\{a_1, b_1\}$, $\{a_1, b_3\}$, $\{b_1, a_3\}$. In the first case, H is edge-wise partitionned into two paths outgoing from A_2 to B_2. Thus H has even length by (21). In the second case H is edge-wise partitionned into two paths outgoing from $B_3 \cup \{a_1\}$ to $B_3 \cup \{a_1\}$, one of them of length 4, the other one included in $V(G)$. Thus H has even length by the definition of cutting 2-joins of type 2. The third case is similar. From now on, we assume that H goes through none of c_1, c_2.

If H goes through both a_1, a_3 then it has length 4. If H goes through a_2 and not through a_3 then H has even length by (22). If H goes through a_3 and not through a_2 then H has even length by (23). Thus, we may assume that H goes through none of a_1, a_3. Similarly, we may assume that H goes through none of b_1, b_3.

Now H may be viewed as a hole of G. In every case, H has even length.

Let us now consider an antihole H of G'. We may assume that H has length at least 7. Let v be a vertex of $V(H) \setminus \{a_1, b_1, c_1, c_2, a_3, b_3\}$. By (26) the antipath $V(H) \setminus v$ does not go through c_1, c_2 and we may assume up to a symmetry that $V(Q) \cap \{b_1, b_3\}$ is empty. If H goes through both a_1, a_3 then H must contains a vertex that sees a_3 and misses a_1, a contradiction. If H goes through a_1 and not through a_3 then H has even length by (24). If H goes through a_3 and not through a_1 then H has even length by (25). If H goes through none of a_1, a_3 then H may be viewed as an antihole of G. In every case, H has even length. This proves (27).

(28) G' has no even skew partition.

Suppose that G' has an even skew partition (E', F') with a split (E'_1, E'_2, F'_1, F'_2). Starting from F', we shall build an even skew cutset F of G which contradicts the properties of G.

By the property k of G, F' cannot be a star cutset centered at one of $a_1, b_1, c_1, c_2, a_3, b_3$. For the same reason, F' cannot be a subset of one of $\{c_1, c_2, a_3, b_3\}$, $\{a_1, c_1, a_3\} \cup A_3$, $\{b_1, c_2, b_3\} \cup B_3$. Thus, $c_1 \notin F'$ and $c_2 \notin F'$. Since a_1, b_1 are non-adjacent with no common neighbors, they are not both in F'. Similarly a_1, b_3 are not both in F' and a_3, b_1 are not both in F'. From now on we assume $b_1 \notin F'$. Up to symmetry we may assume $\{c_1, c_2, b_1\} \subset E'_1$, implying $\{a_1, a_3, c_1, c_2, b_1, b_3\} \cap E' \subset E'_1$.

Let v be any vertex of E'_2. Since $\{a_1, a_3, c_1, c_2, b_1, b_3\} \cap E' \subset E'_1$, we have $v \in X_2$. If one of a_1, a_3 is in F, put $A'_1 = \{a_1\}$, else put $A'_1 = \emptyset$. Now $F = A'_1 \cup F' \setminus \{a_3, b_3\}$ is a skew cutset of G that separates v from the interior vertices of the path induced by X_1. It suffices now to prove that F is an even skew cutset of G.

Let P be an outgoing path of G from F to F. We shall prove that P has even length.

If $a_1, \notin F$, then $F \subset X_2$ and the end-vertices of P are both in X_2. So Lemma 2.6 applies to P. Suppose that the first outcome of Lemma 2.6 is satisfied: $V(P) \subseteq X_2 \cup \{a_1, b_1\}$. Hence, P may be viewed as an outgoing path from F' to F', so P has even length since F' is an even skew cutset of G'. Suppose now that the second outcome of Lemma 2.6 is satisfied: $P = c\text{–}\cdots\text{–}a_2\text{–}a_1\text{–}X_1\text{–}b_1\text{–}b_2\text{–}\cdots\text{–}c'$. Put $P' = c\text{–}P\text{–}a_2\text{–}a_1\text{–}c_1\text{–}c_2\text{–}b_1\text{–}b_2\text{–}P\text{–}c'$. The paths P and P' have same parity and P' is an outgoing path of G' from F' to F'. So P' and P has even length since F' is an even skew cutset of G'.

If $a_1 \in F$ then $F \subset X_2 \cup \{a_1\}$ and Lemma 2.7 applies. If Outcome 1 of the lemma holds, then P has even length. If Outcome 2 of the lemma holds then we may assume $a_1 \in F \setminus F'$ and a_1 is an end of P, since otherwise the proof works like in the paragraph above. This implies $a_3 \in F'$. We build a path P' of G' that is outgoing from F' to F' with same parity than P, by replacing $\{a_1\}$ by $\{a_3, c_1, a_1\}$. So P has even length. If Outcome 3 of the lemma holds then $P = a_1 - X_1 - b_1 - b_2 - \cdots - c$ where $b_2 \in B_2$, $c \in X_2$. Note that one of a_1, a_3 is in F'. If $a_3 \in F'$, then we put $P' = a_3 – c_1 – c_2 – b_1 – b_2 – P – c$. If $a_3 \notin F'$ then $a_1 \in F'$ and we put $P' = a_1 - c_1 - c_2 - b_1 - b_2 - P - c$. In every cases P has even length.

