From exploration to theory-driven tables (and back again). A History of Tables in Number Theory.

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0.1 Beginnings, transmissions and transformations

There are a number of traditions in which integers and their problems are featured and that are known to have nourished and stimulated the body of knowledge that would become number theory.[1] Some of these traditions harbor a number of methods, some lists of facts and even theoretical constructs, but it is often hard to identify unambiguously a coherent and systematic body of knowledge that is recognized as such. There exists such a body in the Arabic tradition in the 10th-13th centuries, in Pierre de Fermat's time in the 17th century, and of course there is 'number theory' as a mathematical discipline with proper textbooks and institutional embedding in the beginning of the 19th century. Though these traditions and bodies of knowledge share quite some elements, they are not identical, but are each framed within a particular context and derive their coherence from that embeddedness in time and space. We will start our story in Ancient Greece with one of the extant texts that deal with integers, their problems and theory, Nicomachus' *Arithmetica* (2nd century CE) [Nicomachus].

Through Nicomachus' text, we can get an idea of the forms and functions of tables when working on integers. Half a dozen tables, rectangular or triangular in form, appear in the *Arithmetica*. Looking at the explication in the text, about three different though affiliate usages may be discerned.[2] Most of the tables display progressions of numbers defined or calculated beforehand in the text itself,[3] the word used in this context is *υποδειξις*, the showing, demonstration or illustration. Often the combination *υποδειξις διαγραμμα* is used, a figure marked out by lines, pointing to the (two-dimensional) structure of what is shown. The table of polygonal numbers [Nicomachus, 97] is a case of *υπομνησις*, the re-memoration or summoning up of the relationships between the polygonal numbers. Indeed, these relationships are explained earlier on in the text, but are now derived from the table.[4] Finally, in one case, the table is described as *πυθμενες*, as the basis for further calculation [Nicomachus, 57].

Only the last table could qualify as an aid to computation (as would a table of multiplication), the other tables are rather illustrations of definitions, properties or procedures. They are used to conveniently display the first few results that follow from a definition or procedure, laying out before our eyes the properties inherent in these definitions. One could easily put "etcetera" after it to indicate that it is but an illustrative sample of the procedure and that the same process can be repeated indefinitely. This might be called the 'inductive' property of tables, they may serve to illustrate and enumerate, but also to show and suggest the next numbers to be generated by the same procedure. Such a table prepares a proof by induction.[5] During the afterlife of Nicomachus' text the small exemplar tables (together with some new ones) would slowly evolve further and acquire a sizeable autonomy, mostly as aids in computation or in demonstration.

0.1.1 Tables as Tools of Demonstration

Nicomachus' text became a stock element in other traditions, sometimes through Boethius' latin *Introductio in Arithmetica*, into which Nicomachus' *Arithmetica* was integrated[6], sometimes with additions by Iamblichus or with some passages from Euclid or Theon of Smyrna. First, there is a revived interest in Nicomachus in the Arabic world after Thabit b. Qurra's translation (10th century), which led to more research into problems

[1] The oldest extant remnant of such a tradition might possibly be found in Babylonia and the famous clay tablet Plimpton 322 is certainly one of the first documented uses of tabular thinking applied to problems involving integers. The interpretation of the content and the context of Plimpton 322 is still a matter of debate, see C. Proust's part in the chapter on Antiquity. Unfortunately, Plimpton 322's trail is now impossible to reconstruct and we can only speculate about its posterity.

[2] Note that in some manuscripts, more tables appear that have been added later. Also, although the procedure now known as Eratosthenes' sieve that will play an important role in the construction of factor tables is explained in Nicomachus, no sample of the output of the sieve, neither in linear nor in tabular form, is given.

[3] This is the case for the tables of doubles and proportions [Nicomachus, 51, 77], and for the table that brings together 'in natural mixture' a number of progressions [Nicomachus, 25].

[4] The term *υπομνησις* is famous for its use in Plato's dialogue *Menon*, where it used to explain how the slave 'remembers' how to construct geometrically square roots of integers. This remembering is an actualisation of something that has been previously recorded or written down, in this case, the memory of the slave is likened to a wax board where mathematical truth is inscribed.

[5] As [Mueller 2012] recently argued. We should to add that there are two forms of mathematical induction. In the case where the table exhausts all (finite) cases, we have a proof by finite (sometimes called: perfect) induction. If the table shows the n first cases and thereby suggests $n + 1$ for an infinite (but enumerable) number of cases we have classic (infinite) induction. Only the second kind appears in Nicomachus. Other kinds of induction exist, but have been developed later. One may think of Francis Bacon's induction through rejection and elimination of items in a tabular list, or the kind of induction that originated probably with Euler and would be used extensively by Gauss, where an infinite number of cases are reduced to a finite number (e.g. by looking only at the remainders modulo a prime).

[6] For a detailed comparison of Nicomachus' text and its variations in Boethius, see pp. 132–137 in D'Ooge's translation of [Nicomachus].

with integers int the following centuries. Second, as part of Boethius' text, Nicomachus also became a classic in the medieval scholastic tradition of Western Europe and a steady source of material that could be recycled in treatises on practical arithmetic and algebra from the 13th century onwards until long into the 17th century.[7] In both cases, profound and complex restructuration of mathematical practices (including notation and algebraic procedures) took place. Within this greater evolution, tables also acquired new forms and functions. Let us look at some examples to measure this change. One example is taken from the Arabic reception of Nicomachus, the other from the European Renaissance reception of the Greek tradition.

Al-Farisi's use of Nicomachus' table of polygonal numbers in the 13th century is a clear-cut case where a table is adapted to show the validity of a procedure in combinatorics.[8] In the context of amicable numbers, an analysis of aliquot divisors of a number had led to combinatorial questions.[9] Earlier, in the 11th century, Al-Kariji had proposed the use of tables as a method of demonstrating combinatorial properties (by 'induction'). Now, Al-Farisi recycled an extended table of polygonal numbers, not as an illustration this time, but as a tool that lays out the inductive properties of figurative numbers (and what we now call binomial coefficients). The table helps, according to Al-Farisi, "to obtain" the numbers, but also "constitutes a model for those who want to determine further numbers starting from these numbers" [Rashed 1983, p. 142]. The display of the procedure of defining the next polygonal number became a device to calculate the next combinatorial coefficient and to demonstrate that this coefficient could be easily obtained in this way ad infinitum. Another case was P.A. Cataldi's prime list (upto 750) from 1603 [Cataldi 1603]. Looking for perfect numbers, another problem of Greek origin, Cataldi needed a list of primes to prove that the 'seed' $2^n - 1$ of possible perfect numbers of the form $2^{n-1}(2^n - 1)$ is prime. By dividing $2^{17} - 1$ and $2^{19} - 1$ exhaustingly by all prime numbers (less than $\sqrt{2}^{17} - 1$ and $\sqrt{2}^{19} - 1$), listing quotients and remainders, he showed them both prime and, as a corollary, $2^{16}(2^{17} - 1)$ and $2^18(2^{19} - 1)$ were perfect numbers. Cataldi claimed that all $2^n - 1$ (with $n = 23, 29, 31, 37$) were prime, but the small prime list did not allow to check beyond $n = 19$.[10] Both examples illustrate a will to extend or go beyond Greek results and show innovation in the techniques of demonstration and computation, tending more towards arithmetical than geometrical means. The first example also shows the impact of algebraic thinking that transforms the problem of aliquot divisors of a number into a combinatorial problem, e.g., a number of the form aab has 6 divisors $(1, a, b, aa, ab, aab)$. As a matter of fact, number theory and combinatorics often appeared together before 1800.[11] Finally, tables appear as aids within a demonstration, exhausting possibilities that may occur and can be 'ticked off' in the table, or showing a pattern that may be repeated ad infinitum thereby providing a blueprint for inductive reasoning.

0.1.2 Explorative and transformative tables

Let us look at two more examples that feature other usages of tables, again one from the Arabic tradition and one from the European Early Modern period. This time problems that stem from Diophantus' *Arithmetica* are focussed upon.[12] The first example comes from a 13th century copy of an anonymous 10th century arabic manuscript [Woepcke 1861].[13] The author gives two rules for computing lists of right-angled triangles with integers whose sides are rational numbers and analyzes the results by means of tables (f. 85 recto–86 recto).[14] The first method starts from the hypotenuse, looks for a decomposition of a square of the hypotenuse into the sum of the squares and so produces all basic triangles "in order, one after the other, without forgetting a single one, and it cannot happen that one misses one out or takes one for another" [Woepcke 1861, p. 8]. The second method starts from the decomposition of odd numbers into the sum of two numbers (3=2+1; 5=2+3=1+4;

[7]For arithmetic in Boethius and in the quadrivium and its subsequent transformations see [Beaujouan 1991].

[8]See [Rashed 1983] and [Chemla and Pahaut 1992, pp. 161–163].

[9]The aliquot divisors of a numbers are all its different divisors, e.g., 12 has as its aliquot divisors 1, 2, 3, 4, 6 and 12.

