Are consistent expectations better than rational expectations ?

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Abstract

In this paper, I argue that agents may prefer learning a misspecified model instead of learning the rational expectation model. I consider an economy with two types of agent. Fundamentalists learn a model where endogenous variables depend on relevant exogenous variables whereas followers learn a model where endogenous variables are function of their lagged values. A Fundamentalist is like a DSGE econometrician and a follower is like a VAR econometrician. If followers (resp. fundamentalists) give more accurate forecasts, a fraction of fundamentalists (resp. followers) switch to the follower model. I apply this algorithm in a linear model. Results are mixed for rational expectations. Followers may dominate in the long run when there are strategic complementarities and high persistence of exogenous variables. When additionnal issues are introduced, like structural breaks or unobservable exogenous variable, followers can have a significant edge on fundamentalists. I apply the algortihm in three economic models a cobweb model, an asset price model and a simple macroeconomic model.

JEL Classification: D83,D84

Keyword: Adaptive learning, cobweb model, naive expectations

Elliot Aurissergues

PHD student, Paris School of Economics and University Paris Pantheon Sorbonne

elliot.aurissergues@gmail.com

Introduction

The Rational expectation hypothesis (REH thereafter) is the dominant theory of expectations in macroeconomics. This hypothesis remains however a source of controversy. Economic agents are supposed to be completely forward looking. Is it a good approximation of reality ? Recent debates, for example about the discrepancy between predicted and actual outcome of forward guidance, have suggested that agents are less forward looking than the theory still assumes.

The main alternative to the REH remains the old adaptive expectation hypothesis. With the REH, agents use all available and relevant information about exogenous variables to forecasts future values of endogenous variables. With the adaptive expectation hypothesis, agents use lagged endogenous variables to forecast future ones. Thus, they do not use all relevant available information contained in exogenous variables and use the irrelevant information contained in lagged endogenous variables. A refinement of adaptive expectations is the consistent expectations concept introduced by Hommes and Sorger (1998). When forming consistent expectations, agents assume that endogenous variables follows an autoregressive process and the coefficient of this process have to be equal to the true value of autocorrelation. However, the concept have not been largely used. Using a variant of expectations is still perceived as less "rational" for economic agents than forming rational expectation.

Indeed, it has taken for granted that rational expectation is the best way to forecast. This paper challenges this idea and shows that consistent expectations may deliver more accurate forecasts under some assumptions. Consistent expectations may also be evolutionnary dominant. When consistent forecasts competes with rational forecasts, agents may progressively swith to the former because they are more accurate.

I consider an economy characterized by a simple univariate linear model. An endogenous variable depends on a peristent exogenous variable, its expectation and a random perturbation. There are two type of agents. Rational agents believe that the endogenous variable is determined by the exogenous variable. Consistent agents believe that the endogenous variable follows a first order autoregressive process.

In the first part of the paper, I study the long run behavior of the economy. I define a long run equilibrium as a situation where both type have a stable estimation of their model and in which one type is dominant. The model of the dominant type should deliver more accurate forecasts in average than the other type. I distinguish between rational and consistent equilibrium. In rational (resp. consistent) equilibrium, rational (resp. consistent) agents dominate and the rational (resp. consistent) forecast is more accurate. I show that for many parameter values, both consistent equilibrium and ratiobal equilibrium exists.

This multiplicity of long run equilibrium invites to study the transitionnal dynamics. To do so, I perform simulations of the economy. Simulation includes both internal and external learning. Both type of agents learn their model in the spirit of Bray and Savin (1986) and Evans and Honkapoja (2001). At each period, they estimate their model through econometric techniques and using historical datas. Once they have estimated it, they use the model to forecast the value of the endogenous variable. Once the value of the endogenous variable have been determined, both compare the accuracy of their respective model. A fraction of agents using the less accurate model switch to the accurate one. These simulations clearly shows that the consistent equilibrium is not a theoretical curiosity. For many parameters, the economy converges toward the consistent equilibrium and not the rational one.

In a third part, I slightly modify the economy by introducing the unobersed exogenous variables and structural breaks for some parameters. It enhances the scope of convergence toward the consistent equilibrium.

1 Framework

Consider a model in which a macroeconomic variable x is given by the equation

$$x_t = \alpha + \beta y_t + \lambda x_t^E + u_t \tag{1}$$

α is a constant.β, λ are parameters.u_t is a white noise of standard deviation σ_u. It is not observable by agents before they make their decision in period t.

y is an exogenous variable which follows an autoregressive process

$$y_t = \theta y_{t-1} + \epsilon_t \tag{2}$$

θ is a parameter and ϵ_t is a white noise of standard deviation σ_y

x_t^E is the expectation of the value of x in t. There is a mass one of agents. Each of them form a forecast.x^E is the aggregate forecast.

$$x_t^E = \int_0^1 x_t^e(i)di \tag{3}$$

$x_t^e(i)$ is the indiviudual expectation formed at period t by agent i.

ζ_t is a white noise.

Agents observe contemporaneous values of exogenous variables and lagged endogenous variables. Formally, their information set can be summarized by

$$\Theta_t = \{y_T, x_{T-1}\}_{T=-\infty}^{T=t}$$

Rational expectations solution It is convenient to compute the rational expectation solution of the model before outlining the main point.

The fixed point value for x, denoted x^* is given by

$$x^* = \frac{\alpha}{1-\lambda}$$

I denote $\hat{x}_t = x_t - x^*$. Equation (1) can be rewritten

$$\hat{x}_t = \beta y_t + \lambda \hat{x}_{E,t} + u_t$$

The rational expectation value for $\hat{x}_t$ can be found easily. One way is to use undetermined coefficient method. I guess that the deviation of x from steady state depends from the last observation of the exogenous variable y_{t-1}.

$$\hat{x}_t = \frac{\beta}{1-\lambda} y_t + u_t$$

The learning hypothesis The rational expectation for x_t is

$$E(x_t) = \frac{\alpha}{1-\lambda} + \frac{\beta}{1-\lambda} y_t$$

To form the rational expectation, the agent have to know the values of $\frac{\alpha}{1-\lambda}$ and $\frac{\beta}{1-\lambda}$. What happens if they do not know them ? The learning literature was developed to answer this question.

An agent will try to learn the values of the two parameters. In period t, She estimates the model

$$x_k = \pi_f y_k + \varphi_f + u_{f,k}$$

where k goes from period 0 (where observables start) to period $t-1$.

After estimating the model, they form a forecast for x_t

$$x_t^e = \pi_{f,t} y_t + \varphi_{f,t}$$

I label these agents fundamentalists or rational agents.

It has been shown by Bray and Savin (1986) that the estimator $(\hat{\pi}, \hat{\varphi})$ converges toward $\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}$ if $\lambda < 1$

The alternative forecast This learning strategy converges toward the "good" solution (according to economic theory). But, does it allows agents to make accurate forecasts ?

I explore the possibility that an alternative strategy provides better forecast. Agents do not learn the "true" model but learns autocorrelation for endogenous variables. I label it the consistent strategy. Agents which adopts thei strategy are called "consistent agents" or "followers"

I now define the consistent learning and the consistent forecast.

At each period, consistent agents estimates the model.

$$x_k = \pi_o x_{k-1} + \varphi_o + u_{o,t}$$

In period t, their forecast for x is

$$x_t^e = \pi_{o,t} x_{t-1} + \varphi_{o,t}$$

Consistent agents behave like VAR econometrician whereas fundamentalists can be viewed as "DSGE" econometrician. Intuitively, it seems difficult to believe that the consistent strategy could deliver more accurate forecasts than the fundamentalist one. Indeed, it does not use all availabe information. In particular, it does not take into account contemporaneous innovation on y whereas it takes into account past innovation on x u_{t-1} which should not be relevant to forecast x_t. This reasoning is true if agents are fully informed about parameter values. But it is more complicated if they should first learn these values for two reasons. First, the fundamental learning is actually misspecified as aknowledged by the literature (Bray and Savin 1986), (Evans and Honkapoja 2004) and this misspecification can give advantages to followers. Second, if there are strategic complementarities, the follower strategy can be self fulfiling.

2 Multiple Equilibria

In this section, I study the long run behavior of the economy. I define long run as a situation in which both type of agents have a stable estimation of their respective models and in which there is only one type of agent remaining.

