Making diagrams speak, in Bhâskara I's commentary of the Âryabhatîya

Agathe Keller

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Abstract

This article is a survey of the numerous questions raised by the presence of diagrams in a VIIth century Sanskrit mathematical commentary. Exploring the links between text, manuscripts and edition of Bhāskara I's *Āryabhaṭīyabhāṣya*, the enquiry ranges from the tools employed to draw geometrical figures to the diverse functions that were assigned to drawings in geometry. Whether technical objects disposed on a working surface or testimonies of oral explanations, diagrams apear as dealing with a part of mathematical reasoning which was not formulated through a written speech.

Cet article examine les questions soulevées par la présence de diagrammes dans un commentaire mathématique en langue sanskrite datant du VIIème siècle après J. C. Explorant les liens entre texte, manuscrit et édition de l'*Āryabhaṭīyabhāṣya* de Bhāskara I, cette enquête se penche tout autant sur les outils utilisés pour dessiner des figures géométriques que sur les diverses fonctions assignées au diagrammes en géométrie. Qu'il s'agisse d'objets techniques disposés sur une surface de travail ou de témoins d'une explication orale, les diagrammes y concernent une part du raisonnement mathématique qui n'tais pas ét'e formulée discursivement.

Keywords: Indian Mathematics, Diagrams, Āryabhaṭa I, Bhāskara I.

MSC classification numbers 01A32, 01A72

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Introduction

As historians, when we try to recover the way mathematical objects were thought of and used in different times and cultures, we should assess in which way the remaining material traces of a mathematical activity inform us on the reasonings and practices they were inserted in . In this respect, diagrams are especially interesting objects [1]. Indeed, although a written artifact, diagrams are not discursive testimonies of mathematical reasonings. Studying mathematical drawings and their functions may thus be a way to reach aspects of mathematical practices that are not transmitted through written speech. This is the reason why diagrams and pictures have attracted recently the attention of historians and epistemologists of mathematics[2].

The specificity of India

Compared to many other civilizations, India has left us with a remarkable amount of mathematical manuscripts[3], some of which transcribing texts that date back approximatively to 600 B.C. This huge amount of manuscripts is surprising for a tradition which values the oral transmission of knowledge.

Treatises in pre-muslim India were often a versified set of *sūtras* (aphoristic rules) in Sanskrit language. These *sūtras* were usually so condensed that they could not be understood on their own and required a commentary. The more important treatises in India were, the more commentaries they gave rise to. Treatises were usually considered as spoken (*uc*-) whereas commen-

[1] In the following we will call 'figure' the abstract idea of mathematical objects and 'diagram' any drawn representation of such ideas. This contrast may not always be relevant: practicing mathematicians will sometime use mental diagrammatic representations of figures. But this fact will not create any difficulties for our understanding of Bhāskara's text.

[2] For recent studies on diagrams in mathematics, one can refer, for instance to [Netz 1999]'s study of Euclidean diagrams, or [Brown 1999] for a philosophical point of view. K. Chemla, in her research seminar 'history of mathematics, history of text' in Paris, has for the past 7 years been pursuing a historical reflection on this theme, which has triggered the writing of the present article.

[3] See [CESS, Volume I, Introduction] where D. Pingree explains his motivations for his monumental and still incomplete (after almost fifty years) census of the mathematical and astronomical texts written in Sanskrit of the Indian subcontinent.

taries were written affairs (*likh-*). Astronomical treatises were no exception to this rule. And most mathematical texts that have come down to us, were somewhat autonomous chapters of astronomical treatises.

Despite the importance given to oral transmissions, mathematics in India was clearly an activity which required writing. Indeed, rules in treatises were given to note numbers and draw diagrams. One of the recorded names of arithmetics was 'dust or finger mathematics'[4], a reference to an existing working surface on which elements were drawn or written.

Thus, in India the tradition itself considers writen texts as a fragment of a more important oral lore, but these fragmentary traces are numerous. In this context, mathematical diagrams seem to be a technical and specific instance of such traces, with the additional peculiarity that diagrams are non-discursive in nature. In the following we will see how diagrams appear in a specific text. We hope to highlight the limits of what written texts tell us of ancient mathematical activities, and question how to make them 'speak' in other manners. We will also have opened a window on to the mathematical world of a VIIth century Indian astronomer.

Bhāskara's commentary

This article will concentrate on a VIIth century Sanskrit commentary written by an astronomer called Bhāskara I[5] on a late Vth century versified astronomical treatise, the *Āryabhaṭīya* (Ab) of Āryabhaṭa. The *Āryabhaṭīya*

[4] [Datta & Singh 1935, 123].

[5] Also called the 'the elder Bhāskara' to distinguish him from the XIIth century astronomer bearing the same name, Bhāskara II or 'the younger Bhāskara'.

has four chapters, the second concentrates on *gaṇita* or mathematics. In the following, we will focus on the contents of the mathematical chapter of the *Ārybhaṭīya* and of Bhāskara's comments on this part (BAB.2), questioning when and how diagrams appear. We then attempt to characterize these drawings, and reconstruct how they were drawn. Finally we examine the different functions that were assigned to diagrams in this text.

1 Where can Diagrams be found? Locating drawings in Bhāskara's text

Diagrams can be found in Bhāskara's commentary on the *Āryabhaṭīya*. This means that they can be seen in the preserved manuscripts of this text, and in its printed edition. Bhāskara not only mentions these drawings as we will see below, but also opens a space, within his written text, where they can be drawn. In the following we will examine the editorial work on the diagrams of the printed edition before describing where and how diagrams appear in the text of Bhāskara's commentary.

1.1 Edition and Manuscripts

A printed version of Bhāskara's commentary was published by K.S. Shukla in 1976 for the Indian National Science Academy (INSA)[6]. We have relied on this edition. However, it has a certain number of limitations.

[6] [Shukla 1976].

The first problem comes from the state of the sources themselves. Only six manuscripts of the commentary are known to us [7]. Five were used to elaborate Shukla's edition. Five belong to the Kerala University Oriental Manuscripts Library (KUOML) in Trivandrum and one to the British Office in London[8]. All the manuscripts used in the edition prepared by K. S. Shukla have the same source[9]. This means they all have the same basic pattern of mistakes, each version having its own additional ones as well. They are all incomplete. Shukla's edition of the text has used a later commentary of the text inspired by Bhāskara's, to provide a commentary on the end of the last chapter of the treatise. We do not know what is the history of these manuscripts (who had them copied, why are they in the present library etc.), nor when they were written. The oldest known palm-leaf manuscripts are generally not more than 500 years old and most paper manuscripts date back to the XIXth century. Therefore, there is probably *more* than a thousand years gap between Bhāskara's text and the remaining manuscripts we have of it. When differences between the written text and what is found in the manuscripts appear, the lack of historical contextualization and reflection on the relation of the manuscripts to the original text will prevent us from

[7] Four of the six manuscritps remaining of this text are made of dried and treated palm leaves which are carved and then inked. The two other are paper manuscripts. Manuscritps do not preserve well in the climate of the subcontinent, as noted in [Pingree 1981, 118].

[8] Shukla has used four manuscripts from the KUOML and the one from the BO ([Shukla 1976, cxi]). A fifth manuscript was uncovered by D. Pingree at the KUOML. As one of the manuscripts of the KUOML is presently lost it is difficult to know if the 'new one' is the misplaced old one or not. Furthermore, this manuscript is so dark that its contents cannot be retrieved anymore. See [CESS, Series A, Volume IV, 297].

[9] [Shukla 1976, cxii].

pinpointing with accuracy what belongs to Bhāskara and what belongs to the manuscripts[10].

While the lack of primary material is a major difficulty, other problems stem from the quality of the edition itself. Editorial choices concerning textual arrangements (such as diagrams and number dispositions) are often, if not systematically, implicit. I have consulted four of the six manuscripts of the text and can testify that dispositions of numbers and diagrams vary from one manuscript to the other. Discrepancy in between the printed text and the manuscripts further deepen the already existing gap between the written text and the manuscripts themselves.

Thus, all analysis of diagrams should be subject to great care: as we will try to unravel what are the elements existing in Bhāskara's text, what belongs to the manuscripts and what belongs to the printed edition.

In 1997, I obtained a photographic copy of one of the manuscripts: Co 1712 of the KUOML (Manuscript D of Shukla's edition). It most probably was written in the XIXth century[11].

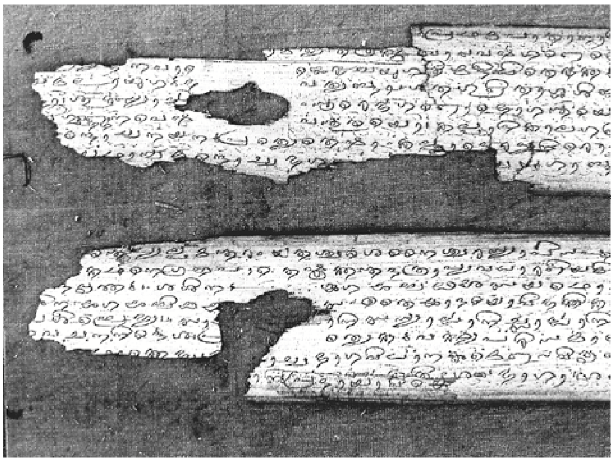
In the following, reproductions of this manuscript will, as far as we can, be given with pictures of the printed text, in order to compare them. Because the manuscript was in quite a bad shape, many of its folios already broken, as

[10] Very few studies have noted this. A noted exceptions is [Sarma 2002] which considers the dispositions of the Rule of Three, underlining discrepancies between numerical dispositions described in a text and those adopted in manuscripts.