[10]Cataldi's claim proved to be wrong except for 31, as Fermat and Euler later showed.

[11]See [Chemla and Pahaut 1992] for the Arabic world and [Coumet 1968] for the 17th century in France, outside of France we have, e.g., van Schooten or Wallis.

[12]Diophantus' *Arithmetica* was known and translated into Arabic by Abul Wafa in the 10th century. In Western Europe, the reception of Diophantus started in the 15th century (by acces to Greek manuscripts), though it was only Xylander's translation (1575) and Bachet's edition (1621) that made Diophantus' work widely accessible [Morse 1981]. Note that Diophantus looks for fractional solutions, whereas most of the authors mentioned here look for integral solutions.

[13]The interpretation of this source owes much to a talk by Katiah Asselah and the ensuing discussion at a workshop on tables in number theory that was organized as part of the HTN-project.

[14]According to Woepcke [Woepcke 1861, p. 23], the use of tables in computations and mathematics in general may be called a characteristic of Arabic mathematics of the time (but it is of course not exclusive of Arabic mathematics). Note also that Indo-Arabic numerals are used in the table, whereas the running text has Arabic numerals.

etc.) and generates not only the basic but also the derived triangles[15], however, one has to "keep in mind the hypotenuse in their proper order" to check for omissions or duplications. Two tables are used to achieve this: A first table maps the computational steps of the second procedure, and a second table displays the results of using this procedure. These results are not presented in the order as calculated (according to the sequence of odd numbers and their decompositions), but the entries are ordered according to the length of the hypotenuse found. The table is interspersed with comments that explain why some hypotenuses are absent (or have been eliminated) or why some decompositions generate two hypotenuses.

A close relative in spirit, though not in time, of this anonymous table, is John Pell's 1668 analysis of a small sample of problems on right-angled triangles and a problem from Diophantus. For the problem no. XXIX (find integer isosceles with same area and perimeter) Pell starts with a critique of the solutions proposed by Frans van Schooten (taken from Descartes), because it only produces one or two but not all the solutions. Then, he puts Descartes' solution method into tabular, near algorithmic form and uses it to generate many solutions. However, Pell wrote, it is "disorderly mixture of Answers in Great Numbers amongst Smaller Numbers" with "inverted repetitions" (solutions (a, b, c) and (c, b, a)) and "confused Anticipations" [Rahn 1668, p. 168]. Therefore, "I required that the Enumeration of them should be orderly, pag. 159. I declared that I would have that Enumeration Complete, giving All the answers that do not exceed 100,000 in their greatest side." [Rahn 1668, p. 168] This orderly enumeration is achieved by a cascade of tables. In a first phase, Pell analyzed the inverted repetitions in the table and then rewrote parts of Descartes' procedure accordingly to avoid the inverted repetitions. In a second phase, the direct repetitions still present in the table had to be removed by hand to get a table without repetitions. In a third stage, the problem of order is addressed, by using the method of finite differences on a part of the table obtained. Pell deduced that the problem of disorder is caused by solutions (a, b, c) that have a common divisor greater than unity.[16] Six divisors appear in this analysis: 1, 3, 4, 8, 12, and 24. For each divisor, Pell wrote a procedure to be inserted in the main computation procedure that thus branched, according to the divisor obtained, into 6 subroutines. Applying this to tables obtained finally led to a table ordered according to the length of the hypotenuse.

The tables of the anonymous Arabic author and of John Pell are not merely displays of results, nor immediately relevant scaffolding of a demonstration, but are an exploration of the relations between the numbers, columns and rows, they are rearrangements and transformations of its contents.[17] This use of tables is typical in research on a mathematical problem. On the one hand, it is an explorative use, searching and sorting through numbers according to the layout of lines, rows, diagonals etc. On the other hand, it is a transformative use, the table serves as a snapshot, an intermediary stage that holds the raw material that can then be refined, reduced, manipulated and ordered, the table that appears here is one of possibly many stages of processing data through successive tabulations and computations. Mostly, these kind of tables are not published, but remain manuscript, only the final result, either table, method or proposition, is offered to the public in most cases. In some sense, these tables, once the results are obtained, can be thrown away. However, for mathematical practitioners who value the traces of doing research, their trial-and-errors[18] and their heuristics, it is worthwhile to share this important, often invisible side of the mathematical trade.

From the 9th to the 15th centuries in the Arabic world and later from the 15th to the 20th centuries in Europe tables grew to be an important form within what became known as number theory. First of all, tables conjugated well with Indo-Arabic numeral notation and, later, with symbolic algebraic notation, so that tables became a favorite format for numeric computation and, later, for symbolic, algebraic computation (in Western Europe). In this sense, they played their part in the slow and complex process of arithmetising and algebrising mathematics and its emancipation from geometry. Furthermore, tables evolved beyond being the mere illustration of definitions and propositions, but acquired a certain autonomy as tools and aids for demonstration and computation. Finally, their double role, as aids for both computation and storage, made tables a versatile medium for exploring number-theoretical problems and processing large sets of numerical data.

[15] These are multiples of the basic triangles.

[16] This closely resembles the problem the anonymous Arabic author faced when the order disappears using the second procedure that produces both basic and derived solutions. In Pell speak, the derived solutions were those with a common divisor greater than unity.

[17] In Pell's case, there is an inluence of F. Bacon's *Novum Organon* (1620) where 'Tables of Discovery' are recommended to obtain induction.

[18] The term 'trial-and-error' might sometimes be misleading and often has a pejorative ring to it. As a matter of fact, 'trial-and-error' is mostly a systematic way of trying out things, informed by previous knowledge and experience. A typical kind of 'trial-and-error', used in some of the examples quoted here, takes the actual form of a 'do-until' loop. Divisions, decompositions in squares etc. are tried out systematically, using a list or table of numbers, until one finds a useful result.

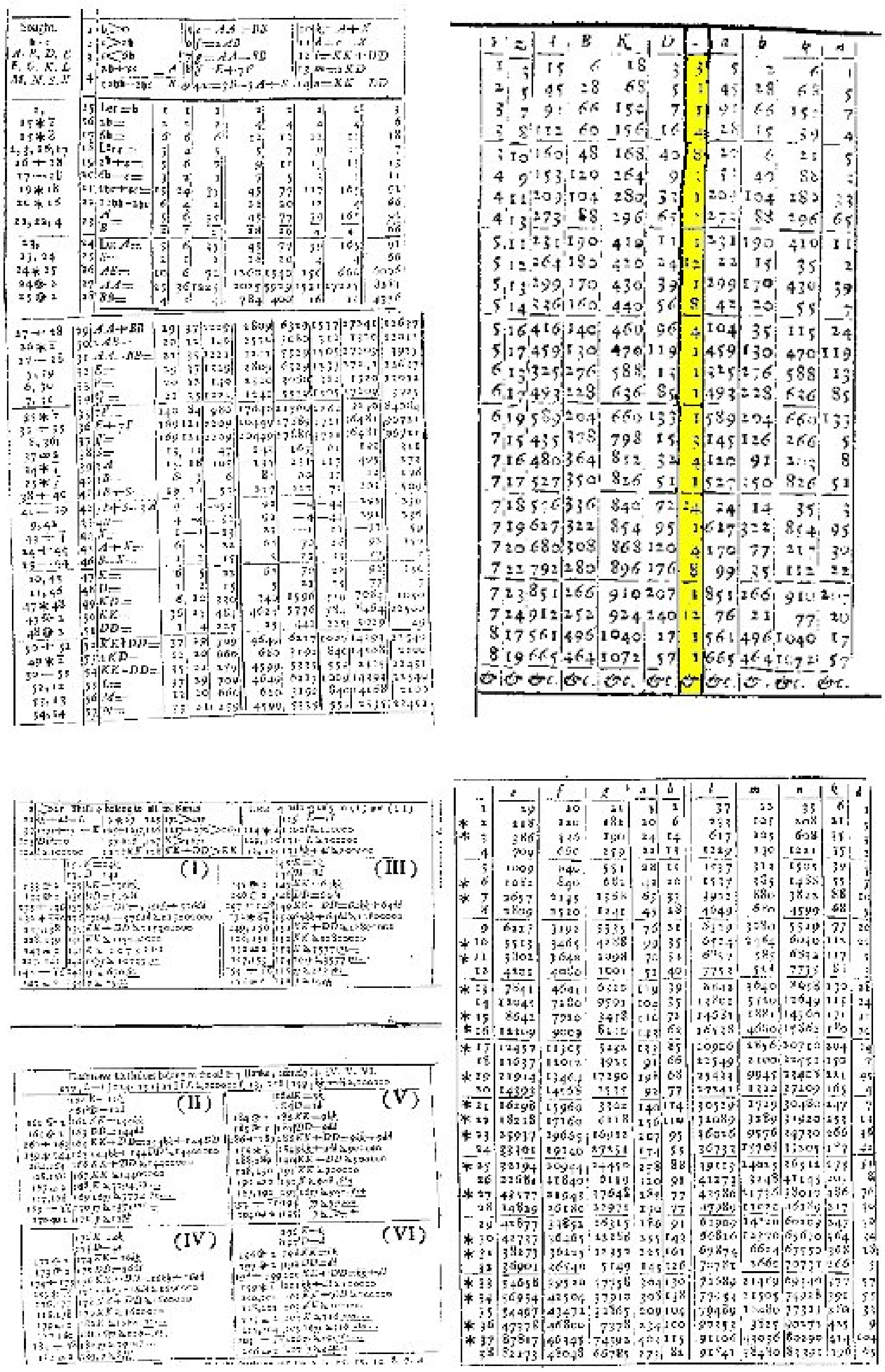

Fig. 0.1 Some tables from Pell's analysis on problem XXIX. In the upper left, some solutions with repetitions. In the upper right, common divisors (in the yellow column) are derived from the results using difference methods. Lower left, 6 sub-procedures for each common divisor 1, 3, 4, 8, 12, and 24. Lower right, the ordered result.