There is a long run equilibrium if the dominant agent makes more accurate forecasts in average than the other type. There are no incentives for the dominant agent to deviate from her model.

Two resuts emerge. First, the situation in which fundamentalists dominate is always an equilibrium. Indeed, if there are only fundamentalists, their model is correctly specified in the long run and their average forecasts errors are equal to standard deviation of the white noise u. Followers have a misspecified model and make forecasts errors in average. I label it the "fundamentalist equilibrium".

A second result is that situations in which followers dominate are also an equilibrium for a large set of parameters. I detail these results in following paragraphs.

Equilbrium definition First I define the two type of equilbriums. In the following definition and propositions, the limit of a sequence of random variables is a random variable toward which the sequence converges in probability. I define the two matrix of observables $Z_{o,T}$ and $Z_{f,T}$. These matrix have T columns and these columns are observations respectively for vectors $\begin{pmatrix} 1 \\ x_{t-1} \end{pmatrix} (1 \quad x_{t-1})$ and $\begin{pmatrix} 1 \\ y_t \end{pmatrix} (1 \quad x_{t-1})$. I also defines the vector X_T which is the vector column for observations of x from 0 to T

Definition 1 *A fundamentalist equilbrium is a couple of vectors (π_o, φ_o), (π_f, φ_f) for which*

1. $(\pi_o, \varphi_o) = \lim_{T\to\infty} \frac{1}{T}(Z'_{o,T} Z_{o,T})^{-1}(Z'_{o,T} X_T)$
2. $(\pi_f, \varphi_f) = \lim_{T\to\infty} \frac{1}{T}(Z'_{f,T} Z_{f,T})^{-1}(Z'_{f,T} X_T)$
3. $\forall T \quad x_T = \alpha + \lambda\varphi_f + (\beta + \lambda\pi_f)y_T + u_T$
4. $\lim_{T\to\infty} E[(x_T - \varphi_f - \pi_f y_T)^2] < \lim_{T\to\infty} E[(x_T - \varphi_o - \pi_o y_T)^2]$

The consistent equilbrium is defined similarly

Definition 2 *A consistent equilbrium is a couple of vectors (π_o, φ_o), (π_f, φ_f), both belonging to $\mathbb{R}^2$ for which*

1. $(\pi_o, \varphi_o) = \lim_{T\to\infty} \frac{1}{T}(Z'_{o,T} Z_{o,T})^{-1}(Z'_{o,T} X_T)$
2. $(\pi_f, \varphi_f) = \lim_{T\to\infty} \frac{1}{T}(Z'_{f,T} Z_{f,T})^{-1}(Z'_{f,T} X_T)$
3. $\forall t \quad x_T = \alpha + \lambda\varphi_o + \beta y_T + \lambda\pi_o)x_{T-1} + u_T$
4. $\lim_{T\to\infty} E[(x_T - \varphi_f - \pi_f y_T)^2] > \lim_{T\to\infty} E[(x_T - \varphi_o - \pi_o y_T)^2]$

In a nutshell, at the equilibrium, algorithm have converged, there is only one type remaining and this type makes more accurate forecasts in average.

Here, I used square errors instead of absolute deviation errors which do not allow analytical results.

2.1 The Fundamentalist Equilibrium

Fundamentalist forecasts and errors The equilibrium equation is given by

$$x_t = \alpha + \lambda\varphi_f + (\beta + \lambda\pi_f)y_t + u_t$$

If λ is inferior to 1. it is known that

$$\pi_f = \frac{\beta}{1-\lambda}$$
$$\varphi_f = \frac{\alpha}{1-\lambda}$$

Errors are straightforward to compute

$$lim_{t\to\infty} E[(x_t - \varphi_f - \pi_f y_t)^2] = \sigma_u^2$$

Follower forecast and errors Followers estimation converges toward

$$\pi_o = lim_{t->\infty} \frac{cov(x_t, x_{t-1})}{V(x_t)}$$
$$\varphi_o = \frac{\alpha(1-\pi_o)}{1-\lambda}$$

Computations of covariance and variance gives

$$\pi_o = \theta - \frac{\sigma_u^2 \theta}{\left(\frac{\beta}{1-\lambda}\right)^2 \frac{\sigma_y^2}{1-\theta^2} + \sigma_u^2}$$

If σ_u is small, θ is a good approximation of π_o. Using this approximaation, forecasts errors of followers are

$$lim_{t\to\infty} E[(x_t - \varphi_o - \pi_o y_t)^2] = (1+\theta^2)\sigma_u^2 + \left(\frac{\beta}{1-\lambda}\right)^2 \sigma_y^2$$

Obviously, consistent squared errors are always bigger than fundamentalists squared errors. It leads to the following proposition

Proposition 1 *For all vector of parameters $(\beta, \alpha, \sigma_u, \sigma_y, \theta)$, there exist a fundamentalist equilibrium*

Whereas fundamentalists make more accurate forecasts in this situation. Squared errors of followers are not necessarily high compared to the variance of X. Indeed, variance of X is

$$V(x) = \left(\frac{\beta}{1-\lambda}\right)^2 \frac{\sigma_y^2}{1-\theta^2} + \sigma_u^2 \quad (4)$$

Suppose that σ_u is small compare to σ_y and that θ is close to one, the variance of x is several times the average squared errors of the autoregressive model. Intuitively, the model is misspecified but still explains a large part of x variations.

2.2 The consistent equilibrium

Follower forecasts and errors I now characterize the consistent equilibrium. If followers dominates, the equilibrium equation is given by

$$x_T = \alpha + \lambda\varphi_o + \beta y_T + \lambda\pi_o x_{T-1} + u_T \tag{5}$$

The coefficient of the autoregressive process is given by

$$\pi_o = \lim_{t->\infty} \frac{Cov(x_t, x_{t-1})}{V(x_t)} \tag{6}$$

However, both the covariance and the variance in the previous formula depends on π_o. However, it is possible to compute an explicit solution for the coefficient.

Proposition 2 *Possible values for π_o are given by the root of the quadratic equation*

$$\pi_o = \frac{\beta^2 V(y)(\theta + \lambda\pi_o) + \lambda\pi_o(1 - \lambda\pi_o\theta)\sigma_u^2}{\beta^2 V(y)(1 + \lambda\pi_o\theta) + (1 - \lambda\pi_o\theta)\sigma_u^2} \tag{7}$$

φ_o is given by $\varphi_o = \frac{\alpha(1-\pi_o)}{1-\lambda}$

It is a quadratic equation and thus an explicit solution for the two roots is possible. One of the root is negative and one positive. Simulations always converge toward the positive root. Thus, I focus on it. The explicit solutions is not very tractable. A simpler solution emerges in the particular case if $\sigma_u^2 = 0$

Proposition 3 *When $\sigma_u^2 = 0$ π_o is given by the polynomial equation*

$$\pi_o^2\lambda\theta + \pi_o(1 - \lambda) - \theta = 0 \tag{8}$$

The positive root of the equation

$$\pi_o = \frac{\sqrt{(1-\lambda)^2 + 4\lambda\theta^2} - (1 - \lambda)}{2\lambda\theta} \tag{9}$$

Knowing π_o consistent forecasts errors can be computed

Proposition 4 *Squared Errors of followers denoted erro are*

$$Erro \equiv \sum_{k=0}^{+\infty} [-\beta y_k + (1 - \lambda)\pi_o x_{k-1} - u(k)]^2 \tag{10}$$

$$Erro = \beta^2 V(y) + (1 - \lambda)^2\pi_o^2 V(x) + \sigma_u^2 + (1 - \lambda)\pi_o\beta\theta cov(x, y) \tag{11}$$

Fundamentalists estimations and errors If consistent equilibrium prevails, rational agents learn a misspecified model. Their coefficient estimates are not equivalent to those of the rational equilibrium.

Proposition 5 *Parameters estimates by fundamentalists converges toward*

$$\pi_f = \frac{\beta}{1-\lambda\pi_o\theta}$$
$$\varphi_f = \frac{\alpha}{1-\lambda}$$

Knowing π_o and π_f, it is possible to compute forecasts errors of fundamentalists.