[11] According to a piece of oral information provided by Dr. F. Voegli of the University of Lausanne, using epigraphical evidence. P. L. Shaji, a scribe of the KUOML, believes the manuscript to be several hundred years older.

illustrated in Figure 1, there is sometimes no manuscript-diagram available for an edition-diagram.

Figure 1: Palm leaf manuscript breaking into pieces



Using reproductions of manuscript-diagrams and edition diagrams together will allow us to keep in mind that we do not know if what we are analyzing is an element of Bhāskara's original text or an editorial innovation of either the manuscripts or the printed edition. The disparities between the diagrams of the manuscript and those of the printed edition will help us assess in which way every element of a diagram can be significant in providing information on mathematical activities and reasonings. Therefore we will not unravel a truth about Bhāskara's practice of diagrams, but raise questions on this practice. Further similar scrutinies might in the end help us to slowly map diagrammatic activities in the Indian sub-continent.

1.2 Diagrams and the structure of Bhāskara's commentary

All the diagrams that can be found in the text belong to the commentary, there is no diagram in the treatise[12]. All the diagrams belong to the commentary on the mathematical chapter, no diagrams can be found in the commentaries on the other chapters of the *Āryabhaṭīya*. Diagrams are therefore mathematical objects in this case[13]. The mathematical chapter of the *Āryabhaṭīya* is made of 33 verses.

Table 1 on page 46 gives an idea of the diversity of the problems that the verses of the mathematical chapter of the *Āryabhaṭīya* engage with, according to Bhāskara's interpretation of them[14].

We can apply our own classification of mathematics to sort the subjects treated in this chapter. For instance, we would consider that verses 6 to 10 deal with geometry while verse 30 is algebra and verse 11 trigonometry. Bhāskara gives his own definition of the subsubjects that form mathematics[15]. This is how he classifies the different topics Āryabhaṭa deals with, ascribing

[12]As we shall see below, verse 13 of the mathematical chapter alludes to methods to construct diagrams. However, the treatise, which is considered an oral text, does not possess any drawn figure.

[13]In the *Māhabhāskarīya* (MBh.5.60 [Shukla 1960, 64]) our commentator refers to diagrams representing eclipses. Diagrams were certainly of current use in astronomy. This is therefore a specific instance.

[14]As underlined in [Hayashi 1997], other interpretations of Āryabhaṭa's verses were made by different commentators.

[15]A first attempt at understanding these different subdivisions can be found in [Keller 2000, Volume I, 2.1] and a second description in [Keller forthcoming]. Both are written in French.

them to the 'mathematics of quantities' (*rāśigaṇita* or 'arithmetics') and to the 'mathematics of fields'[16] (*kṣetragaṇita* or geometry) :

> *apara āha: gaṇitaṃ rāśikṣetraṃ dviddhā'/ (...) gaṇitaṃ dviprakāram rāśigaṇitaṃ kṣetragaṇitam/ anupātakuṭṭākārādayo gaṇitaviśeṣaḥ rāśigaṇite 'bhihitāḥ, średdhīcchāyādayaḥ kṣetragaṇite/*
>
> Another says-'Mathematics is two fold: quantity and field (...) Mathematics (*gaṇita*) is of two kinds: mathematics of fields and mathematics of quantities. Proportions, pulverizers, and so on, which are specific ⟨subjects⟩ of mathematics, are mentioned in the mathematics of quantities; series, shadows, and so on, ⟨are mentioned⟩ in the mathematics of fields.

Even though Bhāskara's statement is elusive, we can understand that the following subjects belong to what he considers to be geometry: measuring segments, areas and volumes belonging to abstract objects such as squares, trilaterals and spheres; considering succession of numbers and assessing their sum or their number (series or *średhi*); measuring the length of the shadow of a sun dial or gnomon (*śaṅku*) or measuring other related segments; applying such procedures to astronomical problems such as eclipses or to traditional problems such as the breaking of a bamboo rod and the sinking of a lotus.

All of the diagrams appearing in the commentary on the mathematical chapter appear when dealing with subjects that Bhāskara understand as

[16]Geometrical figures are generically called fields (*kṣetra*).

belonging to *kṣetragaṇita* or geometry.

Bhāskara's commentary follows a systematical pattern. He glosses Āryabhaṭa's verses in the order in which they appear. He may comment upon half a verse, whole verses or two verses at a time.The structure of each verse commentary is summarized in Table 2 on page 47.

This structure can be found in other mathematical commentaries as well[17].

The commentary of a verse starts by an introductory sentence, followed by a quotation of the text at stake. It is followed by what we call a 'general commentary' of the verse. Its aim is to lift all the ambiguities that arise from the verse because of its elliptic short form. A 'general commentary' is mainly syntactical and grammatical. All the different steps of a general procedure are spelled out in it. This is also the place were debates are staged, and the validity of algorithms and definitions are discussed.

It is followed by a succession of solved examples whose function is to unravel the different realms of application of the algorithm and to give to it substance (the kind of problem it gives an answer to or the larger procedure it can be integrated within, the different type of results it can yield, and the different interpretations it can lead to, etc.). Solved examples also follow a standard pattern. After announcing an example (*uddeśaka*), a versified problem is exposed. It is followed by a 'setting down' (*nyāsa*) of the elements given in the problem on a working surface. This is precisely the part of

[17]See for instance [Jain 1995].

the text which is a window opened on mathematical practices that are not discursive, numbers are noted on it in a tabular fashion, and diagrams are drawn.

The 'setting down' is followed by a reasoning (*karaṇa*, we have translated this as 'procedure') showing how the example is solved.

The printed edition of the text contains 58 diagrammes; the manuscript 48 but many folios are broken.

78% of the diagrams that can be found in the printed edition are in the 'setting down' part of the solved examples, 7% belong to their 'resolution', and 15% belong to the 'general commentary'.

Let us now turn to the diagrams themselves.

2 Looking at diagrams

Diagrams are mentioned in Bhāskara's commentary.

2.1 Vocabulary

Chedyaka is the Sanskrit word of the commentary that we have translated as 'diagram'[18]. Etymologically it means 'what can be cut'. This may be a reference to the process by which several segments are drawn from the

[18]It is used in relation to a specific diagram, the one in the commentary of verse 11, illustrated in Figure 14 on page 37. But since this word is also used in the *Mahābhāskarīya*, on a totally different subject, we can infer that it was used by Bhāskara as a generic name for such technical drawings.

outline of a geometrical figure[19]. Another word is used once, *ālekhya*. It literally means 'what should be written, drawn' and consequently 'drawing'. In most cases, diagrams are referred to by a composite expression, such as 'a drawn field', or 'a field ⟨which⟩ is set down'. The verb used for drawing (*likh-*) is the same for writing and means 'to dig, scratch', maybe an allusion to the way one writes on palm-leaves.

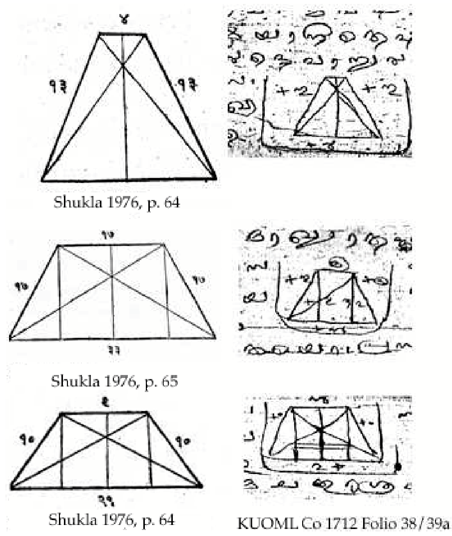
2.2 Diversity

As we go through the commentary we can distinguish several types of diagrams. Some represent simple geometrical figures, like the trapeziums reproduced in Figure 2 on the following page and the triangles in Figure 13 on page 36.

Representations can be more complex and include several geometrical figures, as the square with inner triangles and rectangles in Figure 9 on page 28 and the hexagone and triangles within a circle in Figure 14 on page 37. Some represent tridimensional objects, as the piles of squares and cubes in Figure 12 on page 34. Others are drawn from concrete-like situations, as the 'Hawk and rat' problems illustrated in Figure 5 on page 17. Numbers are noted in these drawings specifying the lengths of certain segments. In the manuscripts, the interior segments do not have numbers. In the printed

[19]This interpretation is suggested by the description of the construction of the diagram in the commentary of verse 11. See BAB.2.11, [Shukla 1976, 78, line 40 sqq.] for the sanskrit and [Keller 2000, Volume II, BAB.2.11] for an English translation. K. Chemla has also suggested that this could refer to the 'cutting out' of a shape, as when we use scissors to cut a piece of paper.

Figure 2: Trapezium



edition, on the contrary, numbers are sometimes ascribed to these inner segments. Some diagrams are endowed with letters in the edition, but not in the manuscript, as in Figure 4 on page 17, Figure 5 on page 17 and Figure 14 on page 37. There are other striking differences between Manuscript KUOML Co 1712 and the printed edition. The diagrams of the manuscript are drawn in little boxes that separate them from the printed text. Strikingly, they are not drawn with accuracy or proportion, they do not have any titles and are not numbered. In the printed edition however, a specific space distinctly separated from the text is allotted to diagrams. As specified in the Introduction of the printed edition[20], they have been labelled and numbered.

In both cases, never are letters (or syllables) used to label the tips of

[20] [Shukla 1976, Introduction, 10.2.iii, cxv].

segments in figures[21]. This has some consequence on the differentiation of diagrams.