0.2 Cataloguing numbers

Although the tables discussed above had acquired some autonomy, they were still subordinate either to a specific theorem or problem, or they could even be collateral waste in a research problem. Fully autonomous tables in number theory gradually appear in the middle of the 17th century. They are tables giving the (greatest) divisor(s) of all (odd) numbers, and tables listing all squares and/or cubes of integers under a certain limit.[19] They appear in the context of the Modern Age as inaugurated by the invention of the printing press that started a golden age for tables. They become a discursively preferred interface for displaying, accessing and deriving knowledge. As a technical object, the table during this 'Classic Period' can rather easily be identified as the typographical and systematical use of the two dimensions of paper to economically bring together and represent data in a form that can be reproduced technically by the printing press and thus communicated within a literate community.[20] Autonomous tables are not tables for single use but can be consulted when needed, e.g., you can look up a value in a table of logarithms if you need one to simplify a multiplication. Autonomous tables appeared mostly at the end of books and were often reprinted, recycled or extracted in other books. For the discourse on problems dealing with integers, these tables were not bound to one specific problem but had a more general application, they could also be used for looking up one entry only or they could provide the mathematician with specific sequences of numbers (e.g. prime numbers, odd squares etc.) that are useful in systematic search or trail-and-error procedures (e.g. primes to test for a factor, squares to find a decomposition of a number in the form $a^2 + nb^2$ etc.).

To my knowledge the oldest such tables can be found in the work of the Jesuit polymath Paul Guldin (1577–1643). In the first book of his series *De Centro Gravitatis*, [Guldin 1635, post p. 228] Guldin added hundred pages tabulating all squares and cubes of the numbers 1 upto 10000. Similarly, at the end of the fourth and last book of the series [Guldin 1641, pp. 383–401], there is a *Tabula ultima*, listing all prime factors (except 5) for all odd numbers upto 9999. At the end of the table, Guldin added, as a kind of conclusion to the table, that, between 1 and 9999, there were 1226 primes and 8773 composites, or about 7 times more composites than primes [Guldin 1641, p. 401].[21]

0.2.1 John Pell's table project[22]

English mathematician John Pell (1611–1685) whom we mentioned earlier was the catalyzing force in the most important project to produce tables on integers and their properties in the 17th century. Pell had been studying Diophantus for a long time and had lectured on the topic once in a while, but he never published his planned translation of Diophantus nor his results on Diophantus' problems. However, when Pell was asked to oversee the translation into English of an algebra-book, *Teutsche Algebra* (1659), that had originally been published in German by a student of Pell, J.H. Rahn, Pell used the occasion to add some hundred or so pages to the original work in which he solved and analysed problems on right-angled triangles (with integer sides) and diophantine problems (see above). Pell also used the translation to open an indirect dialogue with Frans van Schooten's *Exercitationes mathematicae* that had appeared in 1657 and that featured some indeterminate problems in its fifth section on miscellaneous problems. Instead of van Schooten's purely algebraic (or even Cartesian) methods, Pell chose a different approach. Pell had been an advocate of tables for many purposes for a long time. In the vein of the utopian ideas in vogue in the London of the day (Hartlib, Comenius), Pell, in his *Idea on Mathematicks* (1638), proposed to put all mathematical knowledge in tabular form. A condensed table would be a kind of textbook for beginners, a more elaborate one could serve as an encyclopedia through which one could access the whole of mathematics. He also suggested a table useful for solving all mathematical problems. In the same spirit, his own brand of algebra also closely resembled a tabular list of operations, with line numbers on the left, the symbolic-algebraic form in the middle column and an abbreviation of the operation using line numbers in the rightmost column. It is no surprise that tables were his main workhorses for studying indeterminate problems.

However, in the work on indeterminate problems, some tables are used frequently and thus Pell urged the translator, Thomas Brancker, to calculate a "Table of Odd Numbers less than One Hundred Thousand, shewing those that are Incomposit and Resolving the rest into their Factors or Coefficients" to be added to the transla-

[19]Some small lists of primes or squares under 100 already appear in Medieval manuscripts.

[20]In particular, a table differs from a written running text in that the use of two dimensions of paper is not conditioned by reading habits, but can be read left to right, right to left, top to bottom and bottom to top, vertically etc.

[21]Guldin is three off, there are 1229 primes under 10000.

[22]For more details, see [Bullynck 2010, Section 2].

tion.[23] Knowing Guldin's work, Pell aspired to extend it times ten. Although Brancker did the calculations, the table became known as Pell's Table. It is arranged in 21 columns and 40 rows, each row corresponding to the last two digits, the column for the preceding digits. Numbers divisible by 2 and 5 (though not 3) are excluded and only the smallest factor is given. By excluding not only the even numbers, but also the multiples of five, combined with listing only the smallest factor instead of all factors, Pell achieved a considerable economy of tabular presentation in comparison to Guldin, The result is: "a complete and orderly enumeration of all incomposits between 0 and 100000." [Rahn 1668, p. 193] In 1672 Pell followed up on his project of extending Guldin's tables by publishing tables of squares and cubes [Pell 1672]. Apparently he had not found the time and leisure to extend Guldin's tables to 100,000, but he had condensed Guldin's 100 pages of tables into a 20-page booklet that followed the format of the table of incomposits (a 21 by 40 table with columns for the first two digits and rows for the last two digits).

Computing the table had its problems. Brancker wrote in the preface: "I was very sensible of the bad effects of perfunctoriness in Supputating, Transcribing or Printing of it [the Table]. My care therefore was not small, yet pag. 198 is almost filled with Errata, and I dare not warrant that non have escaped unseen." Page 198 lists 96 errors, most of them printing errors. During the publication process of the *Algebra* (Jan.–Feb. 1667), John Wallis (1616–1703) was enlisted to consult on the project. In the correspondence that runs from Wallis, via John Collins (who acted as organizer and mediator), to John Pell and finally to Thomas Brancker, Wallis gave a list of 145 errata, but by the time it reached Brancker, he had already found 19 more.[24] This slow verification process led Brancker to conclude: "I yet doubt its exactness". Wallis published his "Catalogue of Errors" some years later in *A Discourse on Combinations, Alternations and Aliquot Parts* that was appended to his *Treatise of Algebra* [Wallis 1685, pp. 135–136].[25] Wallis claimed to have examined the whole table "in the same method and with the same pains as if I were to Compute it anew", and listed 30 additional errors or misprints. More convinced than Brancker, Wallis added: "the Table will then be very accurate; and (I think) without any Error." [Wallis 1685, p. 135]

0.2.2 J.H. Lambert's table project[26]

Pell's table of divisors would be recycled in the big encyclopedic projects of the 18th century. It was first reprinted by John Harris in the first edition of his *Lexicon Technicum* (1704), following the entry *Incomposite Numbers* and later also added as an appendix to volume XIII (1765) of the famous *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*. The original work, however, was hard to get hold of. Further progress on tables of divisors, squares or cubes had to wait for about a century.[27] This changed decisively in 1770 when Johann Heinrich Lambert (1728–1777) published simultaneously his *Zusätze zu den logarithmischen und trigonometrischen Tafeln* [Lambert 1770] and other articles on the construction of tables in his *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, Band II.

In his *Zusätze*, Lambert wanted to compile numerical information that was frequently needed in mathematics:

> There are numbers, proportions, formulae and calculations that deserve to be done and written down once and for all, because they occur very often, so as to avoid the trouble to find or calculate them over and over again. This is the reason why in all parts of mathematics one has tried to put everything in tables that can possibly be put into tables. [Lambert 1770, p. 1][28]

In his *Zusätze* Lambert had brought together forty-one tables that would be useful in many domains of mathematics, especially in combination with the classic tables of logarithms and of trigonometric functions. Many of these tables were useful for work on problems with integers too. First of all, Lambert recycled Pell's table, but reformatted it in such a way that multiples of three were excluded from the beginning and that multiples of 7 and 11 could be found directly because they are in the diagonals of the table. This not only reduced the size of the table but also helped to find numerous errors still present in Pell's table (even after Wallis's corrections).