Now, let Q be an antipath of G of length at least 5 with all its interior vertices in F and with its end-vertices outside of F. We shall prove that Q has even length.

If $a_1 \notin F$, then $F \subset X_2$ and the interior vertices of Q are all in X_2. So Lemma 2.8 applies: $V(Q) \subseteq X_2 \cup \{a\}$ where $a \in \{a_1, b_1\}$. So Q may be viewed as an antipath of G' that has even length because F' is an even skew cutset of G'.

If $a_1 \in F$, we assume $a_1 \notin F'$ (else the proof is like in the paragraph above). Since $a_1 \notin F'$, we have $a_3 \in F'$. Note that $(A_2 \setminus A_3) \cap F' = (A_2 \setminus A_3) \cap F = \emptyset$. For otherwise consider a vertex a in $(A_2 \setminus A_3) \cap F'$. Then $a – a_1 – c_1 – a_3$ is an outgoing path from F' to F' of odd length, contradicting F' being an even skew cutset. So, if Q is an antipath whose interior is in F, then Q does not go through $A_2 \setminus A_3$. Hence, if we replace a_1 by a_3, we obtain an antipath Q' whose interior is in F' and whose ends are not. Hence, Q has even length.

In every cases, Q has even length. This proves (28).

(29) G' and $\overline{G'}$ have no degenerate 2-join, no degenerate homogeneous 2-join and no star cutset.

If one of G', $\overline{G'}$ has a degenerate proper 2-join, a degenerate homogeneous 2-join or a star cutset, then G' has an even skew partition by Lemma 2.10, 2.15 or 2.2. This contradicts (28). This proves (29).

(30) G' is not basic, not a path-cobipartite graph, not a path-double split graph and has no homogeneous 2-join.

If G' is bipartite then all the vertices of A_2 are of the same color because of a_1. Because of b_1 all the vertices of B_2 have the same color. By the property k of G, there is an outgoing path from A_2 to B_2 that has odd length by (21). Thus G is bipartite, contradicting the properties of G. Hence G' is not bipartite.

The graph $G'[c_2, c_1, a_1, a_3]$ is a claw, so G' is not the line-graph of a bipartite graph. $\overline{G'}[a_1, b_1, a_3, b_3]$ is a diamond, so $\overline{G'}$ is not the line-graph of a bipartite graph.

Note that b_1 has degree at least 3 in G' by the property h of G. So, there exist in G' a stable set of size 3 containning vertices of degree at least 3 ($\{b_1, b_3, c_1\}$), and a vertex of degree 3 whose neighborhood induces a stable set (c_1). Hence, by Lemma 2.16, G' is not a path-cobipartite graph (and in particular, it is not the complement of a bipartite graph), not a path-double split graph (and in particular, it is not a double split graph) and G' has no non-degenerate homogeneous 2-join. Hence by (29), G' has no homogeneous 2-join. This proves (30).

(31) If G' has a proper 2-join (X_1', X_2') then either $\{c_1, c_2, a_3, b_3\} \subset X_1'$ or $\{c_1, c_2, a_3, b_3\} \subset X_2'$.

Suppose not. Up to a symmetry, we have five cases to consider according to $X_1' \cap \{c_1, c_2, a_3, b_3\}$. Each of them leads to a contradiction:

- $\{c_1\} \subset X_1'$ and $\{c_2, a_3, b_3\} \subset X_2'$

  Up to a symmetry, we assume $c_1 \in A_1'$ and $c_2, a_3 \in A_2'$. Note that $A_1' = \{c_1\}$ because c_1 is the only vertex in X_1' that sees both c_2, a_3. Note that a_1 is in X_1' for otherwise c_1 is isolated in X_1'. Also if a vertex x of A_3 is in X_1' then x must be in A_1' since it sees a_3. This is impossible since x misses c_2. Thus $x \in X_2'$. Since x sees $a_1 \in X_1'$, x must be in B_2' and a_1 must be in B_1'. So, a_1 is a vertex of B_1' that is complete to A_1', implying (X_1', X_2') being degenerate, contradicting (29).

- $\{a_3\} \subset X_1'$ and $\{c_1, c_2, b_3\} \subset X_2'$

  This case is like the previous one, we just sketch it. We assume $a_3 \in A_1'$, implying $c_1, b_3 \in A_2'$. Thus $A_1' = \{a_3\}$. There is a x vertex of X_1' in A_3. Also, $a_1 \in X_2'$ for otherwise $a_1 \in A_1'$ while missing b_3, a contradiction. Thus $x \in B_1'$, and x is a vertex of B_1' that is complete to B_1', a contradiction.