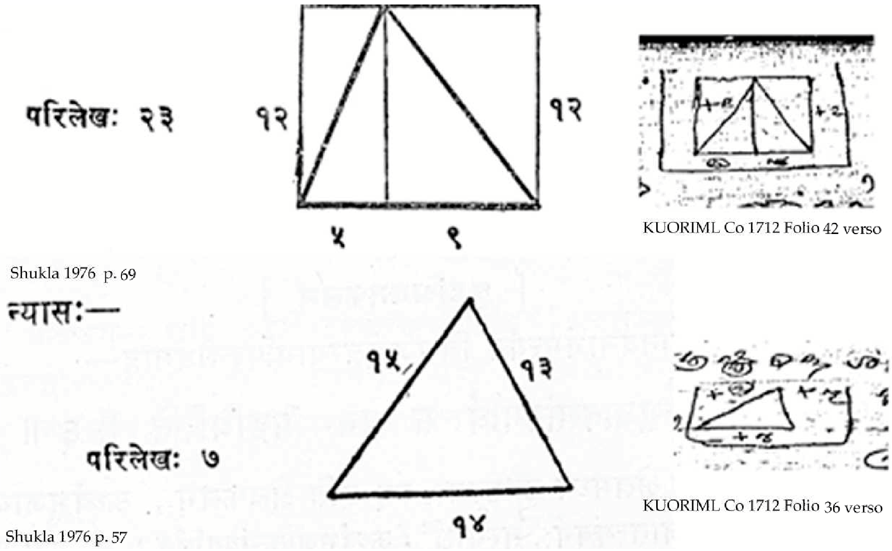
2.3 Differentiating figures

Indeed, none of the different "authors" (that is Bhāskara, the scribes or K. S. Shukla) of the diagrams seem to discriminate between fields that are mirror images of one another. This is what appears as we analyze an apparent 'mistake' in the diagrams of the printed edition. On Figure 3 on the next page, two representations of the same triangle (a scalene triangle whose sides measure respectively, 14, 13, 15) can be seen: one is drawn when computing its area according to the rule given in verse 6, the other when 'verifying' its area according to the rule given in verse 9. They are mirror images of one another in the printed edition.

Because of the vagueness with which the diagrams are drawn in the manuscript, it is not possible to deduce from the reproductions if these triangles were discriminated or not. However, measures of areas and of segments, which are the purpose of Bhāskara's geometry, are not altered by such transformations. It is therefore possible that Bhāskara, like the scribe who wrote the manuscript, considered these two triangles to be the same.

[21] We will see below, that the syllables found in the printed edition are used to indicate cardinal directions.

Figure 3: Two mirror diagrams of a same triangle

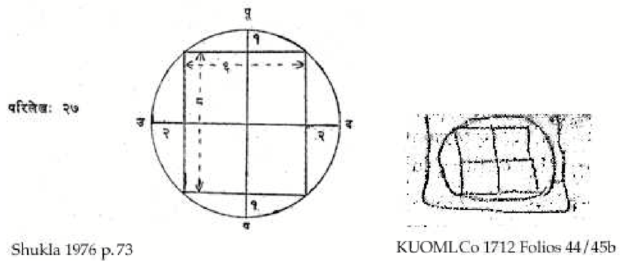


2.4 Orientation

The absence of letters does not, however, prevent Bhāskara from giving an orientation to geometrical figures when he needs it. He uses cardinal directions for this purpose. In Sanskrit, the East (*pūrva*) is 'in front'; the West (*paścāt*) is 'behind'; North (*uttara*) is 'left'; South (*dakṣina*) is 'right'. Figure 14 on page 37, Figure 4 on the next page and Figure 5 on the following page use such an orientation.

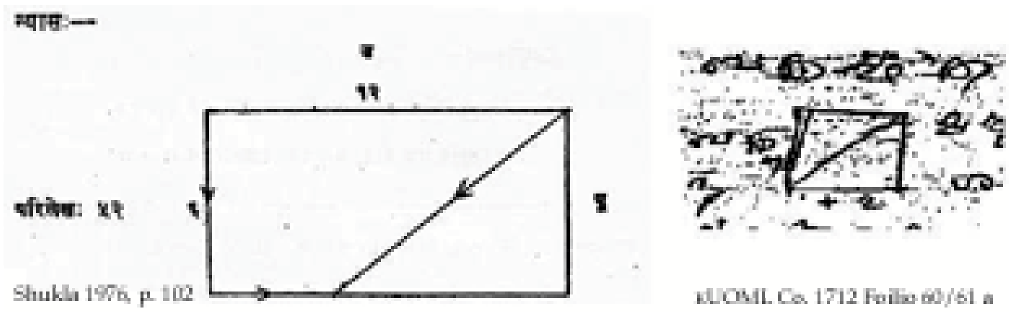
These figures are oriented by indications given *within the written text.* For instance, Bhāskara mentions their eastern part. In the printed edition, these references appear in the diagram itself, as abbreviations for given cardinal points. These letters are not to be found in the diagrams of manuscript D. On Figure 4 on the next page this difference is striking, as the printed edition

Figure 4: A circle, its bow fields and inner rectangle



not only presents initials for the four cardinal points outside of the circle, but also inserts arrows inside the drawing and numbers indicating measures of length; Manuscript D has no such notations at all. In the diagram reproduced in Figure 5, two cardinal directions are noted above and on the right.

Figure 5: Fish and Hawk problem



The North is represented above as conceived of in Europe in the printed edition. No such letters appear in the diagram of the manuscript.

Besides this straightforward orientation, the names given to the sides of a geometrical figure may have confered an implicit positioning of the figure in space. Indeed, the word 'earth' (*bhū*, *dhatrī*, etc.) used for the base of a triangle and a trapezium always seems to refer to a horizontal segment, the lowest possible in a given diagram. Beyond the imagery conveyed by this word, this property of an 'earth' is manifested in two examples where the traditional orientation of a triangle and a trapezium are disturbed.

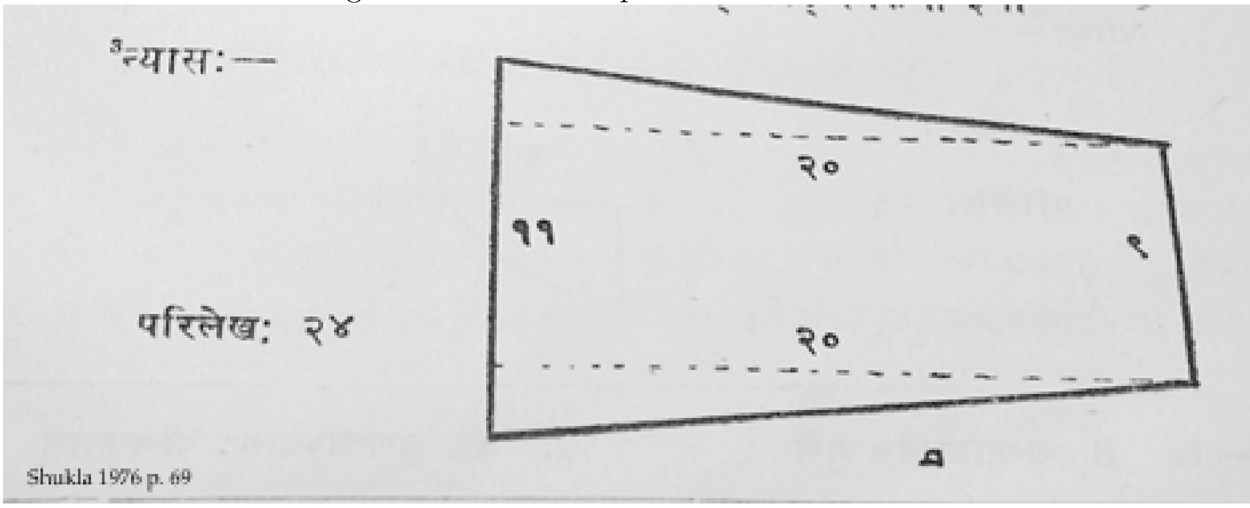
In the first example, what appears as the drawing of a 'tilted trapezium'[22] in Shukla's edition (6 on the next page) in the commentary on the first half of verse 9, has two parallel sides which are not described by Bhāskara as an 'earth' and a 'face', but a face (*mukha*) and a prati-face (*pratimukha*)[23]:

> *anayā diśā prakīrṇakṣetre phalam svadhiyā abhyūhyam — tat yathā*
> *mukham ekādaśa dṛṣṭam pratimukham api ucyate tathā ca nava|*
> *āyāmaḥviśatikaḥphalam asya kiyat bhavet gaṇaka||*
> *nyāsaḥ*
>
> In this way, with one's own intellect the area is infered in miscellaneous fields. It is as follows- The face is seen as eleven and then the opposite face is said to be nine|
> The height is twenty. What should be its area, calculator?|| Setting down:

[22]This is my own description. Bhāskara puts it under the category 'miscelaneous fields' (*prakīrṇakṣetra*) and does not refer to it as a trapezium (*dviṣama/viṣamacaturaśra*). However, the resolution of the problem shows that it has two parallel sides. For more on the name of figures see [Keller 2000, glossary].

[23]See [Shukla 1976, 69] for the text and diagram, translated and reproduced in [Keller 2000, Volume II, BAB.2.9.ab], this diagram is not found in Manuscript D.

Figure 6: A ‘tilted trapezium’

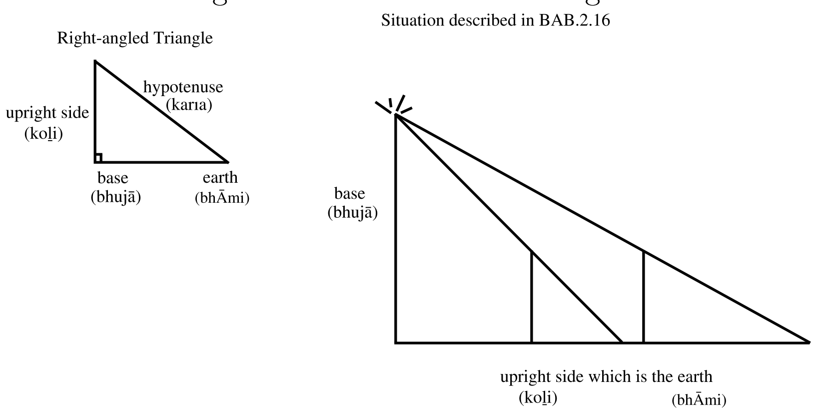


The second example, appears in the commentary of verse 16, which deals with the shadows of two gnomons. In this case, the source of light on a height is called the base (*bhujā*) of a right-triangle. The ‘upright side’ (*koṭī*) which appears lying down is then referred to as ‘the earth’ (*bhūmi*)[24]. Thus even when the right-angle triangle is tilted, as appears in Figure 7 on the following page, the ‘earth’ remains the lowest horizontal segment in the figure.