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[23] In Rahn's German original, a smaller table upto 24,000 is featured.

[24] All letters in July 1668, [Beeley and Scriba 2005, II, pp. 469–470; 525–528; 533–535].

[25] This text, including the error list, is also reprinted in the Latin translation of the *Treatise*. This Latin translation was widely read on the Continent.

[26] For more details, see [Bullynck 2010, Section 3 and 4].

[27] For the exceptions, see [Bullynck 2010, pp. 21–25].

[28] Original: "[es gibt] Zahlen, Verhältnisse, Formeln und Rechnungen, die eben daher, daß sie öfters vorkommen, ein für allemahl gemacht und aufgezeichnet zu werden verdienen, damit man der Mühe, sie immer von neuen zu finden oder zu berechnen, überhoben seyn könne. Dieses ist der Grund, warum man in allen Theilen der Mathematick, was sich in Tabellen bringen liesse, in Tabellen zu bringen gesucht hat."

Fig. 0.2 Four factor tables. Upper left, Guldin's Tabula Ultima (1641); upper right, Pell's table (1668); lower left, Lambert's table (1770); lower right, Burckhardt's table (1817)

The *Zusätze* further contained a list of primes, tables of powers of 2, powers of 3, numbers of the form $2^p3^q5^r7^s$, squares (upto 100,000), cubes (upto 100,000) and figurative numbers.[29]

The fact that this is the first major collection of various tables useful in the study of problems involving only integers numbers is, however, not the main achievement of Lambert's project. For Lambert, tables in general were an important tool for scientific and philosophical research. Especially in the case where a theory was still lacking, or was but fragmentary or lacunary, tables could be systematically deployed to compile what was known, to display what was unknown, to explore what could tie parts together in a theory etc. In the absence of a theory, according to Lambert, a table might often be the next best thing. Lambert turned the publication of the *Zusätze* by a number of communicative strategies into an appeal for a more ambitious, collaborative project. The project had two sides, one practical, viz. the production of new and more extensive tables, and the other theoretical, viz. a plea for the establishment of a more complete and coherent theory of integer numbers, the thing we would now call number theory.[30] As already mentioned, at the same time with the *Zusätze* were published two further treatises appeared in his *Beyträge*, but Lambert also inserted announcements in the widely read review journals of his time and in his personal correspondence to appeal to a wider audience to join him in a project of extending the table collection.[31]

Lambert's appeal did reach parts of the intended audiences. From 1770 until Lambert's death in 1777 a dozen of people or so from various backgrounds computed parts of tables, six of which worked on extending the factor table. Each time (part of) a table was computed, Lambert published an update, either in his own writings or in a journal. Although this aspect of the project was a modest success, the result never got around to being published. By 1774–1775 Oberreit had calculated the smallest factor of all numbers until 500,000 but could not proceed further due to new duties. Therefore Lambert split up the next half-million between von Stamford (500,000-750,000) and Rosenthal (750,000-1,000,000). Although Rosenthal delivered the goods, von Stamford did not. Adding to this, two new people, Anton Felkel and C.F. Hindenburg, wrote to Lambert in 1776 that they would compute a factor table to 1, 2, or even 3 millions from scratch. Independent of each other, Felkel and Hindenburg had developed a kind of mechanization of Eratosthenes' sieve procedure that enabled them to proceed faster than their predecessors. Still counting on von Stamford, Lambert tried to convince them to split the next two millions between them, but their ambitions got the better of them and a priority dispute over their factorization aids developed instead. As a result, Lambert's reformatted version of Pell's factor table to 102,000 remained without competition, though Felkel would publish a book containing a table to 144,000 and later additions to 408,000 in 1776. This book being very rare, it did not get around much, though it was used by Georg von Vega to compile the list of primes in his widely used *Logarithmisch-trigonometrische Tafeln, nebst andern zum Gebrauch der Mathematik eingerichteten Tafeln und Formeln* from 1797.

The more theoretical part of Lambert's appeal also found an echo, mostly among his colleagues at the Berlin Academy though it indirectly motivated Leonhard Euler to write up some more of his (unpublished) results. Perhaps the most salient aspect here is the fact that Lambert's project turned some people on to number theory (Jean III Bernoulli, Nikolaus von Beguelin) and inspired others to go back to their old work and take a new look at it (J.L. Lagrange, L. Euler). Nobody supplied the missing 'theory of numbers', which was left to the next generation (Legendre and Gauss), but between 1772 and 1778 there was a sense of excitement around 'number theory' and some important research papers and many new tables were published. These tables were neither square nor factor tables, but were linked to new, more theoretical issues. Following up on a problem Lambert had worked on, Johann III Bernoulli came up with a table of decimal periods, viz. the decimal expansion of unitary fractions, as an aid for exploring the relationship between the length of the period and the integer of the unitary fraction [Bernoulli 1773]. Lagrange then, in the third rewriting of his theory of binary quadratic

[29] Again, many of those tables were copied, corrected, partially recalculated, reformatted and sometimes extended by Lambert. Lambert's tabular work is an exponent of a typical 18th century practice of using cards and tables to organize and order knowledge, be it in a library, for one's own use or for public use. See [Krajewski 2011, Chapters 2 & 4] for the general practice of cards and catalogues and [Keil 1801] for tables in particular.

[30] "Ich habe mich zu diesem Ende [der Primzahlerkennung] so wie auch zu andern Absichten um die Theorie der Primzahlen näher umgesehen, und da fand ich freylich nur einzelne abgebrochne Stücke, ohne sonderlich Anschein, daß dieselbe so bald sollten zusammengehängt und zum förmlichen System gemacht werden können. Euclid hat wenig, Fermat einzelne meistens unbewiesene Sätze, Euler einzelne Fragmente, die ohnehin von den ersten Anfängen weiter entfernet sind und zwischen sich und den Anfängen Lücken lassen." [Lambert 1770, p. 20]

[31] Da es indessen eine langwierige Arbeit ist, die Tafeln der Theiler der Zahlen von 1 biß 102000 von neuem zu berechnen, so werde ich an die Herren Journalisten und an jede andere Schriftsteller, denen dieses Werckchen vorkommen wird, eine Bitte thun. Sie werden nemlich aus Menschenliebe handeln, und den mathematischen Wissenschaften einen guten Dienst thun, wenn sie zur Bekanntmachung dieses Werkchens so viel möglich beytragen. Denn wer nur auch künftig Lust hat, solche Tafeln zu berechnen, der wird dann immer besser seine Zeit darauf verwenden [...] [die Tabelle] weiter [zu führen], als das bereits berechnete nochmals [zu berechnen]" [Lambert 1765–1772, II, pp. 8–9]

forms “Recherches arithmétiques” [Lagrange 1773 and 1775], included 6 tables to help find divisors of quadratic forms.[32]

0.3 1800: Tables and Theories

In 1798 A.-M. Legendre published his *Essai d’une théorie des nombres*, and three years later, in 1801, C.F. Gauss’ *Disquisitiones Arithmeticae* appeared. Together these two works provided the theoretical background for a new discipline, number theory, and provide for a coherent framework in which problems with integers could be formulated and studied. Tables play quite an important role in the preparation of this transformation and, reciprocally, tables in number theory will rather importantly change, both in function and scope, after 1800. Know-how previously collected in and around tables would further the creation of theory and later even disappear or be obscured under the propositions of a systematic theoretical framework. Conversely, theory will motivate the production of new tables that will presuppose more theoretical background on the part of the user, a process that could be called the ‘internalization’ of theory into a table.

Adrien-Marie Legendre (1752–1833) was a renowned mathematician with a long experience in doing computations and in preparing tables. He was involved in the calculations of the meridian, had helped prepare the algorithms for De Prony’s large logarithm table[33] and he calculated and published extensive tables on elliptic integrals (ref chapter Tournès). His work in number theory also bears witness to his involvement with computations and tables. His earliest publication on number theory in 1785 provided 4 tables that extended Lagrange’s tables of divisors of quadratic forms and Legendre attached 12 different kinds of tables amounting to over 50 pages at the end of his *Essai* [Legendre 1798]. Legendre also studied factor tables and made counts to establish a law for the distribution of prime numbers.

Similarly, Carl Friedrich Gauss (1777–1855) also immersed himself in tabular work. At the young age of fourteen Gauss received a copy J.K. Schulze’s (Lambert’s collaborator’s) table collection and 2 years later Lambert’s *Zusätze*. Both books played an important role in his education in mathematics. In 1793 young Gauss began counting the number of primes per thousand in Lambert’s volume, and started to compute a large table of decimal expansions, a table which he finished October 1795 just before going to Goettingen University. He would continue calculating and tabulating quadratic residues, decompositions of $aa + n$, properties of binary and ternary quadratic forms, etc. but most of this work remained unpublished. His *Nachlass* that was printed in the 2nd volume of the collected works (in 1872–79) contains over 100 pages of (tightly printed) tables relating to number theory [Gauss Werke, vol. II, pp. 399–519]. His *Disquisitiones arithmeticae* contains only 3 very condensed tables [Gauss 1801, Table I–III]: a table of indices, a table of decimal expansions and a table of quadratic residues.