Proposition 6 *Squared Errors of fundamentalists denoted $errf$ are*

$$Errf \equiv \sum_{k=0}^{+\infty}[(\pi_f-\beta)y_k - \lambda\pi_o x_{k-1} - u(k)]^2$$
$$Errf = (\pi_f-\beta)^2 V(y) + (\lambda\pi_o)^2 V(x) + \sigma_u^2 + \lambda\pi_o(\pi_f-\beta)\theta cov(x,y)$$

Existence condition for consistent equilibrium A consistent equilibrium exist if

$$Errf > Erro$$

I can now find for which parameter values the consistent equilibrium exists. Both squared errors can be computed analytically but the solution with respect to deep parameters is not tractable. To get a better idea of the scope of the consistent equilibrium, I compute numerically the difference of squared errors between fundamentalists and followers for different values of the deep parameters λ and θ. In figure 1, I plot the result when $\sigma_u = 0$. The surface is red when rational forecasts are more accurate and in green when consistent forecasts errors are lower. Thus, the green surface defines the scope of the consistent equilibrium. The figure shows that consistent equilibrium is not a curiosity and exists for a large set of parameters. The figure also shows that consistent forecasts tend to be more accurate when the elasticity of x to expectations λ is high or when exogenous variables are very persistent corresponding to a high value of θ. I repeat the same exercise in figure (2) for a higher value of σ_u. The scope of the consistent equilibrium is narrower but remains significant.

3 The simulated economy

Multiple long run equilibrium exists for a large set of parameter values. But, will the economy converge toward the consistent equilibrium. To answer this question, I now perform simulations of this economy. The economy is made of three algortihm. At each period, both consistent and rational agents update their model using their forecasts errors. In the meantime, they compare the forecasting performance of both models. When a model performs better, more agents adopt the model. I describe these features in more details in the following section.

3.1 Overview

The fundamentalist algorithm Fundamentalists believe that the variable x can be forecasted by estimating the equation

$$x_t = \pi_f y_t + \varphi_f + u_{f,t} \tag{12}$$

I shortly describe the recursive algorithm used by fundamentalists to estimate (7).

I define the vector of exogenous variable $z_{f,t}$ and the vector of estimated parameters $\Phi_{f,t}$

$$z_{f,t} \equiv (1 \quad y_t)'$$
$$\Phi_{f,t} \equiv (\varphi_{f,t} \quad \pi_{f,t})'$$

I denote the variance covariance matrix $R_{f,t}$. Parameters estimates are updated by the two recursive equations.

$$R_{f,t+1} = R_{f,t} + \frac{1}{t}(z_{f,t} z_{f,t}^{'} - R_{f,t}) \tag{13a}$$

$$\Phi_{f,t+1} = \Phi_{f,t} + R_{f,t+1} \frac{1}{t} z_t \left(x_t - z_t^{'} \Phi_{f,t} \right) \tag{13b}$$

At period t, the forecast of fundamentalist is

$$x_{f,t} = \pi_{f,t} y_t + \varphi_{f,t} \tag{14}$$

The Follower algorithm Followers have a different strategy. They believe that the variable x is given by

$$x_t = \pi_o x_{t-1} + \varphi_o + \sigma_t \tag{15}$$

Like fundamentalists, they try to learn the value of π_o and the value of φ_o.

I introduce the vector of exogenous variable $z_{o,t}$ and the vector of estimated parameters $\Phi_{o,t}$

$$z_{o,t} \equiv (1 \;\; x_{t-1})'$$
$$\Phi_{f,t} \equiv (\varphi_{o,t} \;\; \pi_{o,t})'$$

The variance covariance matrix is $R_{o,t}$.

The recursive estimation is given by

$$R_{o,t+1} = R_{o,t} + \frac{1}{t}(z_{o,t}z_{o,t}^{'} - R_{foll,t}) \tag{16a}$$

$$\Phi_{o,t+1} = \Phi_{o,t} + R_{o,t+1}\frac{1}{t}z_{o,t}\left(x_t - z_{o,t}^{'}\Phi_t\right) \tag{16b}$$

At period t, the forecast of followers is

$$x_{o,t} = \pi_{o,t}x_{t-1} + \varphi_{o,t} \tag{17}$$

Update of the share of followers Initially, there are $1-\gamma$ fundamentalists and γ followers.

At the end of period t, agents observe forecasts of both types $x_{f,t}, x_{o,t}$ and the actual outcome x_t.

After t simulated periods, they compute the statistics[1]

$$\Delta_t = \frac{1}{t}\sum_{n=0}^{t}\sqrt{(x_{o,t}-x_t)^2} - \frac{1}{t}\sum_{n=0}^{t}\sqrt{(x_{f,t}-x_t)^2} \tag{18}$$

The statistics Δ is simply the average forecasting error of the follower strategy minus the average forecasting error of the fundamentalist strategy

If $\Delta < 0$, the follower strategy was in average more accurate than the fundamentalist one until the period t, a fraction μ of the fundamentalists shifts to the follower strategy. Conversely, if the fundamentalist strategy have been more accurate in average, the same fraction shifts from the follower strategy to the fundamentalist one

Thus, the evolution of γ is given by

$$\gamma_{t+1} = \gamma_t - \mu\gamma_t\mathbb{1}_{\{\Delta_t>0\}} + \mu(1-\gamma_t)\mathbb{1}_{\{\Delta_t<0\}} \tag{19}$$

[1] I also consider the alternative statistics $\Delta_t = \frac{\sum_{n=0}^{t}\mathbb{1}_{\{(\tilde{x}_{o,t}-x_t)^2<(\tilde{x}_{f,t}-x_t)^2\}}}{t} - 0.5$. Which is simply the number of times for which follower strategy have delivered a more accurate forecasts than the fundmaentalist strategy

3.2 Summary of the model and the algorithm

The structure of the model can be summarized by nine equations.

A first bloc of equations is composed of equilibrium equations. It includes the two forecast equation (9) and (12) and the equation giving the equilbrium value of x

$$x_t = \alpha + \beta y_t + \lambda \gamma_t x_{o,t} + \lambda(1 - \gamma_t) x_{f,t} + \zeta_t \tag{20}$$

There are two dynamic blocs.

The first dynamic bloc includes the two recursive estimation algorithm (8) and (11). which gives four equations.

A second dynamic bloc gives the evolution of the share of followers. These are thequations (13) and (14)

Description of the economy algorithm I simulate this economy over a long period. The algorithm may be summarized by the following sequence of events

1. Using the model they have chosen, their past estimates of parameter values and the value of y_t, x_{t-1}, fundamentalists and followers compute their forecasts for x_t
2. The equilibrium value of x_t
3. This value is compared with forecasts of both type of agents.
4. If the model of fundamentalists underperforms (respectively followers) model underperforms, they switch to the other model with probability μ.
5. Once they have chosen their new model, they estimate it using the history of values for x and y

3.3 Initialization

The initial share of fundamentalists is γ_0. I set it at 0.5. Half of agents are initially rational. An higher value would give an initial advantage to fundamentalists (respectively a lower would give an advantage to followers). I perform simulations with different values for γ_0. Its effect on long run outcome seems very limited.

A significant practical issue is the initialization of the two learning algorithms. I have to set priors the two variance-covariance matrix R_f and R_o and the two vectors of parameter estimates ϕ_f and ϕ_o.

I initialize R_f and R_o by using long run values for mean and variance.

The matrix $R_{f,0}$ is given by

$$\begin{pmatrix} 1 & E(y) \\ E(y) & E(y^2) \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{\sigma_y^2}{1-\theta^2} \end{pmatrix}$$

In a similar way, the matrix $R_{o,0}$ is given by

$$\begin{pmatrix} 1 & E(x) \\ E(x) & E(x^2) \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & \frac{\mu}{1-\lambda} \\ \frac{\mu}{1-\lambda} & \left(\frac{\beta}{1-\lambda}\right)^2 \frac{\sigma_y^2}{1-\theta^2} + \sigma_u^2 + \left(\frac{\mu}{1-\lambda}\right)^2 \end{pmatrix}$$

These long run values are the same for fundamentalists and consistent equilibrium and thus do not give advantages to one of the two type of agents. Setting priors to other values is risky because the matrix have to be inverted. Bad initial values may lead to inconsistent or explosive estimations in our simulations.

The long run outcome is sensitive to the initialization of ϕ_f and ϕ_o and especially to the initialization of $\varphi_{f,0}$ and $\varphi_{o,0}$. There are several strategies to deal with this sensitivity issue. A first approach is to use values of the rational expectation equilibrium to set initial guess for ϕ_f and ϕ_o. But, it gives a strong advantage to fundamentalists. I adopt a variant of this srategy. I center the prior around the rational expectation value but I allow for a perturbation, potentially a large one around it.