Thus, we have seen that there seems to be no habit, neither in Bhāskara’s text nor in the manuscript, to give names to the tips of geometrical figures. However, the commentator can provide an oriented diagram when needed. He does so obviously by resorting to cardinal directions. Re-naming segments with technical words (such as *bhumi, mukha*) may also have been a way of providing an orientation. A certain level of confusion remains however, since

[24] *koṭī avasānabhūmiḥ (...) bhujā yaṣṭipradīpocchrāyaḥ.* [Shukla 1976, 93].

Figure 7: The 'earth' in triangles



one cannot always discriminate between mirror images of diagrammes for whom no cardinal orientation is given, a phenomenon which would have been quite indiferent to Bhāskara's geometry which ignores symetries.

This apparent mistake opens the door to the techniques that are described in the text to construct diagrams.

3 How were diagrams drawn?

We have noted already the lack of accuracy and proportion in the diagrams of the manuscript. However, techniques are described in the commentary of verse 13 concerning the construction of trilaterals, quadrilaterals and circles with the help of strings and a pair of compasses[25].

Indeed, the first half of verse 13 of the mathematical chapter of the treatise, the *Āryabhaṭīya*, lists tools that can be used for sketching diagrams:

[25]BAB.2.13, see Appendix 4.5 on page 48.

Ab.2.13.ab *vṛttaṃ bhrameṇa sādhyaṃ tribhujaṃ ca caturbhujaṃ ca karṇābhyām* |
A circle should be brought about with a pair of compasses, and a trilateral and a quadrilateral each ⟨are brought about⟩ with two diagonals|

In the following we will look at how Bhāskara comments on this verse, analyzing seperately what information he gives us on compasses, and then on the construction of trilaterals and quadrilaterals with strings.

3.1 Compasses

Āryabhaṭa's name for compasses is *bhrama* 'a rolling ⟨object⟩'. Bhāskara, his commentator, calls it a *karkaṭa* or *karkaṭaka*, literally a 'crab'. In his commentary on verse 13, Bhāskara gives only a brief explanation on this object[26]:

bhramaśabdena karkaṭakaḥ parigṛhyate| *tena karkaṭakena samavṛttaṃ kṣetraṃ parilekhāpramāṇena parimīyate*|
With the word *bhrama* a pair of compasses (*karkaṭa*) is understood. With that pair of compasses an evenly circular field is delimited by the size of the outline (*parilekhā*).

[26][Shukla 1976,85].

Elsewhere he is slightly more specific. Thus in his commentary of the latter half of verse 9 of the chapter on mathematics, he writes[27]:

> *asmin ca viracitamukhadeśasitavartyaṅkurakarkaṭena ālikhite chedyake...*
> And in this diagram, which is drawn with a compass (*karkaṭa*) for which a sharp stick (*vartyaṅkura*) secured (*sita*) at the mouth spot (*mukhadeśa*) has been arranged....

According to the meanings we give to *vartī* (or *vartikā*; usually the wick of a lamp, a paint-brush or chalk) and to *sita* (has been fastened, white color), different readings of this description are possible, and hence different images of compasses appear. We also do not know what the 'mouth spot' (*mukhadeśa*) of the compass is. The same difficulties arise when we read the short description in Bhāskara's commentary of verse 11[28]:

> *tathā ca paridhiniṣpannaṃ kṣetraṃ karkaṭakena viracitavartikāmukhena likhyate*
> And thus a field produced by a circumference is drawn with a pair of compasses whose opening (*mukha*) has a sharpened stilk (*viracitavartikā*).

We have adopted the improbable reading of *vartī* (or *vartikā* that we have read as a synonym of the first) as 'stick' by accepting Parameśvara's

[27] [Shukla 1976, 71].
[28] [Shukla 1976, 79].

interpretation of the compound *vartikāṅkura*[29]. Figure 8 on the following page illustrates Parameśvara's compasses.

3.2 Ropes or strings

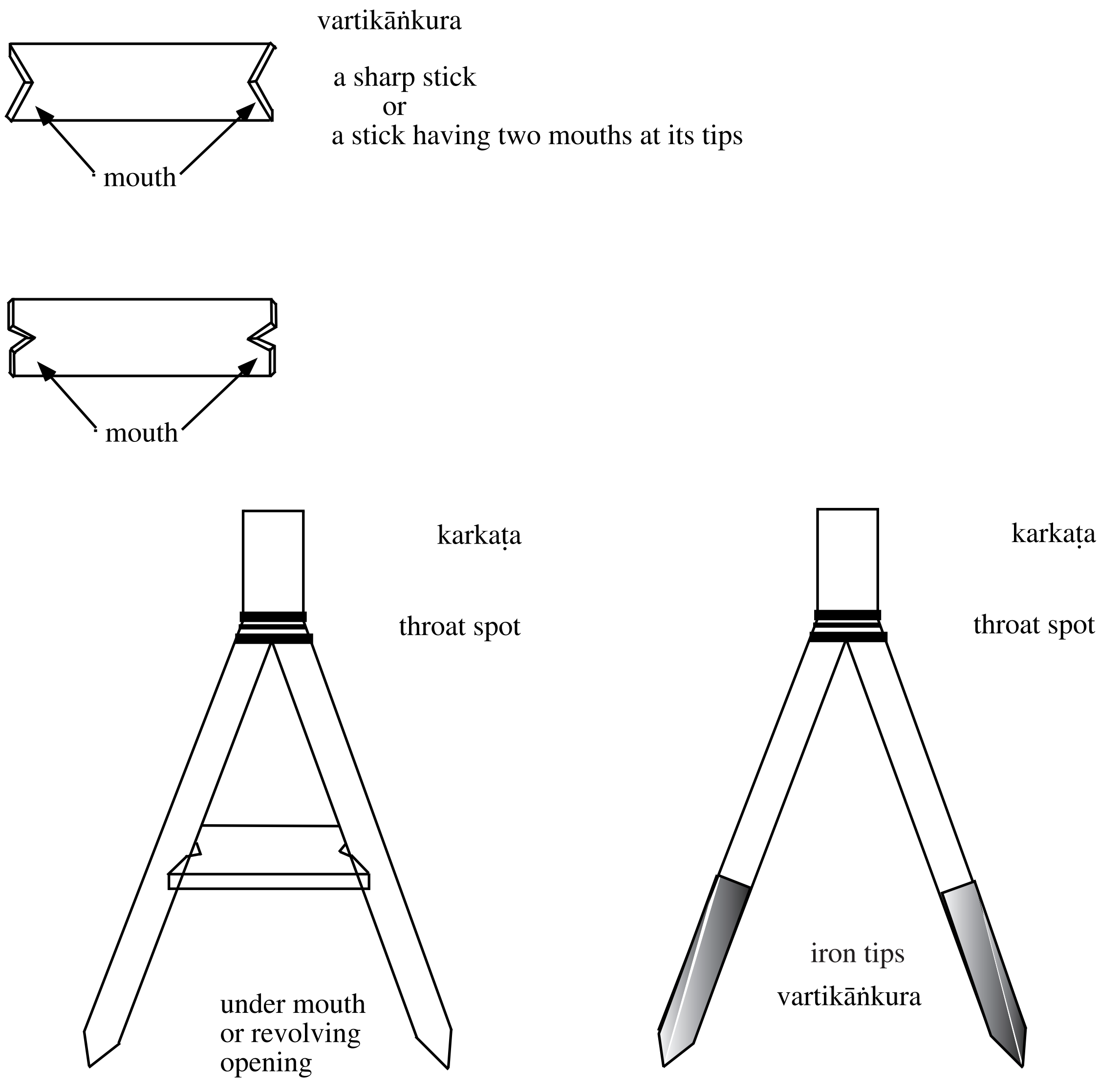
The mentioning of strings (*sūtras*[30]) in Āryabhaṭa's treatise opens the question of the continuity in between two seperate traditions of Indian mathematics. Indeed, the oldest mathematical texts known of in the Indian subcontinent, the *śulbasūtras* [31], described how to delineate sacrificial areas with specific geometrical shapes using sticks and ropes. Apparently, there seems to be little link between this ritual geometry and the *kṣetragaṇita* propounded here a thousand years later, but for the use of this material in geometrical constructions. The question of the posterity of the *śulba* mathematics in Āryabhaṭa's treatise and Bhāskara's commentary remains open to further research. The latter's commentary of verse 13 is quite straightforward when describing the construction of isosceles triangles and rectangles. A transla-

[29]Parameśvara is a well known XVth century astronomical commentator, who has authored many works (See [CESS, Volume IV]). He wrote commentaries on Bhāskara II's works as well as on the *Āryabhaṭīya* (see Kern[1874]). He also wrote a direct and a super commentary on Bhāskara I's *Mahābhāskarīya* and a direct commentary on the same author's *Laghubhāskarīya* ([Sastri 1957]) in which he describes how to make a compass. Almost 800 years separate Parameśvara's from Bhāskara. Furthermore, they belong to distinct regional traditions. Therefore, it is most probable that Parameśvara 's compasses are not the compasses used by Bhāskara. However, rather than letting our imagination run free, we have echoed Parameśvara's compasses in our translation of Bhāskara's descriptions, hoping that by giving more attention to the description of such tools in commentaries, we will one day be able to describe regional differences and chronological evolutions of compasses in the Indian sub-continent.