Especially in Gauss’s work, a new style of number-theoretical tables surfaces. Instead of (rather extensive) straightforward tables that could be used with a minimum of theory, a more condensed (and often smaller) table appeared that demanded more theoretical background knowledge both to read and to use. This trade-off between the accessibility of the table on the one hand and the theory behind its construction on the other, where more theory means less accessibility and vice versa, explains the rather wide range of tables appearing in the 19th and 20th centuries. The range goes from the easily readable tables, often computed by amateurs, to the rather esoteric tables that only mathematicians well-versed in parts of number theory could use. This trade-off in table design will here be illustrated by the evolution in tables on decimal periods and on quadratic forms.

0.3.1 From decimal fractions to the logarithms of number theory[34]

The latter third of the 18th century saw much research on the properties of decimal expansions of common fractions. This was due partly to the fact that decimal fractions had left a specialist discourse to become part of common arithmetic books in the beginning of the 18th century, and partly due to J.H. Lambert who had drawn the attention of the mathematically interested to the regularities occurring in decimal fractions. Indeed, common fractions $\frac{p}{q}$ have a repeating pattern when written as decimal fractions. Sometimes, this pattern has length $q-1$

[32] Euler’s papers have many lists and sometimes tables that bring together the result of computations, but they rather serve either as material for exploration or as inductive evidence for a proposition.

[33] See [Grattan-Guinness 1990] and more recently [Peaucelle] who, by a careful study of the archives, debunks some of the myths around this project.

[34] See [Bullynck 2009a] for more details.

(e.g. $\frac{1}{7} = 0,142857...$), sometimes a length that is a divisor of $q-1$ (e.g. $\frac{1}{13} = 0,076923...$). Moreover, when varying the numerators p, the patterns change only slightly, they are shifts of the digits (e.g. $\frac{2}{7} = 0,285714...$ or $\frac{3}{13} = 0,230769...$). Lambert was the first to note the connection between these phenomena and Fermat's little theorem, viz. $a^{p-1} - 1$ is divisible by p.

This insight is key to the development of a theory of decimal fractions. As a young man, Gauss had computed decimal periods up to the denominator 997 and by assembling and supplementing the bits and pieces of theory that could be found in Euler and Lambert he finally wrought a complete theory of decimal fractions. Essentially, this theory was already complete in 1797, but it was only published in 1801 as Chapter III and part of Chapter VI of the *Disquisitiones Arithmeticae*. The pages and pages of computations did not make it into the the mathematician published version of the book, but the essence of the results took on the much compressed form of tables I and III, two pages all-in-all.

As Gauss himself wrote in Chapter VI, these tables have a similar relationship to decimal fractions (and more in general to congruences and number theory) as logarithm tables have to common arithmetic.

> This means that any number not divisible by p is congruent to some power of a. This remarkable property is of great usefulness, and it can considerably reduce the arithmetic operations relative to congruences in much the same way that the introduction of logarithms reduces the operations in ordinary arithmetic. We will arbitrarily choose some primitive root[35] a as a *base* to which we will refer all numbers not divisible by p. And if $a^e \equiv b$ (mod. p), we will call e the *index* of b [abbreviated as ind. b]. [Gauss 1801, art. 57]

This remark along with its technical terminology (congruence, primitive root...) is the theoretical baggage needed both to construct and to use Table I of indices. For the indices algorithms similar to the logarithmic ones are valid: ind. $ab \equiv$ ind. a + ind. b (mod. $p-1$) and ind. $\frac{a}{b} \equiv$ ind. a - ind. b (mod. $p-1$) (art. 58, 59). The index of a number modulo p, apart from its use in the theory of congruences, can also be used to find the right decimal fraction in Table III. In that case, your index gives you both the right pattern and the right shift that has to be applied to it. For instance, for $\frac{5}{13}$, the index of 5 modulo 13 is 9. There are two possible periods of length 6 for $\frac{n}{13}$, so you have to write 9 as as sum of two's and a rest. Since 9=4.2+1, you have to choose period (1) [i.e. the rest] and shift 4 digits [i.e. the quotient].

A table mapping all decimal periods of the common fractions $\frac{m}{n}$ with $m < n < 50$ would need a 50×50 table with rather large cells. Gauss's reduced form was a 15×15 table with small two-digit cells and a list of about 60 odd entries. This economy comes at a cost. A user would need a minimum of theory to read and use the table and it was necessary to do some number-theoretical calculations. This reliance upon theory affects the public of the table. No-one having an everyday, say commercial (or for that matter astronomical) need for decimal fractions would ever use Gauss's tables, but for his number-theoretical research the mathematician will find it to great use and would end up using only Table I of indices, dismissing table III of periods as of only marginal interest to his research.

0.3.2 From number representations to a classification of forms

Since the rediscovery of Diophantus' work from the 15th to 17th centuries, questions about the representation of particular classes of integers (e.g. odd integers, integers of the form $4n+1$ etc.) as the sum of two squares, as the sum of a square and the double of a square etc. had been investigated by those few interested in probles involving integer numbers only. Early work on these problems and on right-angled triangles with integer sides heavily relied on auxiliary tables of squares and primes.[36] Going through sequences of numbers that are in the table, one can gradually exclude possible representations and hope to find one or more correct ones. This table-aided methods were slowly complemented during the 17th and 18th centuries with a number of theorems that Fermat and Euler managed to prove or to conjecture by induction. Near the end of the 18th century, this work was synthesized by J.L. Lagrange into a proper theory of quadratic forms in two unknowns $ax^2+bxy+cy^2$. In Lagrange's successive moldings of his theory, a change of perspective slowly set in. Whereas computations and tables led Fermat and Euler to a number of solutions and from there to new theorems, Lagrange started to classify not the solutions, but the theorems.

[35] A primitive root of p prime is a number $a < p$ for which the lowest power n that makes a^n congruent to 1 modulo p is the power $p-1$.

[36] Examples would be Pell's method for Diophantine problems, or Frenicle de Bessy's method of exclusion [Goldstein 2008, 61–74].

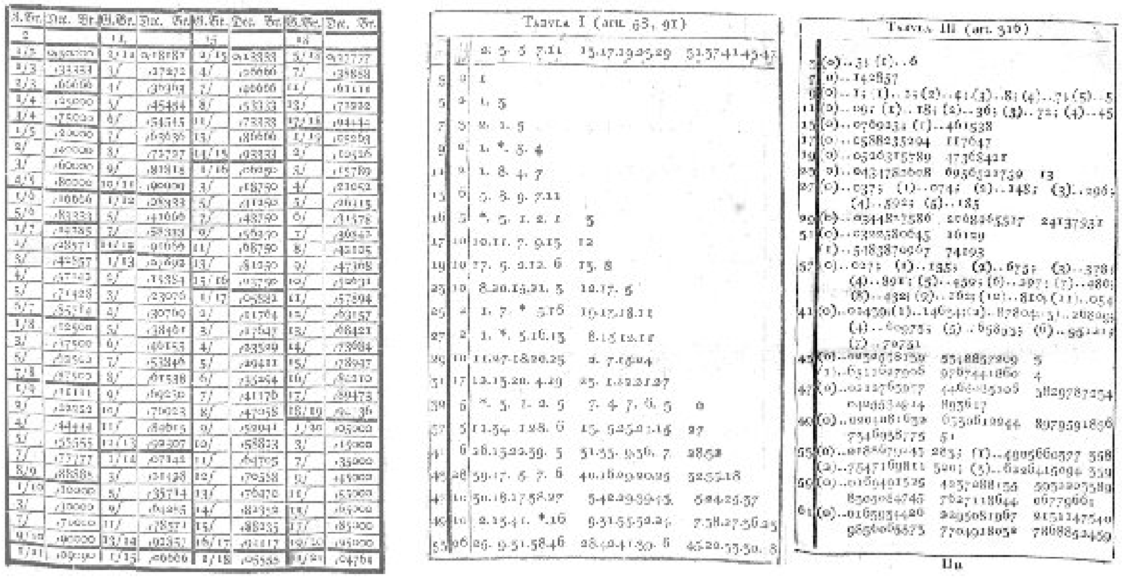

Fig. 0.3 Decimal periods. On the left, a 1796-table by Wucherer giving full decimal expansions for fractions with a denominator < 18. On the right, Gauss's table I of indices modulo a prime (< 97) and table III of decimal periods. The economy of Gauss's table may be appreciated by a comparison: One page of Wucherer's table displays 80 fractions, while Gauss's two pages allow access to the decimal expansions of 4800 fractions.