- $\{c_1, c_2\} \subset X_1'$ and $\{a_3, b_3\} \subset X_2'$

  Up to a symmetry, we assume $c_1 \in A_1'$, $a_3 \in A_2'$, $c_2 \in B_1'$, $b_3 \in B_2'$. Since by (29) (X_1', X_2') is not degenerate, a_3 must have a non-neighbor x in B_2'. Since x must see c_2 we have $x = b_1$ and $b_1 \in B_2'$. Similarly, b_3 must have a non-neighbor in A_2', implying $a_1 \in A_2'$. Now put $Y_1 = X_2 \cap X_1'$ and $Y_2 = X_2 \cap X_2'$. Note that $Y_1 \neq \emptyset$ for otherwise $X_1' = \{c_1, c_2\}$ and (X_1', X_2') is not proper. Also $Y_2 \neq \emptyset$ for otherwise, a_1 is isolated in X_2'. If there is an edge of G' with an end in Y_1 and an end y in Y_2, then y_2 must be in one of A_2', B_2'. This is a contradiction since y misses both c_1, c_2. Thus there is no edge with an end in Y_1 and an end Y_2. This contradicts the property j of G.

- $\{c_1, a_3\} \subset X'_1$ and $\{c_2, b_3\} \subset X'_2$

  Up to a symmetry, we assume $c_1 \in A'_1$, $a_3 \in B'_1$, $c_2 \in A'_2$, $b_3 \in B'_2$. Since by (29) (X'_1, X'_2) is not degenerate, a_3 must have a non-neighbor x in A'_1. Since x must see c_2 we have $x = b_1$ and $b_1 \in A'_1$. Similarly, b_3 must have a non-neighbor in A'_2, implying $a_1 \in A'_2$. So, $b_1 \in A'_1$, $a_1 \in A'_2$ and $a_1b_1 \notin E(G')$, a contradiction.

- $\{c_1, b_3\} \subset X'_1$ and $\{c_2, a_3\} \subset X'_2$

  Up to a symmetry, we assume $c_1 \in A'_1$, $a_3 \in A'_2$, $c_2 \in A'_2$, $b_3 \in A'_1$. There is a vertex x of X'_1 in B_3 for otherwise b_3 is isolated in X'_1. Also, $b_1 \in X'_2$ for otherwise c_2 is isolated in X'_2. But b sees x. Since $b_1 \in A'_2$ is impossible because b_1 misses c_1 we have $b_1 \in B'_2$. Similarly, we prove $a_1 \in B'_1$. So, $b_1 \in B'_2$, $a_1 \in B'_1$ and $a_1b_1 \notin E(G')$, a contradiction.

This proves (31).

(32) G' has no non-path proper 2-join.

By (31), we may assume $\{c_1, c_2, a_3, b_3\} \subset X'_2$. We claim that at most one of c_1, c_2, a_3, b_3 is in $A'_2 \cup B'_2$. For otherwise, up to a symmetry there are three cases. First case, $a_3 \in A'_2$, $b_3 \in B'_2$, implying $A'_1 \subset A_3$ and $B'_1 \subset B_3$, implying (X'_1, X'_2) being degenerate because any vertex of A'_1 is complete to B'_1, contradicting (29). Second case, $a_3 \in A'_2$, $c_1 \in B'_2$ implying $A'_1 \subset A_3$, $a_1 \in B'_1$, implying (X'_1, X'_2) being degenerate because $a_1 \in B'_1$ is to complete to A'_1, contradicting (29). Third case, $a_3 \in A'_2$, $c_2 \in B'_2$ implying $b_1 \in B'_1$. Also $b_3 \in C'_2$ because b_3, c_2 (resp. b_3, a_3) have no common neighbors in X'_1. So $B_3 \subset X'_2$ and because of b_1, $B_3 \subset B'_2$. Because of a_3 there is a vertex a of A'_1 in A_3. Hence a is a vertex of A'_1 that has a neighbor in B'_2, a contradiction. The three cases yield a contradiction, so our claim is proved. Thus up to a symmetry we assume that we are in one of the three cases that we describe below:

- $a_3 \in A'_2$. Moreover, $a_1 \in X'_2$ because $c_1 \in C'_2$. Because of a_3 there is a vertex of X'_1 in A_3, implying $a_1 \in A'_2$ and $B_3 \subset A'_2$.

- $c_1 \in A'_2$. This implies $a_1 \in A'_1$. Since $a_3 \in C'_2$, we have $A_3 \subset X'_2$ and $A_3 \subset A'_2$ because of a_1. Note that $A'_1 = \{a_1\}$ because a_1 is the only neighbor of c_1 in X'_1.

- $a_2 \notin A'_2$ and $c_1 \notin A'_2$. Moreover, $a_1 \in X'_2$ and $A_3 \subset X'_2$.

In either cases, $c_2, b_3 \in C'_2$, implying $\{b_1\} \cup B_3 \subset X'_2$. Note that $X'_1 \subset V(G)$. Let us now put: $X''_1 = X'_1$, $X''_2 = V(G) \setminus X''_1$, $A''_1 = A'_1$, $B''_1 = B'_1$, $B''_2 = B'_2$. If $c_1 \in A'_2$ then put $A''_2 = (A'_2 \cap X_2) \cup (N_G(a_1) \cap X_1)$. If $c_1 \notin A'_2$ then put $A''_2 = A'_2 \setminus \{a_3\}$. From the definitions it follows that (X''_1, X''_2) is a partition of $V(G)$, that $A''_1, B''_1 \subset X''_1$, $A''_2, B''_2 \subset X''_2$, that A''_1 is complete to A''_2, that B''_1 is complete to B''_2 and that there are no other edges between X''_1 and X''_2. So, $(X''_1, X''_2) = (X'_1, V(G) \setminus X'_1)$ is a 2-join of G.