[30]This is indeed the same word as the one used for aphoristic rules.

[31][Bag and Sen 1983].

Figure 8: A pair of compasses as described by Parameśvara



tion of this commentary can be found in Appendix 4.5 on page 48. These techniques, however, seldom seem to have been used or refered to in other parts of the commentary. In most cases, indeed, such methods could not be applied because they required the knowledge of the length of a height or a diagonal, from which the whole figure was then drawn. Versified problems usually did not readily provide these inputs, which were found during the resolution. The question of why these techniques have thus been described remains open.

3.3 Tridimensional objects

Reference is also made to tridimensional artifacts, whether practical objects or abstract figures. Bhāskara in his commentary on verse 14 describes many different gnomons and the way they may be constructed[32].

Solids such as pyramids and spheres described in the treatise are represented as plane figures in diagrams. But Bhāskara while describing a cube, adds (p. 51, line 5) :

aśrayo yasya mṛdānyena vā pradarśayitavyāḥ/

Or its side should be shown with earth or something else

This could be a reference to clay representations of a cube.

Similarly concerning the height of a triangular based regular pyramid he writes (p. 59, lines 25-26) :

[32]For astronomical instruments one can refer to [Ōhashi 1994]. His description of Āryabhaṭa's gnomons is discussed in [Keller 2000, Volume II, Annex of BAB.2.14].

tatra ūrdhvabhujā sūtrakaiḥ śalākādibhir vā pradaśayitavyā/

> In this case one should explain the 'upward side' with strings, sticks, etc.

We can also note that a sphere is sometimes referred to as an iron ball (*ayoguḍa*).

We have seen that we did not know if diagrams needed to be constructed with accuracy or not. Such uncertainty can also be found concerning tridimensional artifacts. Indeed, in some instances, the construction of an accurate tridimensional object is crucial (the gnomon). In other cases, however, it does not seem so important (when visualizing three dimensional figures). The paradox deepens further when we observe what the text tells us about the skill required to draw figures. We will thus turn to what Bhāskara mentions cocerning the expertise required to construct diagrams.

3.4 Expertise and Accuracy

In Bhāskara's text, few constructions of diagrams are described and then drawn within a 'set down' area. More often than not, diagrams are placed in these areas with no comment on the way they were constructed. Alternatively, they are referred to without, seemingly, being 'set down'[33].

Furthermore, constructions are always elusive[34], and as the methods de-

[33]In such cases neither Shukla's edition nor the manuscripts consulted have diagrams. This is for instance the case in BAB.2.8, as quoted below.

[34]Two constructions of diagrams are described with some accuracy in Bhāskara's com-

scribed in BAB.2.13 are of no use in these cases, we cannot reconstruct how the diagrams were to be produced.

Occasionally however, the expertise necessary to draw diagrams is mentioned. For instance in his commentary of verse 8, Bhāskara comments on the computation[35], in a trapezium, of the two segments of the height defined by the point of intersection of its diagonals. These two segments are called in the treatise, *svapātalekhā*, the 'lines on their own falling'[36]. Bhāskara writes[37] :

> *samyagādiṣṭena ālikhite kṣetre svapātalekhāpramāṇaṃ trairāśikagaṇitena pratipādayitavyam/*
>
> The size of the 'lines on their own fallings' is explained with a Rule of Three in a field drawn by a properly instructed person.

Thus in certain occasions a specialist is required to make (and comment on) a diagram. In an other case, however, the use of a mathematical drawing is dispized. Indeed, at the end of the textual description of the diagram reproduced in Figure 9 on the next page, in his commentary of the first half

mentary. One can refer to BAB.2.11 ([Shukla 1976, 78 sqq] for the Sanskrit, [Keller 2000, Volume II, BAB.2.11] for an English translation) whose diagram is reproduced in Figure 14 on page 37, and, in BAB.2.3.ab ([Shukla 1976, 48] for the Sanskrit, and [Keller 2000, Volume II, BAB.2.3.ab] for an English translation), whose diagram is reproduced in Figure 9 on the next page.

[35]In the following, I will systematically call "computation" those calculations dealing with the lengths of geometrical entities. Eventhough it is not relevant to the present article, i would like to maintain the distinction between these calculations that are not solely on numbers, and others.

[36]One can refer to Figure 10 on page 31 for the names of inner segments in a trapezium.

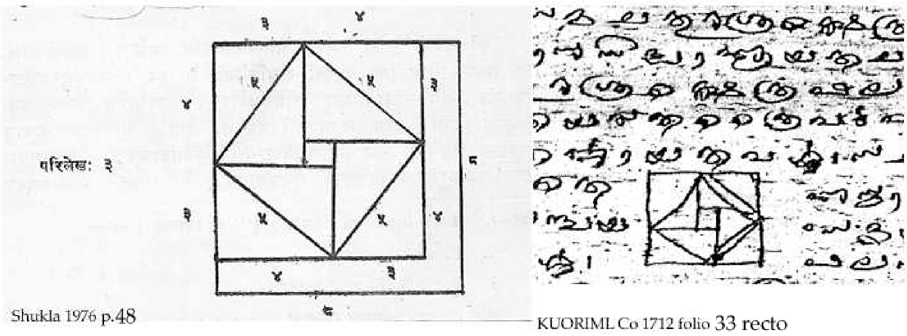
[37][Shukla 1976, 63, line 17].

of verse 3, Bhāskara writes[38]:

durvidagdhapratyāyanāya ca kṣetram ālikhyate

And to convince the dull-minded, a field is drawn

Figure 9: A diagram for the dull minded



In this case, Bhāskara seems to prefer a mental representation of the figure over its actual tracing.

How then should we understand the various standards that can be found in Bhāskara's text concerning both expertise and accuracy?

When diagrams were needed to explain a general idea it may not have been necessary for them to be drawn accurately . Mental representations may then have been prefered over real drawings. This could be one explanation for the lack of precise constructions in Bhāskara's commentary. In other instances, however, to carry out a procedure within a diagram for instance, more accuracy was necessary.

[38][Shukla 1976, 48, line 16].

The situation becomes even more entangled as we turn to the vagueness of the diagrams of Manuscript D. Their inacurateness could spring as much from the reasons mentioned above, than have other origins . For instance, diagrams may have been constructed with accuracy when one had to work with them, but such precision may have not been required when diagrams were transmitted via the manuscripts of copied commentaries. Or maybe scribes didn't have the required expertise to draw good diagrams.

In all cases, examining what exactly is the function of diagrams in Bhāskara's geometry should help discriminating among all the different requirements concerning the accuracy and thus the expertise needed to construct diagrams. We will thus now try to understand what was, according to the commentator, the role of diagrams in geometry.

4 What was the diagram's role?

Most of the diagrams (78%) can be found in the 'setting-down' part of solved examples. Consequently, the function they fulfill, what one is supposed to do with them, is not stated explicitly in these cases. Diagrams indeed appear as common mathematical objects whose status does not require any explanation. By analyzing the context in which diagrams are found, we shall see that numerous functions can be ascribed to diagrams: tools to specify definitions, the summary of a process, an object in which a procedure is carried out or even a proof.

4.1 Specifying a definition

We have mentioned drawings of triangles, trapeziums, etc. Are diagrams representations of geometrical figures? Definitions of geometrical figures can be found in Bhāskara's commentary, where they belong to the 'general commentary' of a verse. A definition is elaborated when the word used in the treatise to name them is discussed by Bhāskara. The commentator then attempts to determine whether the 'right word' is used to designate a given figure[39]. For instance, Āryabhaṭa, when providing a rule to compute the area of triangles in the first half of verse 6, uses the word *samadalakoṭi*, lit. 'equally halving height' for their heights. Does this mean that he only considers equilateral and isosceles triangles? Should this word be understood as meaning technically the height of any triangle? All of these questions are raised and discussed by Bhāskara in his commentary on this verse half[40]. By thus arguing to determine if *samadalakoṭi* is the right word to name the height of a triangle, Bhāskara in fact defines the object at stake.

This reveals that Bhāskara considers that there is an illustrative quality to Āryabhaṭa's technical vocabulary as well as to his own . This picturesque aspect of the mathematical language is striking for the contemporary reader[41]. Indeed, each figure bears specific names for the segments that outline it. For instance, a trapezium is defined by the the earth (*bhū*) parallel to the face

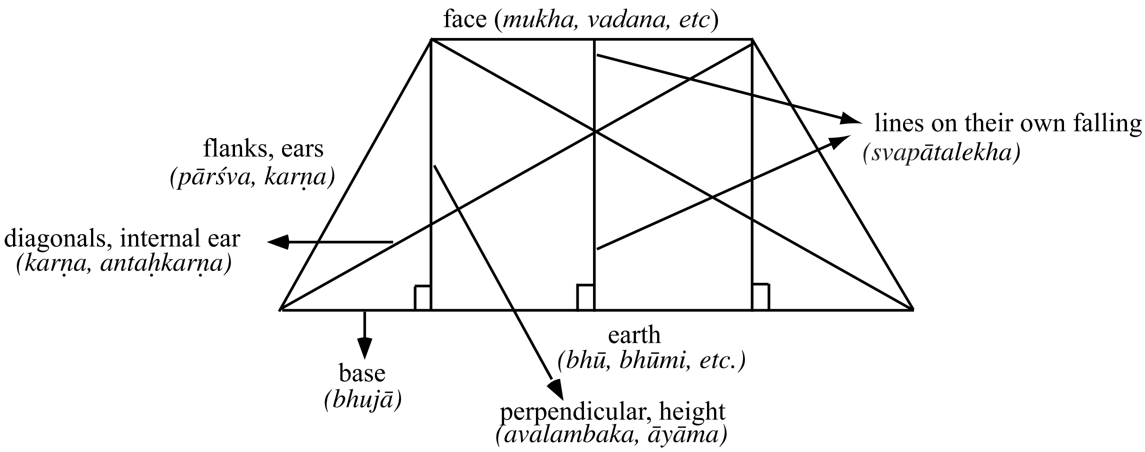
[39][Keller 2000, Volume I, 1.7, 91-101].