This is illustrated most plainly in J.-L. Lagrange's "Recherches arithmétiques" [Lagrange 1773 and 1775]. He included 6 tables in his article, two tabulate equivalent forms[37] with a given discriminant (< 30)[38], 4 tabulate what kind of numbers these forms can express. Combining these 6 tables, Lagrange proudly derived no less than 39 theorems, of which Fermat's and Euler's theorems are but mere particular cases [Lagrange 1773 and 1775, pp. 344–349 & 351–353]! In 1798 Legendre extended Lagrange's table for all discriminants under 100.

In Gauss's *Disquisitiones* there are no tables on quadratic forms. Instead, Gauss's theoretical framework convolutes Lagrange's tables into a systematic classification of quadratic forms. Following Lagrange (and Legendre) Gauss first develops the theory of classes of quadratic forms, where each class contains the equivalent forms that represent the same integers. Then he also gives a theory of genera. For each quadratic form with a determinant D one can specify a number residual characters (i.e. the quadratic residuacity modulo p where p is a factor of D and the residuacity of D modulo 4 and 8). Half of the combinations of residual characters are the genera of D. In Lagrange's or Legendre's vocabulary, the residual characters would amount to forms of integers that are represented by (a genus of) quadratic forms. Gauss gives procedures and tricks to compute tables that give all classes and genera, he also computed their number for all determinants upto 3000 [Gauss 1801, art. 304 footnote], but they do not appear in the *Disquisitiones*. Instead, the focus shifted once more: the question of representations of numbers was replaced by the study of the internal structure of the quadratic forms of a given determinant. This led in its turn to such questions as the class number and (later) the class group belonging to D and inverse question such as what D have class number 1, 2 etc.

0.3.3 Paper machines

A characteristic of number theory is its discreteness, the fact that it deals only with integer numbers.[39] Probably due to this particularity, tables, that are also discrete by nature, seem to have quite frequently given occasion to the development of 'paper machines' in number theory. A 'paper machine' is the systematic cutting and assembling or otherwise transforming of parts, mostly rows and/or columns, of a genuine, printed table. The best known example are Napier's rods. The rods are essentially the columns of a multiplication table, but once liberated from the bounds of its tabular context as a 'free' column, the columns or rods can be freely moved and combined to automatically display a product.

[37] Equivalent quadratic forms are forms that represent exactly the same classes of integers.

[38] For a quadratic form $ax^2 + bxy + cy^2$ the discriminant is defined as $\Delta = b^2 - 4ac$. Gauss used a somewhat different definition: $ax^2 + 2bxy + cy^2$ where the determinant $D = b^2 - ac$.

[39] This is a characteristic shared with combinatorics to which number theory was closely related in its early days, and with algebra upto a certain extent.

Such paper devices seem to have been developed in number theory from the late 18th century onwards until long into the 20th century when machines, mechanical or electronic, would take over. The most famous and perhaps most important of such 'paper machines' in number theory are the factor stencils. The stencils originate in a wooden read-and-write device. You could slide the device over your empty squared paper to measure the distance between the cells, where each paper cell corresponds to an integer. Carl Friedrich Hindenburg, a Leipzig-based mathematician best known for his combinatorial analysis, invented this machine in 1776 to help him compute large factor tables in response to Lambert's call for tables [Hindenburg 1776]. The idea was put into a more practical form by J.C. Burckhardt who had studied under Hindenburg in the 1790ies and had become *astronome-adjoint* at the Bureau des Longitudes in Paris. During his spare time, Burckhardt calculated factor tables of the 2nd and 3rd million and had recomputed the 1st million [Burckhardt 1814-1817]. Burckhardt worked with paper containing 80 times 77 squares per page. For automatically eliminating multiples of 7 and 11 Burckhardt used a copper engraved plate corresponding to one page with holes at the location of multiples of 7 resp. 11. For divisors over 11, say 13, Burckhardt took an empty, squared sheet, started cutting out the squares that were multiples of 13 and stopped after the 13th column, since "this factor will return in the same order [...] because of the distance" [Burckhardt 1814-1817, p. v] This form of paper stencil to make factor tables was used throughout the 19th and 20th centuries, e.g. by J.W.L Glaisher and D.N. Lehmer (see below).

'Paper machines' have been used for other purposes in number theory too. In 1897 R. Haussner described a paper method that helped him construct tables for checking on Goldbach's conjecture that every odd number can be written as the sum of two primes [Haussner 1897]. Derrick N. Lehmer then, in 1924, remarked that Legendre's idea of linear divisors[40] could be transformed into paper stencils useful in factoring [Lehmer 1925-1929]. Each stencil is associated with a number R and contains 56 columns and 100 rows (each cell corresponding to one of the first 5600 primes), a hole is punched for a cell P_n if R is a quadratic residue of P_n. These stencils are then 'translated' into 7 Hollerith punched cards. Superimposing the stencils and locating the holes they have in common gives the arithmetic progressions in which to look for factors.

Finally, the idea fundamental to Napier's rods, using the columns of a table as calculating rods, has found application in number theory too. Anton Felkel, competing with Hindenburg for computing a large factor table, devised factor rods that, by aligning them correctly for a given integer, display a factor [Felkel 1776]. C.F. Gauss, who described a factorization algorithm using quadratic residues, advised cutting out the columns of his quadratic residue tables and gluing them onto wooden or metal rods to obtain "moveable columns" [Gauss 1801, art. 331]. If you know quadratic residues of a number N, align those columns/rods that correspond to those quadratic residues. Since a factor of N must have the same quadratic residues as N, a row with all quadratic residues ticked off will indicate a (possible) factor.

0.4 Disciplined tables

Number theory gradually established itself as a proper discipline within a rapidly professionalizing mathematics during the 19th century [Goldstein et al. 2007]. As a consequence, tables acquired a new position both within number theory as a discipline and within the division of labor of modern mathematics. Now that Gauss and Legendre had laid the fundaments for a theoretical framework, the theory itself would become the focus (and motor) of research and tables became the servants to theory. Also, with the professionalization of mathematics, a divide grew between mathematicians that held research and teaching positions at universities and those who taught in schools or worked in other fields of practical or applied mathematics. As calculation came to be seen more and more as the business of the footmen of mathematics, so tables receded to the background. The factor tables calculated by the astronomer Burckhardt in his 'spare time' or those of the calculating prodigy Zacharias Dase in the 1850s are typical exponents of this tendency.

0.4.1 Number-theoretical tables in the industrial age

The reorganization of tasks in mathematics alluded to above comes down to a certain division of labor.[41] Instead of the projects à la Pell or Lambert, a more systematic and institution-backed approach to table making was taken. It is possible to discern about three forms of table making in number theory (if we disregard the smaller tables that are sometimes included in research articles):

[40] Actually, the idea stems from [Lagrange 1773 and 1775, 334–337].

[41] See de Prony's tables for the classic example of the division of labor in mathematical table making.