The vector $\phi_{f,0}$ is equal to

$$\phi_{f,0} = \begin{pmatrix} \frac{\mu}{1-\lambda}(1+\nu_0) \\ \frac{\beta}{1-\lambda}(1+\nu_1) \end{pmatrix} \tag{21}$$

The vector $\phi_{f,0}$ is set in a similar way

$$\phi_{o,0} = \begin{pmatrix} \frac{\mu(1-\theta)}{1-\lambda}(1+\nu_2) \\ \theta(1+\nu_3) \end{pmatrix} \tag{22}$$

However, I constrained the prior for the autoregressive coefficient to be compatible with a stationnary autoregressive process $\theta(1+\nu_3) < 1$.

I consider two possibilities to set these initialization parameters. In the first one , $\nu_0, \nu_1, \nu_2, \nu_3$ are all equal to a fix value ν. In the second one, they are random variable and follow a uniform distribution.

3.4 Baseline calibration

The model cannot be calibrated by using empirical datas. But I consider a baseline calibration to get benchamerk result. The calibration is summarized in Table 1. The expectation feedback λ is set at 0.5. the persistence of exogenous variable θ is set at 0.8. This can be considered as a high value but quarterly persistence of shocks in macroeconomic models is closer to 0.9 in average. I target a 5 percent standard deviation from the average value in line with macroeconomic volatility. Innovations on y count for eighty percent of that volatility and white noise u count for twenty percent. The initial share of fundamentalists is 0.5. The initialization parameter ν is equal to -1. Both rational and consistent agents start their estimation by guessing a value of the constant equal to 0 and the value of the coefficient equal to 0.

4 Results

4.1 Summary of results

The results of the different simulation can be summarized as follow

1. Consistent equilibrium is not a theoretical curiosity. Simulation converges toward it for a large set of parameters. This is the case of the baseline calibration.

2. The result of a simulation depends on deep structural parameters. High values of λ and θ favor consistent equilibrium. Values of λ and θ for which economy converges to consistent equilibrium seems close to values for which consistent equilibrium exists.

3. The result of a simulation also depends on the initialization parameters, in particular the inital guess of fundamentalists and followers for their respective model. A guess close enough to rational expectation value allows rational agents to dominate.

4. This is the initial guess of the constant which matters, and not the initial guess of the coefficient associated to the exogenous variable or to the lagged endogenous variable.

5. For some calibration, the simulation is path dependant. For similar parameter values, two simulation may give different outcomes. However, the set of parameters for which there is path dependance is small.

4.2 Main result

The main result of my simulations is that the economy does not always converge to the Rational Expectation Equilibrium (REE thereafter) and often converges toward the Consistent Expectation Equilibrium (CEE). I illustrate the result by the figure (2). I simulate the economy for three different calibrations summarized in table (1). Figure (3) represents the evolution of the share of rational agents and Figure (4) the difference between average errors of rational agents and average errors of consistent agents (the error difference thereafter). A value higher than zero implies that rational errors are larger in average than consistent errors. The blue line corresponds to the baseline caibration. Initially, rational agents makes smaller errors (by a very small margin) and their share increases but quickly the error difference reversed and consistent agents become dominant. Error difference seems converging to a value superior to 0, showing that a longer simulation would not lead to a different outcome. The red line is a simulation which favors rational agents. Deep parameters are similar but the initilaization coefficient ν is set at 0. Parameters are initialized at their REE value. Unsurprisingly, rational agents dominates at the beggining, becomes the only type after fifty period and then keep their advantage. The error difference is always inferior to zero. It is interesting however to note that the error difference is not larger in absolute value than the error difference of the baseline calibration. The green line shows that convergence to the CEE may occur for initial parameters much closer to their REE value. The parameter ν is set at 0.3. Components of vectors $\phi_{o,0}$ and $\phi_{f,0}$ are set around 70 percent of their REE value.The feedback parameter λ increases to 0.65 which remains a reasonable value. Similarly to the baseline calibration, rational agents have a short initial edge before being crowded out by consistent agents. Error difference seems converging to a value above zero, confirming the stability of the result.

4.3 Exploration

Results highlighted in the previous section may be particular cases whereas the bulk of simulations converge to the REE. In this paragraph, I show it is not the case. I explore how changes in these parameter values affect the outcome of the simulation. The ouctome depends on many parameters : Four deep structural parameters $(\lambda, \theta, \sigma_y, \sigma_u)$ and two initialization parameter sν, γ_0. Formally, I define the function $G : (\nu, \gamma_0, \lambda, \theta, \sigma_y, \sigma_u) \rightarrow \{0, 1\}$. The value of G is 0 (resp. 1 when the simulation run with parameter values $(\nu, \gamma_0, \lambda, \theta, \sigma_y, \sigma_u)$ converges to the follower equilibrium (resp. the fundamentalist equilibrium). Representing a 7 dimension figure is difficult. I use my baseline calibration for four parameters and make two others varying. I consider succesively variations of (λ, θ), (ν, γ_0) and (σ_u, σ_y). Results are displayed in figure (5), (7) and (8). The surface is in red when the simulation converges to fundamentalist equilibria and in blue if fundamentalists dominates.

The blue surface is important in all of these figures. Figure (5) shows that convergence to CEE occurs more often when the feedback parameter λ or the persistence θ are high. Figure (6)shows error difference with respect to λ and θ. The exercise is an "empirical" counterpart to the experiment performed in figure (1) and (2). It shows than green surface are roughly similar for both the empirical and the theoretical experiment. When a consistent equilibrium exists, the economy have substantial chances to converge toward it.

Figuer (7) shows the sensitivity to standard deviation σ_u and σ_y. Larger standard deviation implies that consistent agents use a noisier information and miss a more important one. Surprisingly, however, they do not seem so important to determine the outcome of the simulation. Convergence for consistent equilibrium occurs for most values.

4.4 The role of initialization

Figure (8) shows the sensitivity to initialization parameters. The outcome seems independant from the initial share of rational agents γ_0 but seems strongly affected by the value of ν. For values close to zero, impying an initialization close to REE value, convergence to REE is sytematic. this result is reversed when initial parameters are set farther from their REE value.

As the outcome seems very sensitive to initialization, I perform additionnal experiments with a different initialization method to better understand what drives the result. I differentiate between the initial guess of the constant and the initial guess of coefficients π_o and π_f. Initial vectors $\phi_{f,0}$ and $\phi_{o,0}$ are respectively equal to $\left(\frac{\mu}{1-\lambda}(1-\nu_1), \frac{\beta}{1-\lambda}(1-\nu_2)\right)$ and $\left(\frac{\mu(1-\theta)}{1-\lambda}(1-\nu_1), \theta(1-\nu_2)\right)$. Figure (9) and (10) represents the evolution of rational agents and difference errors for three different initializations. The blue line represents the baseline initialization. Initial guessed values are equal to zero. The economy converges to the consistent equilibrium. The green line represents a simulation in which agents have accurate intial guess of π_f and π_o but are clueless about the constant. Consistent equilibrium also emerges as a winner. In the last simualtion, agents have an accurate initial guess of the constant but an inaccurate guess of the two coefficients. Unlike the two previous case, the economy converges to the REE. This result suggests that consistent agents forecasts temporary deviations poorly but compensates by a faster estimation of permanent deviations.

4.5 Path dependence and robustness

In what extent the outcome of two simulations may diverge whereas they have the same parametrization ? I perform two experiments to answer this question. In the first one, I represent the percentage of simulations which converges to the REE with respect to λ and θ. For each couple (λ, θ), I repeat the simulation one hundred times and sum the value of γ after 2000 periods. The figure is not very different from figure (5). Surfaces of zeros and ones are nearly unchanged. However, at the frontier, there are values between 0 and 1, indicating path dependancy.

Figure (12) represents the result of a similar experiment with a different method of initialization. Initialization parameters $(\nu_0, \nu_1, \nu_2, \nu_3)$ are random variable and are drawn at each new simulation. The figure is similar to figure (1), (2), (5) and (11) showing the robustness of the main result.