[40]Refer to [Shukla 1976, 55, line 4 sqq] for the Sanskrit, [Keller 2000, Volume II, BAB.2.6.ab] for an English translation.

[41]This has been also underlined in [Filliozat 1988, 257-258].

(*mukha*), and its lateral sides. They are called 'flanks' (*pārśva*) by Āryabhaṭa, 'ears' or 'diagonals' (*karṇa*) by Bhāskara. It is distinguished from any quadrilateral by the fact that its heights (*āyāma*) are equal. As mentioned above, Āryabhaṭa gives a rule to compute the length of the two segments of the height whose extremity is the point of intersection of the diagonals. These segments are called the 'lines on their own falling' (*svapātalekha*). This is illustrated in Figure 10.

Figure 10: Names of segments in a Trapezium



Similarly, there are three classes of trilaterals: those whose sides are all equal (*sama* e.g. equilaterals), those who have two equal sides (*dvisama*, e.g. isosceles), and scalene ones (*viṣama*). The base (*bhujā*) or earth (*bhū*) is distinguished from the other two sides (*pārśva*, flanks or *karṇa* ear, diagonal) by the fact that the height (*avalambaka*) falls on it.

Thus, the text sometimes seems to refer implicitly to a virtual diagram, which can easily be imagined because of the picturesque quality of the vocabulary. If names of geometrical figures and segments suggest a drawing, they can sometimes be deceitful. In one instance specific figures (and maybe their diagrams) are used to invalidate the virtual representation of a figure that words suggests. The word used for geometrical squares in Āryabhaṭa's treatise is *samacaturaśra* or 'equi-quadrilateral'. Bhāskara describes figures of 'equi-quadrilaterals' that are not squares in order to specify the object at hand[42]:

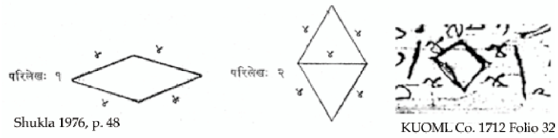
> *kva anyatra aniṣṭasya samacaturaśrakṣetraviśeṣasya vargasaṃjñāprasaṅgaḥ? ucyate asamakarṇasya samacaturaśrakṣetraviśeṣasya asya/ dvisama-tryaśrakṣetrasya samunnatavadavasthitasya asya/* 'When, in the other case, is it possible ⟨to give⟩ the name "square" to an undesirable kind of equi-quadrilateral field? This is said: "This kind of equi-quadrilateral with different diagonals has ⟨that name⟩, and this ⟨field made of⟩ two equi-trilateral fields placed as if upraised, has ⟨that name⟩"[43]

He refers to these non-square figures by using the demonstrative pronoun *ayam*, which designates objects at hand. Does this mean that diagrams were originally included in the text? These figures are illustrated in manuscript D and in the printed edition, as seen in Figure 11 on the next page[44].

Bhāskara ends the discussion by explaining that squares are 'equilateral

[42]BAB.2.3.ab
[43][Shukla 1976, 47-48].
[44][Shukla 1976, 48].

Figure 11: Two ‘equi-quadrilaterals’ which are not squares



quadrilaterals with equal diagonals’. ‘

Therefore, figures to specify (and correct) representations that words suggest are used, by Bhāskara, in definitions. They invalidate a mental visualizing that could be induced by the picturesque quality of the vocabulary. This suggests an environment were diagrams, real or mental, were used.

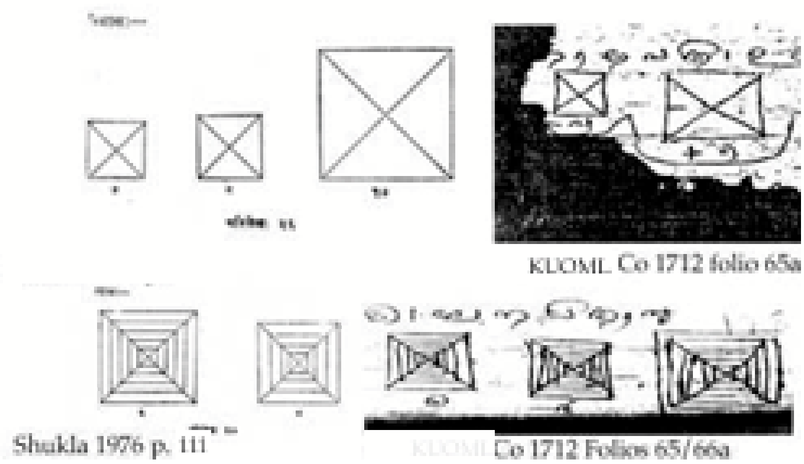
In a different manner, in his commentaries on the verses on series, Bhāskara may have provided diagrams which illustrated the geometrical quality of Āryabhaṭa’s vocabulary. Indeed, Āryabhaṭa describes series as piles (*citi*, *upaciti*) of objects[45]. For example, Āryabhaṭa describes sum of square numbers, as solid piles of square objects (*vargacitighana*): the sum of the square numbers, being the sum of the areas of these squares. Similarly, the sum of cube numbers is seen as a solid pile of cube objects (*ghanacitighana*), the sum of the cubes being the sum of the volumes of each cube. Series, in Bhāskara’s interpretation, are both arithmetical and geometrical objects, and he often gives an arithmetical interpretation of Āryabhaṭa’s vocabulary[46]. However, the examples contained in the verses on series all have ‘setting down’ parts which contain diagrams in the manuscripts. This suggests, unless there has

[45]See BAB.2.21, [Shukla 1976, 109-110]; [Keller 2000, Volume II, BAB.2.21], and BAB.2.22, [Shukla 1976, 110-112]; [Keller 2000, Volume II, BAB.2.22].

[46][Keller 2000, Volume I, 2.4.4, 95-97].

been a distortion of the original text, that numbers were not disposed there, and thus that Bhāskara did illustrate with diagrams Āryabhaṭa's vocabulary. The diagrams that can be found in manuscript D and which are reproduced with more accuracy in the printed edition illustrate these piles by representing them in a two dimensional projection. The case of the 'sum of squares and cubes' is thus illustrated in Figure 12.

Figure 12: Piles of squares and cubes



Note how damaged is the top reproduction of Manuscript D, a testimony of its bad state. Here, the top diagrams present piles of squares and the bottom diagrams piles of cubes.

Diagrams and characterizations of geometrical figures thus seem to have been closely linked, diagrams being a way to invalidate figures described by misleading words or simply a way to represent new geometrical figures.

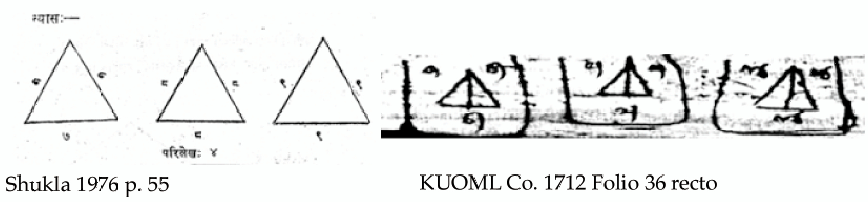
But they can also be linked in another way. As noted above, the differences between various types of quadrilaterals and trilaterals are defined using their inner segments: a square has equal diagonals where a trapezium has equal heights, the heights of an equilateral and isosceles triangles are mediators which is not the case for scalene ones... In the first half of verse 13, Āryabhaṭa indicates that trilaterals and quadrilaterals should be constructed from 'diagonals' (*karṇa*). If a *karṇa* is a hypothenuse or a side in a triangle and a trapezium, in other instances and more generally it refers to the inner diagonals of a quadrilateral figure. Bhāskara's commentary on this part of verse 13 (translated in Appendix 4.5 on page 48) describes construction of fields which thus rest on their inner segments. In the same way a circle is always defined by its semi-diameter (*vyāsārdha*) and circumference (*pariṇāha*). Could it be then that geometrical figures were characterised by their inner segments? If this was the case, then an element which defined a figure (the inner segment) would have been used to construct its diagram.

Definitions of geometrical figures and diagrams, implicitly, seem to complete and confirm one another. We have seen that diagrams could be used to rectify definitions, and conversly definitions would have enabled the construction of correct diagrams. But other functions seem to have been ascribed to diagrams as well.

4.2 Summarizing processes

Solved examples in the commentary of a verse were not only illustrations of a procedure, they also gave a specific meaning to what was explained in an abstract and general manner beforehand[47]. Now, in the 'setting-down' part of solved geometrical (in Bhāskara's sense) examples, diagrams are drawn. Thus, several functions can thus be ascribed to these diagrams. Look at Figure 13.

Figure 13: Triangles



It shows the 'setting-down' part of a problem which requires the area of three equilateral triangles knowing the length of their sides. What is known is stated with numbers. Thus, in the edition as in the manuscripts, the size of the sides is indicated by numbers noted within the diagram. But there is more to the diagram of the manuscript, than just a summary of the problem to be solved: in fact the whole process is illustrated here. Indeed, to compute the area of these triangles one needs the lengths of the heights which are unknown. The diagrams of the manuscript represent the heights without any number. They then show simultaneously what is known and what is sought. They summarize the problem but also illustrate each step of the reasoning

[47][Keller 2000, Volume I, 2.6, 80-91].