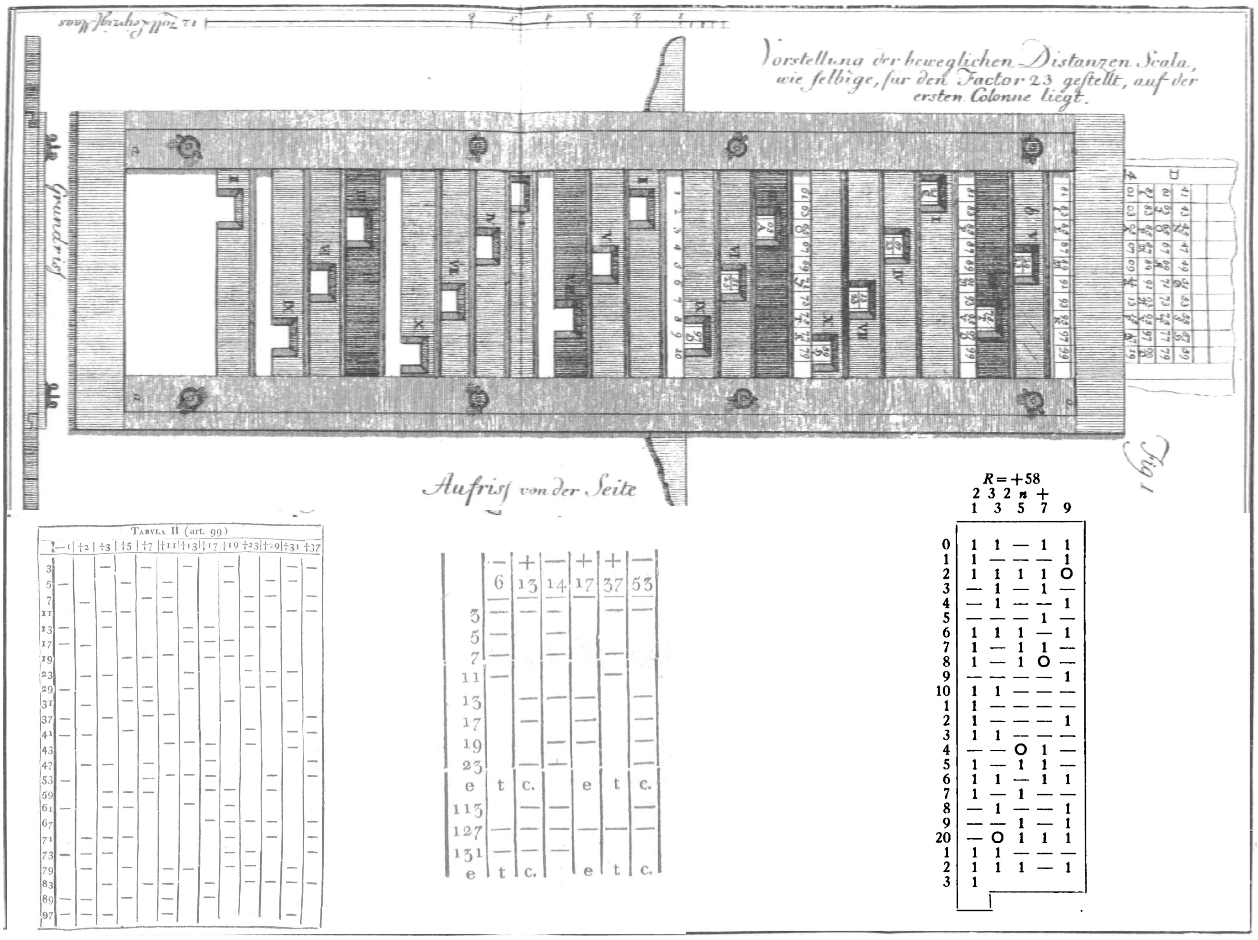

Fig. 0.4 Paper machines. On top, Hindenburg's sieve device. Below, Gauss's moveable columns of quadratic residues; and D.N. Lehmer's example of a stencil for linear divisors (the 1's will become holes on Hollerith cards).

1. Research problem oriented tables involving single or small groups of mathematicians (19th-20th century)
2. Systematic organization by (national) committees on mathematical tables (1870–1950)
3. International networks of (amateur-) mathematicians and table-makers (1900–1950)

Although these forms appeared one after the other chronologically, the older forms do not disappear but remain used throughout. Looking at the number of tables big and small published between 1850 and 1950 (using [Lehmer 1941]), one notices a steady increase that finally peaks in the 1920s and 1930s, a period that could be dubbed the 'Golden Age of Tables' just before the arrival of the digital computer.

0.4.1.1 Tables and research problems

Preparing for the computation of a table that can help explore, elaborate or apply and illustrate a piece of theory or a research problem may take much time and consideration. Executing the table by the development and improvement of algorithms and by long computations demands even a greater investment of time and effort. During the 19th century, the task of (1) planning a table; and (2) computing it, was often divided between two or more mathematicians. The first was mostly an academic research mathematician, the second often a mathematician of a lesser standing. This form of collaboration seems to have thrived mostly in Prussia and most projects can be situated in the tradition of Gauss's *Disquisitiones arithmeticae*. The projects often pursue the study of primitive roots of prime numbers and structures in quadratic forms.

One of the earliest major tables in this tradition was C.G.J. Jacobi's *Canon arithmeticus* [Jacobi 1839] that extended Gauss's table I of indices modulo a prime number until 1000. Jacobi recycled Ostrogradsky's table until 200 (1838) and the part between 200 and 1000 was calculated by a *Kanonier-Unteroffizier* Krämer who was paid by the Berlin Academy. Some extra tables for moduli that are powers of small prime numbers (2, 3, 5) were computed by a school teacher, Wichert. Finally, all the tables were read and corrected by Jacobi's colleagues and friends (Dirichlet, Dirichlet's wife and mother, Dove, Steiner Mädler and some PhD-students, viz. Wolfers, Bremiker and Galle) under direction of J.F. Encke, director of the new observatory and secretary of the Academy of Sciences in Berlin. A similar structure was set up at the Berlin Academy for G.E. Eisenstein's *Tabelle der reducirten positiven ternären quadratischen Formen, nebst den Resultaten neuer Forschungen über diese Formen, in besonderer Rücksicht auf ihre tabellarische Berechnung* (1851). The Academy paid for both extra "computational power" (*"Rechenkraft"*, no names are given!) and for the publication costs.

Tables which required certain mathematical competence and much theoretical background, not to mention no little skill in crafting algorithms, were often produced by mathematicians with a status just under that of university professor. Karl Gustav Reuschle (1812–1875), a professor at the Stuttgarter Gymnasium, was well-known for his tabular work. He published tables on the lengths of decimal periods and on the primitive roots of numbers of the form $2^n \pm 1$ and $10^n \pm 1$ after a correspondence with Jacobi (1856). At the end of his life, a voluminous *Tafeln Complexer Primzahlen Welche Aus Wurzeln Der Einheit Gebildet sind* (1875) appeared for which Reuschle had closely corresponded with E.E. Kummer. Similarly, Peter Friedrich Arndt (1817–1866) who had been a professor in Stralsund and later a *Privatdozent* in Berlin, published tables of reduced cubic forms in 1858. Another case is Gustav Wertheim (1848–1902), a *Realschullehrer* from Frankfurt, who published a number of (smaller) tables of primitive roots between 1893 and 1898 in the *Acta Arithmetica* and in his manuals on number theory.

Perhaps the very last example of this kind of cooperation may have been the tables edited by Heinrich Brandt (1886-1954), a professor at Halle University. Brandt, who himself had performed many calculations during his PhD-research on quaternary quadratic forms, edited Wilhelm Patz's recalculation of Jacobi's *Canon arithmeticus* (published 1956) and helped prepare *Studienrat* O. Intrau's tables on ternary quadratic forms (1959) and compositions of quaternary quadratic forms (1970). It may come as a surprise that these tables were published well into the age of digital computing, but one has to take into account that the first computer in the German Democratic Republic, the OPREMA a slow relay computer for use in the Zeiss-factory in Jena, only became operational in 1955.

0.4.1.2 Tables under committee

An organization even more in tune with the spirit of the Industrial Age was promoted by the mathematical tables committees that were mainly an Anglosaxon phenomenon. The first committee was established after the Edinburgh meeting of the British Association for the Advancement of Science in 1871, where a grant of £ 50 was made to a new committee 'for the purpose of reporting on Mathematical Tables, which it may be desirable to compute or reprint'.[42] Its main organizer proved to be J.W.L. Glaisher (1848-1928) who compiled a first general report on tables in 1873, while Arthur Cayley (1821-1895) added a specific report on tables in number theory in 1875 [Cayley 1875/1876]. The Committee's review of tables helped to identify those tables most urgently required. In the case of number theory, Cayley calculated a number of small tables that were mssing and also reformatted some existing ones (e.g. Arndt's 1858-table), but the main project on the wanted list were the missing three millions of the factor table (between the third and the seventh). J.W.L. Glaisher's father, James Glaisher, was asked to oversee the work of two hired calculators and between 1879 and 1883 they produced the missing millions for the price of £ 250. After 1883 J.W.L. Glaisher stopped being an active member of the Committee but he would continue to play an important role in table making because he became editor-in-chief of the *Messenger of Mathematics* (in 1887) and *The Quarterly Journal of Pure and Applied Mathematics* (in 1897). He edited both journals until to his death in 1928.

The British Committee saw a new surge in the making of number-theoretical tables with the arrival of Allan Cunningham (1842-1928) in 1895. Cunningham, a retired colonel, 'capered' the Committee between 1896 and 1901 when he served as secretary using its funds to publish his own tables: *A binary canon* (1900) and *Quadratic partitions* (1904). After Cunningham leaving (see next section), the Committee was inactive for a while before embarking on the computation of extensive tables of elliptic functions and Bessel functions again. On Cunningham's death in 1928, the Committee received a £ 3272 legacy to publish number-theoretical tables. Asking G.H. Hardy and A.E. Weston for advice, the committee published 5 tables between 1933 and 1940 [Croarken 2003, p. 244].[43]

In the U.S., there were a number of large projects on tables and bibliography in number theory, D.N. Lehmer's *Factor Table for First Ten Millions* (1909) and Dickson's *History of the theory of numbers* (1919-1923). How-

[42] See [Croarken 2003] for its history.

[43] Two tables came from J.W.L. Glaisher's legacy, two others were computed by L.E. Dickson and E.L. Ince.

ever, these were not the work of committees but the work of individuals funded by a single institution, the Carnegie Institute of Washington. In 1930 the National Research Council, following up on the fruitful exploitation of mathematics during the first World War, proposed the establishment of a Committee on Mathematical Tables and Other Aids to Computation[44], commonly abbreviated as MTAC, that would compile a bibliography of tables.[45] When R.C.A. Archibald, mathematics professor at Brown University, became chairman of the Committee in 1939, number-theoretical tables came uner the spotlight. Archibald hired D.H. Lehmer, D.N. Lehmer's son and a number-theoretician himself, as his assistant. In this function, D.H. Lehmer wrote a *Guide to tables in the theory of numbers* [Lehmer 1941]. This *Guide* contained an exhaustive list of number-theoretical tables together with a 50-page list of errata. The committee had a more general and lasting impact on the history of tables (and of computing) through its journal *Mathematical Tables and other Aids to Computation*, first published in 1943 and which became the main medium of communication for early computing before the establishment of the *Communications of the ACM* in 1957.