Let us put $D = B_3 \cup X_1 \cup \{a_1\}$. By the properties above, $D \subset X''_2 \subset V(G)$ and $G[D]$ is connected. We claim that (X''_1, X''_2) is a proper 2-join of G. Every component of X''_1 meets A''_1, B''_1: this follows from $A''_1 = A'_1$, $B''_1 = B'_1$ and from the fact that (X'_1, X'_2) is a proper 2-join of G'. Let E be a connected component of X''_2. If $E \cap D = \emptyset$ then E is a component of $G[(X_2 \cup \{a_1\}) \cap X''_2] = G'[(X_2 \cup \{a_1\}) \cap X''_2]$, so E meets $A''_2 \cap A'_2$ and $B''_2 \cap B'_2$ because (X'_1, X'_2) is a proper 2-join of G'. If $E \cap D \neq \emptyset$ then $D \subset E$ since $G[D]$ is connected. We put $E' = (E \setminus D) \cup \{c_1, c_2, a_3, b_3, b_1\} \cup B_3$. Since E' is a component of X'_2 it meets A'_2, B'_2 because (X'_1, X'_2) is proper. This implies that E meets A''_2 and B''_2. Note that $G[X''_1]$ is not an outgoing path of length 2 or 3 from A''_1 to B''_1, because (X'_1, X'_2) is a proper 2-join of G'. Also $G[X''_2]$ is not an outgoing path from A''_2 to B''_2 because b_1 has at least 2 neighbors in X''_2 (c_2 and one in B_3) while having degree at least 3 by the property h of G. This proves our claim.

Since (X''_1, X''_2) is proper, we know by the properties of G that (X''_1, X''_2) is a path 2-join of G. If X''_2 is the path-side of (X''_1, X''_2) then b_1 is an interior vertex of this path while having degree at least 3, a contradiction. Hence, X''_1 is the path-side of (X''_1, X''_2). Thus (X''_1, X''_2) is a path 2-join of G because $X''_1 = X'_1$. This proves (32).

(33) $\overline{G'}$ has no proper 2-join.

In the proof of (33), the word "neighbor" refers to the neighborhood in $\overline{G'}$. Let (X'_1, X'_2) be a proper 2-join of $\overline{G'}$.

If $c_1 \in C'_1$ then $X'_2 \subset \{a_1, a_3, c_2\}$ implying (X'_1, X'_2) being degenerate or non-proper, contradicting (29). Thus, we may assume $c_1 \in A'_1$. Similarly c_2 must be in one of A'_1, A'_2, B'_1, B'_2. But $c_2 \in A'_2$ is impossible because c_2 is not a neighbor of c_1. Also $c_1 \in A'_1$ is impossible because otherwise $B'_2 = \emptyset$ since no vertex of $\overline{G'}$ can be a non-neighbor of both c_1, c_2. Thus c_2 is in one of B'_1, B'_2.

If $c_2 \in B'_1$ then $A'_2 \subset \{b_1, b_3\}$ because of c_2 and $B'_2 \subset \{a_1, a_3\}$ because of c_1. But b_1 must be in A'_2 because it is a common neighbor of c_1, a_1, a_3.

Thus b_1 is a vertex of A'_2 that is complete to B'_2, implying (X'_1, X'_2) being degenerate, contradicting (29).

If $c_2 \in B'_2$ then there is a non-neighbor of c_2 in A'_2 for otherwise (X'_1, X'_2) is degenerate. Thus at least one of b_1, b_3 is in A'_2. Similarly, because of c_1, at least one of a_1, a_3 must be in B'_1. But since there is no edge of $\overline{G'}$ between B'_1, A'_2, we have $a_3 \in B'_1$, $b_3 \in A'_2$. Since a_3, b_3, c_2 are neighbors of a_1, we know $a_1 \in B'_2$. Now b_1 is a neighbor of $c_1 \in A'_1$, $a_3 \in B'_1$, $a_1 \in B'_2$, $b_3 \in A'_2$, a contradiction. This proves (33).

(34) $\overline{G'}$ is not path-cobipartite, not a path-double split graph, has no homogeneous 2-join and has no flat path of length at least 3.

Else, by Lemma 2.3 there is a contradiction with one of (30), (28) or (33). This proves (34).

(35) $f(G') + f(\overline{G'}) < f(G) + f(\overline{G})$.