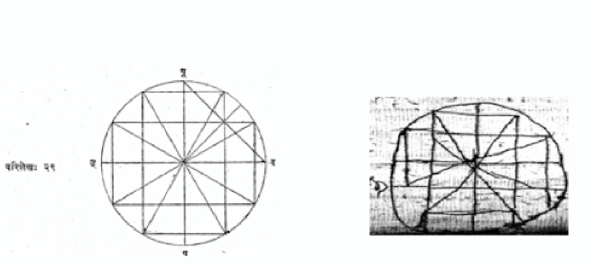
which has to be followed. Additionnally, this reproductioncould confirm that triangles, for Mansucript D as maybe for Bhāskara, where thought of as including their heights.

Diagrams of the 'setting-down' part of solved examples are therefore not simple, transparent transcriptions of a written problem.

4.3 An object to work with

In his commentary on verse 11, Bhāskara describes the construction of a diagram that will be used to derive sines. It is reproduced in Figure 14.

Figure 14: Derivation of half-chords



Shukla 1976, p. 79 KUOML Co. 1712 folio 47 recto

Bhāskara then spells out a computation to derive half-chords within it. This procedures uses the right-angle triangles that can be drawn in a circle, to compute, with the help of the Pythagoras procedure, half-chords or sines[48]. To modulate the values of the sines computed different angular arcs considered. Several uniform subdivisions of the circonference of the circle

[48]Rsines specifically, that is sines multiplied by the value of the radius, R, of the circle. For more on the computation, one can refer to [Keller 2000, BAB.2.11annex].

are used, and different multiples of these subdivisions serve as angular arcs for the sines considered. A classical measure unit is used here to subdivise the circumference: the *rāśi*, which 1/12th of the circumference of the circle . In the diagram whose construction is described by Bhāskara and which is reproduced above, whole *rāśis* are represented. And in the process that Bhāskara describes, pair subdivisions of *rāśis* are used. Thus, immediately after presenting the diagram, Bhāskara describes a process where the arcs considered are multiples of half a *rāśi*. He then considers the same process, but with arcs which are multiples of a quarter of a *rāśi*, and then of one eighth of a *rāśi*.

But let us come back to the construction of the above reproduced diagram. When the description of how to draw it is over.Bhāskara writes[49]:

> *evam ālikhite kṣetre sarvaṃ pradarśayitavyam*
>
> In the field drawn in this way all is to be shown.

The verb used to express 'to show', *pradṛṣ-*, possesses the same ambiguities as its English equivalent: it can mean to see with the eyes, but also to explain. We can then understand that the function of this diagram is to be a visual aid, a place where the process is explained and understood as it is worked out.

Indeed, the diagram that has just been constructed and which is reproduced in Figure 14 on the preceding page does not represent directly the

[49][Shukla 1976, 79]; [Keller 2000, Volume II, BAB.2.11].

computations described. As we have noted above, the drawing presents a circle subdivided in whole *rāśi*s, when the process that Bhāskara describes uses different pair subdivisions of *rāśi*s . In this sense the diagram presented in this commentary is like a general model from which all other specific cases could be understood. It does not show directly the computation to be carried out, but it still can be used to escort the process as a heuristic tool.

The function of such a diagram thus seems hybrid, does it illustrate the process, explain it? The border line between an illustrative diagram and an explanatory diagram is indeed almost impossible to draw.

4.4 Where one understands

Sometimes diagrams of the 'setting-down' part of solved examples not only seem to summarize the algorith to be caried out but also appear as explanations of the process altogether. For instance, as already mentioned (in paragraph 4.1 on page 33) the diagrams illustrating the examples of the commentary on verse 22, reproduced in Figure 12 on page 34, provide immediately the explanation of the geometrical aspect of series: piles of squares and cubes are considered and represent their sum.

In a very different way, in verse 9 of the mathematical chapter, Āryabhaṭa proposes a 'verification' (*pratyayakaraṇa* , lit. 'producing conviction'[50]) of the areas of all geometrical fields. For each verification, and at each stage

[50]A first attempt in analyzing this mode of reasoning can be found in [Keller 2000, Volume 1, I.].

of the reasoning the commentary on this verse uses diagrams. "Setting-down" anouncements can be found continuously. Let us observe one of these diagrams in its context. Bhāskara considers the area of a triangle whose sides are given as 13, 15, 14. This triangle as found in Mansucript D and in the edition is reproduced in Figure 3 on page 16. Bhāskara provides two verifications of the area of this triangle. Concerning the second method, he writes:

> *athavā āyatacaturaśrakṣetrayor ardhakṣetraphalasaṃyogo 'sya phalam| tayor dvayoḥ pañcavistārasya dvādaśāyāmasyaikasya, dvitīyasyāpi navavistārsya dvādaśāyāmasyārdhakṣetraphalasaṃyogo 'sya phalam| tayor dvayoḥ pañcavistārasya dvādaśāyāmasyaikasya dvitīyasyāpi navavistārasya dvādaśāyāmasya nyāsaḥ–*
>
> Or else, its area is the sum of half the areas of two rectangular fields. This area ⟨of the trilateral⟩ is the sum of half the areas of these two ⟨rectangles⟩, the one whose width is five and length twelve, and the second one also, whose width is nine and length twelve.
>
> Setting down these two fields, the first one whose width is five and length twelve, and also the second whose width is nine and length twelve:
>
> (the top figure reproduced in Figure 3 on page 16 is presented here)

The top diagram, reproduced in Figure 3 on page 16, summarizes the problem: it gives the lengths of the known sides. It shows the rectangles whose areas will be computed. It also shows how half the areas of each of these rectangles can add up to give the area of the triangle[51]. Such a diagram, explicitly, is used to explain a reasoning. We have thus seen that the diagrams which illustrate a process or an exemple can also be used to explain this very process. In fact, certain diagrams seem to have only this purpose.

4.5 Where one can see, show or prove

As for the verification mentioned above, Bhāskara's commentary refers to procedures justifying the correctness of Āryabhaṭa's rules[52]. Later Sanskrit commentaries will sometimes include systematical proofs. This is not the case of Bhāskara's commentary, where proofs are more often alluded to than developed. But, all fragments of proof that are found in Bhāskara's commentary have a step where the mathematical properties are represented and 'shown' (with all the ambiguity of such an expression) within a diagram.

The commentary on the second half of verse 9, which states the equality of the chord of one sixth of the circumference of a circle with its radius, for instance, mentions a diagram where an explanation is carried out[53]:

etām eva ṣaḍbhāgajyāṃ pratipādayiṣatā vṛttakṣetre ṣaṭ sama-

[51] Bhāskara in the resolution that follows uses the expression *anupraviṣṭa* , e.g. says that the areas *enter* within the triangle.

[52] [Keller 2000, Volume I, 1.8, 101-117].

[53] [Shukla 1976, 71, lines 12-13]; [Keller 2000, Volume II, BAB.2.9cd].

tryaśrikṣetrāṇi prasaṅgena pradarśitāni |

In a circular field, six equi-trilateral fields have been shown (*pradarśita*) incidentally by one who wishes to explain (*pratipādayiṣatā*) this very chord of the sixth part.

In his commentary on the second half of verse 17, similarly, Bhāskara announces an explanation for a procedure computing segments in a circle within the traditional problem of 'Hawks and rats'[54].

Bhāskara announces an explanation and then immediately produces a diagram[55]:

tat tu pradarśyate

nyāsaḥ

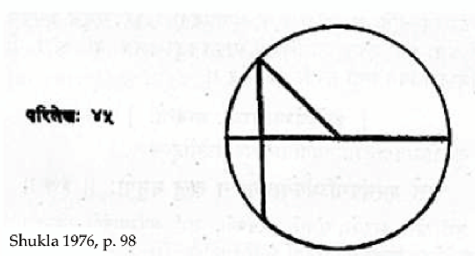
And that is explained:

Setting down:

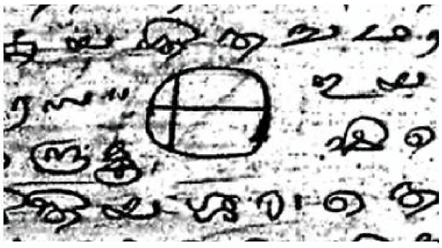
Note that the diagram here reflects its explanatory aspect, for it bears neither numbers nor letters. It just represents the geometrical situation at stake. The only sentence where the word proof (*upapatti*) is used refers to a diagram where it should be carried out. Bhāskara considers two similar

[54]The problem is as follows: A hawk on a height, which is in fact a half chord, sees a rat on a spot, which is in fact the circumference of a circle, whose hole is at the base of the height were the hawk stands. The rat, seeing the hawk attempts to run back to his hole, but the hawk flying along a hypotenuse kills the rat at the center of the circle. Both the distance crossed by the hawk and the distance missing for the rat to reach his hole are sought.

[55][Shukla 1976, 98]; [Keller 2000, Volume II, BAB.2.17.cd]. In this case, the diagram is followed by a reasoning which has been studied in [Keller 2000, Volume I, 1.8.4].



Shukla 1976, p. 98



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triangles and wanting to compute the length of one of the sides of these triangles with a Rule of Three, he writes[56]:

> *trairāśikopapattipradarśanārthaṃ kṣetranyāsaḥ*
>
> In order to show the proof of ⟨that⟩ Rule of Three, a field is set-down: (followed by the diagram)

As in most cases, the diagrams is not followed by any reasoning. It seems that what -to us- is the crucial moment of explanation in geometry, belonged to the oral sphere in Bhāskara's mind. In geometry such oral reasoning would have been based on a diagram. We thus cannot say more about the diagrams or the proofs here, without fictionalizing the process.