0.4.1.3 Tables in (inter)national research networks

In the course of the work of these committees, not unfrequently a small but growing network of collaborators continued working after the main project had been completed, using journals to inform their peers and the general public of their progress. This was the case in the United Kingdom where J.W.L. Glaisher's *Messenger of Mathematics* and *The Quarterly Journal of Pure and Applied Mathematics* served as the principal means of communications from 1900 to 1925. Apart from Glaisher, Cunningham and his collaborators, T.G. Creak and H.J. Woodall, were the most substantial contributors. Similarly, the journal *Mathematical Tables and other Aids to Computation* (MTAC) (1943–1959) became the main medium through which people calculating tables either manually or with the help of machinery in the U.S. and abroad in the 1940s and 1950s could publish or describe their work.

Apart from these communities that originated from committees, there was also another network that was mainly active in France and Belgium. Its origins lie with the French mathematician Edouard Lucas (1842–1891) and with the yearly meetings of the AFAS (Association Française pour l'Avancement des Sciences) (1875–1930), the French equivalent of the British Association for the Advancement of Science [Decaillot 1999]. The network only really grew from 1905 onwards when a number of non-academic mathematicians (A. Gérardin, A. Aubry, E. Lebon, P. Poulet, S. Hoppenot) started discussing number-theoretical topics in a variety of journals such as *Mathesis*, *Nouvelles Annales*, *Enseignement mathématique*, *Intermédiaire des mathématiques* and the proceedings of the AFAS-conferences.[46] These were not the top rank mathematical research journals of the time, but rather mathematical periodicals for a broader public of teachers, engineers, students and other mathematically inclined individuals.[47] The most active members were the Nancy-based mathematics professor André Gérardin (1879–1953) and, from 1920 onwards, the Russian-Belgian mathematician Maurice Kraïtchik (1882–1957).[48] Between 1906 and 1928 Gérardin published a special journal, *Sphinx-Oedipe, journal mensuel de la curiosité et de concours* and later, Kraïtchik edited *Sphinx, revue mensuelle des questions récréatives* (1931–1939). These journals mainly contained puzzles and other forms of mathematical recreations and mathematical problems, but factorisation and small tables featured frequently.

With time, these national networks (British and French) would gradually start to correspond and their work sometimes appeared in each other's publications to become a shortlived international network of people interested in number-theoretical tables. The network would, by correspondence and by publication in Kraïtchik's *Sphinx*, include other national nodes too: D.H. Lehmer (USA), L. Poletti (Italy), N.W.G.H. Beeger (Holland). Taken together they account for the majority of the over 150 small tables that were published between 1910 and 1940. The problem of factorization was the central question around which these researchers gathered and the most popular tables were primitive roots, factorizations of Mersenne numbers or, more generally, binomials ($a^n \pm b^n$), solutions to special cases of Diophantine equations etc.

[44] Under 'Aids to Computation' they had in mind mechanical calculators, graphical methods and the newly developed harmonic and differential analysers.

[45] See chapters 12 to 17 in [Grier 2005] for more on the Committee and on table-making projects in the U.S.

[46] Cunningham also often featured in the Q&A parts of these discussions.

[47] *Mathesis*, the journal of the Belgian mathematical society, is the exception here.

[48] Incidentally, Gérardin and Kraïtchik would also found *Les amis des nombres* a subproject of Paul Otlet's Mundaneum-project that wanted to gather all worldwide documentation in one place [Boyd 1975]. "Le 3 juin 1921, à Bruxelles, MM. A. Gérardin et Kraïtchik ont fondé la société "Les Amis des nombres" dont le but est de réunir les professionnels et les amateurs qui s'intéressent surtout aux nombres. La séance de constitution a eu lieu au Palais mondial, Bruxelles-Cinquantenaire, siège social. Le Bureau est composé de M. Otlet, président, et de MM. Kraïtchik, A. Errera, et A. Gérardin, secrétaire. Le "Sphinx-Oedipe", dirigé par M. Gérardin (Nancy) insérera les communications officielles du Comité."

0.4.1.4 Machinery for Number Theory[49]

Most tables before the Interwar period were still solely calculated by hand, e.g., only upon receiving Cunningham's financial legacy in 1928 did the British committee acquire calculating machines.[50] In the beginning of the 20th century commercial calculating machines and graphical computation methods start to spread. Many of the people mentioned above had a more general interest in computing per se. Reuschle, Kraïtchik, D.H. Lehmer for instance all published on graphical methods of calculation. Some were also interested in the development of computing machinery.

The idea of turning the paper devices from the 18th-19th centuries into a machine with cyclic movement (because of modular repetition) had already been suggested by J.H. Lambert in 1777, by E. Lucas in 1876 and by W.F. Lawrence in 1896. Upon Gérardin's translation of Lawrence's paper in *Sphinx-Oedipe* (1905), Gérardin, Kraïtchik and the brothers Carissan went on to construct their mechanical 'congruence machines' (1912-1920) that could effect a sieve procedure that is in essence Gauss's quadratic residue paper method.

Though the machines did not produce noteworthy results, Gérardin's work did inspire the young D.H. Lehmer to construct a sieve device of his own. First, a motor-driven bicycle chain machine, then, in 1932–1933 with the financial help of the Carnegie Institution, he translated the mechanism into an photo-electric sieve that was exhibited at the Chicago World Fair of 1933.[51] The bicycle chain sieve could canvas 50 numbers per second and the photo-electric device 6000 numbers per seconds, compared to 3 numbers per second by manual paper procedures.

0.4.2 Number-theoretical tables in the digital age: Returns and transformations

It is a nice coincidence that one of the first programs that ever ran on the ENIAC, one of the world's first digital electronic general purpose computers, was number-theoretical. On Labor day 1946, D.H. Lehmer, his wife Emma and their two children had the opportunity to use this computer to calculate a table of exponents of 2 modulo a prime. The idea was to correct one of Kraïtchik's tables and to extend the list of exceptions to the reverse of Fermat's little theorem. The output of the program itself, a series of punched cards, was archived but never published, only the list of errors in Kraïtchik's table and the list of exceptions were printed in a journal. The main body of the article was, however, devoted to a description of the algorithm used.[52]

This was symptomatic of a more general trend. Although the first big calculators and computers had been developed for the computation of tables, once the general-purpose computer arrived, interest in publishing tables waned while the problem of good algorithms and subroutines became the center of attention. This trend is amply documented in the articles of the journal *Mathematical Tables and Other Aids for Computation* (see Durand-Richard and De Mol's chapter). In some cases the table in form of punched cards could be used as an algorithm, just as the factor stencils as the embodiment of a table had been used for computation.[53] However, in general, the tables were rather stored in (working) memory to be used during further computation or to be compressed or hand-picked for publication.

There was much continuity between the table committees and the first computer programs written for number theory. In the U.S. D.H. Lehmer planned further calculations on the ENIAC to obtain tabular material useful for work on Fermat's last theorem and on the Rieman hypothesis. In Great-Britain J.C.P. Miller of the Table Committee tried his hand at factorizations of Mersenne numbers using the EDSAC. Remarkably, the digital computer would soon interest not only those working in computational number theory, but also those working in 'pure' number theory. In the U.S. Emil Artin suggested to John von Neumann that he compute evidence for or against a conjecture of Kummer [von Neumann and Goldstine 1953], and in Great Britain, Miller and Wilkes of the committee asked L.J. Mordell to specify a number of Diophantine problems the EDSAC could work on [Miller 1954].

Though the table has somewhat disappeared from view, it still lives on electronically, hidden in the computer, where its forms and uses have multiplied.[54] D.H. Lehmer's words on a future 'mechanized mathematics' may well stand here as an *envoi*: *Most of the fundamental classical theorems of number theory were discovered from*

[49] See [Bullynck 2013] for more details.

[50] Karl Pearson and his lab were one of the first fervent promoters of calculating machines in the 1910s, cfr. [Grier 2005].

[51] Incidentally, a plan for a relay-based sieve, probably inspired by Lehmer's device, would appear in C.E. Shannon Master's thesis of 1937.

[52] For more details see [Bullynck and De Mol 2010].

[53] This was the case with solutions to linear equations modulo 2 that had been computed by Hans Lewy to simplify the solution of systems of linear equations.

[54] Compare with Durand-Richard and De Mol's chapter.

the inspection of tabular evidence. Today tables can be produced in such quantities as to render their publication or even, in some cases, their inspection economically impossible. In these latter cases it is a simple matter to ask the computer to do the inspection internally. From the computer's report one can make new conjectures which may have some interest and whose proof may not be too difficult. Again, in publishing our results we need not give credit where some credit is due. [Lehmer 1966, p. 746]

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