Every vertex in $\{a_1, b_1\} \cup A_3 \cup B_3$ has degree at least 3 in G'. For a_1, this is a property of G and for vertices in $A_3 \cup B_3$, this is clear. Hence no vertex in $\{a_1, b_1\} \cup A_3 \cup B_3$ can be an interior vertex of a flat path of G', and no vertex in $\{c_1, c_2, a_3, b_3\}$ can be in a maximal flat path of G' of length at least 3. Hence, every maximal flat path of G' of length at least 3 is a maximal flat path of G, implying $f(G') \leq f(G)$. But in fact $f(G') < f(G)$ because X_1 is a flat path of G that is no more a flat path in G'. By (34) we know $0 = f(\overline{G'}) \leq f(\overline{G})$. We add these two inequalities. This proves (35).

Let us now finish the case. By (27), G' is Berge. By (30), G' is not basic, not path-cobipartite, not a path-double split graph, and has no homogeneous 2-join. By (28), G' has no even skew partition. By (32), G' has no proper non-path 2-join. By (33) $\overline{G'}$ has no proper 2-join. By (34), $\overline{G'}$ is not a path-cobipartite graph, a path-double split graph and has no homogeneous 2-join. So, G' is a counter-example to the theorem we are proving now. Hence there is a contradiction between the initial choice of G and (35). This completes the proof in Case 2.

Case 3: We are neither in Case 1 nor in Case 2. In particular, (X_1, X_2) is not a cutting 2-join.

We consider the graph G' obtained from G by replacing X_1 by a path of length $2 - \varepsilon$ from a_1 to b_1. Possibly, this path has length 2. In this case we denote by c_1 its unique interior vertex. Else, this path has length 1, and for convinience we put $c_1 = a_1$ (thus c_1 is a vertex of G' whatever ε). Note that $(V(G') \setminus X_2, X_2)$ is not a proper 2-join of G since $V(G') \setminus X_2$ is a path of length 1 or 2 from a_1 to b_1. Note that $a_1 - c_1 - b_1$ a flat path of G' (possibly of length 1 when $a_1 = c_1$) because if there is a common neighbor c of a_1, b_1, or if $c_1 \neq a_1$ has degree at least 3, then (X_1, X_2) is not a 2-join of G. Note that G' is what we call in section 2 the piece G_2 of G with respect to the 2-join (X_1, X_2).

(36) G' has no even skew partition, and none of G, $\overline{G'}$ has a star cutset, a degenerate proper 2-join or a degenerate homogeneous 2-join.

Since G' is a piece of G, and since (X_1, X_2) is not cutting, by Lemma 2.13, if G' has an even skew partition then so is G, contradicting the properties of G. By Lemma 2.2, 2.10 and 2.15, $G, \overline{G}$ have no star cutset, no degenerate 2-join and no degenerate homogeneous 2-join. This proves (36).

(37) G' is Berge.

Any hole H' of G' yield a hole of G of the same parity after possibly subdivising the flat path $a_1{-}c_1{-}b_1$. Also, a_1, b_1 cannot both be in an antihole of G' because in an antihole of length at least 7, any pair of vertex have a common neighbor. Also, if $c_1 \neq a_1$ then c_1 does not lie in an antihole of G' of length at least 7 because c_1 has degree 2. Thus, any antihole of G' may be viewed as an antihole of G. Thus, every holes and every antiholes in G' are even. This proves (37).

(38) G' has no proper non-path 2-join.

Let $(X'_1, X'_2, A'_1, B'_1, A'_2, B'_2)$ be a split of a proper non-path 2-join of G'. If $a_1 \in X'_1$, $b_1 \in X'_1$ then $c_1 \in X'_1$ since otherwise c_1 is isolated in X'_2. If $c_1 \neq a_1$ then $c_1 \in C'_1$ because c_1 has degree 2. So, by subdivising $a_1{-}c_1{-}b_1$ we obtain a non-path proper 2-join of G, contradicting the properties of G. Thus, since $a_1{-}c_1{-}b_1$ is a flat path of G', up to a symmetry, we may assume $c_1 \in B'_1$, $b_1 \in B'_2$.

Suppose $|B'_2| = 1$. Then no vertex of A'_2 has a neighbor in B'_2 for otherwise, (X_1, X_2) is degenerate. Thus, $(X'_1 \cup B'_2, X'_2 \setminus B'_2)$ is a non-path proper 2-join of G', and by subvising a_1b_1, we obtain a non-path proper 2-join of G, contradicting the properties of G. Thus, $|B'_2| \geq 2$. In particular, $c_1 = a_1$, and similarly $|B'_1| \geq 2$.

In G, a_1 is complete to $B'_2 \setminus \{b_1\}$, and b_1 is complete to $B'_1 \setminus \{a_1\}$. We put $A_3 = B'_2 \setminus \{b_1\}$, $B_3 = B'_1 \setminus \{a_1\}$. In G, X_1 is a path from a_1 to b_1, $A_3 \subset A_2$ and $B_3 =\subset B_2$ and A_3 is complete to B_3. We claim that every path of G outgoing from $A_3 \cup \{b_1\}$ to $A_3 \cup \{b_1\}$ has even length. Note that after possibly deleting c_1, c_2, such a path may be view as a path P' of G' that has same parity than P. In G', P' is an outgoing path from B'_1 to B'_1 and by Lemma 2.5, P has even length as claimed. We claim that every outgoing antipath of G whose interior is in $A_3 \cup \{b_1\}$ and whose ends are outside of $A_3 \cup \{b_1\}$ has even length. Let Q be such an antipath of length at least 5. Note that c_1, c_2 are not in Q since every vertex in Q have degree at least 3. Thus Q is an outgoing path of G' whose interior is in B'_1 and whose ends are not in B'_1 and by Lemma 2.5, P has even length as claimed. The same properties hold with $B_3 \cup \{a_1\}$. Now, A_3, B_3 show that (X_1, X_2) satisfies the items 1–5 of the definition of cutting 2-joins of type 2, contradicting that we are not in Case 2 of the proof of our theorem. This proves (38).