5 Extensions: misspecification and structural breaks

5.1 Intuition

Learning an autoregressive process for endogenous variables may be better than learning the true Rational Expectations model when agents know the "true" model of the economy. The advantage of the adaptative behavior could be even bigger if agents have a misspecified model of the economy. For example, the variable x may be affected by unobserved variables or the parameters of the equation (1) can be subject to structural breaks. Intuitively, past values of x can carry information about structural breaks or unobersved variables and an adapative behavior can capture it whereas purely exogenosu variables do not carry anything.

5.2 Adding persistent unobservables

A first misspecification is the existence of an unobserved exogenous variable. For example, the equation (1) becomes

$$x_t = \alpha + \beta y_t + \lambda x_t^E + u_t + v_t \tag{23}$$

with

$$v_t = \rho_v v_{t-1} + \epsilon_t^v$$

v is not observable by agents and fundamentalists continue to estimate the model $x_t = \varphi_f + \pi_f y_t$. Because v is persistent, past values of x carries information about the current value of v, giving an edge to followers over fundamentalists.

5.3 Adding structural breaks on constant

a second misspecification is that some parameters are not constant but time varying and follow for example a markov chain.

The equation becomes

$$x_t = \alpha_t + \beta y_t + \lambda x_t^E + u_t + v_t \tag{24}$$

α_t is a random variable whose support is the vector $\{\alpha_l, \alpha_h\}$, where both α_l and α_h are real numbers.

α_t evolves according to a markov chain. In the "h" state, the probability to remain in the high state is p_h whereas the probability to remain in the low state is p_l.

Fundamentalist still estimates the misspecified model $x_t = \varphi_f + \pi_f y_t$

5.4 Results

I simulate an economy where bot misspecification are present. I calibrate the markov chain to have a structural breaks every 100 periods in average. α_l and α_h are three percent deviation from the average value of α. I set σ_v atv the same level than σ_u but introduces a small persistance coefficient with $\rho_v = 0.3$. I display the convergence with respect to $(\nu, \lambda$ in figure (8). The two misspecification significantly enhances the dominance of the follower equilibrium.

5.5 Why do fundamentalists misspecify their model

It seems implausible to assume that fundamentalists will not detect the misspecification. However, I have two reasons to keep assuming they estimate the misspecified model.

First, I consider small deviations from the original model. For example, α_l and α_h are three percent deviation from the average value of α and the autoregressive coefficient ρ_v is only 0.3.

Second, even if they detect a misspecification, they could have serious trouble to identify and estimate the true model. In case of structural breaks, they should estimate no less than five parameters $\pi_f, \alpha_l, \alpha_h, p_h, p_l$. The number is the same if there is an unobserved variable ($\sigma_v, \sigma_u, \rho_v, \varphi_f, \pi_f$. If both misspecification are present, They have 8 parameters to estimate. In every case, they still observe two variables.

6 Literature

In this section, I describe in more details the relation between the paper and the previous literature.

Rational expectations and learning literature Rational expectations were introduced by Muth (1961). The learning literature was developed to adress the issue of limited knowledge of parameter values. A classical exposition can be found in Evans and Honkapoja (2001). Convergence theorem are due to Bray and Savin (1986), a result refined by Marcet and Sargent (1989)

Consistent expectations Our paper is more directly related to three approaches. The first one is the Consistent Expectation Equilibrium (CEE) literature developed by Brock and Hommes since their seminal paper (1997) and refined in a recent textbook by Hommes (2012). Consistent Equilibrium Expectation departs from rational expectation by imposing much weaker condition for expectations. Expectations should simply be consistent with observed autocorrelations. The link with the behavior of our followers is obvious. In the Brock and Hommes original paper, agents switch between rational and naive expectations according to a performance/cost comparison.

There are however several difference between my paper and this branch of the literature. I am interested by the convergence toward one type of expectations, either rational expectations or more adpatative ones. Brock and Hommes (1997, 1998) and Hommes (2012) are more interested by the cyclical dynamic, or even the chaotic one, induced by the coexistence of naive and rational expectations.

A second important difference is the learning behavior. Learning is very simple in most of the CEE literature. In Hommes (2012), the equivalent of our followers are endowed with *given* forecasting rules and do not learn parameters of the forecasting rules. Fundamentalists know the true model of the economy and have not to learn it. By contrast, Our approach heavily borrows from Evans and Honkapoja learning.

A third difference is the evolutionnary criteria. In Brock and Hommes, agents use a discrete choice model to choose between the different forecasting rules. I choose a more intuitive criteria which also allows for convergence toward one rule more easily.

Self fulfiling prophecies Rational expectations were challenged by the sunspot literature initiated by cass and Shell (1977) and refined by Azariadis and Guesnerie (1982). These two papers have shown that, in some class of models, exogenous variables completely unrelated to endogenous variable may affect them simply because agents believe they do. Our idea is quite close. Lagged endogenous variable does not affect directly current ones but may through beliefs. The difference is that lagged endogenous variable are correlated to current ones through the persistence of fundamental exogenous variables. Intuitively, they may play a role in a larger class of models whereas pure sunspots needs strict conditions to emerge (see Guesnerie 2001 for a review).

Evolutionnary theory and economics: Nelson and Winter, Saint Paul The evolutionnary viewpoint has a long history in economics. Some intuitions may be found in Schumpeter (1926) and in the austrian school. Friedman (1953) has defended the rationality assumption by suggesting that "rational" agents will eliminate "irrationnal" ones in markets. The volutionnary viewpoint was formalized in a more rigorous way by Nelson and Winter (1982) and more recently by Saint Paul (2015).

Agent based modelling Close to the CEE literature is agent based modelling. Agent based models takes the opposite approach to standard theory. Instead of deriving the optimal behavior from a well defined maximization problem, they impose given behavioral rules to agents. A good example of agent based model for asset market can be found in Lebaron (2005). The drawback of this approach is the high number of possible behavioral rules and the large degree of freedom it gives to the modeller. Our contribution is closer to the standard theory. I look at behavioral rules which can outperform "rational" ones in environment with limited knowledge.

Adaptive asset pricing Another paper closely related to mine is a recent paper by Adam and Marcet (2011). In this paper, they compare two learning strategies in a Lucas asset market. In the first one, agents learn the relation between price and current dividends. In the second one, agents learn the relation between current and past prices. they show that the second learning strategy offers a simple explanation to many asset pricing puzzle. There are two differences with my paper. First, in their model both strategies converges to the rational expectation solution. This is because dividends follows a very simple process in which past dividends are a sufficient statistics to forecast future ones. Because with rational expectations past prices are function of past dividends, past prices are also a sufficient statistics for future asset price. If dividend equation are more complex, past prices are no more a sufficient statistics to forecast future ones. Agents miss available information. Thus, we are not convinced that the second learning strategy proposed by Adam and Marcet is compatible with rational expectations in a more general setup.

A second difference is the role of evolution in our model. Adam and marcet compares the ability of two learning models to explain stylized facts. We look at the selection process between different learning strategies.

7 Conclusion

In this paper I have shown that rational expectations may not be evolutionnary dominant. Agents using a misspecified model can forecast more accurately, leading rational agents to adopt the misspecified model. This result suggests that the Consistent Expectations Hypothesis proposed by Hommes and Sorger are a genuine alternative to the Rational Expectation Hypothesis. This is particularly true when there is a large positive feedback of expectations and when exogenous variables are persistent. This result is obtained in a simple univariate linear model. In more complex models, learning the rational expectation model may be extremely complicated. But I show that it is probably better to learn a misspecified model based on lagged endogenous variable rather than a misspecifed model based on exogenous variables. The interesting point is that large positive feedback of expectations or high persistence of exogenous is common in macroeconomics and finance. the next step of the research agenda is to verify that consistent expectations may be evolutionnary dominant in a simple asset price model or a simple macroeconomic model.