Conclusion

We have seen that diagrams were common objects of Bhāskara's mathematical practice. They are used in a matter-of-fact way, mostly in the 'setting-down' part of solved examples. As such their functions is not explicitly given by the commentator.

[56] [Shukla 1976, 59, line 3]; [Keller 2000, Volume II, BAB.2.6.cd].

However, a diagram is not the transparent unambiguous transposition of a written process in Bhāskara's commentary.

We saw that diagrams of Bhāskara's commentary as represented in Manuscript D could not only summarize a problem to be solved, but also indicate the process to be carried out. They could thus be seen as tools to understand a process or to explain it. In Bhāskara's text, diagrams have an explicatory, even a demonstrative function. However, because these explanations were probably oral, the diagram is the only trace of explanation we are left with. Considering that the diagrams we can examine belong to manuscripts which are more than a thousand years older than the original text itself, this means that, in the case of Bhāskara's text, these explanations are impossible to recover.

We do not know with what degree of precision diagrams would have had to be drawn. Indeed, there is an apparent contradiction between specific rules of constructions given in the commentary on verse 13 on the one hand and the fact that, in most cases, these methods could not be applied. If diagrams were meant to visualize a process or provide an explanation to it, they did not need to be drawn with accuracy. This leaves open the question why a precise but ineffectual process to draw diagrams was given in the treatise and expounded in the commentary. One answer to this problem could be to consider that the type of window opened by the 'setting down' parts of solved examples as represented in manuscript, differed from what was effectively done on the working surface itself. In the manuscripts, diagrams are representations of working objects, not the working objects themselves. The commentary does not display the real working out of a reasoning, in all its different stages, but just remodeled bits of the process, for the sake of transmission. As such, it could be less precise than the working object itself.

We have noted how remote diagrams where from the words of the problems they were associated to. Such a distance is but an aspect of the complex relationship that diagrams in Bhāskara's commentary seem to have with speech. Thus the vocabulary associated with geometrical figures is full of imagery, but can induce false representations. Diagrams could then be used to contradict such false mental visualizations. We have also seen how difficult it was to separate the definition of a geometrical figure from the fact it is represented in diagrams. They both complete one another.

Finally, diagrams appear as windows openning the mathematical text onto the larger context of the people and place in which they were used. Who was the person who could provide the explanation that was associated with a diagram? To whom was the explanation given? On what surfaces did people work? Within what institution? Such are some of the questions, among many others, which remain to be explored.

Table 1: Contents of the Chapter on Mathematics (*gaṇitapāda*)

Verse 1	Prayer
Verse 2	Definition of the decimal place value notation
Verse 3	Geometrical and arithmetical definition of the square and the cube
Verse 4	Square root extraction
Verse 5	Cube root extraction
Verse 6	Area of the triangle, volume of an equilateral tetrahedron
Verse 7	Area of the circle, volume of the sphere
Verse 8	Area of a trapezium, length of inner segments
Verse 9	Area of all plane figures and chord subtending the sixth part of a circle
Verse 10	Approximate ratio in a circle, of a given diameter to its circumference
Verses 11-12	Derivation of sine and sine differences tables
Verse 13	Tools to construct circle, quadrilaterals and triangles, vertical and horizontal lines
Verses 14-16	Gnomons
Verse 17	Pythagoras Theorem and inner segments in a circle
Verse 18	Intersection of two circles
Verses 19-22	Series
Verses 23-24	Finding two quantities knowing their sum and squares or product and difference
Verse 25	Commercial Problem
Verse 26	Rule of Three
Verse 27	Computations with fractions
Verse 28	Inverting procedures
Verse 29	Series/First degree equation with several unknowns
Verse 30	First degree equation with one unknown
Verse 31	Time of meeting
Verses 32-33	Pulverizer (Indeterminate analysis)

Table 2: Structure of a verse commentary

Introductory sentence
Quotation of the half, whole, one and a half or two verses to be commented
'General Commentary'
Word to word gloss, staged discussions, general explanations and verifications
'Solved examples' (*uddeśaka*)
Versified Problem
'Setting-down' (*nyāsa*)
'Procedure' (*karaṇa*)

APPENDIX

Bhāskara's description of the construction of trilaterals and quadrilaterals

In his commentary on verse 13 of the mathematical chapter of the *Āryabhaṭīya*, Bhāskara describes two methods to construct geometrical figures[57].

> Having stretched a string (*sūtra*) on level ground one should make a line (*rekha*). And that is:
>
> Figure 30[58]
>
> Here, with a pair of compasses (*karkaṭaka*) which is placed on both tips ⟨of the line⟩, a fish should be produced.
>
> A perpendicular is a second string which goes from the mouth to the tail of this ⟨fish⟩:
>
> Having appointed one tip of a string on the extremity ⟨of the fish⟩, having appointed the second tip ⟨of the string⟩ firmly on the tip of the base, one should make a line. On the second tip ⟨of the base⟩, too, it is just in that way. In this way, there are two

[57][Shukla 1976, 84-87]

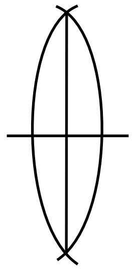


Figure 31

diagonal strings. With those two diagonal strings a trilateral is brought about:

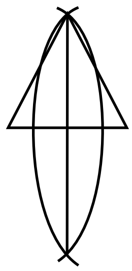


Figure 32

In ⟨the case of⟩ a quadrilateral, one should stretch obliquely a string which is equal to [the diagonal of] the desired quadrilateral. And that string is:

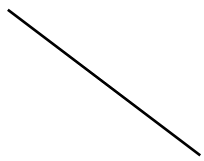
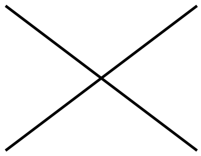
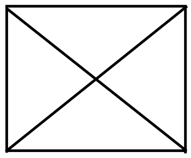


Figure 33

One should stretch obliquely the second ⟨string⟩ too, a cross (*svastika*) is produced from the middle of that ⟨first string⟩. And therefore there are two diagonal strings:

Figure 34 

The sides (*pārśva*) of these two ⟨strings⟩ are filled in, ⟨and⟩ a quadrilateral field is produced:

Figure 35 

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Bibliography

- Bag A. K. & Sen S. N. 1983. *The śulbasūtras.* New-Delhi: INSA.

- Brown, James. 1999. *Philosophy of Mathematics: An Introduction to the world of Proofs and Pictures.* New-York/London: Routledge.

- Pingree, David. 1970-1995. *Census of the Exact Sciences in Sanskrit, Series A, 5 volumes* (CESS), 5 volumes. American Philosophical Society: Philadelphia.

- Datta, B & Singh, S. N. 1935. *History of Hindu Mathematics*, Motilal Banarsi Das: Lahore; Reprinted in Delhi: Bharatiya Kala Prakashan, 2001.

- Filliozat, Pierre-Sylvain. 1988. 'Calculs de demi-cordes d'arcs par Āryabhaṭa et Bhāskara I'. *Bulletin des Études Indiennes* **6** : 255-274.

- Jain, Pushpa Kumari . 1995. *A Critical edition, English translation and commentary of the Upodghāta Ṣaḍvidhaprakaraṇa and Kuṭṭakādhikāra of the Sūryaprakāsa of Sūryadāsa.* Simon Fraser University, Canada: PhD Thesis.

- Hayashi, Takao. 1995. *The Bhakhshālī Manuscript, An ancient Indian mathematical treatise.* Groningen: Egbert Forsten.

- –. 1997. 'Āryabhaṭa's rule and table for sine-differences' in *Historia Mathematica* **24** : 396-406.

- Keller, Agathe. 2000. 'Un commentaire indien du VIIe siècle : Bhāskara et le *gaṇitapāda* de l'*Āryabhaṭīya*'. Paris VII University, France: PhD Thesis.

- Kern, H. 1874. *The Āryabhaṭīya, with the commentary Bhaṭadīpikā of Parameśvara.* Leiden: E. J. Brill.

- Netz, Reviel. 1999. *The Shaping of Deduction in Greek Mathematics : A Study in Cognitive History.* Ideas in Context. Cambridge, UK: Cambridge University Press.

- Ōhashi, Yukio. 1994. Astronomical Instruments in Classical Siddhāntas. *Indian Journal of History of Science*, **29** (2) : 155-313, 1994.

- Pingree, David. 1981. *Jyotiḥśāstra*. A History of Inidan litterature, Volume VI. Fasc. 4. Wiesbaden: Otto Harrossowitz.

- Sarasvati Amma, T. A. 1979. *Geometry in Ancient and Medieval India* . Delhi: Motilal Banarsidass.

- Sarma, S. R. 2002. Rule of Three and its Variations in India. In *From China to Paris: 2000 Years of Mathematical Transmission*, Dauben, Dold, Van Dalen eds. 133-156. Stuttgart, RFA: Steiner Verlag.

- Sastri, K. 1957. *Mahābhāskarīya* of Bhāskarācārya with the *Bhāṣya* of Govindasvāmin and Supercommentary *Siddhāntadīpikā* of Parameśvara. Madras, India: *Madras Govt. Oriental Series*, no. cxxx.

- Shukla, K. S. 1960. *Mahābhāskarīya, Edited and Translated into English, with Explanatory and Critical Notes, and Comments, etc.* Lucknow, India: Department of Mathematics, Lucknow University.

- – 1976. *Āryabhaṭīya of Āryabhaṭa, with the commentary of Bhāskara I and Someśvara.* New-Delhi, India: Indian National Science Academy (INSA).

- Srinivas, M. D. 1990. The Methodology of Indian Mathematics and its contemporary relevance. In *History of Science and Technology in India*; Vol. I, Mathematics and Astronomy. 28-86. Delhi, India: Sundeep Prakashan.