(39) $\overline{G'}$ *has no proper 2-join.*

Let us consider a proper 2-join of $\overline{G'}$ with a split $(X'_1, X'_2, A'_1, B'_1, A'_2, B'_2)$. If $c_1 \neq a_1$ then c_1 has degree $n-2$ in $\overline{G'}$. Thus, up to a symmetry, we may assume $c_1 \in B'_1$. Since (X'_1, X'_2) is not degenerate, c_1 must have a non-neighbor in A'_1. Thus, up to a symmetry, we may assume $a_1 \in A'_1$, $b_1 \in A'_2$. Now, since (X'_1, X'_2) is not degenerate, there exists a vertex of B'_2 that is a common neighbor of a_1, b_1 in G, contradicting $a_1 - c_1 - b_1$ being a flat path of G. We proved $a_1 = c_1$.

Since a_1, b_1 form a flat edge of G', they must be non-adjacent in $\overline{G'}$ with no common non-neighbor. Thus, up to a symmetry we have to deal with three cases:

- $a_1 \in C'_1$, $b_1 \in X'_2$.

  Since in G' a_1b_1 is flat, in $\overline{G'}$ a_1 is complete to $A'_1 \cup B'_1$ or up to a symmetry $b_1 \in A'_2$ while being complete to B'_2. Thus, (X'_1, X'_2) is a degenerate 2-join, a contradiction.

- $a_1 \in A'_1$, $b_1 \in B'_2$.

  Since in G', a_1b_1 is flat, in $\overline{G'}$, a_1 must be complete to $(A'_1 \cup C'_1) \setminus \{a_1\}$.

  Suppose first $C'_1 \neq \emptyset$. There is at least a vertex of C'_1 that has a neighbor in B'_1 for otherwise $A'_1 \cup A'_2$ is a skew cutset of $\overline{G'}$, implying (X'_1, X'_2) being degenerate. If a_1 has a neighbor in B_1 then by Lemma 2.4 every outgoing path from A'_1 to B'_1 has odd length. Thus, a_1 must see every vertex of B'_1 that has a neighbor in C'_1. This implies that $A'_1 \cup (N(a_1) \cap B'_1)$ is a star cutset of G', centered at a_1 and separating C'_1 and separarting from X'_2. Thus, a_1 has no neighbor in B_1. Hence, there is at least an outgoing path of even length from A'_1 to B'_1, implying that no vertex in A'_1 has a neighbor in B'_1. If $|A'_1| \geq 2$ then $\{a_1\} \cup C'_1 \cup B'_2$ is a star cutset centered at a_1 that separates $A'_1 \setminus \{a_1\}$ from B'_2. Thus, $|A_1| = 1$. Since, every outgoing path from A'_1 to B'_1

has even length, we know that every outgoing path from A'_2 to B'_2 has even length. Thus, $C'_2 \neq \emptyset$. By the same proof than above, this implies $B'_2 = \{b_1\}$. Note that every vertex in C'_1 has a neighbor in B'_1 because a vertex of C'_1 with no neighbor in B'_1 can be separated from the rest of the graph by a star cutset centered at a_1. Every vertex in C'_1 has a non-neighbor in B'_1 because a vertex of C'_1 complete to B'_1 would imply (X'_1, X'_2) being degenerate. Note also that every vertex in B'_1 has a neighbor in C'_1 for otherwise (X'_1, X'_2) is degenerate. Every vertex in B'_1 has a non-neighbor in C'_1 because if there is a vertex $b \in B'_1$ complete to C'_1 then $|B'_1| \geq 2$ implyies that $\{b\} \cup C'_1 \cup B'_2$ is a star cutset separating $B'_1 \setminus \{b\}$ from A'_2, and $|B'_1| = 1$ implies that every vertex in C'_1 is complete to $A'_1 \cup B'_1$, a case already treated. Let us come back to G: in G, X_1 is a path from a_1 to b_1. Let us denote by E its interior. We observe that $(C'_1, B'_1, \{b_1\}, \{a_1\}, E, A'_2 \cup C'_2)$ is an homogeneous 2-join of G, contradicting the properties of G.

We proved $C'_1 = \emptyset$. By the same way, $C'_2 = \emptyset$. Thus, $(A'_1 \cup B'_2, A'_2 \cup B'_1)$ is a non-path proper 2-join of G', contradicting (38).

- $a_1 \in A'_1, b_1 \in B'_1$.