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A Multiple equilibra

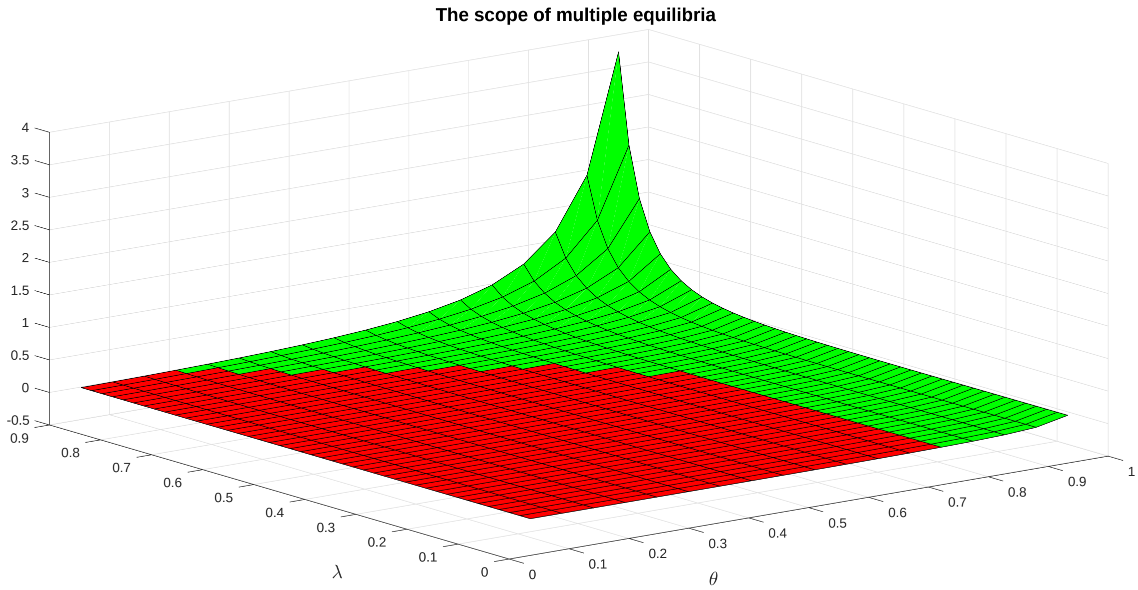

Figure 1: Convergence for several parametrization

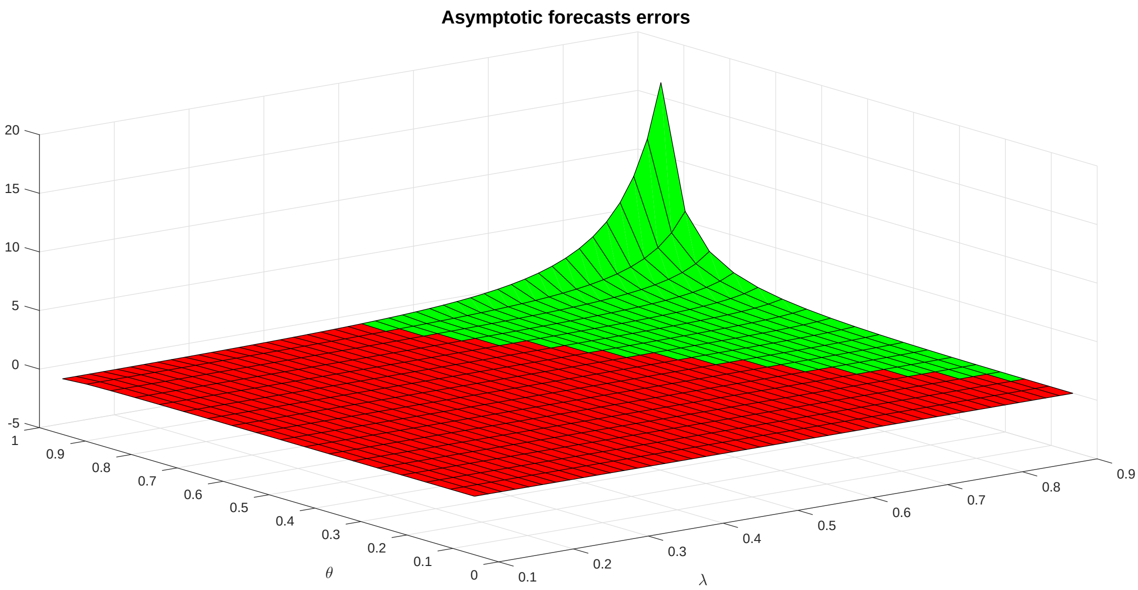

Figure 2: Convergence for several parametrization

B Calibration tables baseline model

Variable	Baseline	Fundamentalist	Follower
λ	0.5	0.5	0.75
ν	1	0	0.3
γ_0	0.5	0.5	0.5
θ	0.8	0.8	0.8
σ_y	$0.04p_1$	$0.04p_1$	$0.04p_1$
σ_u	$0.01p_2$	$0.01p_2$	$0.01p_2$
α	0.5	0.5	0.5
β	1	1	1

Table 1: calibration

with $p_1 = (1-\theta) * \frac{\mu}{1-\lambda}$ and $p_2 = \frac{\mu}{1-\lambda}$

C Results baseline model

C.1 Main result

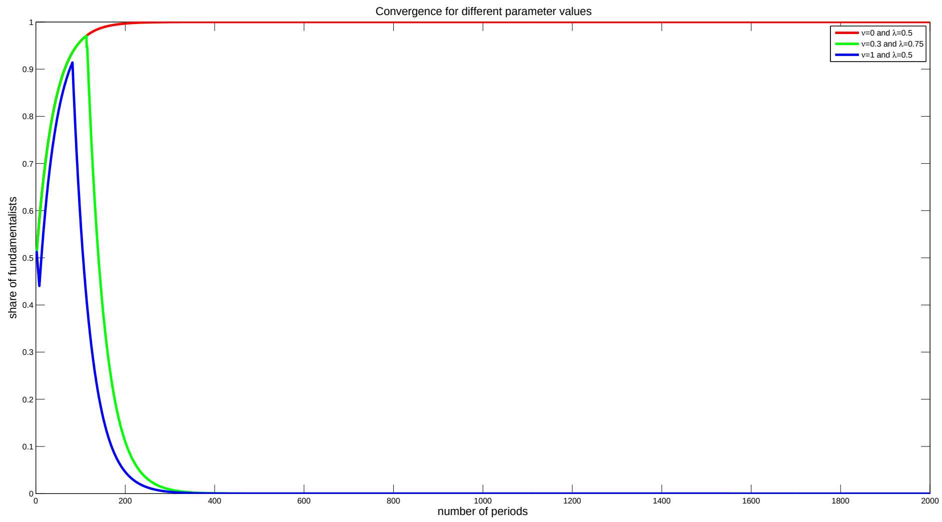

Figure 3: Convergence for several parametrization

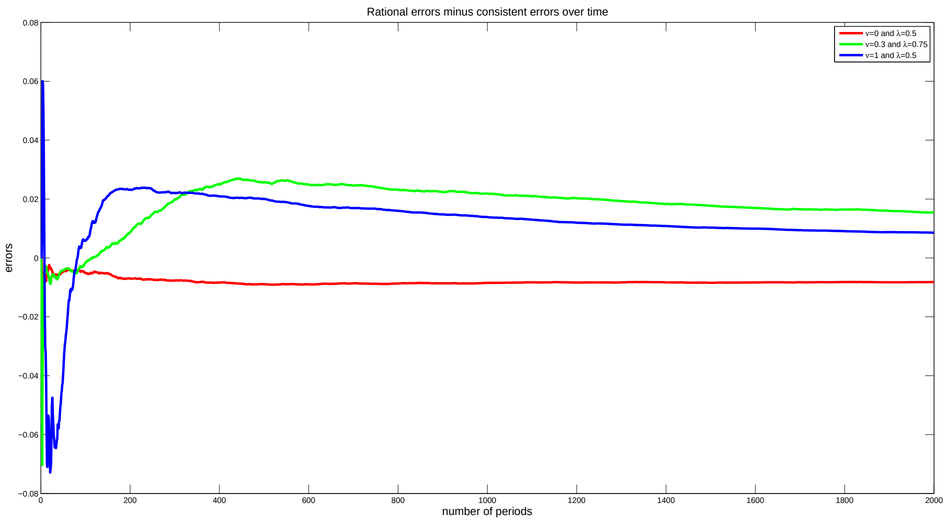

Figure 4: Spread between rational and consistent errors for several parametrization