  Since $a_1 - b_1$ is a flat edge of G', $C'_2 = \emptyset$. If $C'_1 = \emptyset$, then just like above $(A'_1 \cup B'_2, A'_2 \cup B'_1)$ is a non-path proper 2-join of G', contradicting (38). So, $C'_1 \neq \emptyset$. Hence, $(A'_2, B'_2, B'_1, A'_1, X_1 \setminus \{a_1, b_1\}, C'_1)$ is an homogeneous 2-join of G, contradicting the properties of G.

This proves (39).

(40) G' is neither a bipartite graph nor the line-graph of a bipartite graph.

Subdivising flat paths of a line-graph of a bipartite graph (resp. of a bipartite graph) into a path of the same parity yields a line-graph of a bipartite graph (resp. a bipartite graph). Thus, if G' is the line-graph of a bipartite graph or a bipartite graph, then so is G, contradicting the properties of G. This proves (40).

(41) $\overline{G'}$ is not the line-graph of a bipartite graph.

Suppose that $\overline{G'}$ is the line-graph of bipartite graph. If $c_1 \neq a_1$ then by the properties of G there exists an outgoing path of even length from A_2 to B_2 whose interior is in C_2. Thus, there is a vertex $c \in C_2$. Since (X_1, X_2) is not degenerate, c_2 has at least a non-neighbor b in one of A_2, B_2, say B_2 up to symmetry. Now $\{a_1, c_1, c, b\}$ induces a diamond of $\overline{G'}$, a contradiction. We prove $a_1 = c_1$.

Let B be a bipartite graph such that $G = \overline{L(B)}$. Let (X, Y) be a bipartition of B. So, a_1, b_1 may be seen as edges of B. Let us suppose $a_1 = a_X a_Y$ and $b_1 = b_X b_Y$ where $a_X, b_X \in X$ and $a_Y, b_Y \in Y$. Note that these four vertices of B are pairwise distinct since in $L(B) = \overline{G'}$, a_1 misses b_1. Since $a_1 b_1$ is flat in G', every edge of B is either adjacent to a_X, a_Y, b_X or b_Y. Thus, the vertices of $L(B) = \overline{G'}$ different of a_1, b_1 partition into six sets:

- A_X, the sets of the edges of B seing a_X and missing b_Y;
- A_Y, the sets of the edges of B seing a_Y and missing b_X;
- B_X, the sets of the edges of B seing b_X and missing a_Y;
- B_Y, the sets of the edges of B seing b_Y and missing a_X;
- possibly a single vertex c representing the edge $a_X b_Y$;
- possibly a single vertex d representing the edge $a_Y b_X$.

Suppose $|A_X| \geq 2$. Then, $|B_X| \geq 1$ for otherwise one of $\{a_1\}$, $\{a_1, c\}$ is a star cutset of $\overline{G'}$ separating A_X from b_1. We observe that $(A_X \cup B_X, V(G') \setminus (A_X \cup B_X))$ is a 2-join of $\overline{G'}$. By (39), this 2-join is not proper. Since $|A_X| \geq 2$, since $V(G') \setminus (A_X \cup B_X)$ does not induce a path of $\overline{G'}$ of length at most 2, there is either a component of $A_X \cup B_X$ that does not meet A_X and B_X or a component of $V(G') \setminus (A_X \cup B_X)$ that does not meet a_1 and b_1. In both cases, there is a star cutset of $\overline{G'}$ centered at one of a_1, b_1, a contradiction. Thus, $|A_X| \leq 1$, and similarly $|B_X| \leq 1$, $|A_X| \leq 1$, $|B_Y| \leq 1$. In the case when $|A_X| = |B_X| = |A_Y| = |B_Y| = 1$ and when c, d are both vertices of G', we observe that $\overline{G'}$ is the self-complementary graph $L(K_{3,3} \setminus e)$. Hence, G' is an induced subgraph of the line-graph of a bipartite graph, and G' is the line-graph of a bipartite graph, contradicting (40). This proves (41).

(42) G' is not a path-cobipartite graph.

If G' is a path-cobipartite graph then it is partionned into two cliques A and B and a path P joinning, like in the definition, a vertex a of A to a vertex b of B. Suppose first $P = \emptyset$. If $a_1 \in A, b_1 \in A$, then since $a_1 b_1$ is a flat edge of G' we have $|A| = 2$. If a vertex c of B sees none of a_1, b_1 then $B \setminus c$ is a star-cutset of G' separating c from $a_1 b_1$. Thus $\{a_1\} \cup N(a_1)$ and $\{b_1\} \cup N(b_1)$ are two cliques of G' that partition $V(G')$. Thus, we may always assume that $a_1 \in A$, $b_1 \in B$. So, G is obtained by subdivising $a_1 b_1$

implying G being a path-cobipartite graph, contradicting the properties of G.

Thus $P \neq \emptyset$. Note that $(P \cup \{a, b\}, A \setminus \{a\} \cup B \setminus \{b\})$ is a 2-join of G'. Also, $G'[A \setminus \{a\} \cup B \setminus \{b\}]$ is connected because otherwise, any vertex of P is a star cutset of G', contradicting. Also, $G'[A \setminus \{a\} \cup B \setminus \{b\}]$ is not a single edge, for otherwise G' is bipartite, a case already treated. Thus this 2-join is proper, and so it is not degenerate. In particular, every vertex in $A \setminus \{a\}$ has a neighbor and a non-neighbor in $B \setminus \{b\}$, implying $|A| \geq 3$, $|B| \geq 3$. If at least one a_1, b_1 is on P then the graph G obtained by subdivising a_1b_1 is again a path-cobipartite graph, contradicting the properties of G. Thus since a_1b_1 is a flat edge of G', we may assume $a_1 \in A \setminus \{a\}$, $b_1 \in B \setminus \{b\}$. The graph G is obtained by subdivising a_1b_1 into a path Q. Now $(P \cup Q \cup \{a, b\}, V(G') \setminus (P \cup Q \cup \{a, b\})$ is a 2-join of G. By the properties of G this 2-join must be either a path 2-join or a non-proper 2-join, meaning that $V(G') \setminus (P \cup Q \cup \{a, b\})$ is a single edge. Now we observe that G is the line-graph of a bipartite graph (it is in fact a graph called *prism* in [4]), contradicting the properties if G. This proves (42).

(43) G' is not a path-double split graph.

Suppose that G' is a path-double split graph. Let $A' = \{a'_1, \dots, a'_m\}, B' = \{b'_1, \dots, b'_m\}, C' = \{c'_1, \dots, c'_n\}, D' = \{d'_1, \dots, d'_n\}$ and E' be sets of vertices of G' that are like in the definition. If $a_1 \in A' \cup E'$ and $b_1 \in B' \cup E'$, then G is obtained from G' by subdivising the flat path $a_1 - c_1 - b_1$. If this yields a path of even length between a vertex a'_i and b'_i, then this path together with a neighbor of a'_i in $C' \cup D'$ and a neighbor of b'_i in $C' \cup D'$ that are adjacent, yields an odd hole of G. Thus every path with an end in A', and end in B' and interior in E has odd length, and G is a path-double split graph contradicting the properties of G. The case when $a_1 \in B' \cup E, b_1 \in A' \cup E$ is symmetric. Since $a_1 - c_1 - b_1$ is a flat path of G', there is only one case left up to a symmetry: $a_1 = c_1$, $|C'| = |D'| = 2$, $a_1 = c'_1$, $b_1 = c'_2$ and for every $i \in \{1, \dots, m\}$, a'_i sees c'_1, d'_2 and b'_i sees d'_1, c'_2. So, G is obtained by subdivising $c'_1c'_2$ into a path P. We see that $(P \cup \{d'_1, d'_2\}, A' \cup B')$ is a proper non-path 2-join of G, contradicting the properties of G. This proves (43).

(44) G' has no homogeneous 2-join.

Suppose that G' has an homogeneous 2-join (A, B, C, D, E, F). If $c_1 \neq a_1$ then since c_1 has degree 2, c_1 must be in E. Thus, by subdivising $a_1 - c_1 - b_1$ we obtain a graph G with an homogeneous 2-join. If $c_1 = a_1$ then a_1b_1 is a flat edge of G', thus, up to a symmetry, either $a_1 \in C$, $b_1 \in E \cup D$

or $a_1 \in C$, $b_1 \in A$. But the last case is impossible since a_1b_1 being flat implies $N(a_1) \subset A \cup D \cup E$, implying (A, B, C, D, E, F) being degenerate, contradicting (36). Hence, $a_1 \in C$ and $b_1 \in D \cup E$. So, by subdivising a_1b_1 we obtain a graph G that has an homogeneous 2-join. This proves (44).

(45) $\overline{G'}$ is not a path-cobipartite graph, not a path-double split graph, has no homogeneous 2-join and no flat path of length at least 3.

Else, by Lemma 2.3 either $\overline{G'}$ has a proper 2-join, contradicting (39) or $\overline{G'}$ has an even skew partition contradicting (36), or $\overline{G'}$ is bipartite contradicting (42), or G' is bipartite contradicting (40), or $\overline{G'}$ is a double split graph and so is G', contradicting (43). This proves (45).

(46) $f(G') + f(\overline{G'}) < f(G) + f(\overline{G})$.

Every flat path of G' is a flat path of G thus $f(G') \leq f(G)$. But in fact $f(G') < f(G)$ since X_1 is a flat path of G and not of G'. By (45) $0 = f(\overline{G'}) \leq f(\overline{G})$. We add these two inequalities. This proves (46).

Let us now finish the proof. By (37), G' is Berge. By (36), G' has no even skew partition. By (38), G' has no proper non-path 2-join. By (39) $\overline{G'}$ has no proper 2-join. By (40, 41), none of $G', \overline{G'}$ is the line-graph of a bipartite graph and G' is not bipartite. By (42) G' is not a path-cobipartite graph. By (43) G' is not a path-double split graph. By (44) G' has no homogeneous 2-join. By (45), $\overline{G'}$ is not a path-cobipartite graph, not a path-double split graph and has no homogeneous 2-join. So, G' is a counter-example to the theorem we are proving now. Hence there is a contradiction between the initial choice of G and (46). This completes the proof.

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