C.2 Exploration

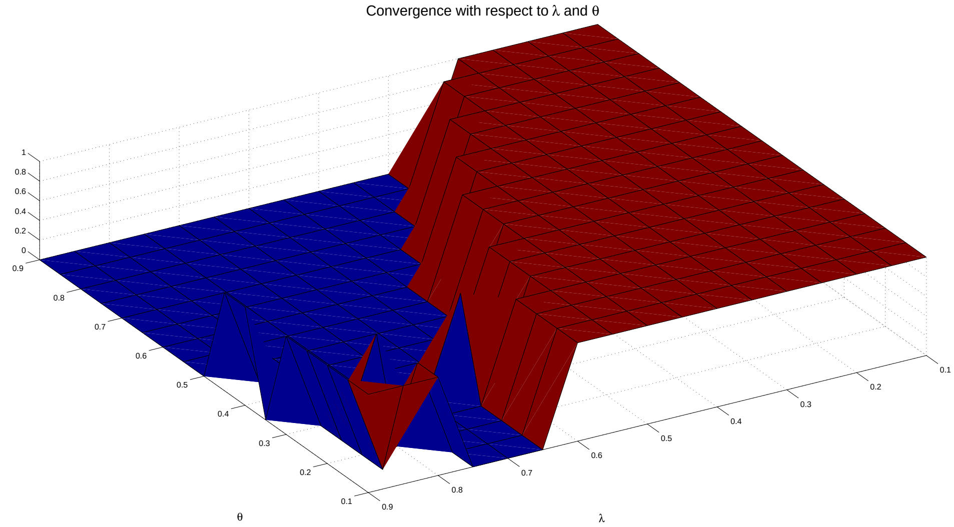

Figure 5: Sensitivity to λ and θ

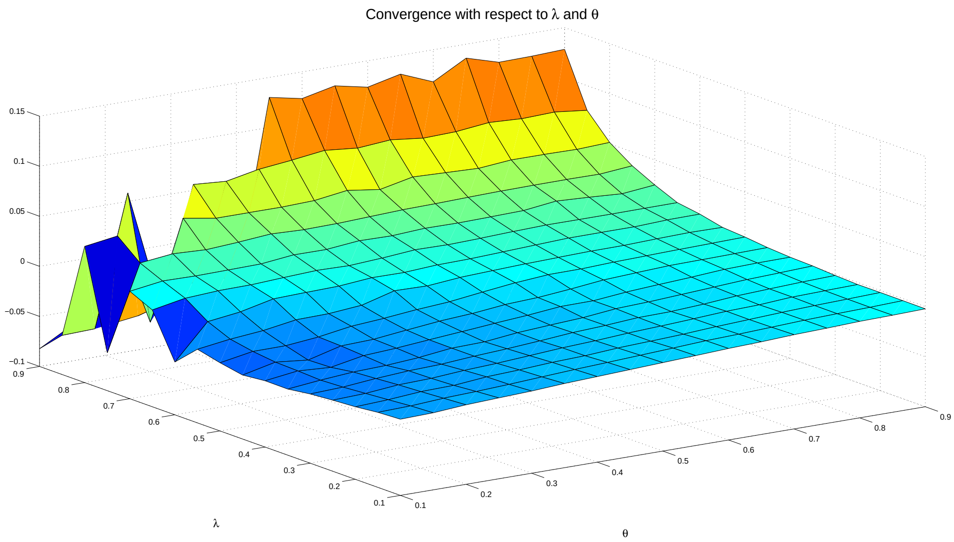

Figure 6: Error spread with respect to λ and ν

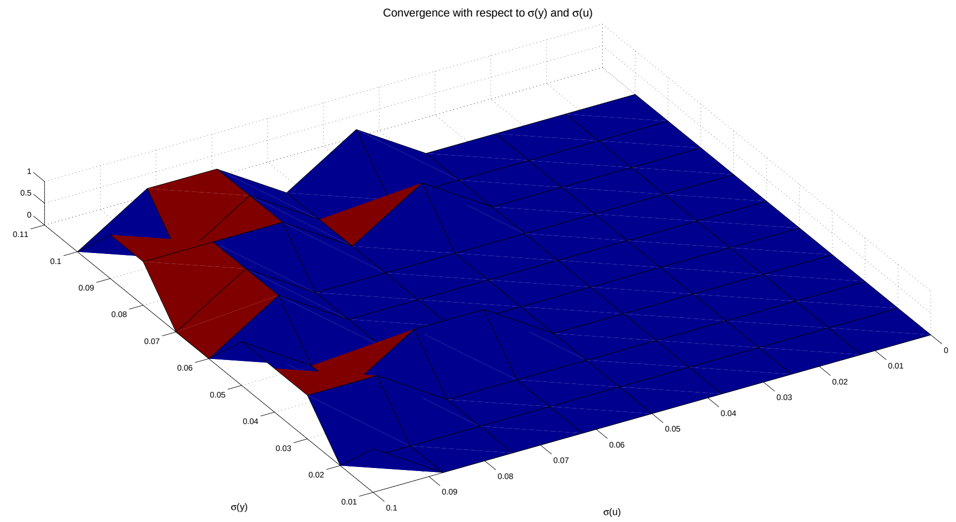

Figure 7: Sensitivity to σ_y and σ_u

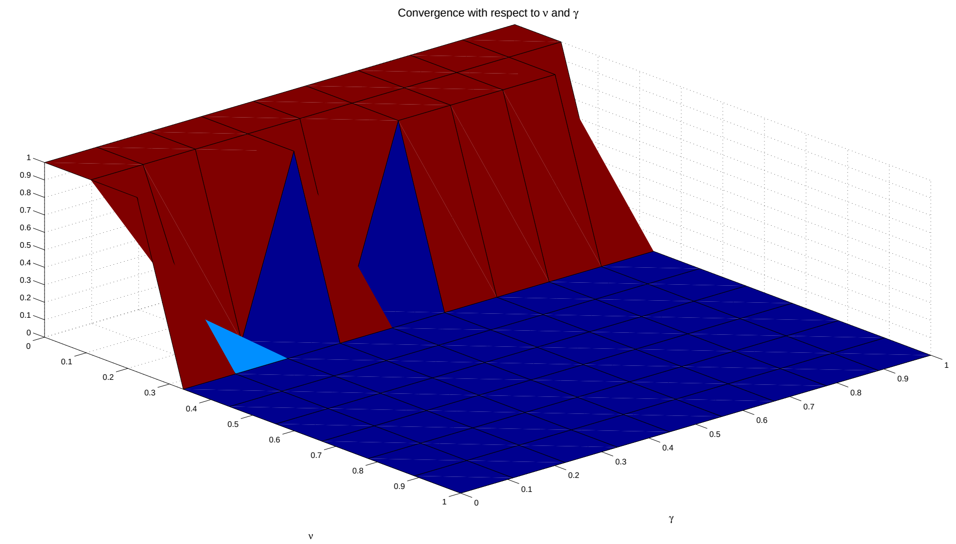

Figure 8: Sensitivity to γ_0 and ν

C.3 The role of initialization

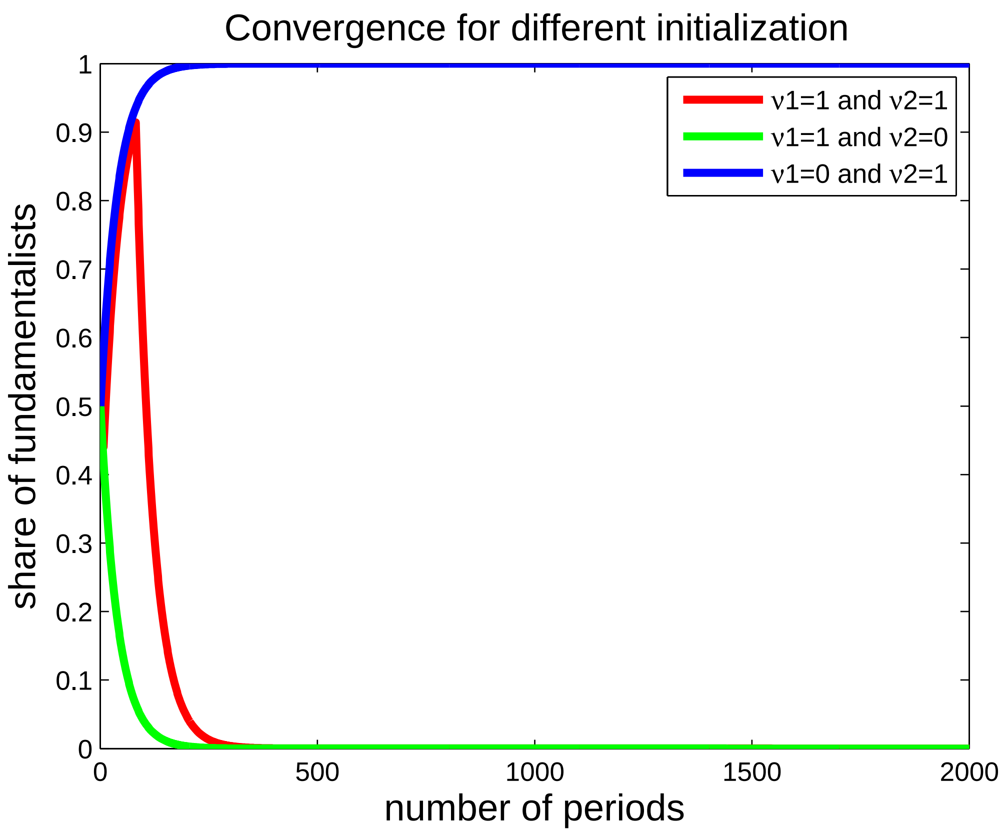

Figure 9: Convergence for different initialization

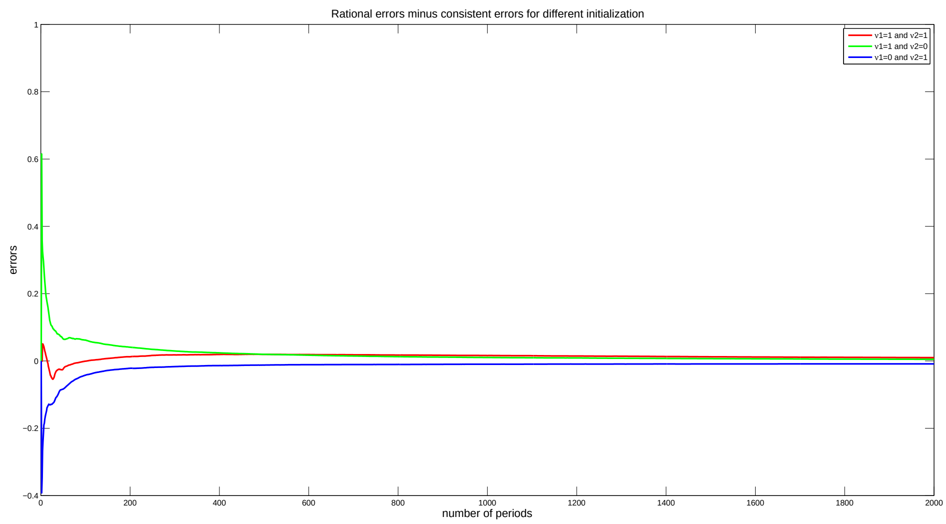

Figure 10: Error spread for different initialization

C.4 Path dependance

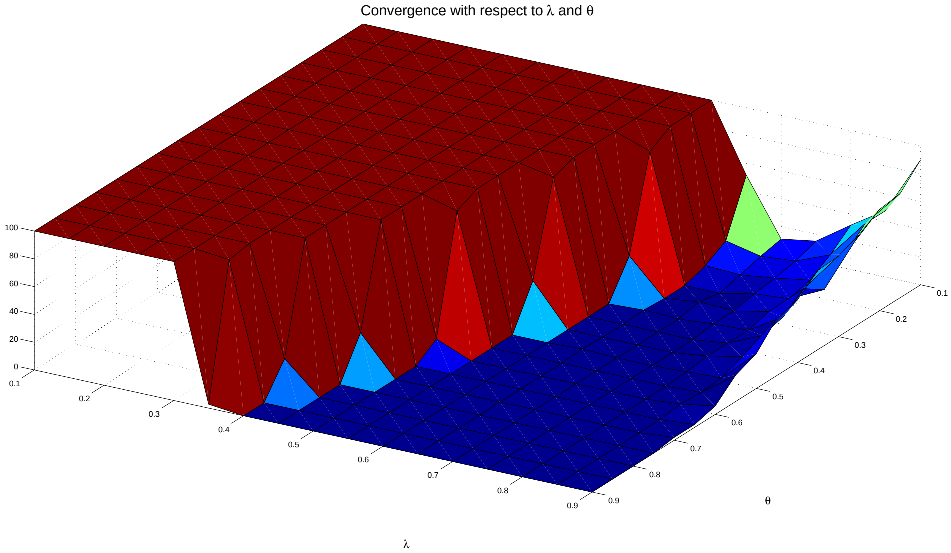

Figure 11: Sensitivity to λ and θ when initialization parameters : multiple simulation

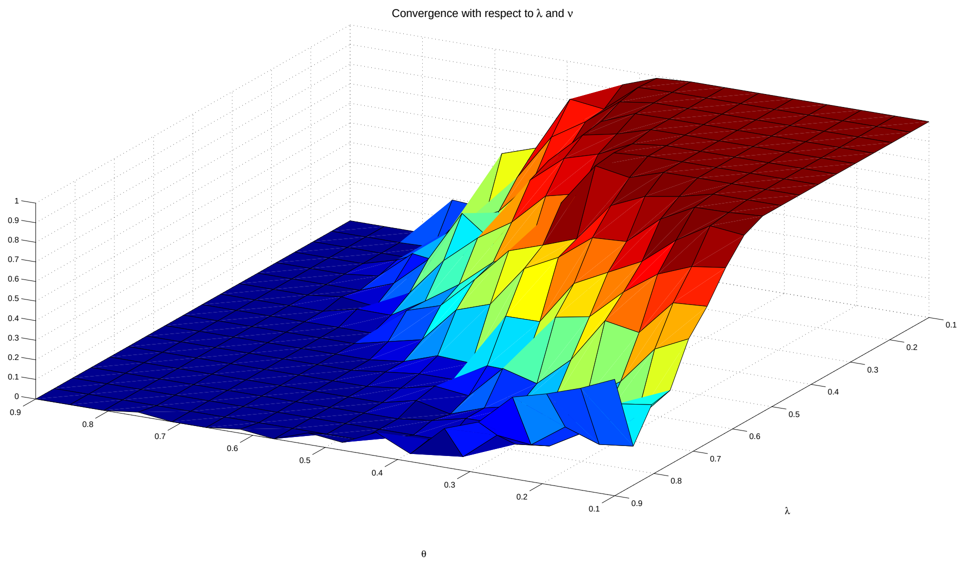

Figure 12: Sensitivity to λ and θ when initialization parameters are random

D Results for structural breaks and unobervables

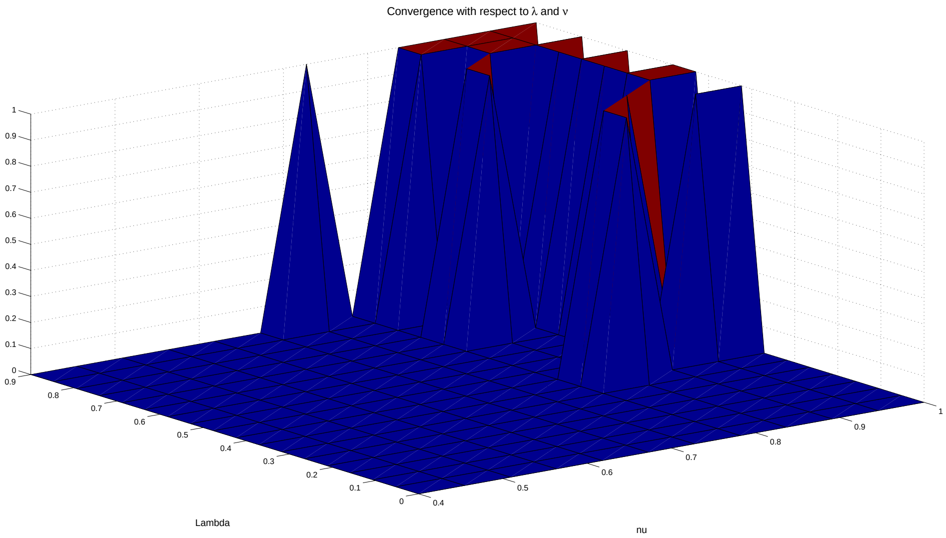

Figure 13: Sensitivity of the response to λ and ν with structural breaks